CYCLOTOMIC GAUDIN MODELS WITH IRREGULAR SINGULARITIES

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Abstract. Generalizing the construction of the cyclotomic Gaudin algebra from [VY16a], we define the *universal cyclotomic Gaudin algebra*. It is a cyclotomic generalization of the Gaudin models with irregular singularities defined in [FFT10].

We go on to solve, by Bethe ansatz, the special case in which the Lax matrix has simple poles at the origin and arbitrarily many finite points, and a double pole at infinity.

Contents

Keywords: Quantum integrable model; cyclotomic Gaudin model; irregular singularities; coinvariants; Bethe ansatz

1. Introduction

Pick a primitive Tth root of unity $\omega \in \mathbb{C}^\times$, for some non-negative integer T and let $\Gamma := \langle \omega \rangle \subset \mathbb{C}^\times$ denote a copy of the cyclic group $\mathbb{Z}/T\mathbb{Z}$. Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra and $\sigma : \mathfrak{g} \to \mathfrak{g}$ an automorphism whose order divides T.

Associated to these data is a *cyclotomic Gaudin algebra* [VY16a]. It is a large commutative subalgebra of $U(\mathfrak{g}^{\oplus N})^{\mathfrak{g}^\sigma}$, depending on a choice of non-zero marked points $\boldsymbol{z} = \{z_1, \ldots, z_N\}$ in the complex plane whose Γ-orbits are pairwise disjoint. It is generated by a hierarchy of Hamiltonians, among which are quadratic Hamiltonians $\mathcal{H}_1, \ldots, \mathcal{H}_N$ that have appeared previously in [Skr06, Skr13] – see also [CY07] – and, in the context of cyclotomic KZ equations, in [Bro10]. It defines a quantum integrable model generalizing the quantum Gaudin model [Gau76], to which it reduces in the special case $T = 1$.

The cyclotomic Gaudin algebra was constructed in [VY16a] using the technology of coinvariants/conformal blocks of $\widehat{\mathfrak{g}}$-modules of critical level, following [FFR94]. The relevant coinvariants in this case are Γ-equivariant; see [VY16b].

Now, in fact, this approach using coinvariants naturally gives commutative subalgebras not just of $U(\mathfrak{g}^{\oplus N})$ but of the larger algebra $U\left(\bigoplus_{i=1}^N \mathfrak{g}[[t-z_i]]\right)$, where $\mathfrak{g}[[t-z]] \cong \mathfrak{g}[[t]]$ is the half loop algebra. Moreover, in the cyclotomic setting it is natural to include also 0 and ∞ as marked points. These are the fixed points of the action of Γ, and to them one attaches twisted half loop algebras, respectively $\mathfrak{g}[[t]]^\Gamma$ and $(t^{-1}\mathfrak{g}^{\mathsf{op}}[[t^{-1}]])^\Gamma$ (see §2.2). The first main result of the present sequel to [VY16a] is thus to construct, in §2, a large commutative subalgebra

$$\mathscr{Z}_{\infty,\boldsymbol{z},0}(\mathfrak{g},\sigma)^\Gamma \subset U\left((t^{-1}\mathfrak{g}^{\mathsf{op}}[[t^{-1}]])^\Gamma \oplus \bigoplus_{i=1}^N \mathfrak{g}[[t-z_i]] \oplus (\mathfrak{g}[[t]])^\Gamma\right)^{\mathfrak{g}^\sigma}.$$

It is the cyclotomic generalization of the *universal Gaudin algebra* defined in [FFT10].

Quotients of half loop algebras of the form $\mathfrak{g}[[t]]/t^n\mathfrak{g}[[t]]$ are called *(generalized) Takiff* algebras. Taking such quotients of $\mathscr{Z}_{\infty,\boldsymbol{z},0}(\mathfrak{g},\sigma)^\Gamma$ one obtains commutative algebras $\mathscr{Z}^{n_\infty,\boldsymbol{n},n_0}_{\infty,\boldsymbol{z},0}(\mathfrak{g},\sigma)^\Gamma$.[1] In particular, one recovers the cyclotomic Gaudin algebra of [VY16a] as the special case

$$\mathscr{Z}^{1,(1),0}_{\infty,\boldsymbol{z},0}(\mathfrak{g},\sigma)^\Gamma \subset U(0 \oplus \mathfrak{g}^{\oplus N} \oplus 0)^{\mathfrak{g}^\sigma}.$$

More generally, following [FFT10] it is natural to call the integrable models defined by representing $\mathscr{Z}^{n_\infty,\boldsymbol{n},n_0}_{\infty,\boldsymbol{z},0}(\mathfrak{g},\sigma)^\Gamma$ on tensor products of modules over Takiff algebras, cyclotomic Gaudin models *with irregular singularities*. Among the simplest possibilities is to introduce one irregular singularity, as mild as possible, at ∞; that is, to consider

$$\mathscr{Z}^{2,(1),1}_{\infty,\boldsymbol{z},0}(\mathfrak{g},\sigma)^\Gamma \subset U\left((\Pi_{-1}\mathfrak{g}) \oplus \mathfrak{g}^{\oplus N} \oplus \mathfrak{g}^\sigma\right)^{\mathfrak{g}^\sigma}.$$

Here $\Pi_{-1}\mathfrak{g} \subset \mathfrak{g}$ denotes the ω^{-1}-eigenspace of σ. It is to be regarded here as a commutative Lie algebra: it arises as the quotient $\Pi_{-1}\mathfrak{g} \cong_{\mathbb{C}} (t^{-1}\mathfrak{g}[[t^{-1}]])^\Gamma / (t^{-2}\mathfrak{g}[[t^{-1}]])^\Gamma$. Suppose we now pick a one-dimensional representation of this commutative Lie algebra *i.e.* a linear map $\chi : \Pi_{-1}\mathfrak{g} \to \mathbb{C}$. Applying this map we obtain a commutative subalgebra

$$\mathcal{A}^{1,(1),1}_{\infty,\boldsymbol{z},0}(\mathfrak{g},\sigma,\chi)^\Gamma \subset \left(U(\mathfrak{g})^{\otimes N} \otimes U(\mathfrak{g}^\sigma)\right)^{\mathfrak{g}^\sigma_\chi},$$

where $\mathfrak{g}^\sigma_\chi$ denotes the centraliser of χ under the coadjoint action of $\mathfrak{g}^\sigma$ on $(\Pi_{-1}\mathfrak{g})^*$. In the special case of $N=0$ (*i.e.* only one marked point, at the origin) this is a cyclotomic generalization of the *quantum shift-of-argument* subalgebra of [Ryb06]; see also [FFRb10]. The latter is a quantisation of the *shift-of-argument* subalgebra $\mathcal{A}_\chi$ [MF78], the Poisson commutative subalgebra of $S(\mathfrak{g}) \simeq P(\mathfrak{g}^*)$ generated by all derivatives of every element of $S(\mathfrak{g})^{\mathfrak{g}}$ in the direction of some fixed $\chi \in \mathfrak{g}^*$. The quantum shift-of-argument subalgebra has important connections to $\mathfrak{g}$-crystals, certain limits of $U_q(\mathfrak{g})$-modules, and cactus group actions on these, see *e.g.* [Ryb16]. It would be interesting to investigate these various connections in the cyclotomic setting. (Note also that quantum KdV theory is closely related to an affine shift-of-argument subalgebra [FF07].)

In the remainder of the paper we go on to diagonalize the Hamiltonians generating $\mathcal{A}^{1,(1),1}_{\infty,\boldsymbol{z},0}(\mathfrak{g},\sigma,\chi)^\Gamma$ on tensor products of Verma modules, by means of a Bethe ansatz. We assume that χ, and the highest weights $\lambda_1,\ldots,\lambda_N,\lambda_0$ of these Verma modules, all belong to the dual of a single Cartan subalgebra, and that this Cartan subalgebra is stable under σ. Under these assumptions one can apply

[1] Actually, we define these $\mathscr{Z}^{n_\infty,\boldsymbol{n},n_0}_{\infty,\boldsymbol{z},0}(\mathfrak{g},\sigma)^\Gamma$ first, and then the universal algebra $\mathscr{Z}_{\infty,\boldsymbol{z},0}(\mathfrak{g},\sigma)^\Gamma$ is their inverse limit.

the approach to the Bethe ansatz for Gaudin models from [FFR94, Fre05], which uses coinvariants of a particular class of $\widehat{\mathfrak{g}}$-modules at critical level called *Wakimoto modules*. See §3, Theorem 3.2, for the precise statement of the result.

Finally, in the special case $\chi = 0$ we prove that the Bethe vectors are singular. See Theorem 3.4.

Let us conclude this introduction with some remarks and open questions.

As discussed in [FFT10], the origin of the term *irregular singularities* comes from the description of the spectrum of Gaudin algebras in terms of *opers*. The notion of opers with regular singularities was recently extended to the cyclotomic setting in [LV06], and it was conjectured that the spectrum of the cyclotomic Gaudin algebra $\mathscr{Z}^{1,(1),0}_{\infty,\boldsymbol{z},0}(\mathfrak{g},\sigma)^\Gamma$ admits a description in terms of such cyclotomic opers, or Γ-equivariant opers. It would be interesting to extend the definition of cyclotomic opers to include the case of irregular singularities and relate these to the spectrum of $\mathscr{Z}^{n_\infty,\boldsymbol{n},n_0}_{\infty,\boldsymbol{z},0}(\mathfrak{g},\sigma)^\Gamma$ defined in the present paper.

The quadratic Hamiltonians of the algebra $\mathcal{A}^{1,(1),1}_{\infty,\boldsymbol{z},0}(\mathfrak{g},\sigma,\chi)^\Gamma$ are the cyclotomic analogs of the Gaudin models considered in [FMTV00], which exhibit a certain bispectrality property. It would be interesting to investigate bispectrality in the cyclotomic setting, in the spirit of that paper.

Cyclotomic analogs of the KP hierarchy were defined recently in [CS]. This construction involves a generalization of (the completion of) Calogero-Moser phase space, which can be seen as a quiver variety whose underlying quiver has a single loop, to quiver varieties for cyclic quivers. Calogero-Moser space is known to be related to Gaudin algebras (Bethe algebras) [MTV14], so it is natural to hope for a similar relation in the cyclotomic setting.

2. The cyclotomic Gaudin model

2.1. **Rational functions and formal series.** We work over $\mathbb{C}$. For any formal variable t, we have the ring of polynomials $\mathbb{C}[t]$, the ring of formal power series $\mathbb{C}[[t]]$, and the field of formal Laurent series $\mathbb{C}((t))$. Given a finite collection of points $\boldsymbol{x} = \{x_1, \dots, x_p\} \subset \mathbb{C}$ let $\mathbb{C}_{\infty,\boldsymbol{x}}(t)$ denote the localization of $\mathbb{C}[t]$ by the multiplicative subset generated by $t - x_1, \dots, t - x_p$. Elements of $\mathbb{C}_{\infty,\boldsymbol{x}}(t)$ are rational functions in t with poles at most at the points $x_1, \dots, x_p$ and at infinity.

For any $z \in \mathbb{C}$ we have the map $\iota_{t-z} : \mathbb{C}_{\infty,\boldsymbol{x}}(t) \to \mathbb{C}((t-z))$ which returns the Laurent expansion $\iota_{t-z} f(t)$ of a rational function $f(t)$ about $t = z$. We have also $\iota_{t^{-1}} : \mathbb{C}_{\infty,\boldsymbol{x}}(t) \to \mathbb{C}((t^{-1}))$ which returns the Laurent expansion $\iota_{t^{-1}} f(t)$ of $f(t)$ in powers of t^{-1}. The maps ι_{t-z} and $\iota_{t^{-1}}$ are both injective homomorphisms of $\mathbb{C}$-algebras.

Let $\mathrm{res}_t : \mathbb{C}((t)) \to \mathbb{C}$ be the map which returns the coefficient of t^{-1}. For any $f(t) \in \mathbb{C}_{\infty,\boldsymbol{x}}(t)$ we have

$$-\mathrm{res}_{t^{-1}}\, t^2 \iota_{t^{-1}} f(t) + \sum_{i=1}^{p} \mathrm{res}_{t-x_i}\, \iota_{t-x_i} f(t) = 0. \tag{2.1}$$

(This is equivalent to the statement that the sum of the residues of a meromorphic one-form $f(t)dt$ on $\mathbb{C}P^1$ vanishes.)

2.2. **Opposite Lie algebras and left vs. right modules.** Given a complex Lie algebra $\mathfrak{a}$ with Lie product $[\cdot,\cdot] : \mathfrak{a} \otimes \mathfrak{a} \to \mathfrak{a}$ we write $\mathfrak{a}^{\mathsf{op}}$ for the opposite Lie algebra, namely the vector space $\mathfrak{a}$ endowed with the Lie product $[X, Y]^{\mathsf{op}} := [Y, X]$. The Lie algebras $\mathfrak{a}$ and $\mathfrak{a}^{\mathsf{op}}$ are isomorphic (by e.g. $X \mapsto -X$) but it will be useful to regard them as two distinct Lie algebra structures on the same underlying vector space. Modules over $\mathfrak{a}$ are naturally identified with left modules over the envelope $U(\mathfrak{a})$; modules over $\mathfrak{a}^{\mathsf{op}}$ are naturally identified with right modules over the envelope $U(\mathfrak{a})$.

2.3. **Marked points and the group Γ.** Let ω be a root of unity of order $T \in \mathbb{Z}_{\geq 1}$. The cyclic group $\Gamma := \langle \omega \rangle \cong \mathbb{Z}/T\mathbb{Z}$ acts on the Riemann sphere $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ by multiplication. The fixed points of this action are 0 and ∞. Pick $N \in \mathbb{Z}_{\geq 0}$ points $z_1, \ldots, z_N \in \mathbb{C}P^1 \setminus \{0, \infty\}$ whose Γ-orbits are disjoint: $\Gamma z_i \cap \Gamma z_j = \emptyset$ whenever $i \neq j$. We write $\boldsymbol{z} = \{z_1, \ldots, z_N\}$.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$ and $\sigma : \mathfrak{g} \to \mathfrak{g}$ an automorphism of $\mathfrak{g}$ whose order divides T. Let $\langle \cdot, \cdot \rangle$ denote the Killing form on $\mathfrak{g}$ normalised such that long roots have square length 2. This is σ-invariant since for any pair of elements $X, Y \in \mathfrak{g}$ we have $\mathrm{tr}\big(\mathrm{ad}(\sigma X) \circ \mathrm{ad}(\sigma Y)\big) = \mathrm{tr}\big(\sigma \circ \mathrm{ad} X \circ \mathrm{ad} Y \circ \sigma^{-1}\big) = \mathrm{tr}\big(\mathrm{ad} X \circ \mathrm{ad} Y\big)$ using the fact that $\mathrm{ad}(\sigma X) = \sigma \circ \mathrm{ad} X \circ \sigma^{-1}$. Let Π_k, $k \in \mathbb{Z}/T\mathbb{Z}$, be the projectors

$$\Pi_k := \frac{1}{T} \sum_{m=0}^{T-1} \omega^{-mk} \sigma^m : \mathfrak{g} \to \mathfrak{g} \tag{2.2}$$

onto the eigenspaces of σ. They obey $\sum_{k \in \mathbb{Z}/T\mathbb{Z}} \Pi_k = \mathrm{id}$. We write $\mathfrak{g}^\sigma$ for the subalgebra of invariants,

$$\mathfrak{g}^\sigma := \Pi_0 \mathfrak{g}.$$

Denote by $\mathfrak{g}^{\Gamma,k}_{\infty,\boldsymbol{z},0}$, $k \in \mathbb{Z}/T\mathbb{Z}$, the Lie algebra of those $\mathfrak{g}$-valued rational functions $f(t)$ of a formal variable t that have no poles outside the set of points $\{0, \infty\} \cup \Gamma \boldsymbol{z}$ and that obey the equivariance condition $\omega^* f = \omega^k \sigma f$, *i.e.*

$$f(\omega t) = \omega^k \sigma f(t).$$

Let also $(\mathfrak{g} \otimes \mathbb{C}((t^{\pm 1})))^{\Gamma,k} := \{f(t^{\pm 1}) \in \mathfrak{g} \otimes \mathbb{C}((t^{\pm 1})) : f(\omega^{\pm 1} t^{\pm 1}) = \omega^k \sigma f(t^{\pm 1})\}$. For brevity we write $\mathfrak{g}^{\Gamma}_{\infty,\boldsymbol{z},0} := \mathfrak{g}^{\Gamma,0}_{\infty,\boldsymbol{z},0}$ and $(\mathfrak{g} \otimes \mathbb{C}((t^{\pm 1})))^{\Gamma} := (\mathfrak{g} \otimes \mathbb{C}((t^{\pm 1})))^{\Gamma,0}$, etc.

There is an injective homomorphism of Lie algebras

$$\mathfrak{g} \otimes \mathbb{C}_{\infty,\boldsymbol{z},0}(t) \longrightarrow \mathfrak{g}^{\mathsf{op}} \otimes \mathbb{C}((t^{-1})) \oplus \bigoplus_{i=1}^N \mathfrak{g} \otimes \mathbb{C}((t - z_i)) \oplus \mathfrak{g} \otimes \mathbb{C}((t))$$

defined by

$$f(t) \longmapsto (-\iota_{t^{-1}} f(t); \iota_{t-z_1} f(t), \ldots, \iota_{t-z_N} f(t); \iota_t(t))$$

(note the op and minus sign in our conventions).

Lemma 2.1 (Γ-equivariant Strong residue theorem)**.** *A tuple of formal series*

$$(f_\infty; f_{z_1}, \ldots, f_{z_N}; f_0) \in (\mathfrak{g}^{\mathsf{op}} \otimes \mathbb{C}((t^{-1})))^{\Gamma,k} \oplus \bigoplus_{i=1}^N \mathfrak{g} \otimes \mathbb{C}((t - z_i)) \oplus (\mathfrak{g} \otimes \mathbb{C}((t)))^{\Gamma,k}$$

belongs to $(-\iota_{t^{-1}}; \iota_{t-z_1}, \ldots, \iota_{t-z_N}; \iota_t)(\mathfrak{g}^{\Gamma,k}_{\infty,\boldsymbol{z},0})$, *i.e. they are the Laurent expansions of some rational function in* $\mathfrak{g}^{\Gamma,k}_{\infty,\boldsymbol{z},0}$, *if and only if*

$$-\frac{1}{T} \mathrm{res}_{t^{-1}}\, t^2 \big\langle f_\infty, -\iota_{t^{-1}} g(t) \big\rangle + \sum_{i=1}^N \mathrm{res}_{t-z_i} \big\langle f_{z_i}, \iota_{t-z_i} g(t) \big\rangle + \frac{1}{T} \mathrm{res}_t \big\langle f_0, \iota_t(g) \big\rangle = 0 \tag{2.3}$$

for all $g(t) \in \mathfrak{g}^{\Gamma,-k-1}_{\infty,\boldsymbol{z},0}$.

Proof. The proof is as in [VY16a, Lemma A.1], but including the poles at ∞. Compare (2.1). □

Let $\widehat{\mathfrak{g}}_{z_i}$ denote the extension of $\mathfrak{g} \otimes \mathbb{C}((t-z_i))$ by a one-dimensional centre $\mathbb{C}K_{z_i}$, defined by the cocycle

$$\Omega_{z_i}(f_{z_i}, g_{z_i}) := \operatorname{res}_{t-z_i} \langle f_{z_i}, \partial_{t-z_i} g_{z_i} \rangle K_{z_i}, \qquad f_{z_i}, g_{z_i} \in \mathfrak{g} \otimes \mathbb{C}((t-z_i)). \tag{2.4}$$

Thus, each $\widehat{\mathfrak{g}}_{z_i}$, $i = 1, \ldots, N$, is a copy of the (untwisted) affine Lie algebra $\widehat{\mathfrak{g}}$.

Let $\widehat{\mathfrak{g}}_0^\Gamma$ denote the extension of $(\mathfrak{g} \otimes \mathbb{C}((t)))^\Gamma$ by a one-dimensional centre $\mathbb{C}K_0$, defined by the cocycle

$$\Omega_0(f_0, g_0) := \operatorname{res}_t \langle f_0, \partial_t g_0 \rangle K_0, \qquad f_0, g_0 \in (\mathfrak{g} \otimes \mathbb{C}((t)))^\Gamma. \tag{2.5}$$

Let $\widehat{\mathfrak{g}}_\infty^{\mathsf{op},\Gamma}$ denote the extension of $(\mathfrak{g}^{\mathsf{op}} \otimes \mathbb{C}((t^{-1})))^\Gamma$ by a one-dimensional centre $\mathbb{C}K_\infty$, defined by the cocycle

$$\begin{aligned}\Omega_\infty(f_\infty, g_\infty) &:= \operatorname{res}_{t^{-1}} \langle f_\infty, \partial_{t^{-1}} g_\infty \rangle K_\infty \\ &= -\operatorname{res}_{t^{-1}} t^2 \langle f_\infty, \partial_t g_\infty \rangle K_\infty, \qquad f_\infty, g_\infty \in (\mathfrak{g}^{\mathsf{op}} \otimes \mathbb{C}((t^{-1})))^\Gamma.\end{aligned} \tag{2.6}$$

Given any $X \in \mathfrak{g}$, $n \in \mathbb{Z}$ we introduce the notations

$$X[n]_{z_i} := X \otimes (t-z_i)^n \in \widehat{\mathfrak{g}}_{z_i}, \qquad X[n]_0 := X \otimes t^n \in \widehat{\mathfrak{g}}_0, \qquad X[n]_\infty := X \otimes t^n \in \widehat{\mathfrak{g}}_\infty.$$

Note in particular our conventions for the n^{th}-modes at ∞.

The algebras $\widehat{\mathfrak{g}}_0^\Gamma$ and $\widehat{\mathfrak{g}}_\infty^\Gamma$ are both copies of an algebra $\widehat{\mathfrak{g}}^\Gamma$ which is either a twisted affine Lie algebra (if σ is an outer automorphism) or else isomorphic to $\widehat{\mathfrak{g}}$ (if σ is an inner automorphism).

Let $\widehat{\mathfrak{g}}_{\infty,N,0}$ denote the extension of $(\mathfrak{g}^{\mathsf{op}} \otimes \mathbb{C}((t^{-1})))^\Gamma \oplus \bigoplus_{i=1}^N \mathfrak{g} \otimes \mathbb{C}((t-z_i)) \oplus (\mathfrak{g} \otimes \mathbb{C}((t)))^\Gamma$, by a one-dimensional centre $\mathbb{C}K$, defined by the cocycle

$$\Omega(f,g) := \left(-\frac{1}{T} \operatorname{res}_{t^{-1}} t^2 \langle f_\infty, \partial_t g_\infty \rangle + \sum_{i=1}^N \operatorname{res}_{t-z_i} \langle f_{z_i}, \partial_t g_{z_i} \rangle + \frac{1}{T} \operatorname{res}_t \langle f_0, \partial_t g_0 \rangle \right) K, \tag{2.7}$$

where $f = (f_\infty; f_{z_1}, \ldots, f_{z_N}; f_0)$ and $g = (g_\infty; g_{z_1}, \ldots, g_{z_N}; g_0)$ are in $(\mathfrak{g}^{\mathsf{op}} \otimes \mathbb{C}((t^{-1})))^\Gamma \oplus \bigoplus_{i=1}^N \mathfrak{g} \otimes \mathbb{C}((t-z_i)) \oplus (\mathfrak{g} \otimes \mathbb{C}((t)))^\Gamma$. In other words, $\widehat{\mathfrak{g}}_{\infty,N,0}$ is the quotient of the direct sum $\widehat{\mathfrak{g}}_\infty^{\mathsf{op},\Gamma} \oplus \bigoplus_{i=1}^N \widehat{\mathfrak{g}}_{z_i} \oplus \widehat{\mathfrak{g}}_0^\Gamma$ by the ideal spanned by $K_{z_i} - TK_0$, $i = 1, \ldots, N$, and $K_\infty - K_0$, leaving one central generator, say K_{z_1}, which we call K.

We have an embedding of Lie algebras

$$(-\iota_{t^{-1}}; \iota_{t-z_1}, \ldots, \iota_{t-z_N}; \iota_t) : \mathfrak{g}^\Gamma_{\infty,\boldsymbol{z},0} \longrightarrow (\mathfrak{g}^{\mathsf{op}} \otimes \mathbb{C}((t^{-1})))^\Gamma \oplus \bigoplus_{i=1}^N \mathfrak{g} \otimes \mathbb{C}((t-z_i)) \oplus (\mathfrak{g} \otimes \mathbb{C}((t)))^\Gamma.$$

By Lemma 2.1 the restriction of the cocycle Ω to the image of $\mathfrak{g}^\Gamma_{\infty,\boldsymbol{z},0}$ under this embedding vanishes. Therefore the embedding lifts to an embedding

$$\mathfrak{g}^\Gamma_{\infty,\boldsymbol{z},0} \hookrightarrow \widehat{\mathfrak{g}}_{\infty,N,0}. \tag{2.8}$$

2.4. **Induced $\widehat{\mathfrak{g}}_{\infty,N,0}$-modules.** Let $\mathcal{M}_{z_i}$ be a module over $\mathfrak{g} \otimes \mathbb{C}[[t-z_i]]$, for each $i = 1, \ldots, N$. We then make it into a module over $\mathfrak{g} \otimes \mathbb{C}[[t-z_i]] \oplus \mathbb{C}K_{z_i}$ by declaring that K_{z_i} acts by multiplication by $k \in \mathbb{C}$. Then we have the induced left $U(\widehat{\mathfrak{g}}_{z_i})$-module,

$$\mathbb{M}^k_{z_i} := U(\widehat{\mathfrak{g}}_{z_i}) \otimes_{U(\mathfrak{g} \otimes \mathbb{C}[[t-z_i]] \oplus \mathbb{C}K_{z_i})} \mathcal{M}_{z_i}. \tag{2.9}$$

Let $\mathcal{M}_0$ be a module over $(\mathfrak{g} \otimes \mathbb{C}[[t]])^\Gamma$. We make it into a module over $(\mathfrak{g} \otimes \mathbb{C}[[t]])^\Gamma \oplus \mathbb{C}K_0$ by declaring that K_0 acts by multiplication by $k/T \in \mathbb{C}$. Then we have the induced left $U(\widehat{\mathfrak{g}}_0^\Gamma)$-module,

$$\mathbb{M}_0^{k/T} := U(\widehat{\mathfrak{g}}_0^\Gamma) \otimes_{U((\mathfrak{g}\otimes\mathbb{C}[[t]])^\Gamma \oplus \mathbb{C}K_0)} \mathcal{M}_0. \tag{2.10}$$

Let $\mathcal{M}_\infty$ be a module over $(\mathfrak{g}^{\mathsf{op}} \otimes t^{-1}\mathbb{C}[[t^{-1}]])^\Gamma$. We make it into a module over $(\mathfrak{g}^{\mathsf{op}} \otimes t^{-1}\mathbb{C}[[t^{-1}]])^\Gamma \oplus \mathbb{C}K_\infty$ by declaring that K_∞ acts by multiplication by $k/T \in \mathbb{C}$. We have the induced left $U(\widehat{\mathfrak{g}}_\infty^{\mathsf{op},\Gamma})$-module,

$$\mathbb{M}_\infty^{k/T} := U(\widehat{\mathfrak{g}}_\infty^{\mathsf{op},\Gamma}) \otimes_{U((\mathfrak{g}^{\mathsf{op}}\otimes t^{-1}\mathbb{C}[[t^{-1}]])^\Gamma \oplus \mathbb{C}K_\infty)} \mathcal{M}_\infty \tag{2.11}$$

Let us write

$$\mathcal{M} := \mathcal{M}_\infty \otimes \bigotimes_{i=1}^N \mathcal{M}_{z_i} \otimes \mathcal{M}_0 \tag{2.12}$$

and

$$\widehat{\mathfrak{g}}^{++}_{\infty,N,0} := (\mathfrak{g}^{\mathsf{op}} \otimes t^{-1}\mathbb{C}[[t^{-1}]])^\Gamma \oplus \bigoplus_{i=1}^N \mathfrak{g} \otimes \mathbb{C}[[t - z_i]] \oplus (\mathfrak{g} \otimes \mathbb{C}[[t]])^\Gamma \oplus \mathbb{C}K. \tag{2.13}$$

Then the tensor product

$$\mathbb{M} := \mathbb{M}_\infty^{k/T} \otimes \bigotimes_{i=1}^N \mathbb{M}_{z_i}^k \otimes \mathbb{M}_0^{k/T} = U(\widehat{\mathfrak{g}}_{\infty,N,0}) \otimes_{U(\widehat{\mathfrak{g}}^{++}_{\infty,N,0})} \mathcal{M} \tag{2.14}$$

is a module over $\widehat{\mathfrak{g}}_{\infty,N,0}$ on which K acts as k. Pulling back by the embedding (2.8), we have that $\mathbb{M}$ becomes a module over $\mathfrak{g}^\Gamma_{\infty,\boldsymbol{z},0}$ and we can form the space of *coinvariants*,

$$\mathbb{M}/\mathfrak{g}^\Gamma_{\infty,\boldsymbol{z},0} := \mathbb{M}/(\mathfrak{g}^\Gamma_{\infty,\boldsymbol{z},0}.\mathbb{M}).$$

Proposition 2.2. *The Lie algebras $\widehat{\mathfrak{g}}^{++}_{\infty,N,0}$ and $\mathfrak{g}^\Gamma_{\infty,\boldsymbol{z},0}$ embed as a pair of complementary Lie subalgebras in $\widehat{\mathfrak{g}}_{\infty,N,0}$, i.e.*

$$\widehat{\mathfrak{g}}_{\infty,N,0} = \widehat{\mathfrak{g}}^{++}_{\infty,N,0} \dotplus \mathfrak{g}^\Gamma_{\infty,\boldsymbol{z},0}$$

as vector spaces. Therefore there is a canonical isomorphism of vector spaces

$$\mathbb{M}/\mathfrak{g}^\Gamma_{\infty,\boldsymbol{z},0} \cong_{\mathbb{C}} \mathcal{M}. \tag{2.15}$$

Proof. As in [VY16a], Lemma 2.1 and Corollary 2.4. □

2.5. **Vacuum verma module $\mathbb{V}_0^k$.** Now let $u \in \mathbb{C}^\times$ be an additional non-zero marked point, whose orbit Γu is disjoint from $\Gamma\boldsymbol{z}$. Then we have the algebras $\mathfrak{g}^\Gamma_{\infty,\boldsymbol{z},u,0}$, $\widehat{\mathfrak{g}}_{\infty,N+1,0}$, etc. defined as above but with the point u included. To the point u we assign a copy of the vacuum Verma module $\mathbb{V}_0^k$ over the local copy $\widehat{\mathfrak{g}}_u$ of the affine Lie algebra $\widehat{\mathfrak{g}}$. Recall that by definition $\mathbb{V}_0^k$ is the induced module

$$\mathbb{V}_0^k := U(\widehat{\mathfrak{g}}_u) \otimes_{U(\mathfrak{g}\otimes\mathbb{C}[[t-u]]\oplus\mathbb{C}K_u)} \mathbb{C}v_0. \tag{2.16}$$

Here $\mathbb{C}v_0$ denotes the one-dimensional module over $\mathfrak{g} \otimes \mathbb{C}[[t-u]] \oplus \mathbb{C}K_u$ on which $\mathfrak{g} \otimes \mathbb{C}[[t-u]]$ acts trivially and K_u acts by multiplication by $k \in \mathbb{C}$.

A vector $X \in \mathbb{V}_0^k$ is *singular* if $A.X = 0$ for all $A \in \mathfrak{g} \otimes \mathbb{C}[t]$. The singular vectors form a linear subspace of $\mathbb{V}_0^k$ denoted $\mathfrak{z}(\widehat{\mathfrak{g}})$.

Proposition 2.3. *There is a canonical isomorphism of vector spaces*

$$(\mathbb{M} \otimes \mathbb{V}_0^k)\big/\mathfrak{g}^{\Gamma}_{\infty,\boldsymbol{z},u,0} \cong_{\mathbb{C}} \mathcal{M} \otimes \mathbb{C}v_0 \cong_{\mathbb{C}} \mathcal{M}. \qquad \square$$

It follows that, given any $X \in \mathbb{V}_0^k$, there is a linear map $X(u) : \mathcal{M} \to \mathcal{M}$ defined by

$$\mathcal{M} \hookrightarrow \mathbb{M} \xrightarrow{\cdot \otimes X} \mathbb{M} \otimes \mathbb{V}_0^k \twoheadrightarrow (\mathbb{M} \otimes \mathbb{V}_0^k)\big/\mathfrak{g}^{\Gamma}_{\infty,\boldsymbol{z},u,0} \xrightarrow{\sim} \mathcal{M} \tag{2.17}$$

where $\mathcal{M} \hookrightarrow \mathbb{M}$ is the natural embedding. The map $X(u)$ depends rationally on u, with poles at most at the points 0, $\omega^k z_i$ ($1 \le i \le N$ and $k \in \mathbb{Z}/T\mathbb{Z}$) and ∞.

2.6. **Generalized Takiff algebras.** For any $n \in \mathbb{Z}_{\geq 1}$ there is an ideal $\mathfrak{g} \otimes t^n \mathbb{C}[t] \subset \mathfrak{g} \otimes \mathbb{C}[t]$. Define the Lie algebra $\mathcal{T}_n\mathfrak{g}$ to be the quotient

$$\begin{aligned} \mathcal{T}_n\mathfrak{g} &:= (\mathfrak{g} \otimes \mathbb{C}[t]) \big/ (\mathfrak{g} \otimes t^n \mathbb{C}[t]) \\ &\cong_{\mathbb{C}} \mathfrak{g} \oplus t\mathfrak{g} \cdots \oplus t^{n-1}\mathfrak{g}. \end{aligned}$$

Thus $\mathcal{T}_1\mathfrak{g} = \mathfrak{g}$. The Lie algebra $\mathcal{T}_n\mathfrak{g}$ is known as a *(generalized) Takiff algebra.*

The Lie algebras $\mathcal{T}_n\mathfrak{g}$ together with the canonical projections $\mathcal{T}_n\mathfrak{g} \twoheadrightarrow \mathcal{T}_m\mathfrak{g}$, $n > m$, form an inverse system, and $\mathfrak{g} \otimes \mathbb{C}[[t]]$ is the inverse limit $\varprojlim \mathcal{T}_n\mathfrak{g}$.

Define also the *twisted Takiff algebra* $\mathcal{T}_n\mathfrak{g}^{\Gamma}$:

$$\begin{aligned} \mathcal{T}_n\mathfrak{g}^{\Gamma} &:= (\mathfrak{g} \otimes \mathbb{C}[t])^{\Gamma} \big/ (\mathfrak{g} \otimes t^n \mathbb{C}[t])^{\Gamma} \\ &\cong_{\mathbb{C}} \mathfrak{g}^{\sigma} \oplus t\Pi_1\mathfrak{g} \cdots \oplus t^{n-1}\Pi_{n-1}\mathfrak{g}. \end{aligned}$$

In particular $\mathcal{T}_1\mathfrak{g}^{\Gamma} = \mathfrak{g}^{\sigma}$.

We use the notation $X_{\underline{p}}$ for the class of the element $t^p X = X \otimes t^p$ in $\mathcal{T}_n\mathfrak{g}$.

For any $z \in \mathbb{C}$ we have the naive isomorphism $\mathcal{T}_n\mathfrak{g} \cong (\mathfrak{g} \otimes \mathbb{C}[[t-z]]) \big/ (\mathfrak{g} \otimes (t-z)^n \mathbb{C}[[t-z]])$ which sends $X_{\underline{p}}$ to the class of $X \otimes (t-z)^p$. By means of this isomorphism, modules over $\mathcal{T}_n\mathfrak{g}$ pull back to modules over $\mathfrak{g} \otimes \mathbb{C}[[t-z]]$.

2.7. **Universal Cyclotomic Gaudin Algebra.** Given any $n_0, n_{z_1}, \ldots, n_{z_N}, n_\infty \in \mathbb{Z}_{\geq 1}$ we write $\boldsymbol{n} = \{n_{z_1}, \ldots, n_{z_N}\}$. Let $\mathscr{I}_{n_\infty,\boldsymbol{n},n_0} \subset U(\widehat{\mathfrak{g}}^{++}_{\infty,N,0})$ denote the two-sided ideal in $U(\widehat{\mathfrak{g}}^{++}_{\infty,N,0})$ generated by $\big(\mathfrak{g}^{\mathsf{op}} \otimes t^{-n_\infty}\mathbb{C}[[t^{-1}]]\big)^{\Gamma}$, $\mathfrak{g} \otimes (t-z_i)^{n_{z_i}}\mathbb{C}[[t-z_i]]$, $i = 1, \ldots, N$, and $(\mathfrak{g} \otimes t^{n_0}\mathbb{C}[[t]])^{\Gamma}$. Define

$$U(\widehat{\mathfrak{g}}^{++}_{\infty,N,0})_{n_\infty,\boldsymbol{n},n_0} := U(\widehat{\mathfrak{g}}^{++}_{\infty,N,0})/\mathscr{I}_{n_\infty,\boldsymbol{n},n_0}. \tag{2.18}$$

These form an inverse system whose inverse limit is $U(\widehat{\mathfrak{g}}^{++}_{\infty,N,0})$.

Let us now take the module $\mathcal{M}$ in (2.12) to be a copy of $U(\widehat{\mathfrak{g}}^{++}_{\infty,N,0})_{n_\infty,\boldsymbol{n},n_0}$, regarded as a left module over itself. For any $X \in \mathbb{V}_0^k$ we have a map $X(u) : U(\widehat{\mathfrak{g}}^{++}_{\infty,N,0})_{n_\infty,\boldsymbol{n},n_0} \to U(\widehat{\mathfrak{g}}^{++}_{\infty,N,0})_{n_\infty,\boldsymbol{n},n_0}$ as in (2.17). By construction this can be written in terms of the left action of $U(\widehat{\mathfrak{g}}^{++}_{\infty,N,0})_{n_\infty,\boldsymbol{n},n_0}$, which commutes with the right action of $U(\widehat{\mathfrak{g}}^{++}_{\infty,N,0})_{n_\infty,\boldsymbol{n},n_0}$. So $X(u)$ commutes with the right action of $U(\widehat{\mathfrak{g}}^{++}_{\infty,N,0})_{n_\infty,\boldsymbol{n},n_0}$. Hence for all $a \in U(\widehat{\mathfrak{g}}^{++}_{\infty,N,0})_{n_\infty,\boldsymbol{n},n_0}$, $X(u).a = X(u).(1a) = (X(u).1)a$. That is, $X(u)$ acts by left-multiplication by the element $X(u).1 \in U(\widehat{\mathfrak{g}}^{++}_{\infty,N,0})_{n_\infty,\boldsymbol{n},n_0}$. Since the latter depends on the choice of $n_\infty, \boldsymbol{n}, n_0$, we will denote it by $X(u)_{n_\infty,\boldsymbol{n},n_0}$. When the choice of $n_\infty, \boldsymbol{n}, n_0$ is clear from the context we will write $X(u)_{n_\infty,\boldsymbol{n},n_0}$ simply as $X(u)$. By construction, whenever

$n_0' > n_0$, $n_{z_i}' > n_{z_i}$ and $n_\infty' > n_\infty$ then

$$X(u)_{n_\infty,\boldsymbol{n},n_0} = X(u)_{n_0',\boldsymbol{n}',n_\infty'} + \mathscr{I}_{n_\infty,\boldsymbol{n},n_0}. \tag{2.19}$$

In other words, the elements $X(u)_{n_\infty,\boldsymbol{n},n_0} \in U(\widehat{\mathfrak{g}}^{++}_{\infty,N,0})_{n_\infty,\boldsymbol{n},n_0}$ are compatible with the above inverse system and hence define an element of the inverse limit $U(\widehat{\mathfrak{g}}^{++}_{\infty,N,0})$. By a slight abuse of notation we will also call this element simply $X(u) \in U(\widehat{\mathfrak{g}}^{++}_{\infty,N,0})$.

We have the natural inclusion $\mathfrak{g}^\sigma \hookrightarrow \mathfrak{g}$ and hence the "diagonal" embedding

$$\begin{aligned} &\mathfrak{g}^\sigma \longhookrightarrow \mathfrak{g}^{\sigma,\mathsf{op}} \oplus \bigoplus_{i=1}^N \mathfrak{g} \oplus \mathfrak{g}^\sigma \longhookrightarrow \widehat{\mathfrak{g}}_{\infty,N,0}, \\ &X \longmapsto (-X; X, \ldots, X; X) \longmapsto (-X[0]_\infty; X[0]_{z_1}, \ldots, X[0]_{z_N}; X[0]_0). \end{aligned} \tag{2.20}$$

Identifying $\mathfrak{g}^\sigma$ with its image under this embedding, this gives an action of $\mathfrak{g}^\sigma$ on $U(\widehat{\mathfrak{g}}_{\infty,N,0})$ by left- and right-multiplication. In particular, we can define the adjoint action of $\mathfrak{g}^\sigma$ on $U(\widehat{\mathfrak{g}}_{\infty,N,0})$. Note that the adjoint action stabilises $U(\widehat{\mathfrak{g}}^{++}_{\infty,N,0})$ but the actions by left- and right-multiplication do not, because the zero-modes at ∞ are not present in $U(\widehat{\mathfrak{g}}^{++}_{\infty,N,0})$. Let us write

$$U(\widehat{\mathfrak{g}}^{++}_{\infty,N,0})^{\mathfrak{g}^\sigma} := \{x \in U(\widehat{\mathfrak{g}}^{++}_{\infty,N,0}) : [a,x] = 0 \text{ for all } a \in \mathfrak{g}^\sigma\} \tag{2.21}$$

for the invariant subspace of the adjoint action of $\mathfrak{g}^\sigma$ on $U(\widehat{\mathfrak{g}}^{++}_{\infty,N,0})$. Define $U(\widehat{\mathfrak{g}}^{++}_{\infty,N,0})^{\mathfrak{g}^\sigma}_{n_\infty,\boldsymbol{n},n_0}$ likewise.

Now suppose X is a singular vector, $X \in \mathfrak{z}(\widehat{\mathfrak{g}}) \subset \mathbb{V}_0^k$. Then X is in particular $\mathfrak{g}^\sigma$-invariant, $a.X = 0$ for all $a \in \mathfrak{g}^\sigma$. Hence we have

$$0 = [1 \otimes a.X] = -[a.1 \otimes X] = -[X(u)a.1 \otimes v_0] = -[X(u)a.1]$$

where we "swapped using the constant rational function a", *i.e.* used $[a.(1 \otimes X)] = 0$, in the second equality and used the definition of $X(u)$ in the third. On the other hand, in the space of coinvariants $\mathbb{M}/\mathfrak{g}^\Gamma_{\infty,\boldsymbol{z},0}$ we have

$$0 = [a.(X(u).1)] = [aX(u).1].$$

Taking the difference of the two equalities above, we get

$$0 = \big[[a, X(u)].1\big] = [a, X(u)]$$

where in the last equality we can use the identification $\mathbb{M}/\mathfrak{g}^\Gamma_{\infty,\boldsymbol{z},0} \cong \mathcal{M}$. (The point is that neither $aX(u)$ nor $X(u)a$ need belong to $\mathcal{M} = U(\widehat{\mathfrak{g}}^{++}_{\infty,N,0})_{n_\infty,\boldsymbol{n},n_0}$, but the commutator $[a, X(u)]$ does, as we noted above.) This shows that if X is singular then $X(u) \in U(\widehat{\mathfrak{g}}^{++}_{\infty,N,0})^{\mathfrak{g}^\sigma}_{n_\infty,\boldsymbol{n},n_0}$.

For each $X \in \mathbb{V}_0^k$, the element $X(u)$ depends rationally on u with poles at most at $0, \omega^k z_i, \infty$, $i = 1, \ldots, N$. Define the algebra $\mathscr{Z}^{n_\infty,\boldsymbol{n},n_0}_{\infty,\boldsymbol{z},0}(\mathfrak{g},\sigma)^\Gamma$ to be the span, in $U(\widehat{\mathfrak{g}}^{++}_{\infty,N,0})^{\mathfrak{g}^\sigma}_{n_\infty,\boldsymbol{n},n_0}$, of all the coefficients of singular terms of Laurent expansions of the elements $Z(u)$ as Z varies in the space of singular vectors $\mathfrak{z}(\widehat{\mathfrak{g}}) \subset \mathbb{V}_0^{-h^\vee}$.

By virtue of (2.19), the algebras $\mathscr{Z}^{n_\infty,\boldsymbol{n},n_0}_{\infty,\boldsymbol{z},0}(\mathfrak{g},\sigma)^\Gamma$ form an inverse system. Define the *universal cyclotomic Gaudin algebra* $\mathscr{Z}_{\infty,\boldsymbol{z},0}(\mathfrak{g},\sigma)^\Gamma$ to be the inverse limit,

$$\mathscr{Z}_{\infty,\boldsymbol{z},0}(\mathfrak{g},\sigma)^\Gamma := \varprojlim \mathscr{Z}^{n_\infty,\boldsymbol{n},n_0}_{\infty,\boldsymbol{z},0}(\mathfrak{g},\sigma)^\Gamma.$$

By the argument in [VY16a], following [FFR94], we have

Theorem 2.4. *Each* $\mathscr{Z}^{n_\infty,\boldsymbol{n},n_0}_{\infty,\boldsymbol{z},0}(\mathfrak{g},\sigma)^\Gamma$ *is a commutative subalgebra of* $U(\widehat{\mathfrak{g}}^{++}_{\infty,N,0})^{\mathfrak{g}^\sigma}_{n_\infty,\boldsymbol{n},n_0}$. *Hence* $\mathscr{Z}_{\infty,\boldsymbol{z},0}(\mathfrak{g},\sigma)^\Gamma$ *is a commutative subalgebra of* $U(\widehat{\mathfrak{g}}^{++}_{\infty,N,0})^{\mathfrak{g}^\sigma}$. □

2.8. **Quadratic cyclotomic Hamiltonians.** Let $I_a \in \mathfrak{g}$ and $I^a \in \mathfrak{g}$, $a = 1,\dots,\dim\mathfrak{g}$, be dual bases of $\mathfrak{g}$ with respect to $\langle\cdot,\cdot\rangle$, *i.e.* $\langle I_a, I^b\rangle = \delta^b_a$. Let $\mathcal{C} := \frac{1}{2}I^aI_a \in Z(U(\mathfrak{g}))$, the quadratic Casimir of $\mathfrak{g}$. Here and below we employ summation convention on the index $a = 1,\dots,\dim\mathfrak{g}$. Define an element $F \in \mathfrak{g}^\sigma$ and number $K \in \mathbb{C}$ by

$$F := \frac{1}{2}\sum_{p=1}^{T-1}\frac{\omega^p[\sigma^pI^a, I_a]}{\omega^p - 1}, \qquad K := \frac{1}{2}\sum_{p=1}^{T-1}\frac{\omega^p\langle\sigma^pI^a, I_a\rangle k}{(\omega^p-1)^2}. \tag{2.22}$$

The *quadratic Segal-Sugawara* vector S is by definition

$$S := \frac{1}{2}I^a[-1]I_a[-1]v_0 \in \mathbb{V}^k_0 \tag{2.23}$$

At the critical level $k = -h^\vee$, the vector S is singular.

Proposition 2.5. *The corresponding element* $S(u) \in U(\widehat{\mathfrak{g}}^{++}_{\infty,N,0})^{\mathfrak{g}^\sigma}$ *is given by*

$$S(u) - \frac{1}{u^2}K = \sum_{i=1}^N\sum_{k=0}^{T-1}\sum_{p=0}^\infty \frac{\omega^{-kp+k}\mathcal{H}_{i,p}}{(u-\omega^{-k}z_i)^{p+1}} + \sum_{\substack{p=0\\ p\equiv 1 \bmod T}}^\infty \frac{\mathcal{H}_{0,p}}{u^{p+1}} + \sum_{\substack{p=0\\ p\equiv -2 \bmod T}}^\infty u^p\mathcal{H}_{\infty,p} \tag{2.24}$$

where

$$\begin{aligned}\mathcal{H}_{i,p} = &\sum_{\substack{j=1\\ j\neq i}}^N\sum_{l=0}^{T-1}\sum_{n,m=0}^\infty \frac{(-1)^n\left[{n+m \atop m}\right]}{(z_i-\omega^{-l}z_j)^{n+m+1}}I_a[n+p]_{z_i}\omega^{-lm}(\sigma^lI^a)[m]_{z_j}\\ &+\sum_{l=1}^{T-1}\sum_{r,m=0}^\infty\frac{\omega^{-lm}(-1)^r\left[{r+m \atop m}\right]}{((1-\omega^{-l})z_i)^{r+m+1}}\frac{1}{2}\Big\{I_a[r+p]_{z_i},(\sigma^lI^a)[m]_{z_i}\Big\} + \sum_{n=0}^{p-1}\frac{1}{2}I_a[n]_{z_i}I^a[p-n-1]_{z_i}\\ &+T\sum_{n,m=0}^\infty\frac{(-1)^n\left[{n+m \atop m}\right]}{z_i^{n+m+1}}I_a[n+p]_{z_i}(\Pi_mI^a)[m]_0 + \sum_{n=0}^\infty\frac{(-1)^n}{z_i^{n+1}}F[n+p]_{z_i}\\ &+T\sum_{n,m=0}^\infty z_i^n\begin{bmatrix}n+m\\ m\end{bmatrix}(\Pi_{-n-m-1}I_a)[-n-m-1]_\infty I^a[p+m]_{z_i},\end{aligned} \tag{2.25}$$

for $i = 1,\dots,N$, *and*

$$\begin{aligned}\mathcal{H}_{0,p} = T^2&\sum_{m=0}^\infty\sum_{n=0}^\infty\sum_{i=1}^N\frac{(-1)^{n+1}\left[{n+m \atop n}\right]}{z_i^{n+m+1}}(\Pi_{-m-1}I_a)[n]_{z_i}(\Pi_{m+p}I^a)[m+p]_0\\ &+T^2\sum_{n=p}^\infty(\Pi_nI_a)[n]_0(\Pi_{-n+p-1}I^a)[-n+p-1]_\infty\\ &+\frac{T^2}{2}\sum_{n=0}^{p-1}(\Pi_nI_a)[n]_0(\Pi_{p-n-1}I^a)[p-n-1]_0 + T(\Pi_{p-1}F)[p-1]_0,\end{aligned} \tag{2.26}$$

and where finally

$$\mathcal{H}_{\infty,p} = T^2 \sum_{i=1}^{N} \sum_{n=0}^{\infty} \sum_{m=p+n+1}^{\infty} z_i^{m-n-1-p} \begin{bmatrix} m-1-p \\ n \end{bmatrix} (\Pi_{-m-1} I_a)[-m-1]_\infty (\Pi_{m-p-1} I^a)[n]_{z_i}$$
$$+ T^2 \sum_{n=0}^{\infty} (\Pi_n I_a)[n]_0 (\Pi_{-p-n-2} I^a)[-p-n-2]_\infty$$
$$+ \frac{T^2}{2} \sum_{n=0}^{p} (\Pi_{-n-1} I_a)[-n-1]_\infty (\Pi_{-p+n-1} I^a)[-p+n-1]_\infty + T(\Pi_{-p-2} F)[-p-2]_\infty. \quad (2.27)$$

Proof. The proof is given in Appendix A. □

Remark 2.6. The expression (2.26) for $\mathcal{H}_{0,p}$ vanishes unless $p \equiv 1 \bmod T$, in accordance with the restriction in the second sum on the right hand side of (2.24). Indeed, the first three terms involve expressions of the form $(\Pi_r I_a)[n](\Pi_s I^a)[m]$ for some $r, s \in \mathbb{Z}_T$ and $m, n \in \mathbb{Z}_{\geq 0}$. However, using the σ-invariance of the bilinear pairing on $\mathfrak{g}$ it follows that this is equal to $\omega^{r+s}(\Pi_r I_a)[n](\Pi_s I^a)[m]$ and therefore vanishes unless $r + s \equiv 0 \bmod T$. Likewise, for the last term, $(\Pi_{p-1} F)[p-1]$ vanishes unless $p = 1$ since $F \in \mathfrak{g}^\sigma$. The expression (2.27) for $\mathcal{H}_{\infty,p}$ vanishes unless $p \equiv -2 \bmod T$ for similar reasons, in accordance with the restriction in the third sum on the right hand side of (2.24).

2.9. **Regular singularities and shift-of-argument.** In the special case when $n_{z_i} = 1$ for $i = 1, \ldots, N$ and $n_0 = 1$, we obtain commutative subalgebras

$$\mathscr{Z}^{n_\infty,(1),1}_{\infty,\boldsymbol{z},0}(\mathfrak{g},\sigma)^\Gamma \subset \Big(U\Big(\big(\mathfrak{g}^{\mathsf{op}} \otimes t^{-1}\mathbb{C}[t^{-1}]\big)^\Gamma \big/ \big(\mathfrak{g}^{\mathsf{op}} \otimes t^{-n_\infty}\mathbb{C}[t^{-1}]\big)^\Gamma \Big) \otimes U(\mathfrak{g})^{\otimes N} \otimes U(\mathfrak{g}^\sigma) \Big)^{\mathfrak{g}^\sigma}.$$

If furthermore we set $n_\infty = 1$ then we obtain the commutative subalgebra

$$\mathscr{Z}^{1,(1),1}_{\infty,\boldsymbol{z},0}(\mathfrak{g},\sigma)^\Gamma \subset (U(\mathfrak{g})^{\otimes N} \otimes U(\mathfrak{g}^\sigma))^{\mathfrak{g}^\sigma}.$$

Now consider setting $n_\infty = 2$. Pick any linear map

$$\chi : \Pi_{-1}\mathfrak{g} \to \mathbb{C}.$$

The Lie algebra $\big(\mathfrak{g}^{\mathsf{op}} \otimes t^{-1}\mathbb{C}[t^{-1}]\big)^\Gamma \big/ \big(\mathfrak{g}^{\mathsf{op}} \otimes t^{-2}\mathbb{C}[t^{-1}]\big)^\Gamma$ is commutative, and is canonically isomorphic to $\Pi_{-1}\mathfrak{g}$ as a vector space. We may therefore regard χ as an algebra homomorphism $\chi : U\big(\big(\mathfrak{g}^{\mathsf{op}} \otimes t^{-1}\mathbb{C}[t^{-1}]\big)^\Gamma \big/ \big(\mathfrak{g}^{\mathsf{op}} \otimes t^{-2}\mathbb{C}[t^{-1}]\big)^\Gamma \big) \to \mathbb{C}$. Let then

$$\mathcal{A}^{1,(1),1}_{\infty,\boldsymbol{z},0}(\mathfrak{g},\sigma,\chi)^\Gamma := (\chi \otimes \mathrm{id}^{\otimes N} \otimes \mathrm{id}) \left(\mathscr{Z}^{2,(1),1}_{\infty,\boldsymbol{z},0}(\mathfrak{g},\sigma)^\Gamma \right).$$

This defines a commutative subalgebra

$$\mathcal{A}^{1,(1),1}_{\infty,\boldsymbol{z},0}(\mathfrak{g},\sigma,\chi)^\Gamma \subset \big(U(\mathfrak{g})^{\otimes N} \otimes U(\mathfrak{g}^\sigma)\big)^{\mathfrak{g}^\sigma_\chi},$$

where $\mathfrak{g}^\sigma_\chi = \{X \in \mathfrak{g}^\sigma : \chi([X,Y]) = 0 \text{ for all } Y \in \Pi_{-1}\mathfrak{g}\}$ denotes the centralizer of χ under the coadjoint action. Note that in the special case $\chi = 0$ we recover $\mathscr{Z}^{1,(1),1}_{\infty,\boldsymbol{z},0}(\mathfrak{g},\sigma)^\Gamma$.

For any $X \in \mathfrak{g}$ and $i = 1, \ldots, N$ we let $X^{(i)}$ denote the element of $U(\mathfrak{g})^{\otimes N} \otimes U(\mathfrak{g}^\sigma)$ with X in the i^{th} tensor factor and a 1 everywhere else. Similarly, for $X \in \mathfrak{g}^\sigma$ we let $X^{(0)}$ be the element

$1^{\otimes N} \otimes X$ of $U(\mathfrak{g})^{\otimes N} \otimes U(\mathfrak{g}^\sigma)$. The only non-zero Hamiltonians of Proposition 2.5 above are then

$$\mathcal{H}_{i,0} = \sum_{\substack{j=1 \\ j\neq i}}^{N} \sum_{l=0}^{T-1} \frac{I_a^{(i)} (\sigma^l I^a)^{(j)}}{(z_i - \omega^{-l} z_j)} + \sum_{l=1}^{T-1} \frac{(\sigma^l I_a)^{(i)} I^{a(i)}}{(1-\omega^{-l}) z_i} + T \frac{1}{z_i} I_a^{(i)} (\Pi_0 I^a)^{(0)} + T I^{a(i)} \chi(\Pi_{-1} I_a),$$

$$\mathcal{H}_{i,1} = \frac{1}{2} I_a^{(i)} I^{a(i)},$$

$$\mathcal{H}_{0,0} = T^2 \sum_{i=1}^{N} \frac{(-1)}{z_i} (\Pi_{-1} I_a)^{(i)} (\Pi_0 I^a)^{(0)} + T^2 (\Pi_0 I_a)^{(0)} \chi(\Pi_{-1} I^a),$$

$$\mathcal{H}_{0,1} = \frac{T^2}{2} (\Pi_0 I_a)^{(0)} (\Pi_0 I^a)^{(0)} + T F^{(0)},$$

and

$$\mathcal{H}_{\infty,0} = \frac{T^2}{2} \chi(\Pi_{-1} I_a) \chi(\Pi_{-1} I^a).$$

Note that $\mathcal{H}_{0,0} = 0$ unless $T = 1$ and $\mathcal{H}_{\infty,0} = 0$ unless $T = 1$ or 2, cf. Remark 2.6.

Remark 2.7. The cyclotomic Gaudin algebra introduced in [VY16a] is the commutative subalgebra $\mathscr{Z}^{1,(1),0}_{\infty,\boldsymbol{z},0}(\mathfrak{g},\sigma)^\Gamma \subset (U(\mathfrak{g})^{\otimes N})^{\mathfrak{g}^\sigma}$. It is the image of $\mathscr{Z}^{1,(1),1}_{\infty,\boldsymbol{z},0}(\mathfrak{g},\sigma)^\Gamma$ under $\mathrm{id}^{\otimes N} \otimes \epsilon$, where $\epsilon : U(\mathfrak{g}^\sigma) \to \mathbb{C}$ is the counit.

Remark 2.8. The algebra $\mathcal{A}^{1,(1),1}_{\infty,\boldsymbol{z},0}(\mathfrak{g},\sigma,\chi)^\Gamma$ is a cyclotomic generalisation of the *quantum shift-of-argument subalgebra*; see [Ryb06, FFT10, FFRb10].

Remark 2.9. Sometimes setting $\chi \neq 0$ is called adding *twisted boundary conditions*. The name comes from the Heisenberg XXX spin chain of which the usual Gaudin model is a limit.

3. Statement of main results

3.1. **Cartan data and Verma modules.** We fix a Cartan decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$ of $\mathfrak{g}$. Let $\Delta^+ \subset \mathfrak{h}^*$ be the set of positive roots of $\mathfrak{g}$ and $\{\alpha_i\}_{i\in I} \subset \Delta^+$ the set of simple roots, where i runs over the set I of nodes of the Dynkin diagram of $\mathfrak{g}$. Let E_α (resp. F_α) be a root vector of weight α (resp. $-\alpha$) for each root $\alpha \in \Delta^+$, and $H_\alpha \equiv \alpha^\vee := [E_\alpha, F_\alpha]$ the corresponding coroot. Overloading notation somewhat, we write $H_i := H_{\alpha_i}$, $i \in I$. Then $\{H_i\}_{i\in I} \cup \{E_\alpha, F_\alpha\}_{\alpha\in\Delta^+}$ is a Cartan-Weyl basis of $\mathfrak{g}$. We shall assume the Cartan decomposition has been chosen to be compatible with the automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$, in the sense that $\sigma(\mathfrak{h}) = \mathfrak{h}$, $\sigma(\mathfrak{n}) = \mathfrak{n}$ and $\sigma(\mathfrak{n}^-) = \mathfrak{n}^-$. (Such a choice is always possible [Kac83].)

Let M_λ denote the *Verma module* over $\mathfrak{g}$ with highest weight $\lambda \in \mathfrak{h}^*$, namely

$$M_\lambda := U(\mathfrak{g}) \otimes_{U(\mathfrak{h}\oplus\mathfrak{n})} \mathbb{C} v_\lambda, \tag{3.1}$$

where $\mathbb{C}v_\lambda$ is the one-dimensional module over $\mathfrak{h} \oplus \mathfrak{n}$ generated by a vector v_λ with $\mathfrak{n}.v_\lambda = 0$ and $h.v_\lambda = \lambda(h) v_\lambda$ for all $h \in \mathfrak{h}$. Similarly, let M^σ_λ denote the Verma module over $\mathfrak{g}^\sigma$ with highest weight $\lambda \in \mathfrak{h}^{*,\sigma}$,

$$M^\sigma_\lambda := U(\mathfrak{g}^\sigma) \otimes_{U(\mathfrak{h}^\sigma\oplus\mathfrak{n}^\sigma)} \mathbb{C} v_\lambda, \tag{3.2}$$

3.2. **The weight function.** Let $\lambda_1, \ldots, \lambda_N \in \mathfrak{h}^*$ be $\mathfrak{g}$-weights. Let $\lambda_0 \in \mathfrak{h}^{*,\sigma}$ be a $\mathfrak{g}^\sigma$-weight.

As above, let $z_1, \ldots, z_N$ be a collection of $N \in \mathbb{Z}_{\geq 0}$ non-zero points in $\mathbb{C}$ such that $\Gamma z_i \cap \Gamma z_j = \emptyset$ for all $1 \leq i < j \leq N$. In addition, let $w_1, \ldots, w_m$ be a collection of $m \in \mathbb{Z}_{\geq 0}$ non-zero points in $\mathbb{C}$ such that $\Gamma w_i \cap \Gamma w_j = \emptyset$ for all $1 \leq i < j \leq m$ and such that $\Gamma w_i \cap \Gamma z_j = \emptyset$ for all $1 \leq i \leq m$ and $1 \leq j \leq N$. Let $c(1), \ldots, c(m)$ be elements of I. We call $c(i)$ the *colour* of the variable w_i.

We now define the *(cyclotomic) weight function* ψ associated to these data.

Recall the projectors Π_k from (2.2), and in particular Π_0. Define linear maps

$$\theta_s : \bigotimes_{i=1}^N M_{\lambda_i} \otimes M^\sigma_{\lambda_0} \otimes \underbrace{\mathfrak{n}_- \otimes \ldots \otimes \mathfrak{n}_-}_{s} \longrightarrow \bigotimes_{i=1}^N M_{\lambda_i} \otimes M^\sigma_{\lambda_0} \otimes \underbrace{\mathfrak{n}_- \otimes \ldots \otimes \mathfrak{n}_-}_{s-1}$$

for $s = 1, 2, \ldots, m$, by

$$\begin{aligned}
&\theta_s(x_1 \otimes \ldots \otimes x_N \otimes x_0 \otimes y_1 \otimes \ldots \otimes y_s) \\
&\qquad = \frac{x_1 \otimes \ldots \otimes \ldots \otimes x_N \otimes (T\Pi_0 y_s)x_0 \otimes y_1 \otimes \ldots \otimes y_{s-1})}{w_s} \\
&\qquad + \sum_{i=1}^N \sum_{j \in \mathbb{Z}_T} \frac{x_1 \otimes \ldots \otimes x_{i-1} \otimes (\sigma^j y_s)x_i \otimes x_{i+1} \otimes \ldots \otimes x_N \otimes x_0 \otimes y_1 \otimes \ldots \otimes y_{s-1})}{w_s - \omega^{-j} z_i} \\
&\qquad + \sum_{i=1}^{s-1} \sum_{j \in \mathbb{Z}_T} \frac{x_1 \otimes \ldots \otimes x_N \otimes x_0 \otimes y_1 \otimes \ldots \otimes y_{i-1} \otimes [\sigma^j y_s, y_i] \otimes y_{i+1} \otimes \ldots \otimes y_{s-1}}{w_s - \omega^{-j} w_i}
\end{aligned}$$

Then the weight function ψ is by definition the element

$$\psi := (-1)^m (\theta_1 \circ \cdots \circ \theta_m)(v_{\lambda_1} \otimes \ldots \otimes v_{\lambda_N} \otimes v_{\lambda_0} \otimes F_{\alpha_{c(1)}} \otimes F_{\alpha_{c(2)}} \otimes \ldots \otimes F_{\alpha_{c(m)}}). \tag{3.3}$$

For $\lambda \in \mathfrak{h}^*$ define $\Pi_0 \lambda \in \mathfrak{h}^{*,\sigma}$ by $(\Pi_0\lambda)(\Pi_0 h) := \lambda(\Pi_0 h)$ for $h \in \mathfrak{h}$.

Lemma 3.1. *The weight function ψ is an element of $\bigotimes_{i=1}^N M_{\lambda_i} \otimes M^\sigma_{\lambda_0}$ of $\mathfrak{g}^\sigma$-weight*

$$\lambda_\infty := \lambda_0 + \sum_{i=1}^N \Pi_0 \lambda_i - \sum_{j=1}^m \Pi_0 \alpha_{c(j)} \in \mathfrak{h}^{*,\sigma}. \tag{3.4}$$

Proof. We can regard $\mathfrak{n}_-$ as a module over $\mathfrak{h}^\sigma$, under the adjoint action. Then $\bigotimes_{i=1}^N M_{\lambda_i} \otimes M^\sigma_{\lambda_0} \otimes \mathfrak{n}_-^{\otimes s}$ is a module over $\mathfrak{h}^\sigma$, for every $s \in \mathbb{Z}_{\geq 0}$. Pick any element $\Pi_0 h \in \mathfrak{h}^\sigma$. For each $i \in I$ we have $(\Pi_0 h).F_{\alpha_i} := [\Pi_0 h, F_{\alpha_i}] = -\alpha_i(\Pi_0 h) F_{\alpha_i} = -(\Pi_0 \alpha_i)(h) F_{\alpha_i}$. That is, F_{α_i} has $\mathfrak{g}^\sigma$-weight $-\Pi_0 \alpha_i$. Therefore $v_{\lambda_1} \otimes \ldots \otimes v_{\lambda_N} \otimes v_{\lambda_0} \otimes F_{\alpha_{c(1)}} \otimes F_{\alpha_{c(2)}} \otimes \ldots \otimes F_{\alpha_{c(m)}} \in \bigotimes_{i=1}^N M_{\lambda_i} \otimes M^\sigma_{\lambda_0} \otimes \mathfrak{n}_-^{\otimes m}$ has $\mathfrak{g}^\sigma$-weight λ_∞.

Now, if an element $y \in \mathfrak{n}_-$ has $\mathfrak{g}^\sigma$-weight λ then so does $\sigma^j y$ for any $j \in \mathbb{Z}_T$. Indeed, $\Pi_0 = \sigma^j \Pi_0$ so that $[\Pi_0 h, \sigma^j y] = [\sigma^j \Pi_0 h, \sigma^j y] = \sigma^j [\Pi_0 h, y] = \sigma^j y \, \lambda(\Pi_0 h)$. It follows that the maps θ_s commute with the action of $\mathfrak{h}^\sigma$, *i.e.* preserve $\mathfrak{g}^\sigma$-weight. Hence the result. □

3.3. **The weight Λ_0.** Define a weight $\Lambda_0 \in \mathfrak{h}^{*,\sigma}$ by

$$\Lambda_0(h) := \sum_{r=1}^{T-1} \frac{\mathrm{tr}_{\mathfrak{n}}(\sigma^{-r} \mathrm{ad}_h)}{1 - \omega^r} \tag{3.5}$$

where $\mathrm{ad}_h : \mathfrak{n} \to \mathfrak{n}; X \mapsto [h, X]$ is the adjoint action of $\mathfrak{h}$ on $\mathfrak{n}$.

For a more explicit expression for Λ_0, note that [Kac83, §8.6]

$$\sigma(E_\alpha) = \tau_\alpha E_{\sigma(\alpha)}, \qquad \sigma(H_i) = H_{\sigma(i)}, \qquad \sigma(F_\alpha) = \tau_\alpha^{-1} F_{\sigma(\alpha)}. \tag{3.6}$$

Here, by overloading notation, we write $\sigma : \Delta^+ \to \Delta^+$ for the symmetry of the root system, coming in turn from a symmetry $\sigma : I \to I$ of the Dynkin diagram. The numbers τ_α, $\alpha \in \Delta^+$, are certain roots of unity in $\Gamma = \omega^{\mathbb{Z}}$. (So the "inner part" of the automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ is encoded in the choice of τ_{α_i}, $i \in I$; the remaining τ_α are fixed by this choice.)

Then

$$\Lambda_0 = \sum_{r=1}^{T-1} \frac{1}{1-\omega^r} \sum_{\substack{\alpha \in \Delta^+ \\ \sigma^r(\alpha)=\alpha}} \left(\prod_{p=0}^{r-1} \tau_{\sigma^p(\alpha)}^{-1} \right) \alpha. \tag{3.7}$$

3.4. **The cyclotomic Bethe equations.** Given a complex vector space A on which σ acts, for any linear map $\eta : A \to \mathbb{C}$ we define $L_\sigma \eta := \eta \circ \sigma^{-1}$. Let $\chi \in \mathfrak{h}^*$ be such that $L_\sigma \chi = \omega \chi$. The *cyclotomic Bethe equations (with twisted boundaries)* are

$$\begin{aligned} 0 = \sum_{r=0}^{T-1} \sum_{i=1}^{N} \frac{\langle \alpha_{c(j)}, L_\sigma^r \lambda_i \rangle}{w_j - \omega^r z_i} - \sum_{r=0}^{T-1} \sum_{\substack{k=1 \\ k \neq j}}^{m} \frac{\langle \alpha_{c(j)}, L_\sigma^r \alpha_{c(k)} \rangle}{w_j - \omega^r w_k} \\ + \frac{1}{w_j} \left(-\frac{1}{2} \sum_{r=1}^{T-1} \langle \alpha_{c(j)}, L_\sigma^r \alpha_{c(j)} \rangle + \langle \alpha_{c(j)}, T\lambda_0 + \Lambda_0 \rangle \right) + \langle \alpha_{c(j)}, \chi \rangle. \end{aligned} \tag{3.8}$$

3.5. **Eigenvectors of the cyclotomic Gaudin algebra.** Let $\chi \in \mathfrak{h}^*$ be such that $L_\sigma \chi = \omega \chi$. Extend χ to an element of $\mathfrak{g}^*$ by setting $\chi(\mathfrak{n}) = \chi(\mathfrak{n}_-) = 0$. Then a χ is a linear map $\Pi_{-1}\mathfrak{g} \to \mathbb{C}$.

Suppose $(w_1, \ldots, w_m; c(1), \ldots, c(m))$ are such that cyclotomic Bethe equations (3.8) are satisfied.

Theorem 3.2. *The weight function is an eigenvector of the algebra* $\mathcal{A}^{1,(1),1}_{\infty,\boldsymbol{z},0}(\mathfrak{g}, \sigma, \chi)^\Gamma$. *In particular, the eigenvalues* E_i *of the quadratic cyclotomic Gaudin Hamiltonians* $\mathcal{H}_{i,0}$ *are given by*

$$E_i := \sum_{\substack{j=1 \\ j \neq i}}^{N} \sum_{s=0}^{T-1} \frac{\langle \lambda_i, L_\sigma^s \lambda_j \rangle}{z_i - \omega^s z_j} - \sum_{j=1}^{m} \sum_{s=0}^{T-1} \frac{\langle \lambda_i, L_\sigma^s \alpha_{c(j)} \rangle}{z_i - \omega^s w_j} + \frac{1}{z_i} \left(\langle \lambda_i, T\lambda_0 + \Lambda_0 \rangle + \frac{1}{2} \sum_{s=1}^{T-1} \langle \lambda_i, L_\sigma^s \lambda_i \rangle \right) + \langle \lambda_i, \chi \rangle.$$

Moreover the eigenvalues of $\mathcal{H}_{0,0}$, $\mathcal{H}_{0,1}$ *and* $\mathcal{H}_{\infty,0}$ *are, respectively,*

$$\big\langle \chi, T\lambda_0 + \Lambda_0 \big\rangle, \quad \frac{1}{2}\big\langle T\lambda_0 + \Lambda_0, T\lambda_0 + \Lambda_0 \big\rangle + \big\langle T\lambda_0 + \Lambda_0, \varrho \big\rangle - K, \quad \frac{1}{2}\big\langle \chi, \chi \big\rangle,$$

where $\varrho := \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$.

Remark 3.3. The eigenvalue of $\mathcal{H}_{0,0}$ is zero unless $T = 1$, on $\mathbb{Z}/T\mathbb{Z}$-grading grounds, as it should be in view of Remark 2.6. Similarly, the eigenvalue of $\mathcal{H}_{\infty,0}$ vanishes unless $T = 1$ or 2.

In particular, when $\chi = 0$ the weight function is an eigenvector of the algebra $\mathscr{Z}^{1,(1),1}_{\infty,\boldsymbol{z},0}(\mathfrak{g}, \sigma)^\Gamma$. Moreover, in that case we have the following.

Theorem 3.4. *In the special case* $\chi = 0$, *the weight function* ψ *belongs to* $\big(\bigotimes_{i=1}^N M_{\lambda_i} \otimes M^\sigma_{\lambda_0} \big)^{\mathfrak{n}^\sigma}_{\lambda_\infty}$, *the subspace of* $\mathfrak{n}^\sigma$*-singular vectors of* $\mathfrak{g}^\sigma$*-weight* λ_∞ *in* $\bigotimes_{i=1}^N M_{\lambda_i} \otimes M^\sigma_{\lambda_0}$.

Remark 3.5. We have not shown that the weight function is non-zero. When σ is a diagram automorphism, this is proved in [VY].

4. Proofs

4.1. **Restricted duals and contragredient Verma modules.** Given a module M over $\mathfrak{g}$ we write $(M)_\mu$ for the subspace of weight $\mu \in \mathfrak{h}^*$,

$$(M)_\mu := \{v \in M : \text{ there exists } n \in \mathbb{Z}_{\geq 1} \text{ such that } (h - \mu(h)1)^n.v = 0 \text{ for all } h \in \mathfrak{h}\}. \tag{4.1}$$

The module M is a *weight module* if $M = \bigoplus_{\mu \in \mathfrak{h}^*} (M)_\mu$. In this paper we work with weight modules all of whose weight subspaces are of finite dimension. If M, N are two such modules, then by $\mathrm{Hom}_{\mathbb{C}}(M, N)$ we shall always mean the restricted space of linear maps

$$\mathrm{Hom}_{\mathbb{C}}(M, N) := \bigoplus_{\mu,\nu \in \mathfrak{h}^*} \mathrm{Hom}_{\mathbb{C}}\left((M)_\mu, (N)_\nu\right).$$

In particular we shall write $M^* := \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C}) = \bigoplus_{\mu \in \mathfrak{h}^*} ((M)_\mu)^*$, *i.e.* our duals are restricted duals. We have $\mathrm{Hom}_{\mathbb{C}}(M, N) = \mathrm{Hom}_{\mathbb{C}}(M \otimes N^*, \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, M^* \otimes N) = M^* \otimes N$.

The restricted dual M^*_λ of the Verma module M_λ is naturally a right $U(\mathfrak{g})$-module. We may twist by any anti-automorphism of $U(\mathfrak{g})$ to obtain a left module. The Cartan anti-automorphism $\varphi : U(\mathfrak{g}) \to U(\mathfrak{g})$ is defined by

$$\varphi(H_i) = H_i, \quad i \in I, \quad \text{and} \quad \varphi(E_\alpha) = F_\alpha, \quad \varphi(F_\alpha) = E_\alpha, \quad \alpha \in \Delta^+. \tag{4.2}$$

It obeys $\varphi^2 = \mathrm{id}$. The twist of M^*_λ by φ is the left $U(\mathfrak{g})$-module called the *contragredient Verma module*. Henceforth by M^*_λ we shall always mean the restricted dual equipped with this left $U(\mathfrak{g})$-module structure. That is

$$(x.f)(v) := f(\varphi(x).v), \qquad f \in M^*_\lambda, \quad x \in \mathfrak{g}, \quad v \in M_\lambda.$$

See e.g. [Hu08, §3.3].

Let $S : U(\mathfrak{g}) \to U(\mathfrak{g})$ be the antipode map, *i.e.* the extension of the map $\mathfrak{g} \to \mathfrak{g}; X \to -X$ to an anti-automorphism of $U(\mathfrak{g})$. We have the automorphism $\varphi \circ S = S \circ \varphi$ of $\mathfrak{g}$. Let us write $(M^*_\lambda)^{\varphi\circ S}$ for the left $U(\mathfrak{g})$-module obtained by twisting by this automorphism. In other words $(M^*_\lambda)^{\varphi\circ S}$ is the dual of M_λ in the usual Hopf-algebraic sense. Hence we have

$$\mathrm{Hom}_{\mathfrak{g}}(A, B \otimes M_\lambda) = \mathrm{Hom}_{\mathfrak{g}}(A \otimes (M^*_\lambda)^{\varphi\circ S}, B). \tag{4.3}$$

Similarly, one has the notion of weight modules, contragredient Verma modules, etc., over $\mathfrak{g}^\sigma$.

Given a module V over a Lie algebra $\mathfrak{a}$, we denote by $V^{\mathfrak{a}}$ the space of invariants $V^{\mathfrak{a}} := \{x \in V : a.x = 0 \text{ for all } a \in \mathfrak{a}\}$.

4.2. **Heisenberg algebras at the marked points.** Let $z_1, \ldots, z_N$, $w_1, \ldots, w_m$, be as in §3. For brevity we introduce $p := N+m$ and $(x_1, \ldots, x_p) := (z_1, \ldots, z_N, w_1, \ldots, w_m)$. Let $\boldsymbol{x} := \{x_1, \ldots, x_p\}$.

Let $\mathfrak{n}_{\mathbb{C}}$ (resp. $\mathfrak{n}^*_{\mathbb{C}}$) denote the vector space $\mathfrak{n}$ (resp. $\mathfrak{n}^*$) endowed with the structure of a commutative Lie algebra. On the commutative Lie algebra $\mathfrak{n}_{\mathbb{C}} \oplus \mathfrak{n}^*_{\mathbb{C}}$ there is a non-degenerate bilinear skew-symmetric form $\langle\cdot,\cdot\rangle$ defined by

$$\big\langle (X,\eta),(Y,\psi)\big\rangle = \psi(X) - \eta(Y), \tag{4.4}$$

for any $(X,\eta),(Y,\psi) \in \mathfrak{n}_{\mathbb{C}} \oplus \mathfrak{n}^*_{\mathbb{C}}$ and an action by automorphisms of the group Γ given by

$$\omega.(X,\eta) := \big(\sigma(X), L_\sigma(\eta)\big). \tag{4.5}$$

Let H_{x_i}, $i = 1,\ldots,p$, denote the central extension of the commutative Lie algebra $(\mathfrak{n}_{\mathbb{C}} \oplus \mathfrak{n}^*_{\mathbb{C}}) \otimes \mathbb{C}((t-x_i))$, by a one-dimensional centre $\mathbb{C}\mathbf{1}_{x_i}$, defined by the cocycle $\mathrm{res}_{t-x_i}\langle f,g\rangle \mathbf{1}_{x_i}$. Let H_0^Γ denote the extension of $(\mathfrak{n}_{\mathbb{C}} \otimes \mathbb{C}((t)))^{\Gamma,0} \oplus (\mathfrak{n}^*_{\mathbb{C}} \otimes \mathbb{C}((t)))^{\Gamma,-1}$ by a one-dimensional centre $\mathbb{C}\mathbf{1}_0$ defined by the cocycle $\mathrm{res}_t\langle f,g\rangle\mathbf{1}_0$. Let H_∞^Γ denote the extension of $(\mathfrak{n}_{\mathbb{C}} \otimes \mathbb{C}((t^{-1})))^{\Gamma,0} \oplus (\mathfrak{n}^*_{\mathbb{C}} \otimes \mathbb{C}((t^{-1})))^{\Gamma,-1}$ by a one-dimensional centre $\mathbb{C}\mathbf{1}_\infty$ defined by the cocycle $\mathrm{res}_{t^{-1}}\, t^2\langle f,g\rangle\mathbf{1}_\infty$.

Let us give a more explicit description of these Lie algebras in terms of generators and relations. To do so, we first construct bases of $\mathfrak{n}$ and $\mathfrak{n}^*$ adapted to the automorphism σ. Recall the projectors Π_k, $k \in \mathbb{Z}/T\mathbb{Z}$, from (2.2). By the adjoint action, $\mathfrak{g}$ is a module over itself. In particular, it is a module over its Lie subalgebra $\mathfrak{g}^\sigma = \Pi_0\mathfrak{g}$. As a $\mathfrak{g}^\sigma$-module, $\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}/T\mathbb{Z}} \Pi_k\mathfrak{g}$. Let Δ_k^+ denote the set of $\mathfrak{g}^\sigma$-weights of $\Pi_k\mathfrak{n}$ and for $\alpha \in \Delta_k^+$ let $\mathfrak{n}_{(k,\alpha)}$ denote the corresponding weight subspace of $\Pi_k\mathfrak{n}$. We may pick a basis of $\mathfrak{n}$ consisting of vectors $E_{(k,\alpha)} \in \mathfrak{n}_{(k,\alpha)}$, where $k \in \mathbb{Z}/T\mathbb{Z}, \alpha \in \Delta_k^+$.[2]

We now have two bases of $\mathfrak{n}$, namely E_α, $\alpha \in \Delta^+$, and $E_{(k,\alpha)}$, $k \in \mathbb{Z}/T\mathbb{Z}$, $\alpha \in \Delta_k^+$. We write E^*_α, $\alpha \in \Delta^+$, and $E^*_{(k,\alpha)}$, $k \in \mathbb{Z}/T\mathbb{Z}$, $\alpha \in \Delta_k^+$ for their respective dual bases of $\mathfrak{n}^*$.

Then H_{x_i} has the following explicit set of generators:

$$a_\alpha[n]_{x_i} := E_\alpha \otimes (t-x_i)^n, \quad a^*_\alpha[n]_{x_i} := E^*_\alpha \otimes (t-x_i)^{n-1},$$

where $\alpha \in \Delta^+$ and $n \in \mathbb{Z}$; while explicit sets of generators for H_0^Γ and H_∞^Γ are

$$\begin{aligned} a_{(k,\alpha)}[nT+k]_0 &:= E_{(k,\alpha)} \otimes t^{nT+k} \quad\;\; \in (\mathfrak{n}_{\mathbb{C}} \otimes \mathbb{C}((t)))^{\Gamma,0}, \\ a^*_{(k,\alpha)}[nT-k]_0 &:= E^*_{(k,\alpha)} \otimes t^{nT-k-1} \in (\mathfrak{n}^*_{\mathbb{C}} \otimes \mathbb{C}((t)))^{\Gamma,-1}, \end{aligned}$$

and

$$\begin{aligned} a_{(k,\alpha)}[nT+k]_\infty &:= E_{(k,\alpha)} \otimes t^{nT+k} \quad\;\; \in (\mathfrak{n}_{\mathbb{C}} \otimes \mathbb{C}((t^{-1})))^{\Gamma,0}, \\ a^*_{(k,\alpha)}[nT-k]_\infty &:= E^*_{(k,\alpha)} \otimes t^{nT-k-1} \in (\mathfrak{n}^*_{\mathbb{C}} \otimes \mathbb{C}((t^{-1})))^{\Gamma,-1}, \end{aligned}$$

respectively, where $k \in \mathbb{Z}/T\mathbb{Z}$, $\alpha \in \Delta_k^+$, and $n \in \mathbb{Z}$.

Remark 4.1. The above notation is in accordance with the notations for the modes $a[n]$, $n \in \mathbb{Z}$ of an element $a \in \mathscr{L}$ in a vertex Lie algebra $\mathscr{L}$ used in [VY16b] and recalled in appendix B below. In particular, the shift by 1 in the power of t for the modes of a^*_α has to do with the fact that this is an element of degree 0 in the vertex Lie algebra associated to the Heisenberg algebra, cf. [VY16b, Example 2.6]. Note also our conventions for the modes at ∞.

[2]Indeed, suppose $\alpha \in \Delta^+$ is a root of $\mathfrak{g}$ such that the orbit $\sigma^{\mathbb{Z}}\alpha$ has t elements, where $t \in \mathbb{Z}_{\geq 1}$ divides T. Then $\sigma^t E_\alpha = \omega^{tm} E_\alpha$ for some unique $m \in \mathbb{Z}/T\mathbb{Z}$. Let $E_{(m-kT/t,\bar{\alpha})} := \sum_{j=0}^{t-1} \omega^{-(m-kT/t)j} \sigma^j E_\alpha \in \mathfrak{n}_{(m-kT/t,\bar{\alpha})}$ for $k = 0,1,\ldots,t-1$. By picking one root from each σ-orbit, we construct a basis of the required form.

These generators obey the relations:

$$
\begin{aligned}
[a_\alpha[n]_{x_i}, a^*_\beta[m]_{x_j}] &= \delta_{ij}\delta_{\alpha\beta}\delta_{n,-m}\mathbf{1}_{x_i},\\
[a_{(i,\alpha)}[n]_0, a^*_{(j,\beta)}[m]_0] &= \delta_{ij}\delta_{\alpha\beta}\delta_{n,-m}\mathbf{1}_0,\\
[a_{(i,\alpha)}[n]_\infty, a^*_{(j,\beta)}[m]_\infty] &= \delta_{ij}\delta_{\alpha\beta}\delta_{n,-m}\mathbf{1}_\infty,
\end{aligned}
$$

with all other commutators vanishing.

Each H_{x_i} is isomorphic to the *Heisenberg Lie algebra* $H(\mathfrak{g})$, while H_0^Γ and H_∞^Γ are isomorphic to a subalgebra $H(\mathfrak{g})^\Gamma$. Note that the opposite Lie algebra $H_\infty^{\Gamma,\mathsf{op}}$ differs from H_∞^Γ only in the sign of the central extension.

Let $H_{\infty,p,0}$ denote the extension of

$$
\begin{aligned}
&(\mathfrak{n}_\mathbb{C}\otimes\mathbb{C}((t^{-1})))^{\Gamma,0}\oplus(\mathfrak{n}^*_\mathbb{C}\otimes\mathbb{C}((t^{-1})))^{\Gamma,-1}\\
&\qquad\oplus\bigoplus_{i=1}^p(\mathfrak{n}_\mathbb{C}\oplus\mathfrak{n}^*_\mathbb{C})\otimes\mathbb{C}((t-x_i))\\
&\qquad\qquad\oplus(\mathfrak{n}_\mathbb{C}\otimes\mathbb{C}((t)))^{\Gamma,0}\oplus(\mathfrak{n}^*_\mathbb{C}\otimes\mathbb{C}((t)))^{\Gamma,-1}
\end{aligned}
\tag{4.6}
$$

by a one-dimensional centre $\mathbb{C}\mathbf{1}$, defined by the cocycle

$$
\Omega(f,g) := \left(\frac{1}{T}\operatorname{res}_t\langle f_0,g_0\rangle + \sum_{i=1}^p \operatorname{res}_{t-x_i}\langle f_{x_i},g_{x_i}\rangle - \frac{1}{T}\operatorname{res}_{t^{-1}} t^2\langle f_\infty,g_\infty\rangle\right)\mathbf{1} \tag{4.7}
$$

where $f=(f_\infty;f_{x_1},\dots,f_{x_p};f_0)$ and $g=(g_\infty;g_{x_1},\dots,g_{x_p};g_0)$ are elements of the Lie algebra (4.6). Equivalently, $H_{\infty,p,0}$ is the quotient of the direct sum $H_\infty^{\Gamma,\mathsf{op}}\oplus\bigoplus_{i=1}^p H_{x_i}\oplus H_0^\Gamma$ by the ideal generated by $\mathbf{1}_{x_i}-T\mathbf{1}_0$, $i=1,\dots,p$, and $\mathbf{1}_\infty-\mathbf{1}_0$. Then $\mathbf{1}:=\mathbf{1}_{x_i}$.

Define also

$$
\mathfrak{h}_{\infty,p,0} := (\mathfrak{h}\otimes\mathbb{C}((t^{-1})))^{\Gamma,0}\oplus\bigoplus_{i=1}^p\mathfrak{h}\otimes\mathbb{C}((t-x_i))\oplus(\mathfrak{h}\otimes\mathbb{C}((t)))^{\Gamma,0}
$$

Let us give a set of explicit generators for this commutative Lie algebra $\mathfrak{h}_{\infty,p,0}$. Let $H_{(k,a)}$, $a=1,\dots,\dim(\Pi_k\mathfrak{h})$, be a basis of $\Pi_k\mathfrak{h}$ for each $k\in\mathbb{Z}/T\mathbb{Z}$. Then $\mathfrak{h}_{0,p,\infty}$ is the commutative Lie algebra with basis

$$
\begin{aligned}
b_j[n]_{x_i} &:= H_j\otimes(t-x_i)^n\in\mathfrak{h}\otimes\mathbb{C}((t-x_i)),\\
b_{(k,a)}[nT+k]_0 &:= H_{(k,a)}\otimes t^{nT+k}\in(\mathfrak{h}\otimes\mathbb{C}((t)))^{\Gamma,0},\\
b_{(k,a)}[nT+k]_\infty &:= H_{(k,a)}\otimes t^{nT+k}\in(\mathfrak{h}\otimes\mathbb{C}((t^{-1})))^{\Gamma,0},
\end{aligned}
\tag{4.8}
$$

for $j\in I$, $k\in\mathbb{Z}/T\mathbb{Z}$, $a=1,\dots,\dim(\Pi_k\mathfrak{h})$ and $n\in\mathbb{Z}$.

4.3. **Wakimoto modules at the marked points.** For each $i=1,\dots,p$, let $\mathbb{C}|\rangle_{x_i}$ denote the one-dimensional left module over $U((\mathfrak{n}_\mathbb{C}\oplus\mathfrak{n}^*_\mathbb{C})\otimes\mathbb{C}[[t-x_i]]\oplus\mathbb{C}\mathbf{1}_{x_i})$ on which $\mathbf{1}_{x_i}$ acts as 1 and $(\mathfrak{n}_\mathbb{C}\oplus\mathfrak{n}^*_\mathbb{C})\otimes\mathbb{C}[[t]]$ acts as zero. Define M_{x_i} to be the induced module over H_{x_i},

$$
\mathsf{M}_{x_i} := U(H_{x_i})\otimes_{U((\mathfrak{n}_\mathbb{C}\oplus\mathfrak{n}^*_\mathbb{C})\otimes\mathbb{C}[[t-x_i]]\oplus\mathbb{C}\mathbf{1}_{x_i})}\mathbb{C}|\rangle_{x_i}.
$$

Suppose we are given an $\mathfrak{h}^*$-valued Laurent series $\nu_i \in \mathfrak{h}^* \otimes \mathbb{C}((t-x_i))$ in the local coordinate $t-x_i$ about the point x_i. Let $\mathbb{C}v_{\nu_i}$ denotes the one-dimensional module over $\mathfrak{h} \otimes \mathbb{C}((t-x_i))$ on which $f.v_{\nu_i} = v_{\nu_i} \operatorname{res}_{t-x_i} \nu_i(f)$, for any $f \in \mathfrak{h} \otimes \mathbb{C}((t-x_i))$. Then the *Wakimoto module* W_{ν_i} is the module over $H_{x_i} \oplus \mathfrak{h} \otimes \mathbb{C}((t-x_i))$ given by

$$W_{\nu_i} := \mathsf{M}_{x_i} \otimes \mathbb{C}v_{\nu_i}.$$

Explicitly, W_{ν_i} is the Fock module generated by a vacuum vector $|\rangle_{x_i}$ such that $\mathbf{1}_{x_i}|\rangle_{x_i} = |\rangle_{x_i}$,

$$a_\alpha[n]|\rangle_{x_i} = 0, \quad n \geq 0, \qquad a^*_\alpha[n]|\rangle_{x_i} = 0, \quad n \geq 1,$$

and

$$b_k[n]|\rangle_{x_i} = |\rangle_{x_i}\, \nu_{i,-n-1}(H_k)$$

where $\nu_i(t-x_i) =: \sum_{s=-S}^{\infty} \nu_{i,s}(t-x_i)^s$ for some $S \in \mathbb{Z}$ and $\nu_{i,s} \in \mathfrak{h}^*$. Here and in what follows we use the obvious shorthand $a_\alpha[n]|\rangle_{x_i}$ to denote $a_\alpha[n]_{x_i}|\rangle_{x_i}$, etc.

Similarly, let $\mathbb{C}|\rangle_0$ denote the one-dimensional module over $(\mathfrak{n}_\mathbb{C} \otimes \mathbb{C}[[t]])^{\Gamma,0} \oplus (\mathfrak{n}^*_\mathbb{C} \otimes \mathbb{C}[[t]])^{\Gamma,-1} \oplus \mathbb{C}\mathbf{1}_0$ on which $\mathbf{1}_0$ acts as $\frac{1}{T}$ and the first two summands act as zero. Define M_0^Γ to be the induced module over H_0^Γ:

$$\mathsf{M}_0^\Gamma := U(H_0^\Gamma) \otimes_{U((\mathfrak{n}_\mathbb{C} \otimes \mathbb{C}[[t]])^{\Gamma,0} \oplus (\mathfrak{n}^*_\mathbb{C} \otimes \mathbb{C}[[t]])^{\Gamma,-1} \oplus \mathbb{C}\mathbf{1}_0)} \mathbb{C}|\rangle_0.$$

Suppose we are given an element $\nu_0 \in (\mathfrak{h}^* \otimes \mathbb{C}((t)))^{\Gamma,-1}$. That is, ν_0 is a $\mathfrak{h}^*$-valued Laurent series in t such that $\nu_0(\omega t) = \omega^{-1}\sigma\nu_0(t)$. Let $\mathbb{C}v_{\nu_0}$ denote the one-dimensional module over $(\mathfrak{h} \otimes \mathbb{C}((t)))^{\Gamma,0}$ given by

$$f.v_{\nu_0} = v_{\nu_0} \operatorname{res}_t \nu_0(f), \tag{4.9}$$

for any $f \in (\mathfrak{h} \otimes \mathbb{C}((t)))^{\Gamma,0}$. We may then consider the *twisted Wakimoto module* $W^\Gamma_{\nu_0}$ defined as the $H_0^\Gamma \oplus (\mathfrak{h} \otimes \mathbb{C}((t)))^{\Gamma,0}$-module

$$W^\Gamma_{\nu_0} := \mathsf{M}_0^\Gamma \otimes \mathbb{C}v_{\nu_0}.$$

Explicitly, it is the Fock module generated by a vacuum vector $|\rangle_0$ such that $\mathbf{1}_0|\rangle_0 = |\rangle_0 \frac{1}{T}$,

$$a_{(k,\alpha)}[n]|\rangle_0 = 0, \quad n \geq 0, \qquad a^*_{(k,\alpha)}[n]|\rangle_0 = 0, \quad n \geq 1, \tag{4.10}$$

and

$$b_{(k,a)}[n]|\rangle_0 = |\rangle_0\, \nu_{0,-n-1}(H_{(k,a)})$$

where $\nu_0(t) =: \sum_{s=-S}^{\infty} \nu_{0,s} t^s$ for some $S \in \mathbb{Z}$ and $\nu_{0,s} \in \mathfrak{h}^*$.

Finally, define

$$H_\infty^{\Gamma,+} := (\mathfrak{n}_\mathbb{C} \otimes t^{-1}\mathbb{C}[[t^{-1}]])^{\Gamma,0} \oplus (\mathfrak{n}^*_\mathbb{C} \otimes t^{-1}\mathbb{C}[[t^{-1}]])^{\Gamma,-1} \oplus \mathbb{C}\mathbf{1}_\infty,$$

and let $\mathsf{M}_\infty^{\Gamma,\vee}$ denote the right module over $U(H_\infty^\Gamma)$ induced from the trivial one-dimensional right module $\mathbb{C}\langle|$ over $U(H_\infty^{\Gamma,+})$ on which $\mathbf{1}_\infty$ acts as $\frac{1}{T}$ and the first two summands act as zero:

$$\mathsf{M}_\infty^{\Gamma,\vee} := \mathbb{C}\langle| \otimes_{U(H_\infty^{\Gamma,+})} U(H_\infty^\Gamma).$$

Suppose $\nu_\infty \in (\mathfrak{h}^* \otimes \mathbb{C}((t^{-1})))^{\Gamma,-1}$. That is, ν_∞ is a $\mathfrak{h}^*$-valued Laurent series in t^{-1} such that $\nu_\infty(\omega^{-1}t^{-1}) = \omega^{-1}\sigma\nu_\infty(t^{-1})$. Let $\mathbb{C}v^*_{\nu_\infty}$ denote the one-dimensional module over $(\mathfrak{h} \otimes \mathbb{C}((t^{-1})))^{\Gamma,0}$ given by

$$f.v^*_{\nu_\infty} = -v^*_{\nu_\infty} \operatorname{res}_{t^{-1}} t^2 \nu_\infty(f), \tag{4.11}$$

for $f \in (\mathfrak{h} \otimes \mathbb{C}((t^{-1})))^{\Gamma,0}$. Then we have the right $U(H^\Gamma_\infty \oplus (\mathfrak{h} \otimes \mathbb{C}((t^{-1})))^{\Gamma,0})$-module

$$W^{\Gamma,\vee}_{\nu_\infty} := \mathsf{M}^{\Gamma,\vee}_\infty \otimes \mathbb{C} v^*_{\nu_\infty}.$$

Explicitly, $W^{\Gamma,\vee}_{\nu_\infty}$ is the Fock module generated by a vacuum vector $\langle|$ such that $\langle|\mathbf{1}_\infty = \frac{1}{T}\langle|$,

$$\langle| a_{(k,\alpha)}[n]_\infty = 0, \quad n < 0, \qquad \langle| a^*_{(k,\alpha)}[n]_\infty = 0, \quad n \leq 0, \tag{4.12}$$

and

$$\langle| b_{(k,a)}[n]_\infty = -\nu_{\infty,-n-1}(H_{(k,a)})\langle| \tag{4.13}$$

where $\nu_\infty(t) =: \sum_{s=-\infty}^{S} \nu_{\infty,s} t^s$ for some $S \in \mathbb{Z}$ and $\nu_{\infty,s} \in \mathfrak{h}^*$.

4.4. **Free field realization.** The modules W_{ν_i} are *smooth*. That means, by definition, that for each $v \in W_{\nu_i}$

$$0 = a_\alpha[n]_{x_i} v = a^*_\alpha[n]_{x_i} v = b_k[n]_{x_i} v \quad \text{for all} \quad n \gg 0,$$

for all $\alpha \in \Delta^+$ and $k \in I$.[3] Similarly $W^\Gamma_{\nu_0}$ is smooth. The module $W^{\Gamma,\vee}_{\nu_\infty}$ is *co-smooth*. By that we mean that for each $v \in W^{\Gamma,\vee}_{\nu_\infty}$,

$$0 = v\, a_{(k,\alpha)}[n]_\infty = v\, a^*_{(k,\alpha)}[n]_\infty = v\, b_{(k,a)}[n]_\infty \quad \text{for all} \quad n \ll 0,$$

for all $k \in \mathbb{Z}/T\mathbb{Z}$, $\alpha \in \Delta^+_k$ and $a \in \{1, \ldots, \dim \Pi_k \mathfrak{h}\}$.

Let now $\pi_0 \simeq \mathbb{C}[b_i[n]]_{i\in I;\, n\leq -1}$ be the induced representation of $\mathfrak{h} \otimes \mathbb{C}((t))$ in which $b_i[n]$ acts as 0 for all $i \in I$ and all $n \in \mathbb{Z}_{\geq 0}$. Let M be the induced module over the Heisenberg Lie algebra with generators $a_\alpha[n]$, $a^*_\alpha[n]$, and $\mathbf{1}$, which we denote $H(\mathfrak{g})$ as in [VY16a, §3.5]. Then

$$\mathbb{W}_0 := \mathsf{M} \otimes \pi_0, \tag{4.14}$$

is an induced representation of $H(\mathfrak{g}) \oplus \mathfrak{h} \otimes \mathbb{C}((t))$. Explicitly, $\mathbb{W}_0$ is the Fock module generated by a vacuum vector $|\rangle$ such that $\mathbf{1}|\rangle = |\rangle$,

$$a_\alpha[n]|\rangle = 0, \quad n \geq 0, \qquad a^*_\alpha[n]|\rangle = 0, \quad n \geq 1,$$

and

$$b_k[n]|\rangle = 0, \quad n \geq 0.$$

Let us recall some facts about the free-field realization of $\widehat{\mathfrak{g}}$.

The $H(\mathfrak{g}) \oplus \mathfrak{h} \otimes \mathbb{C}((t))$-module $\mathbb{W}_0$ is endowed with the structure of a *vertex algebra*, see [FFR94] or *e.g.* [VY16a, equation (3.24)]. In the notation of [VY16b], see also appendix B, it corresponds to the vertex algebra $\mathbb{V}(\mathscr{M})$ where $\mathscr{M}$ is the vertex Lie algebra underlying the Lie algebra $H(\mathfrak{g}) \oplus \mathfrak{h} \otimes \mathbb{C}((t))$, given for instance in [VY16b, Example 2.6]. Specifically, $\mathscr{M}$ is generated by the finite dimensional vector space $M^o = \mathfrak{n}_\mathbb{C} \oplus \mathfrak{n}^*_\mathbb{C} \oplus \mathfrak{h}$ with non-trivial n^{th}-products given by

$$a_{(0)} b = \langle a, b\rangle \mathbf{1},$$

for any $a, b \in \mathfrak{n}_\mathbb{C} \oplus \mathfrak{n}^*_\mathbb{C}$, where the skew-symmetric form $\langle \cdot, \cdot \rangle$ on the right hand side was defined in (4.4). As a vertex algebra, $\mathbb{W}_0$ is in particular also a vertex Lie algebra. We may therefore consider the associated "big" Lie algebra $\mathsf{L}(\mathbb{W}_0)$, cf. §B.1, consisting of all formal modes of states in $\mathbb{W}_0$.

[3]That is, for each $v \in W_{\nu_i}$ $\alpha \in \Delta^+$ and $k \in I$ there exists an $n \in \mathbb{Z}$ such that $0 = a_\alpha[m]_{x_i} v = a^*_\alpha[m]_{x_i} v = b_k[m]_{x_i} v$ for all $m \geq n$.

For each of the marked points x_i, $i = 1, \ldots, p$, there is a "local" copy $\mathsf{L}(\mathbb{W}_0)_{x_i}$ of $\mathsf{L}(\mathbb{W}_0)$, which contains $\mathsf{L}(\mathscr{M})_{x_i} = H_{x_i} \oplus \mathfrak{h} \otimes \mathbb{C}((t - x_i))$ as a subalgebra. Moreover, every smooth module over $H_{x_i} \oplus \mathfrak{h} \otimes \mathbb{C}((t - x_i))$ on which $\mathbf{1}_{x_i}$ acts as 1 becomes a smooth $\mathsf{L}(\mathbb{W}_0)_{x_i}$-module in a canonical way. See *e.g.* [VY16b, Proposition 5.9].

At the fixed-points 0 and ∞ we have the local copies $\mathsf{L}(\mathbb{W}_0)_0$ and $\mathsf{L}(\mathbb{W}_0)_\infty$ of the big Lie algebra $\mathsf{L}(\mathbb{W}_0)$. The automorphism σ of the Lie algebra $\mathsf{L}(\mathscr{M}) = H(\mathfrak{g}) \oplus \mathfrak{h} \otimes \mathbb{C}((t))$ extends in a unique way to an automorphism of $\mathbb{W}_0$ as a vertex algebra. So we get an action by automorphisms of the group Γ on both local Lie algebras $\mathsf{L}(\mathbb{W}_0)_0$ and $\mathsf{L}(\mathbb{W}_0)_\infty$. Let $\mathsf{L}(\mathbb{W}_0)_0^\Gamma$ and $\mathsf{L}(\mathbb{W}_0)_\infty^\Gamma$ denote the respective fixed-point subalgebras. They contain the local Lie algebras $\mathsf{L}(\mathscr{M})_0^\Gamma = H_0^\Gamma \oplus (\mathfrak{h} \otimes \mathbb{C}((t)))^{\Gamma,0}$ and $\mathsf{L}(\mathscr{M})_\infty^\Gamma = H_\infty^\Gamma \oplus (\mathfrak{h} \otimes \mathbb{C}((t^{-1})))^{\Gamma,0}$ as subalgebras, respectively, using the notation of appendix B.2.

Every smooth module over $\mathsf{L}(\mathscr{M})_0^\Gamma$ on which $\mathbf{1}_0$ acts as $\frac{1}{T}$ becomes a smooth module over $\mathsf{L}(\mathbb{W}_0)_0^\Gamma$. See [VY16b, Proposition 5.8]. Likewise, every co-smooth module over $\mathsf{L}(\mathscr{M})_\infty^\Gamma$ on which $\mathbf{1}_\infty$ acts as $\frac{1}{T}$ becomes a co-smooth module over $\mathsf{L}(\mathbb{W}_0)_\infty^\Gamma$. See Proposition B.3.

Now the vacuum Verma module $\mathbb{V}_0^{-h^\vee}$ also has the structure of a vertex algebra. In the notation of [VY16b] recalled in appendix B, it corresponds to the vertex algebra $\mathbb{V}(\mathscr{L})$ where $\mathscr{L}$ is the vertex Lie algebra underlying the affine Lie algebra $\widehat{\mathfrak{g}}$, generated by $L^o = \mathfrak{g}$ with non-trivial n^{th}-products given by (see *e.g.* [VY16b, Example 2.5])

$$a_{(0)}b = [a, b], \qquad a_{(1)}b = -h^\vee \langle a, b\rangle K,$$

for any $a, b \in \mathfrak{g}$. The normalisation of the 1^{st}-product by $-h^\vee$ is chosen so as to conform with the convention adopted in [VY16b] that the central element $K \in \mathsf{L}(\mathscr{L})$ should act as 1 on all modules over $\mathsf{L}(\mathscr{L})$, including $\mathbb{V}(\mathscr{L})$.

The associated "big" Lie algebra $\mathsf{L}(\mathbb{V}_0^{-h^\vee})$ contains $\widehat{\mathfrak{g}} = \mathsf{L}(\mathscr{L})$ as a subalgebra, via the embedding sending $K \mapsto v_0(-1)$ and $A[n] \mapsto (A[-1]v_0)(n)$ for any $A \in \mathfrak{g}$ and $n \in \mathbb{Z}$. In this way, every smooth module over $\mathsf{L}(\mathbb{V}_0^{-h^\vee})$ pull back to a smooth module over $\widehat{\mathfrak{g}}$ of level $-h^\vee$. In the same way, $\mathsf{L}(\mathbb{V}_0^{-h^\vee})^\Gamma$ contains the twisted affine algebra $\widehat{\mathfrak{g}}^\Gamma$ as a subalgebra, so that smooth (resp. co-smooth right) modules over $\mathsf{L}(\mathbb{V}_0^{-h^\vee})^\Gamma$ pulls back to smooth (resp. co-smooth right) module over $\widehat{\mathfrak{g}}^\Gamma$ of level $-h^\vee/T$.

The *Feigin-Frenkel homomorphism*, or free field realisation of $\mathbb{V}_0^{-h^\vee}$, [FF90] (see also [Fre07]) is a homomorphism of vertex algebras

$$\rho : \mathbb{V}_0^{-h^\vee} \longrightarrow \mathbb{W}_0. \tag{4.15}$$

There is a $\mathbb{Z}$-grading on $\mathbb{W}_0$ defined by $\deg |\rangle = 0$ and $\deg a_\alpha[n] = \deg a_\alpha^*[n] = \deg b_i[n] = n$. There is a $\mathbb{Z}$-grading on $\mathbb{V}_0^{-h^\vee}$ defined by $\deg v_0 = 0$ and $\deg X[n] = n$ for $X \in \mathfrak{g}$. The homomorphism ρ respects these $\mathbb{Z}$-gradations (see e.g. [Fre07, §6.2.4]).

The homomorphism ρ induces a homomorphism of big Lie algebras $\mathsf{L}(\mathbb{V}_0^{-h^\vee})_{x_i} \to \mathsf{L}(\mathbb{W}_0)_{x_i}$ for every $i = 1, \ldots, p$. Moreover, by the equivariance of the homomorphism with respect to the action of σ on both vertex algebras [Sz02], it also induces homomorphisms of Γ-invariant subalgebras $\mathsf{L}(\mathbb{V}_0^{-h^\vee})_0^\Gamma \to \mathsf{L}(\mathbb{W}_0)_0^\Gamma$ and $\mathsf{L}(\mathbb{V}_0^{-h^\vee})_\infty^\Gamma \to \mathsf{L}(\mathbb{W}_0)_\infty^\Gamma$.

By means of this homomorphism, every smooth module over $\mathsf{L}(\mathscr{M})_{x_i} = H_{x_i} \oplus \mathfrak{h} \otimes \mathbb{C}((t - x_i))$ on which $\mathbf{1}_{x_i}$ acts as 1 becomes a smooth module of level $-h^\vee$ over the local copy $\widehat{\mathfrak{g}}_{x_i}$ of $\widehat{\mathfrak{g}}$ at x_i, for each $i = 1, \ldots, p$. Likewise, every smooth module over $\mathsf{L}(\mathscr{M})_0^\Gamma = H_0^\Gamma \oplus (\mathfrak{h} \otimes \mathbb{C}((t)))^{\Gamma,0}$ on which

$\mathbf{1}_0$ acts as $\frac{1}{T}$ becomes a smooth module of level $-h^\vee/T$ over $\widehat{\mathfrak{g}}_0^\Gamma$, and every co-smooth module over $\mathsf{L}(\mathscr{M}^{\mathsf{op}})^\Gamma_\infty = H^{\mathsf{op},\Gamma}_\infty \oplus (\mathfrak{h} \otimes \mathbb{C}((t^{-1})))^{\Gamma,0}$ on which $\mathbf{1}_\infty$ acts as $\frac{1}{T}$ becomes a co-smooth module over $\widehat{\mathfrak{g}}_\infty^{\mathsf{op},\Gamma}$ of level $-h^\vee/T$.

In particular, these statements apply to the Wakimoto modules W_{ν_i}, $i = 1, \ldots, p$, $W^\Gamma_{\nu_0}$ and $W^{\Gamma,\vee}_{\nu_\infty}$. We shall need the following facts about the structure of these modules.

For each $i = 1, \ldots, p$, let $\widetilde{W}_{\nu_i}$ denote the linear span of states of the form $a^*_{\alpha_1}[0] \ldots a^*_{\alpha_k}[0]|\rangle_{x_i}$, $k \in \mathbb{Z}_{\geq 0}$.

Lemma 4.2. *Suppose ν_i has at most a simple pole. Then the subspace $\widetilde{W}_{\nu_i}$ is stable under the action of the subalgebra $\mathfrak{g} \subset \widehat{\mathfrak{g}}_{x_i}$ generated by zero-modes, $X[0]_{x_i}$ with $X \in \mathfrak{g}$, and there is an isomorphism of left $U(\mathfrak{g})$-modules*

$$\widetilde{W}_{\nu_i} \cong M^*_{\mathrm{res}_{t-x_i} \nu_i}. \tag{4.16}$$

Moreover the subspace $\widetilde{W}_{\nu_i}$ is annihilated by all strictly positive modes $X[n]_{x_i}$, $n > 0$, $X \in \mathfrak{g}$.

Proof. See [FFR94, §5] for the isomorphism (4.16). Note that $a_\alpha[n]$ and $a^*_\alpha[n]$ act as zero on $\widetilde{W}_{\nu_i}$ for all $n > 0$. The fact that ν_i has at most a simple pole means that $b_k[n]$ acts as zero on $\widetilde{W}_{\nu_i}$ for all $n > 0$. The "moreover" part follows since Feigin-Frenkel homomorphism respects the $\mathbb{Z}$-grading of $\mathbb{V}_0^{-h^\vee}$ and $\mathbb{W}_0$ and their big Lie algebras. □

Let $\widetilde{W}^\Gamma_{\nu_0} \subset W^\Gamma_{\nu_0}$ denote the linear span of states of the form $a^*_{(0,\alpha_1)}[0] \ldots a^*_{(0,\alpha_k)}[0]|\rangle_0$, $k \in \mathbb{Z}_{\geq 0}$, $\alpha_1, \ldots, \alpha_k \in \Delta_0^+$.

Lemma 4.3. *Suppose ν_0 has at most a simple pole. Then the subspace $\widetilde{W}^\Gamma_{\nu_0}$ is stable under the action of the subalgebra $\mathfrak{g}^\sigma \subset \widehat{\mathfrak{g}}_0^\Gamma$ generated by zero-modes, $X[0]_0$ with $X \in \mathfrak{g}^\sigma$, and there is an isomorphism of left $U(\mathfrak{g}^\sigma)$-modules*

$$\widetilde{W}^\Gamma_{\nu_0} \cong M^{*,\sigma}_{\frac{1}{T}(\mathrm{res}_t(\nu_0) - \Lambda_0)} \tag{4.17}$$

where for $\lambda \in \mathfrak{h}^{*,\sigma}$ we denote by $M^{*,\sigma}_\lambda$ the contragredient Verma module over $\mathfrak{g}^\sigma$ of highest weight λ, and where Λ_0 is the weight given in (3.7).

Moreover the subspace $\widetilde{W}^\Gamma_{\nu_0}$ is annihilated by all strictly positive modes $X[n]_0$, $n > 0$, $X \in \mathfrak{g}$.

Proof. The first part is [VY16a, Proposition 4.4], and the "moreover" part is again on $\mathbb{Z}$-grading grounds. □

Concerning the Wakimoto module at infinity, we shall need two results. Recall the definition of χ in §3.5. The first result applies only to the case $\chi = 0$, so that $t^2\nu_\infty$ has at most a simple pole in t^{-1}.

Let $\widetilde{W}^{\Gamma,\vee}_{\nu_\infty} \subset W^{\Gamma,\vee}_{\nu_\infty}$ denote the span of vectors of the form $\langle|a_{(0,\alpha_1)}[0] \ldots a_{(0,\alpha_k)}[0]$, $k \in \mathbb{Z}_{\geq 0}$, $\alpha_1, \ldots, \alpha_k \in \Delta_0^+$.

Lemma 4.4. *Suppose $t^2\nu_\infty$ has at most a simple pole in t^{-1}. Then the subspace $\widetilde{W}^{\Gamma,\vee}_{\nu_\infty}$ is stable under the action of the subalgebra $\mathfrak{g}^\sigma \subset \widehat{\mathfrak{g}}^\Gamma_\infty$ generated by zero-modes, $X[0]_\infty$ with $X \in \mathfrak{g}^\sigma$, and there is an isomorphism of right $U(\mathfrak{g}^\sigma)$-modules (or equivalently left $U(\mathfrak{g}^{\sigma,\mathsf{op}})$ modules)*

$$\widetilde{W}^{\Gamma,\vee}_{\nu_\infty} \cong \mathrm{Hom}_\mathbb{C}(M^{*,\sigma}_{\frac{1}{T}(-\mathrm{res}_{t^{-1}}(t^2\nu_\infty) - \Lambda_0)}, \mathbb{C}) \cong M^{\sigma,\varphi}_{\frac{1}{T}(-\mathrm{res}_{t^{-1}}(t^2\nu_\infty) - \Lambda_0)}. \tag{4.18}$$

Moreover the subspace $\widetilde{W}^{\Gamma,\vee}_{\nu_\infty}$ *is annihilated by all strictly negative modes* $X[n]_\infty$*,* $n<0$*,* $X\in\mathfrak{g}^\sigma$*.*

Proof. The condition that $t^2\nu_\infty$ has at most a simple pole in t^{-1} means that $\langle| b_{(k,\alpha)}[n]=0$ for all $n<0$. We have $\langle| a_{(k,\alpha)}[n]=0$ and $\langle| a^*_{(k,\alpha)}[n]=0$ for all $n<0$. Therefore on $\mathbb{Z}$-grading grounds, for all $X\in\mathfrak{g}$, $X[n]$ acts as zero on $\widetilde{W}^{\Gamma,\vee}_{\nu_\infty}$ for all $n<0$, and the only terms in the free-field expressions for the zero-modes $X[0]$ which survive on vectors in $\widetilde{W}^{\Gamma,\vee}_{\nu_\infty}$ are those involving only the zero-modes of a's, a^*'s, and b's. In particular since only the zero-modes of b's contribute,

$$\widetilde{W}^{\Gamma,\vee}_{\nu_\infty}\cong\widetilde{W}^{\Gamma,\vee}_{-\lambda/t}\quad\text{where}\quad\lambda:=-\operatorname{res}_{t^{-1}}(t^2\nu_\infty),$$

as modules over the copy of algebra $\mathfrak{g}^\sigma$ generated by zero-modes.

Also on $\mathbb{Z}$-grading grounds, the only terms in the free-field expressions for the zero modes $X[0]$ which survive on vectors in $\widetilde{W}^{\Gamma}_{\lambda/t}$ are those involving only the zero modes of a's, a^*'s, and b's.

Now $W^{\Gamma,\vee}_{-\lambda/t}$ and $W^{\Gamma}_{\lambda/t}$ are Fock modules over a copy of $H(\mathfrak{g})^\Gamma$ as in (4.12) and (4.10). We have the natural pairing between them, namely the bilinear map $B: W^{\Gamma,\vee}_{-\lambda/t}\times W^{\Gamma}_{\lambda/t}\to\mathbb{C}$ defined by $B(\langle|,|\rangle)=1$ and $B(\langle|X,Y|\rangle)=B(\langle|,XY|\rangle)=B(\langle|XY,|\rangle)$ for $X,Y\in U(H(\mathfrak{g})^\Gamma)$. This pairing is non-degenerate. In this way the right $U(H(\mathfrak{g})^\Gamma)$-module $W^{\Gamma,\vee}_{-\lambda/t}$ is the (suitably restricted) dual of the left $U(H(\mathfrak{g})^\Gamma)$-module $W^{\Gamma}_{\lambda/t}$. The pairing restricts to a non-degenerate pairing of the grade zero subspaces, $\widetilde{W}^{\Gamma,\vee}_{-\lambda/t}$ and $\widetilde{W}^{\Gamma}_{\lambda/t}$. So $\widetilde{W}^{\Gamma,\vee}_{-\lambda/t}$ is the dual of $\widetilde{W}^{\Gamma}_{\lambda/t}$ as a module over the algebra of zero-modes of a's and a^*'s, and also of b's since $\langle| b_{(k,a)}[0]=\lambda(H_{(k,a)})\langle|$ and $b_{(k,a)}[0]|\rangle=|\rangle\lambda(H_{(k,a)})$.

We conclude that $\widetilde{W}^{\Gamma,\vee}_{-\lambda/t}$ is the dual $\operatorname{Hom}_{\mathbb{C}}(\widetilde{W}^{\Gamma}_{\lambda/t},\mathbb{C})$ of $\widetilde{W}^{\Gamma}_{\lambda/t}$ as a module over the algebra of zero-modes $X[0]$, $X\in\mathfrak{g}^\sigma$. And Lemma 4.3 shows that $\widetilde{W}^{\Gamma}_{\lambda/t}\cong M^{*,\sigma}_{\frac{1}{T}(\lambda-\Lambda_0)}$. The result follows. □

Second, we need a result valid for all χ.

Let $\mathbb{C}_\chi$ denote the one-dimensional representation of $(\mathfrak{g}^{\mathsf{op}}\otimes t^{-1}\mathbb{C}[[t^{-1}]])^\Gamma$ defined by

$$(\mathfrak{g}^{\mathsf{op}}\otimes t^{-1}\mathbb{C}[[t^{-1}]])^\Gamma\longrightarrow(\mathfrak{g}^{\mathsf{op}}\otimes t^{-1}\mathbb{C}[[t^{-1}]])^\Gamma/(\mathfrak{g}^{\mathsf{op}}\otimes t^{-2}\mathbb{C}[[t^{-1}]])^\Gamma\simeq_{\mathbb{C}}\Pi_{-1}\mathfrak{g}\xrightarrow{\chi}\mathbb{C}.$$

Lemma 4.5. *Suppose that* $t^2\nu_\infty$ *has at most a double pole in* t^{-1}*, with 2-residue* $-\chi$*. That is, suppose that*

$$\nu_\infty(t)=-\chi+\mathcal{O}(1/t)$$

Then there is an isomorphism of $(\mathfrak{g}^{\mathsf{op}}\otimes t^{-1}\mathbb{C}[[t^{-1}]])^\Gamma$*-modules*

$$\mathbb{C}\langle|\cong\mathbb{C}_\chi.\tag{4.19}$$

Proof. It follows from the explicit form of the Feigin-Frenkel homomorphism and the definition of the quasi-module map Y_W, cf. appendix B and in particular Proposition B.3, that the vector $\langle|$ of (4.13) obeys

$$\langle|(\mathfrak{g}\otimes t^{-2}\mathbb{C}[[t^{-1}]])^\Gamma=0,\qquad\langle|(\Pi_{-1}A)[-1]=\frac{1}{T}\chi(A)\langle|.$$

□

4.5. **The generators** $G_\alpha[n]$**.** The Verma module $M_\lambda=U(\mathfrak{g})\otimes_{U(\mathfrak{h}\oplus\mathfrak{n})}\mathbb{C}v_\lambda\cong_{\mathbb{C}}U(\mathfrak{n}_-)\otimes_{\mathbb{C}}\mathbb{C}v_\lambda$, §3.1, is by definition a left module over $U(\mathfrak{g})$. In particular it is a left module over the subalgebra $U(\mathfrak{n}_-)\subset U(\mathfrak{g})$. But M_λ also admits a second left action, call it $\triangleright$, of this subalgebra $U(\mathfrak{n}_-)$. These two left actions of $U(\mathfrak{n}_-)$ are mutually commuting. They are given by

$$X.(n \otimes v_\lambda) = Xn \otimes v_\lambda \quad \text{and} \quad X \triangleright (n \otimes v_\lambda) = -nX \otimes v_\lambda, \qquad X \in \mathfrak{n}_-, n \in U(\mathfrak{n}_-).$$

In particular

$$X.(1 \otimes v_\lambda) = -X \triangleright (1 \otimes v_\lambda). \tag{4.20}$$

Correspondingly the contragredient Verma module admits a second left action of $U(\mathfrak{n})$. We write G_α, $\alpha \in \Delta^+$, for the generators of this second copy of $\mathfrak{n}$, which we denote $\mathfrak{n}_{\langle G\rangle}$. So we have $[G_\alpha, E_\beta] = 0$ for all $\alpha, \beta \in \Delta^+$, where E_α are the generators of $\mathfrak{n}$ from §3.1.

We have the vacuum Verma module of $\mathfrak{n}_{\langle G\rangle} \otimes \mathbb{C}((t))$,

$$U(\mathfrak{n}_{\langle G\rangle} \otimes \mathbb{C}((t))) \otimes_{U(\mathfrak{n}_{\langle G\rangle} \otimes \mathbb{C}[[t]])} \mathbb{C}v_0 \tag{4.21}$$

where v_0 is a nonzero vector such that $\mathfrak{n}_{\langle G\rangle} \otimes \mathbb{C}[[t]]v_0 = 0$. It has a natural vertex algebra structure, in which $Y(A[-1]v_0, x) = \sum_{n\in\mathbb{Z}} A[n]x^{-n-1}$ for all $A \in \mathfrak{n}_{\langle G\rangle}$. There is a free-field realization of this vertex algebra, *i.e.* there is an injective homomorphism from this vertex algebra into $\mathbb{W}_0$. It is defined by

$$G_\alpha[-1]|\rangle = \sum_{\beta\in\Delta^+} R_\alpha^\beta(a^*[0])a_\beta[-1]|\rangle, \qquad \alpha \in \Delta^+,$$

for certain polynomials R_α^β homogeneous of grade $\beta - \alpha$.

By means of this free-field realization, the Wakimoto module W_{ν_i} becomes a module not only over $\widehat{\mathfrak{g}}_{x_i}$ but also over a copy of $\mathfrak{n}_{\langle G\rangle} \otimes \mathbb{C}((t-x_i))$ whose generators we denote $G_\alpha[n]_{x_i} := G_\alpha \otimes (t-x_i)^n$. With the obvious modifications, the statement of Lemma 4.2 holds for this copy of $\mathfrak{n}_{\langle G\rangle} \otimes \mathbb{C}((t-x_i))$. Namely, the subspace $\widetilde{W}_\nu$ is stable under the subalgebra $U(\mathfrak{n}_{\langle G\rangle}) \subset U(\mathfrak{n}_{\langle G\rangle} \otimes \mathbb{C}((t)))$ generated by zero modes, and (4.16) is an isomorphism of modules over $\mathfrak{n}_{\langle G\rangle}$, *i.e.* $G_\alpha[0]$ acts as G_α.

The analogous statements hold for $\mathfrak{n}^\sigma_{\langle G\rangle}$ too. Namely we have generators $G_{(0,\alpha)}$, $\alpha \in \Delta_0^+$ which act from the left on contragredient Verma modules $M_\lambda^{*,\sigma}$ over $\mathfrak{n}^\sigma$. These generators commute with the generators $E_{(0,\alpha)}$ of the standard left action of $U(\mathfrak{n}^\sigma)$.

Lemma 4.6. *The subspace $\widetilde{W}^\Gamma_{\nu_0}$ is stable under the action of the subalgebra $\mathfrak{n}^\sigma_{\langle G\rangle} \subset (\mathfrak{n}_{\langle G\rangle} \otimes \mathbb{C}((t)))^\Gamma$ generated by zero-modes, $X[0]_0$ with $X \in \mathfrak{n}^\sigma_{\langle G\rangle}$, and (4.17) is an isomorphism of left $U(\mathfrak{n}^\sigma_{\langle G\rangle})$-modules. Moreover the subspace $\widetilde{W}^\Gamma_{\nu_0}$ is annihilated by all strictly positive modes $X[n]_0$, $n > 0$, $X \in \mathfrak{n}_{\langle G\rangle}$.*

Proof. The first part of the proof of [VY16a, Proposition 4.4] shows that (4.17) is an isomorphism of left $U(\mathfrak{n}^\sigma)$ modules. The argument that it is an isomorphism of $U(\mathfrak{n}^\sigma_{\langle G\rangle})$-modules is line-by-line identical but with the polynomials P_α^β (which define the free-field realization of E's) replaced by R_α^β. □

4.6. Global Heisenberg algebra and coinvariants.

Let $H^\Gamma_{\infty,\boldsymbol{x},0}$ be the commutative Lie algebra

$$H^\Gamma_{\infty,\boldsymbol{x},0} := (\mathfrak{n}_\mathbb{C} \otimes \mathbb{C}_{\infty,\Gamma\boldsymbol{x},0}(t))^{\Gamma,0} \oplus (\mathfrak{n}^*_\mathbb{C} \otimes \mathbb{C}_{\infty,\Gamma\boldsymbol{x},0}(t))^{\Gamma,-1}.$$

That is, an element of $H^\Gamma_{\infty,\boldsymbol{x},0}$ is a pair $(f(t), g(t))$ where $f(t)$ is a rational function valued in $\mathfrak{n}_\mathbb{C}$ with poles at most at the points $0, x_1, \ldots, x_p, \infty$ and obeying the equivariance condition $f(\omega t) = \sigma f(t)$, and where $g(t)$ is a rational function valued in $\mathfrak{n}^*_\mathbb{C}$ with poles at most at the points $0, x_1, \ldots, x_p, \infty$

and obeying the equivariance condition $g(\omega t) = \omega^{-1} L_\sigma g(t)$. By virtue of the residue theorem there is an embedding of Lie algebras,

$$H^\Gamma_{\infty,\boldsymbol{x},0} \hookrightarrow H_{\infty,p,0},$$

by taking Laurent expansions, $f(t) \mapsto (-\iota_{t^{-1}}; \iota_{t-x_1}, \ldots, \iota_{t-x_p}; \iota_t) f(t)$. Cf. (2.8).

Let us write

$$\mathfrak{h}^\Gamma_{\infty,\boldsymbol{x},0} := \left(\mathfrak{h} \otimes \mathbb{C}_{0,\Gamma\boldsymbol{x},\infty}(t)\right)^{\Gamma,0} \quad \text{and} \quad \mathfrak{h}^{*,\Gamma}_{\infty,\boldsymbol{x},0} := \left(\mathfrak{h}^* \otimes \mathbb{C}_{0,\Gamma\boldsymbol{x},\infty}(t)\right)^{\Gamma,-1}.$$

There is an embedding of commutative Lie algebras

$$\mathfrak{h}^\Gamma_{\infty,\boldsymbol{x},0} \hookrightarrow \mathfrak{h}_{\infty,p,0}$$

and we have the following, which is [VY16a, Proposition 4.3] but now including the pole at ∞.

Lemma 4.7. *The space of coinvariants $\mathbb{C}v_{\nu_\infty} \otimes \bigotimes_{i=1}^p \mathbb{C}v_{\nu_i} \otimes \mathbb{C}v_{\nu_0} / \mathfrak{h}^\Gamma_{\infty,\boldsymbol{x},0}$ is one-dimensional if and only if there exists a $\nu(t) \in \mathfrak{h}^{*,\Gamma}_{\infty,\boldsymbol{x},0}$ such that*

$$(\nu_\infty; \nu_1, \ldots, \nu_p; \nu_0) = (-\iota_{t^{-1}}; \iota_{t-x_1}, \ldots, \iota_{t-x_p}; \iota_t)\nu(t).$$

Otherwise it is zero-dimensional. □

Define

$$H^+_{\infty,p,0} := \left(\mathfrak{n}_\mathbb{C} \otimes \mathbb{C}[[t^{-1}]]\right)^{\Gamma,0} \oplus \left(\mathfrak{n}^*_\mathbb{C} \otimes \mathbb{C}[[t^{-1}]]\right)^{\Gamma,-1} \oplus \bigoplus_{i=1}^p (\mathfrak{n}_\mathbb{C} \oplus \mathfrak{n}^*_\mathbb{C}) \otimes \mathbb{C}[[t-x_i]] \oplus \left(\mathfrak{n}_\mathbb{C} \otimes \mathbb{C}[[t]]\right)^{\Gamma,0} \oplus \left(\mathfrak{n}^*_\mathbb{C} \otimes \mathbb{C}[[t]]\right)^{\Gamma,-1} \oplus \mathbb{C}\mathbf{1}$$

Then

$$H_{\infty,p,0} = H^+_{\infty,p,0} + H^\Gamma_{\infty,\boldsymbol{x},0} \qquad \text{and} \qquad H^+_{\infty,p,0} \cap H^\Gamma_{\infty,\boldsymbol{x},0} = \{0\}.$$

We may regard $M^{\Gamma,\vee}_\infty$ as a left module over $U(H^{\Gamma,\mathsf{op}}_\infty)$ and then

$$\left(\mathsf{M}^{\Gamma,\vee}_\infty \otimes \bigotimes_{i=1}^p \mathsf{M}_{x_i} \otimes \mathsf{M}^\Gamma_0\right) = U(H_{\infty,p,0}) \otimes_{U(H^+_{\infty,p,0})} \mathbb{C}\, \langle| \otimes |\rangle_{x_1} \otimes \ldots \otimes |\rangle_{x_p} \otimes |\rangle_0 \tag{4.22}$$

Hence

$$\left(\mathsf{M}^{\Gamma,\vee}_\infty \otimes \bigotimes_{i=1}^p \mathsf{M}_{x_i} \otimes \mathsf{M}^\Gamma_0\right) / H^\Gamma_{\infty,\boldsymbol{x},0} \cong_\mathbb{C} \mathbb{C}\, \langle| \otimes |\rangle_{x_1} \otimes \ldots \otimes |\rangle_{x_p} \otimes |\rangle_0 \cong_\mathbb{C} \mathbb{C}.$$

Therefore for any $\nu(t) \in \mathfrak{h}^{*,\Gamma}_{\infty,\boldsymbol{x},0}$ the space of coinvariants

$$\left(W^{\Gamma,\vee}_{-\iota_{t^{-1}}\nu(t)} \otimes \bigotimes_{i=1}^p W_{\iota_{t-x_i}\nu(t)} \otimes W^\Gamma_{\iota_t\nu(t)}\right) / \left(H^\Gamma_{\infty,\boldsymbol{x},0} \oplus \mathfrak{h}^\Gamma_{\infty,\boldsymbol{x},0}\right) \tag{4.23}$$

has dimension one. That means there is a unique $\left(H^\Gamma_{\infty,\boldsymbol{x},0} \oplus \mathfrak{h}^\Gamma_{\infty,\boldsymbol{x},0}\right)$-invariant linear functional, call it $\tau_{\nu(t)}$,

$$\tau_{\nu(t)} : W^{\Gamma,\vee}_{-\iota_{t^{-1}}\nu(t)} \otimes \bigotimes_{i=1}^p W_{\iota_{t-x_i}\nu(t)} \otimes W^\Gamma_{\iota_t\nu(t)} \longrightarrow \mathbb{C} \tag{4.24}$$

normalised such that $\tau_{\nu(t)}(\,\langle| \otimes |\rangle_{x_1} \otimes \ldots \otimes |\rangle_{x_p} \otimes |\rangle_0) = 1$.

By functoriality, *i.e.* [VY16b, Theorem 6.2] and [VY16b, Corollary 6.6] generalised to the situation of Appendix B for coinvariants of a tensor product of modules including one attached to infinity, the functional $\tau_{\nu(t)}$ is also invariant under $\mathfrak{g}^{\Gamma}_{\infty,\boldsymbol{x},0}$.

4.7. **Proof of Theorem 3.2.** Let $c(1),\ldots,c(m)\in I$, $\lambda_1,\ldots,\lambda_N\in\mathfrak{h}^*$, $\lambda_0\in\mathfrak{h}^{*,\sigma}$ and χ be as in §3. Recall

$$(x_1,\ldots,x_p)=(z_1,\ldots,z_N,w_1,\ldots,w_m).$$

We now fix

$$\nu(t):=\chi+\sum_{r\in\mathbb{Z}_T}\left(\sum_{i=1}^{N}\frac{L^r_\sigma\lambda_i}{t-\omega^r z_i}-\sum_{j=1}^{m}\frac{L^r_\sigma\alpha_{c(j)}}{t-\omega^r w_j}\right)+\frac{T\lambda_0+\Lambda_0}{t}\in\mathfrak{h}^{*,\Gamma}_{\infty,\boldsymbol{x},0}, \tag{4.25}$$

where Λ_0 is as in (3.5).

The following is [FFR94, Lemma 2]. We write $G_i:=G_{\alpha_i}$. (Recall G_α for $\alpha\in\Delta^+$ a positive root was defined in §4.5.)

Lemma 4.8. *Let $\mu(t)$ be a highest weight of the form*

$$\mu(t)=-\frac{\alpha_i}{t}+\sum_{n=0}^{\infty}\mu^{(n)}t^n,$$

$\mu^{(n)}\in\mathfrak{h}^*$. Then the vector $G_i[-1]|\rangle\in W_{\mu(t)}$ is singular for $\widehat{\mathfrak{g}}$ (i.e. $X[n]G_i[-1]|\rangle=0$ for all $n\geq 0$ and all $X\in\mathfrak{g}$) if and only if

$$\langle\alpha_i,\mu^{(0)}\rangle=0.$$

□

Let $\nu_j^{(0)}$ be the constant term in the Laurent expansion of $\nu(t)$ about w_j, for $j=1,\ldots,m$. Note that the Bethe equations (3.8) are equivalent to the statement that

$$\langle\alpha_{c(j)},\nu_j^{(0)}\rangle=0,\quad j=1,\ldots,m. \tag{4.26}$$

By Lemma 4.8, under this condition the vector $G_{c(j)}[-1]|\rangle_{w_j}\in W_{\iota_{t-w_j}\nu(t)}$ is singular for $\widehat{\mathfrak{g}}_{(w_j)}$.

Henceforth, suppose that the Bethe equations are indeed satisfied. It follows that the linear functional

$$\psi_{\nu(t)}:W^{\Gamma,\vee}_{-\iota_{t^{-1}}\nu(t)}\otimes\bigotimes_{i=1}^{N}W_{\iota_{t-z_i}\nu(t)}\otimes W^{\Gamma}_{\iota_t\nu(t)}\to\mathbb{C} \tag{4.27}$$

defined by

$$\psi_{\nu(t)}(v_\infty,v_1,\ldots,v_N,v_0)=\tau_{\nu(t)}\big(v_\infty,v_1,\ldots,v_N,G_{c(1)}[-1]|\rangle_{w_1},\ldots,G_{c(m)}[-1]|\rangle_{w_m},v_0\big)$$

is invariant under the Lie subalgebra $\mathfrak{g}^{\Gamma}_{\infty,\boldsymbol{z},0}\subset\mathfrak{g}^{\Gamma}_{\infty,\boldsymbol{x},0}$.

Now we use the set-up of §2.4 and specialize by setting

$$\begin{aligned}
\mathcal{M}_{z_i}&=M^*_{\lambda_i}&&\cong\widetilde{W}_{\iota_{t-z_i}\nu(t)}\subset W_{\iota_{t-z_i}\nu(t)},\qquad i=1,\ldots,N,\\
\mathcal{M}_0&=M^{*,\sigma}_{\lambda_0}&&\cong\widetilde{W}^{\Gamma}_{\iota_t\nu(t)}\quad\subset W^{\Gamma}_{\iota_t\nu(t)},\\
\mathcal{M}_\infty&=\mathbb{C}_\chi&&\cong\mathbb{C}\,\langle|\qquad\subset W^{\Gamma,\vee}_{-\iota_{t^{-1}}\nu(t)}.
\end{aligned} \tag{4.28}$$

Here the isomorphisms are as in Lemmas 4.2, 4.3 and 4.5, given that we have $\operatorname{res}_{t-z_i}\nu(t)=\lambda_i$ and $\frac{1}{T}(\operatorname{res}_t \nu(t)-\Lambda_0)=\lambda_0$.

Let ψ denote the restriction of the linear functional $\psi_{\nu(t)}$ of (4.27) to the subspace

$$\mathbb{C}_\chi \otimes \bigotimes_{i=1}^N M^*_{\lambda_i} \otimes M^{*,\sigma}_{\lambda_0} \cong \mathbb{C}\langle| \otimes \bigotimes_{i=1}^N \widetilde{W}_{\iota_{t-z_i}\nu(t)} \otimes \widetilde{W}^{\Gamma}_{\iota_t\nu(t)}. \tag{4.29}$$

Such a functional is the same thing as a vector

$$\psi \in \bigotimes_{i=1}^N M_{\lambda_i} \otimes M^{\sigma}_{\lambda_0}. \tag{4.30}$$

To complete the proof of Theorem 3.2, we shall now establish that

(i) ψ is the Bethe vector defined in (3.3), and
(ii) it is an eigenstate of the Gaudin algebra with the given eigenvalue for the quadratic Hamiltonians.

We consider (i) first.

For notational convenience let us reorder tensor factors in the argument of $\tau_{\nu(t)}$ in such a way that

$$\tau_{\nu(t)} : W^{\Gamma,\vee}_{-\iota_{t^{-1}}\nu(t)} \otimes \bigotimes_{i=1}^N W_{\iota_{t-z_i}\nu(t)} \otimes W^{\Gamma}_{\iota_t\nu(t)} \otimes \bigotimes_{j=1}^m W_{\iota_{t-w_j}\nu(t)} \longrightarrow \mathbb{C} \tag{4.31}$$

Lemma 4.9. *For any $\boldsymbol{v}\otimes v_0 \in \bigotimes_{i=1}^N \widetilde{W}_{\iota_{t-z_i}\nu(t)} \otimes \widetilde{W}^{\Gamma}_{\iota_t\nu(t)}$ and any $X_1,\dots,X_s\in\mathfrak{n}_{\langle G\rangle}$, we have*

$$\begin{aligned}
&\tau_{\nu(t)}(\,\langle|, \boldsymbol{v}, v_0, X_1[-1]|\rangle_{w_1},\dots,X_{s-1}[-1]|\rangle_{w_{s-1}}, X_s[-1]|\rangle_{w_s}, |\rangle_{w_{s+1}},\dots,|\rangle_{w_m})\\
&\quad= \sum_{i=1}^N\sum_{k\in\mathbb{Z}_T} \frac{\tau_{\nu(t)}(\,\langle|, (\sigma^k X_s)[0]_{z_i}\boldsymbol{v}, v_0, X_1[-1]|\rangle_{w_1},\dots,X_{s-1}[-1]|\rangle_{w_{s-1}}, |\rangle_{w_s},\dots,|\rangle_{w_m})}{w_s-\omega^{-k}z_i}\\
&\qquad+\frac{\tau_{\nu(t)}(\,\langle|, \boldsymbol{v}, (T\Pi_0X_s)^W_{(0)}v_0, X_1[-1]|\rangle_{w_1},\dots,X_{s-1}[-1]|\rangle_{w_{s-1}}, |\rangle_{w_s},\dots,|\rangle_{w_m})}{w_s}\\
&\qquad+\sum_{j=1}^{s-1}\sum_{k\in\mathbb{Z}_T}\frac{1}{w_s-\omega^{-k}w_j}\tau_{\nu(t)}(\,\langle|, \boldsymbol{v}, v_0, X_1[-1]|\rangle_{w_1},\dots,X_{j-1}[-1]|\rangle_{w_{j-1}}, [\sigma^k(X_s),X_j][-1]|\rangle_{w_j},\\
&\qquad\qquad X_{j+1}[-1]|\rangle_{w_{j+1}},\dots,X_{s-1}[-1]|\rangle_{w_{s-1}}, |\rangle_{w_s},\dots,|\rangle_{w_m}).
\end{aligned}$$

Proof. Let us write $\boldsymbol{y}=X_1[-1]|\rangle_{w_1}\otimes\ldots\otimes X_{s-1}[-1]|\rangle_{w_{s-1}}\otimes|\rangle_{w_s}\otimes\ldots\otimes|\rangle_{w_m}\in\bigotimes_{j=1}^m W_{\iota_{t-w_j}\nu(t)}$ for brevity. We have

$$0=\sum_{k\in\mathbb{Z}_T}\left[\frac{\sigma^k X_s[-1]|\rangle}{\omega^{-k}t-w_s}.(\,\langle|\otimes\boldsymbol{v}\otimes v_0\otimes\boldsymbol{y})\right]. \tag{4.32}$$

The square brackets $[\cdot]$ on the right hand side denotes the class in the space of coinvariants

$$\left(W^{\Gamma,\vee}_{-\iota_{t^{-1}}\nu(t)} \otimes \bigotimes_{i=1}^N W_{\iota_{t-z_i}\nu(t)} \otimes W^{\Gamma}_{\iota_t\nu(t)} \otimes \bigotimes_{j=1}^m W_{\iota_{t-w_j}\nu(t)}\right)\Big/\left(H^{\Gamma}_{\infty,\boldsymbol{z},0,\boldsymbol{w}}\oplus\mathfrak{h}^{\Gamma}_{\infty,\boldsymbol{z},0,\boldsymbol{w}}\right),$$

and the equality in (4.32) follows from the straightforward generalisation of the isomorphism in [VY16b, Theorem 6.2] including the point at infinity. Now

$$\sum_{k\in\mathbb{Z}_T} \frac{\sigma^k X_s[-1]|\rangle}{\omega^{-k}t - w_s}.(\,\langle| \otimes \boldsymbol{v} \otimes v_0 \otimes \boldsymbol{y})$$
$$= \sum_{k\in\mathbb{Z}_T} \bigg(- \mathrm{res}_{t^{-1}}\, t^2 \iota_{t^{-1}} \frac{1}{\omega^{-k}t - w_s} \,\langle| Y_W(\sigma^k X_s[-1]|\rangle, t) \otimes \boldsymbol{v} \otimes v_0 \otimes \boldsymbol{y}$$
$$+ \,\langle| \otimes \sum_{i=1}^N \mathrm{res}_{t-z_i}\, \iota_{t-z_i} \frac{1}{\omega^{-k}t - w_s} Y_M(\sigma^k X_s[-1]|\rangle, t-z_i)_{z_i} \boldsymbol{v} \otimes v_0 \otimes \boldsymbol{y}$$
$$+ \,\langle| \otimes \boldsymbol{v} \otimes \mathrm{res}_t\, \iota_t \frac{1}{\omega^{-k}t - w_s} Y_W(\sigma^k X_s[-1]|\rangle, t) v_0 \otimes \boldsymbol{y}$$
$$+ \,\langle| \otimes \boldsymbol{v} \otimes v_0 \otimes \sum_{j=1}^m \mathrm{res}_{t-w_j}\, \iota_{t-w_j} \frac{1}{\omega^{-k}t - w_s} Y_M(\sigma^k X_s[-1]|\rangle, t-w_j)_{w_j} \boldsymbol{y} \bigg). \quad (4.33)$$

The res_{t-w_s} term here is $\langle|\otimes\boldsymbol{v}\otimes v_0 \otimes X_1[-1]|\rangle_{w_1} \otimes \ldots \otimes X_{s-1}[-1]|\rangle_{w_{s-1}} \otimes X_s[-1]|\rangle_{w_s} \otimes |\rangle_{w_{s+1}} \otimes \ldots \otimes |\rangle_{w_m}$. (The expansion of $1/(\omega^{-k}t - w_s)$ at $t = w_s$ has a regular part but it does not contribute because $(X_s[-1]|\rangle)(n)|\rangle = X_s[n]|\rangle$ is zero for all $n \geq 0$.)

Consider the first term on the right. We have

$$\mathrm{res}_{t^{-1}}\, t^2 \iota_{t^{-1}} \frac{1}{\omega^{-k}t - w_s} \,\langle| Y_W(\sigma^k X_s[-1]|\rangle, t) = \sum_{n\geq 0} w_s^n \omega^{k(n+1)} \,\langle| (\sigma^k X_s[-1]|\rangle)^W_{(-n-1)} \quad (4.34)$$

and this vanishes on $\mathbb{Z}$-grading grounds. (The free-field generators of $\mathfrak{n}_{\langle G\rangle}$ involve a, a^* but not b.) Consider the third term. We have

$$\mathrm{res}_t\, \iota_t \frac{1}{\omega^{-k}t - w_s} Y_W(\sigma^k X_s[-1]|\rangle, t) v_0 = -\sum_{n\geq 0} w_s^{-n-1} \omega^{-kn} (\sigma^k X_s[-1]|\rangle)^W_{(n)} v_0 \quad (4.35)$$

and on grading grounds only the term $n = 0$ in this sum contributes, since $v_0 \in \widetilde{W}_{\iota_t \nu(t)}$. The remaining terms are similar. □

Recall that we can regard an element of a Verma module M_λ as a linear map $M^*_\lambda \to \mathbb{C}$.

Lemma 4.10. *The restriction of the linear functional*

$$\tau_{\nu(t)}(\,\langle|, \cdot, \ldots, \cdot, \cdot, |\rangle_{w_1}, \ldots, |\rangle_{w_m}) : \bigotimes_{i=1}^N W_{\iota_{t-z_i}\nu(t)} \otimes W^\Gamma_{\iota_t \nu(t)} \to \mathbb{C}$$

to the subspace

$$\bigotimes_{i=1}^N \widetilde{W}_{\iota_{t-z_i}\nu(t)} \otimes \widetilde{W}^\Gamma_{\iota_t\nu(t)} \cong_{\mathfrak{g}^\sigma} \bigotimes_{i=1}^N M^*_{\lambda_i} \otimes M^{*,\sigma}_{\lambda_0}.$$

is equal to the tensor product

$$v_{\lambda_1} \otimes \ldots v_{\lambda_N} \otimes v_{\lambda_0}$$

of the highest weight vectors $v_{\lambda_i} \in M_{\lambda_i}$, $i = 1, \ldots, N$, and $v_{\lambda_0} \in M^\sigma_{\lambda_0}$.

Proof. The functional $\tau_{\nu(t)}(\,\langle|,\cdot,\ldots,\cdot,\cdot,|\rangle_{w_1},\ldots,|\rangle_{w_m})$ agrees with $v_{\lambda_1}\otimes\ldots v_{\lambda_N}\otimes v_{\lambda_0}$ on the vector $|\rangle_{z_1}\otimes\ldots\otimes|\rangle_{z_N}\otimes|\rangle_{z_0}$. Indeed, $\tau_{\nu(t)}$ was normalized such that

$$\tau_{\nu(t)}(\,\langle|,|\rangle_{z_1},\ldots,|\rangle_{z_N},|\rangle_{z_0},|\rangle_{w_1},\ldots,|\rangle_{w_m})=1$$

and by definition

$$(v_{\lambda_1}\otimes\ldots v_{\lambda_N}\otimes v_{\lambda_0})\,(|\rangle_{z_1},\ldots,|\rangle_{z_N},|\rangle_{z_0})=1.$$

It remains to show that the two agree on all other vectors in $\bigotimes_{i=1}^N\widetilde{W}_{\iota_{t-z_i}\nu(t)}\otimes\widetilde{W}^\Gamma_{\iota_t\nu(t)}$. For that it is enough to show that, for any $\boldsymbol{v}\otimes v_0\in\bigotimes_{i=1}^N\widetilde{W}_{\iota_{t-z_i}\nu(t)}\otimes\widetilde{W}^\Gamma_{\iota_t\nu(t)}$,

$$\tau_{\nu(t)}(\,\langle|,\boldsymbol{v},a^*_{(0,\alpha)}[0]v_0,|\rangle_{w_1},\ldots,|\rangle_{w_m})=0, \tag{4.36}$$

for any $\alpha\in\Delta_0^+$, and

$$\tau_{\nu(t)}(\,\langle|,a^*_\alpha[0]_{(z_i)}\boldsymbol{v},v_0,|\rangle_{w_1},\ldots,|\rangle_{w_m})=0,\qquad i=1,\ldots,N, \tag{4.37}$$

for any $\alpha\in\Delta^+$.

To establish the equality (4.36) we use the invariance of $\tau_{\nu(t)}$ under the Γ-equivariant function $E^*_{(0,\alpha)}\otimes t^{-1}\in(\mathfrak{n}^*_{\mathbb{C}}\otimes\mathbb{C}_{0,\Gamma\boldsymbol{x},\infty}(t))^{\Gamma,-1}$. Namely, we have

$$0=\tau_{\nu(t)}\left(\frac{E^*_{(0,\alpha)}}{t}.(\,\langle|\otimes\boldsymbol{v}\otimes v_0\otimes|\rangle_{w_1}\otimes\ldots\otimes|\rangle_{w_m})\right)$$

and here

$$\begin{aligned}
&\frac{E^*_{(0,\alpha)}}{t}.(\,\langle|\otimes\boldsymbol{v}\otimes v_0\otimes|\rangle_{w_1}\otimes\ldots\otimes|\rangle_{w_m})\\
&\qquad=\ \langle|a^*_{(0,\alpha)}[0]\otimes\boldsymbol{v}\otimes v_0\otimes|\rangle_{w_1}\otimes\ldots\otimes|\rangle_{w_m}\\
&\qquad+\sum_{i=1}^N\ \langle|\otimes\iota_{t-z_i}\frac{(E^*_{(0,\alpha)})^{(i)}}{t}\boldsymbol{v}\otimes v_0\otimes|\rangle_{w_1}\otimes\ldots\otimes|\rangle_{w_m}\\
&\qquad+\ \langle|\otimes\boldsymbol{v}\otimes a^*_{(0,\alpha)}[0]v_0\otimes|\rangle_{w_1}\otimes\ldots\otimes|\rangle_{w_m}\\
&\qquad+\sum_{i=1}^m\ \langle|\otimes\boldsymbol{v}\otimes v_0\otimes|\rangle_{w_1}\otimes\ldots\otimes\iota_{t-w_i}\frac{E^*_{(0,\alpha)}}{t}|\rangle_{w_i}\otimes\ldots\otimes|\rangle_{w_m}.
\end{aligned}\tag{4.38}$$

The first line is zero by definition of $\langle|$. The second and fourth lines are zero on $\mathbb{Z}$-grading grounds (recall that $E^*_\alpha\otimes t^0=a^*_\alpha[1]$). This leaves only the third line, and thus we have (4.36) as required. The proof of (4.37) is similar (and is as in [VY16a, FFR94, ATY91]). □

Recall the definition of G_α and $G_{(0,\alpha)}$ from §4.5. Let $F_{(0,\alpha)}:=\varphi(E_{(0,\alpha)})\in\mathfrak{n}^\sigma_-$, for $\alpha\in\Delta_0^+$, be the negative root vectors of $\mathfrak{g}^\sigma$.

Lemma 4.11.

(1) Let $v_\lambda\in M_\lambda$ be a highest weight vector. For any roots $\alpha^{(1)},\ldots,\alpha^{(k)}\in\Delta^+$ we have

$$v_\lambda(G_{\alpha^{(1)}}G_{\alpha^{(2)}}\ldots G_{\alpha^{(k)}}\cdot)=(-1)^k(F_{\alpha^{(1)}}F_{\alpha^{(2)}}\ldots F_{\alpha^{(k)}}v_\lambda)(\cdot).$$

(2) Let $v_{\lambda_0} \in M^\sigma_{\lambda_0}$ be a highest weight vector. For any $\mathfrak{g}^\sigma$-weights $\alpha^{(1)}, \ldots, \alpha^{(k)} \in \Delta_0^+$ we have

$$v_{\lambda_0}(G_{(0,\alpha^{(1)})} G_{(0,\alpha^{(2)})} \ldots G_{(0,\alpha^{(k)})} \cdot) = (-1)^k (F_{(0,\alpha^{(1)})} F_{(0,\alpha^{(2)})} \ldots F_{(0,\alpha^{(k)})} v_\lambda)(\cdot).$$

Proof. Consider part (1). It follows from (4.20) that $v_{\lambda_i}(E_\alpha \cdot) = -v_{\lambda_i}(G_\alpha \cdot)$. Because the two actions of $U(\mathfrak{n})$ commute, we therefore have

$$\begin{aligned} v_\lambda(G_{\alpha^{(1)}} G_{\alpha^{(2)}} \ldots G_{\alpha^{(k)}} \cdot) &= -v_\lambda(E_{\alpha^{(1)}} G_{\alpha^{(2)}} \ldots G_{\alpha^{(k)}} \cdot) \\ &= -v_\lambda(G_{\alpha^{(2)}} \ldots G_{\alpha^{(k)}} E_{\alpha^{(1)}} \cdot) = \cdots = (-1)^k v_\lambda(E_{\alpha^{(k)}} \ldots E_{\alpha^{(2)}} E_{\alpha^{(1)}} \cdot) \end{aligned}$$

for any roots $\alpha^{(1)}, \ldots, \alpha^{(k)} \in \Delta^+$. Part (1) follows by definition of the contragredient dual. For part (2) the argument is the same. □

Lemmas 4.9, 4.10 and 4.11 together imply that the weight function ψ of (4.30) is given recursively as in (3.3). Indeed, repeated use of the relations in Lemma 4.9 allow us to express

$$\tau_{\nu(t)}\big(\langle |, v_1, \ldots, v_N, v_0, G_{c(1)}[-1] |\rangle_{w_1}, \ldots, G_{c(m)}[-1] |\rangle_{w_m} \big)$$

as a linear combination of terms of the form

$$\tau_{\nu(t)}\big(\langle |, v_1', \ldots, v_N', v_0', |\rangle_{w_1}, \ldots, |\rangle_{w_m} \big)$$

for certain $v_0', v_1', \ldots, v_N'$, with each v_i', $1 \le i \le N$, of the form $G_{\alpha^{(1)}} \ldots G_{\alpha^{(k)}} |\rangle$ and v_0' of the form $G_{(0,\alpha^{(1)})} \ldots G_{(0,\alpha^{(k)})} |\rangle$. Then Lemma 4.10 tells us that $\tau_{\nu(t)}\big(\langle |, v_1', \ldots, v_N', |\rangle_{w_1}, \ldots, |\rangle_{w_m}, v_0 \big) = v_{\lambda_0}(v_0') \prod_{i=1}^N v_{\lambda_i}(v_i')$. So we can use Lemma 4.11 to exchange G's for F's. It follows that the vectors ψ in (3.3) and (4.30) coincide, because the structure of the recursive definition of ψ in (3.3) is chosen so as to match the relations of Lemma 4.9.

It remains to consider (ii), that is to show that ψ is an eigenstate of the Gaudin algebra, and to compute its eigenvalue for the quadratic Hamiltonians. The argument is exactly as in [VY16a, FFT10] following [FFR94]. For completeness, let us recall it. Let $u \in \mathbb{C}^\times$ be an additional nonzero point, with Γ-orbit disjoint from those of the x_i, $1 \le i \le p$. To the point u we assign a copy of the $H_u \oplus \mathfrak{h} \otimes \mathbb{C}((t-u))$-module $\mathbb{W}_0 = \mathsf{M} \otimes \pi_0$, cf. (4.14). Then we have a unique $\big(H^\Gamma_{\infty,\boldsymbol{x},u,0} \oplus \mathfrak{h}^\Gamma_{\infty,\boldsymbol{x},u,0}\big)$-invariant linear functional

$$\tau_{\nu(t)} : W^{\Gamma,\vee}_{-\iota_{t^{-1}}\nu(t)} \otimes \bigotimes_{i=1}^N W_{\iota_{t-z_i}\nu(t)} \otimes W^\Gamma_{\iota_t \nu(t)} \otimes \bigotimes_{j=1}^m W_{\iota_{t-w_j}\nu(t)} \otimes \mathbb{W}_0 \longrightarrow \mathbb{C}$$

normalised such that

$$\tau_{\nu(t)}(\langle | \otimes |\rangle_{z_1} \otimes \ldots \otimes |\rangle_{z_p} \otimes |\rangle_0 \otimes |\rangle_{w_1} \otimes \ldots \otimes |\rangle_{w_p} \otimes |\rangle_u) = 1.$$

(Compare (4.24) and (4.31).)

Let $Z \in \mathfrak{z}(\widehat{\mathfrak{g}}) \subset \mathbb{V}_0^k$ be a singular vector in the vacuum Verma module. Recall the Feigin-Frenkel homomorphism of vertex algebras, $\rho : \mathbb{V}_0^{-h^\vee} \to \mathbb{W}_0$ from (4.15). Now, for any vector

$$v \in \mathbb{C}_\chi \otimes \bigotimes_{i=1}^N M^*_{\lambda_i} \otimes M^{*,\sigma}_{\lambda_0}$$

we may evaluate the quantity

$$\tau_{\nu(t)}(v, G_{c(1)}[-1]|\rangle, \dots, G_{c(m)}[-1]|\rangle, \rho(Z)). \tag{4.39}$$

in two ways. First, the functional $\tau_{\nu(t)}$ is $\mathfrak{g}^{\Gamma}_{\infty,\boldsymbol{x},u,0}$-invariant. If the cyclotomic Bethe equations (3.8) hold then for each j, $1 \leq j \leq m$, the vector $G_{c(j)}[-1]|\rangle \in W_{\mu_j}$ is singular for the action of $\widehat{\mathfrak{g}}_{(w_j)}$, as in Lemma 4.8. Therefore

$$\begin{aligned}\tau_{\nu(t)}(v, G_{c(1)}[-1]|\rangle, \dots, G_{c(m)}[-1]|\rangle, \rho(Z)) &= \tau_{\nu(t)}(Z(u).v, G_{c(1)}[-1]|\rangle, \dots, G_{c(m)}[-1]|\rangle, |\rangle) \\ &= \psi(Z(u).v) = \left(\varphi(Z(u)).\psi\right)(v). \end{aligned}\tag{4.40}$$

On the other hand, using the invariance of the functional $\tau_{\nu(t)}$ under the rational function

$$\frac{1}{(n-1)!}\left(\frac{\partial}{\partial u}\right)^{n-1}\sum_{r\in\mathbb{Z}_T}\frac{\sigma^r b_s}{\omega^{-r}t-u} \in \mathfrak{h}^{\Gamma}_{\infty,\boldsymbol{x},u,0},$$

and the fact that $[b_s[n], G_{c(i)}[-1]] = 0$, we have that for all $w \in \mathbb{W}_0$,

$$\begin{aligned}&\tau_{\nu(t)}(v, G_{c(1)}[-1]|\rangle, \dots, G_{c(m)}[-1]|\rangle, b_s[-n]w) \\ &\qquad = \tau_{\nu(t)}(v, G_{c(1)}[-1]|\rangle, \dots, G_{c(m)}[-1]|\rangle, w)\left(\frac{1}{(n-1)!}\left(\frac{\partial}{\partial u}\right)^{n-1}\left\langle\alpha_s^\vee, \nu(u)\right\rangle\right). \end{aligned}\tag{4.41}$$

Here, cf. (4.25),

$$\left\langle\alpha_s^\vee, \nu(u)\right\rangle = \left\langle\alpha_s^\vee, \chi\right\rangle + \sum_{r\in\mathbb{Z}_T}\left(\sum_{i=1}^N\frac{\left\langle\alpha_s^\vee, L_\sigma^r\lambda_i\right\rangle}{u-\omega^r z_i} - \sum_{j=1}^m\frac{\left\langle\alpha_s^\vee, L_\sigma^r\alpha_{c(j)}\right\rangle}{u-\omega^r w_j}\right) + \frac{\left\langle\alpha_s^\vee, T\lambda_0+\Lambda_0\right\rangle}{u}.$$

It is known that $\rho(\mathfrak{Z}(\widehat{\mathfrak{g}})) \subset \pi_0$ [FFR94]. That means $\rho(Z)$ is a linear combination of terms of the form $b_{s_1}[-n_1]\dots b_{s_M}[-n_M]|\rangle$, $s_1, \dots, s_M \in I$, $n_1, \dots, n_M \in \mathbb{Z}_{\geq 1}$. Therefore, by repeated use of formula (4.41) one has that $\tau_{\nu(t)}(v, G_{c(1)}[-1]|\rangle, \dots, G_{c(m)}[-1]|\rangle, \rho(Z))$ is proportional to $\tau_{\nu(t)}(v, G_{c(1)}[-1]|\rangle, \dots, G_{c(m)}[-1]|\rangle, |\rangle)$. Together with (4.40), this establishes that ψ is an eigenvector of $\varphi(Z(u))$, as required.

In particular, consider the quadratic singular vector $S \in \mathfrak{Z}(\widehat{\mathfrak{g}})$ from (2.23). It is known (see e.g., [Fre07, §8.1.4]) that

$$\rho(S) = \frac{1}{2}\sum_{s,t\in I}\langle\omega_s, \omega_t\rangle b_s[-1]b_t[-1]|\rangle - \sum_{\alpha\in\Delta^+}\frac{\langle\alpha,\alpha\rangle}{4}b_\alpha[-2]|\rangle.$$

where ω_s, $s \in I$, are the fundamental weights. Hence

$$\begin{aligned}\varphi(S(u)).\psi &= \left(\frac{1}{2}\sum_{s,t\in I}\langle\omega_s, \omega_t\rangle\langle\alpha_s^\vee, \nu(u)\rangle\langle\alpha_t^\vee, \nu(u)\rangle - \sum_{\alpha\in\Delta^+}\frac{\langle\alpha,\alpha\rangle}{4}\langle\alpha^\vee, \nu'(u)\rangle\right)\psi \\ &= \left(\frac{1}{2}\langle\nu(u), \nu(u)\rangle - \langle\nu'(u), \varrho\rangle\right)\psi \end{aligned}\tag{4.42}$$

where $\varrho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$. Taking the residue in $u - z_i$ (which comes only from the term $\langle\nu(u), \nu(u)\rangle$) we obtain the eigenvalue of the quadratic Hamiltonians $\mathcal{H}_{i,0}$ given in Theorem 3.2. (In this calculation one needs the following steps

$$\sum_{s=1}^{T-1} \frac{\langle \lambda_k, L_\sigma^s \lambda_k \rangle}{z_k(1-\omega^s)} = \frac{1}{2} \sum_{s=1}^{T-1} \left(\frac{\langle \lambda_k, L_\sigma^s \lambda_k \rangle}{z_k(1-\omega^s)} + \frac{\langle L_\sigma^{-s} \lambda_k, \lambda_k \rangle \omega^{-s}}{z_k(\omega^{-s}-1)} \right)$$
$$= \frac{1}{2} \sum_{s=1}^{T-1} \left(\frac{\langle \lambda_k, L_\sigma^s \lambda_k \rangle}{z_k(1-\omega^s)} + \frac{\langle L_\sigma^{s} \lambda_k, \lambda_k \rangle \omega^{s}}{z_k(\omega^{s}-1)} \right) = \frac{1}{2} \sum_{s=1}^{T-1} \frac{\langle \lambda_k, L_\sigma^s \lambda_k \rangle}{z_k}$$

where the change of variable $s \mapsto -s$ was performed in the second sum.) This completes the proof of Theorem 3.2.

4.8. **Proof of Theorem 3.4.** We keep the notations of the previous section. Let us now suppose that $\chi = 0$. Then we have from (4.25) that

$$\iota_{t^{-1}} \nu(t) = 0 + \frac{T}{t} \left(\sum_{i=1}^N \Pi_0 \lambda_i - \sum_{j=1}^m \Pi_0 \alpha_{c(j)} + \lambda_0 + \Lambda_0 / T \right) + \mathcal{O}\left(\frac{1}{t^2}\right)$$
$$= 0 + \frac{T}{t} \left(\lambda_\infty + \Lambda_0/T \right) + \mathcal{O}\left(\frac{1}{t^2}\right)$$

where λ_∞ is as in (3.4).

Now we modify the construction of §2.4 as follows. Let $\mathcal{M}_\infty$ be a module over $(\mathfrak{g}^{\mathsf{op}} \otimes \mathbb{C}[[t^{-1}]])^\Gamma$ (rather than $(\mathfrak{g}^{\mathsf{op}} \otimes t^{-1}\mathbb{C}[[t^{-1}]])^\Gamma$ as in §2.4). We make it into a module $\mathcal{M}_\infty^{k/T}$ over $(\mathfrak{g}^{\mathsf{op}} \otimes \mathbb{C}[[t^{-1}]])^\Gamma \oplus \mathbb{C}K_\infty$ by declaring that K_∞ acts by multiplication by $k/T \in \mathbb{C}$. We have the induced right module $U(\widehat{\mathfrak{g}}_\infty^\Gamma)$ module of level k/T,

$$\mathbb{M}_\infty^{k/T} := \mathcal{M}_\infty^{k/T} \otimes_{U((\mathfrak{g} \otimes \mathbb{C}[[t^{-1}]])^\Gamma \oplus \mathbb{C}K_\infty)} U(\widehat{\mathfrak{g}}_\infty^\Gamma). \tag{4.43}$$

The tensor product

$$\mathbb{M} := \mathbb{M}_\infty^{k/T} \otimes \bigotimes_{i=1}^N \mathbb{M}_{z_i}^k \otimes \mathbb{M}_0^{k/T} \tag{4.44}$$

is again a module over $\widehat{\mathfrak{g}}_{\infty,N,0}$ on which K acts as k. Pulling back by the embedding (2.8), we have that $\mathbb{M}$ becomes a module over $\mathfrak{g}_{\infty,\boldsymbol{z},0}^\Gamma$ and we can form the space of coinvariants $\mathbb{M}/\mathfrak{g}_{\infty,\boldsymbol{z},0}^\Gamma$. Let us write

$$\widehat{\mathfrak{g}}_{\infty,N,0}^+ := (\mathfrak{g}^{\mathsf{op}} \otimes \mathbb{C}[[t^{-1}]])^\Gamma \oplus \bigoplus_{i=1}^N \mathfrak{g} \otimes \mathbb{C}[[t-z_i]] \oplus (\mathfrak{g} \otimes \mathbb{C}[[t]])^\Gamma. \tag{4.45}$$

We have the natural inclusion $\mathfrak{g}^\sigma \hookrightarrow \mathfrak{g}$ and hence the embedding

$$\mathfrak{g}^\sigma \longhookrightarrow \widehat{\mathfrak{g}}_{\infty,N,0}^+; \quad X \longmapsto (-X[0]_\infty; X[0]_{z_1}, \ldots, X[0]_{z_N}; X[0]_0). \tag{4.46}$$

Pulling back by this embedding, the tensor product

$$\mathcal{M} := \mathcal{M}_\infty \otimes \bigotimes_{i=1}^N \mathcal{M}_{z_i} \otimes \mathcal{M}_0 \tag{4.47}$$

is a module over $\mathfrak{g}^\sigma$ and we have the space of coinvariants $\mathcal{M}/\mathfrak{g}^\sigma$.

Proposition 4.12. *There is a canonical isomorphism of vector spaces*

$$\mathbb{M}/\mathfrak{g}_{\infty,\boldsymbol{z},0}^\Gamma \cong_{\mathbb{C}} \mathcal{M}/\mathfrak{g}^\sigma.$$

Proof. We have
$$\mathbb{M} = U(\widehat{\mathfrak{g}}_{\infty,N,0}) \otimes_{U(\widehat{\mathfrak{g}}^+_{\infty,N,0})} \mathcal{M}.$$
Here we regard $\mathcal{M}_\infty$ as a left module over $U(\mathfrak{g}^{\sigma,\mathsf{op}})$.

There are natural embeddings of Lie algebras $\mathfrak{g}^\sigma \hookrightarrow \widehat{\mathfrak{g}}^+_{\infty,N,0}$ and $\mathfrak{g}^\sigma \hookrightarrow \mathfrak{g}^\Gamma_{\infty,\boldsymbol{z},0}$; the first is given in (4.46); for the second, $X \in \mathfrak{g}^\sigma$ embeds as the constant function $X(t) = X$ in $\mathfrak{g}^\Gamma_{\infty,\boldsymbol{z},0}$. In turn, both $\widehat{\mathfrak{g}}^+_{\infty,N,0}$ and $\mathfrak{g}^\Gamma_{\infty,\boldsymbol{z},0}$ embed into $\widehat{\mathfrak{g}}_{\infty,N,0}$ – see (2.8) – and the following diagram of embeddings commutes
$$\begin{tikzcd} \mathfrak{g}^\sigma \arrow[r, hook] \arrow[d, hook] & \widehat{\mathfrak{g}}^+_{\infty,N,0} \arrow[d, hook] \\ \mathfrak{g}^\Gamma_{\infty,\boldsymbol{z},0} \arrow[r, hook] & \widehat{\mathfrak{g}}_{\infty,N,0}. \end{tikzcd}$$

We may identify $\mathfrak{g}^\sigma$, $\widehat{\mathfrak{g}}^+_{\infty,N,0}$ and $\mathfrak{g}^\Gamma_{\infty,\boldsymbol{z},0}$ with their images in $\widehat{\mathfrak{g}}_{\infty,N,0}$. Then
$$\widehat{\mathfrak{g}}_{\infty,N,0} = \widehat{\mathfrak{g}}^+_{\infty,N,0} + \mathfrak{g}^\Gamma_{\infty,\boldsymbol{z},0} \quad \text{and} \quad \widehat{\mathfrak{g}}^+_{\infty,N,0} \cap \mathfrak{g}^\Gamma_{\infty,\boldsymbol{z},0} = \mathfrak{g}^\sigma.$$
Therefore [Di74, Proposition 2.2.9] there is an isomorphism
$$U(\widehat{\mathfrak{g}}_{\infty,N,0}) \cong U(\mathfrak{g}^\Gamma_{\infty,\boldsymbol{z},0}) \otimes_{U(\mathfrak{g}^\sigma)} U(\widehat{\mathfrak{g}}^+_{\infty,N,0})$$
of vector spaces and in fact of left $U(\mathfrak{g}^\Gamma_{\infty,\boldsymbol{z},0})$-modules. Hence we have an isomorphism of left $U(\mathfrak{g}^\Gamma_{\infty,\boldsymbol{z},0})$-modules,
$$\mathbb{M} \cong U(\mathfrak{g}^\Gamma_{\infty,\boldsymbol{z},0}) \otimes_{U(\mathfrak{g}^\sigma)} U(\widehat{\mathfrak{g}}^+_{\infty,N,0}) \otimes_{U(\widehat{\mathfrak{g}}^+_{\infty,N,0})} \mathcal{M} = U(\mathfrak{g}^\Gamma_{\infty,\boldsymbol{z},0}) \otimes_{U(\mathfrak{g}^\sigma)} \mathcal{M}.$$
The result follows by the following elementary lemma. □

Lemma 4.13. *Suppose $\mathfrak{b} \hookrightarrow \mathfrak{a}$ is an embedding of complex Lie algebras. Let V be a $\mathfrak{b}$-module and M an $\mathfrak{a}$-module. Then $((U(\mathfrak{a}) \otimes_{U(\mathfrak{b})} V) \otimes M)/\mathfrak{a} \cong_{\mathbb{C}} (V \otimes M)/\mathfrak{b}$.*

In particular (taking M to be the trivial one-dimensional module), $(U(\mathfrak{a}) \otimes_{U(\mathfrak{b})} V)/\mathfrak{a} \cong_{\mathbb{C}} V/\mathfrak{b}$.

Proof. Consider the linear map $V \otimes M \cong_{\mathbb{C}} \big(U(\mathfrak{b}) \otimes_{U(\mathfrak{b})} V\big) \otimes M \hookrightarrow \big(U(\mathfrak{a}) \otimes_{U(\mathfrak{b})} V\big) \otimes M \twoheadrightarrow \big(\big(U(\mathfrak{a}) \otimes_{U(\mathfrak{b})} V\big) \otimes M\big)/\mathfrak{a}$ sending $v \otimes m \mapsto [(1 \otimes v) \otimes m]$. This map is surjective. We must show that it has kernel $\mathfrak{b}.(V \otimes M)$. And indeed, the kernel is the intersection of $\big(U(\mathfrak{b}) \otimes_{U(\mathfrak{b})} V\big) \otimes M$ with $\mathfrak{a}.\big(\big(U(\mathfrak{a}) \otimes_{U(\mathfrak{b})} V\big) \otimes M\big)$, which is $\mathfrak{b}.\big(\big(U(\mathfrak{b}) \otimes_{U(\mathfrak{b})} V\big) \otimes M\big) \cong_{\mathbb{C}} \mathfrak{b}.(V \otimes M)$. □

We may now choose modules – cf. (4.28) –
$$\begin{aligned} \mathcal{M}_{z_i} &= M^*_{\lambda_i} \cong \widetilde{W}_{\iota_{t-z_i}\nu(t)}, \qquad i = 1, \ldots, N, \\ \mathcal{M}_0 &= M^{*,\sigma}_{\lambda_0} \cong \widetilde{W}^\Gamma_{\iota_t\nu(t)}, \\ \mathcal{M}_\infty &= M^{\sigma,\varphi}_{\lambda_\infty} \cong \widetilde{W}^{\Gamma,\vee}_{-\iota_{t^{-1}}\nu(t)}, \end{aligned} \tag{4.48}$$
where the isomorphisms are as in Lemma 4.2, Lemma 4.3 and, now, Lemma 4.4. The embedding of $\mathfrak{g}$-modules $M^*_{\lambda_i} \cong \widetilde{W}_{\iota_{t-z_i}\nu(t)} \hookrightarrow W_{\iota_{t-z_i}\nu(t)}$ extends to an embedding of modules over $\mathfrak{g}\otimes\mathbb{C}[[t-z_i]]\oplus\mathbb{C}K$ of level $-h^\vee$. (Positive modes $A[n]_{z_i}$, $n > 0$, $A \in \mathfrak{g}$, act as zero on $\widetilde{W}_{\iota_{t-z_i}\nu(t)}$ as in Lemma 4.2.) $\mathbb{M}^{-h^\vee}_{z_i}$ is by definition the $\widehat{\mathfrak{g}}_{x_i}$-module of level $-h^\vee$ induced from $\mathcal{M}_{z_i} = M^*_{\lambda_i}$. So we get a canonical homomorphism $\mathbb{M}^{-h^\vee}_{z_i} \to W_{\iota_{t-z_i}\nu(t)}$ of $\widehat{\mathfrak{g}}_{z_i}$-modules of level $-h^\vee$. The marked points 0 and ∞ work

similarly. In total we get a canonical homomorphism of $\widehat{\mathfrak{g}}_{\infty,p,0}$-modules

$$\mathbb{M}_{\infty}^{-h^\vee/T} \otimes \bigotimes_{i=1}^{N} \mathbb{M}_{z_i}^{-h^\vee} \otimes \mathbb{M}_0^{-h^\vee/T} \to W^{\Gamma,\vee}_{-\iota_{t^{-1}}\nu(t)} \otimes \bigotimes_{i=1}^{N} W_{\iota_{t-z_i}\nu(t)} \otimes W^{\Gamma}_{\iota_t\nu(t)}.$$

Hence, from the $\mathfrak{g}^{\Gamma}_{\infty,\boldsymbol{z},0}$-invariant functional $\psi_{\nu(t)}$ of (4.27), we obtain a $\mathfrak{g}^{\Gamma}_{\infty,\boldsymbol{z},0}$-invariant functional $\mathbb{M}_{\infty}^{-h^\vee/T} \otimes \bigotimes_{i=1}^{N} \mathbb{M}_{z_i}^{-h^\vee} \otimes \mathbb{M}_0^{-h^\vee/T} \to \mathbb{C}$. By Proposition 4.12 this is the same thing as a $\mathfrak{g}^{\sigma}$-invariant functional

$$M^{\sigma,\varphi}_{\lambda_\infty} \otimes \bigotimes_{i=1}^{N} M^*_{\lambda_i} \otimes M^{*,\sigma}_{\lambda_0} \to \mathbb{C}. \tag{4.49}$$

Recall that we are embedding $\mathfrak{g}^\sigma \hookrightarrow \mathfrak{g}^{\sigma,\mathsf{op}} \oplus \bigoplus_{i=1}^N \mathfrak{g} \oplus \mathfrak{g}^\sigma$ by $X \mapsto (-X, X, \ldots, X, X)$ and that the antipode map $S : U(\mathfrak{g}^\sigma) \to U(\mathfrak{g}^\sigma)$ is defined by $S(X) = -X$ for $X \in \mathfrak{g}^\sigma$. So the space of $\mathfrak{g}^\sigma$-invariant functionals $M^{\sigma,\varphi}_{\lambda_\infty} \otimes \bigotimes_{i=1}^{N} M^*_{\lambda_i} \otimes M^{*,\sigma}_{\lambda_0} \to \mathbb{C}$ is the space

$$\mathrm{Hom}_{\mathfrak{g}^\sigma}\left((M^{\sigma}_{\lambda_\infty})^{\varphi\circ S} \otimes \bigotimes_{i=1}^{N} M^*_{\lambda_i} \otimes M^{*,\sigma}_{\lambda_0}, \mathbb{C}\right).$$

Proposition 4.14. *There is an isomorphism of vector spaces, and in fact of left $(U(\mathfrak{g})^{\otimes N} \otimes U(\mathfrak{g}^\sigma))^{\mathfrak{g}^\sigma}$-modules,*

$$\left(\bigotimes_{i=1}^{N} M_{\lambda_i} \otimes M^{\sigma}_{\lambda_0}\right)^{\mathfrak{n}^\sigma}_{\lambda_\infty} \cong \mathrm{Hom}_{\mathfrak{g}^\sigma}\left((M^{\sigma}_{\lambda_\infty})^{\varphi\circ S} \otimes \bigotimes_{i=1}^{N} M^*_{\lambda_i} \otimes M^{*,\sigma}_{\lambda_0}, \mathbb{C}\right) \tag{4.50}$$

Proof. We have

$$\begin{aligned}
\left(\bigotimes_{i=1}^{N} M_{\lambda_i} \otimes M^{\sigma}_{\lambda_0}\right)^{\mathfrak{n}^\sigma}_{\lambda_\infty} &= \mathrm{Hom}_{\mathfrak{h}^\sigma\oplus\mathfrak{n}^\sigma}\left(\mathbb{C}v_{\lambda_\infty}, \bigotimes_{i=1}^{N} M_{\lambda_i} \otimes M^{\sigma}_{\lambda_0}\right)\\
&\cong \mathrm{Hom}_{\mathfrak{g}^\sigma}\left(U(\mathfrak{g}^\sigma) \otimes_{U(\mathfrak{h}^\sigma\oplus\mathfrak{n}^\sigma)} \mathbb{C}v_{\lambda_\infty}, \bigotimes_{i=1}^{N} M_{\lambda_i} \otimes M^{\sigma}_{\lambda_0}\right)\\
&= \mathrm{Hom}_{\mathfrak{g}^\sigma}\left(M^{\sigma}_{\lambda_\infty}, \bigotimes_{i=1}^{N} M_{\lambda_i} \otimes M^{\sigma}_{\lambda_0}\right).
\end{aligned} \tag{4.51}$$

This is the space of those maps $\phi : M^{\sigma}_{\lambda_\infty} \to \bigotimes_{i=1}^{N} M_{\lambda_i} \otimes M^{\sigma}_{\lambda_0}$ such that

$$\phi(x.v) = (\Delta^{N+1}x).\phi(v), \quad v \in M^{\sigma}_{\lambda_\infty}, x \in \mathfrak{g}^\sigma. \tag{4.52}$$

It carries a natural left action of $(U(\mathfrak{g})^{\otimes N} \otimes U(\mathfrak{g}^\sigma))^{\mathfrak{g}^\sigma}$: given one such a map ϕ, the map $v \mapsto X\phi(v)$ is another, for any $X \in (U(\mathfrak{g})^{\otimes N} \otimes U(\mathfrak{g}^\sigma))^{\mathfrak{g}^\sigma}$.

Then

$$\mathrm{Hom}_{\mathfrak{g}^\sigma}\left(M^{\sigma}_{\lambda_\infty}, \bigotimes_{i=1}^{N} M_{\lambda_i} \otimes M^{\sigma}_{\lambda_0}\right) \cong \mathrm{Hom}_{\mathfrak{g}^\sigma}\left(M_{\lambda_\infty} \otimes \left(\bigotimes_{i=1}^{N} M_{\lambda_i} \otimes M^{\sigma}_{\lambda_0}\right)^{\vee,S}, \mathbb{C}\right) \tag{4.53}$$

where for a finitely-weighted left $U(\mathfrak{g}^\sigma)$-module M we denote by $M^{\vee,S}$ its restricted dual made into a left $U(\mathfrak{g}^\sigma)$-module by twisting by the antipode map S. Indeed, given a map ϕ such that (4.52) holds we define a map $\Phi : M_{\lambda_\infty} \otimes \left(\bigotimes_{i=1}^N M_{\lambda_i} \otimes M^\sigma_{\lambda_0}\right)^{\vee,S} \to \mathbb{C}$ by $\Phi(v\otimes\mu) = \mu(\phi(v))$, $v \in M_{\lambda_\infty}$, $\mu \in \left(\bigotimes_{i=1}^N M_{\lambda_i} \otimes M^\sigma_{\lambda_0}\right)^{\vee,S}$. It obeys $\Phi(x.(v\otimes\mu)) = 0$ where $x.(v\otimes\mu) := (x.v)\otimes\mu + v\otimes(\mu\circ(-\Delta^{N+1}x.))$, for $x \in \mathfrak{g}^\sigma$.

Finally, we may twist the action of $\mathfrak{g}^\sigma$ on $M_{\lambda_\infty} \otimes \left(\bigotimes_{i=1}^N M_{\lambda_i} \otimes M^\sigma_{\lambda_0}\right)^{\vee,S}$ by the automorphism $S\circ\varphi = \varphi\circ S$. Since $S^2 = \mathrm{id}$ here, $(M_\lambda^{\vee,S})^{S\circ\varphi} = M_\lambda^{\vee,\varphi} =: M^*_\lambda$. Thus

$$\mathrm{Hom}_{\mathfrak{g}^\sigma}\left(M^\sigma_{\lambda_\infty} \otimes \left(\bigotimes_{i=1}^N M_{\lambda_i} \otimes M^\sigma_{\lambda_0}\right)^{\vee,S}, \mathbb{C}\right) \cong \mathrm{Hom}_{\mathfrak{g}^\sigma}\left((M^\sigma_{\lambda_\infty})^{\varphi\circ S} \otimes \bigotimes_{i=1}^N M^*_{\lambda_i} \otimes M^{*,\sigma}_{\lambda_0}, \mathbb{C}\right) \tag{4.54}$$

as required. □

Thus, the functional of (4.49) defines an element of $\left(\bigotimes_{i=1}^N M_{\lambda_i} \otimes M^\sigma_{\lambda_0}\right)^{\mathfrak{n}^\sigma}_{\lambda_\infty}$. At the same time, by (4.51), this functional is uniquely determined by its restriction to the subspace

$$\mathbb{C}v_{\lambda_\infty} \otimes \bigotimes_{i=1}^N M^*_{\lambda_i} \otimes M^{*,\sigma}_{\lambda_0} \cong \mathbb{C}\langle| \otimes \bigotimes_{i=1}^N \widetilde{W}_{\iota_{t-z_i}\nu(t)} \otimes \widetilde{W}^\Gamma_{\iota_t\nu(t)}.$$

By definition, on this subspace it agrees with the functional of (4.29). In other words, it defines the same vector $\psi \in \bigotimes_{i=1}^N M_{\lambda_i} \otimes M^\sigma_{\lambda_0}$ as in (4.30). We saw in the previous section that ψ vector of (4.30) is the weight function defined in (3.3). So we have shown that the weight function ψ is singular when $\chi = 0$, completing the proof of Theorem 3.4.

Appendix A. Proof of Proposition 2.5

Let us write

$$\partial_x^{(n)} := \frac{1}{n!}\left(\frac{\partial}{\partial x}\right)^n$$

Consider the Γ-equivariant rational function

$$f(t) = \sum_{k=0}^{T-1} \frac{\sigma^k A}{(\omega^{-k}t - u)^{p+1}} \in \mathfrak{g}^\Gamma_{\infty,\boldsymbol{z},u,0}. \tag{A.1}$$

The expansion of (A.1) at u is

$$\iota_{t-u} f(t) = A[-p-1]_u + \sum_{k=1}^{T-1}\sum_{n=0}^{\infty} \begin{bmatrix} p+n \\ n \end{bmatrix} \frac{(-1)^{p+1}\omega^{k(p+1)}}{(\omega^k-1)^{n+p+1}u^{n+p+1}} (\sigma^k A)[n]_u \in \mathfrak{g}\otimes\mathbb{C}((t-u)). \tag{A.2}$$

Its expansions at z_i, $i = 1, \dots, N$, and at 0 are given by

$$\iota_{t-z_i} f(t) = -\partial_u^{(p)} \sum_{k=0}^{T-1}\sum_{n=0}^{\infty} \frac{\omega^{-kn}}{(u-\omega^{-k}z_i)^{n+1}} (\sigma^k A)[n]_{z_i} \in \mathfrak{g}\otimes\mathbb{C}[[t-z_i]], \tag{A.3}$$

$$\iota_t f(t) = -T\partial_u^{(p)} \sum_{n=0}^{\infty} \frac{1}{u^{n+1}} (\Pi_n A)[n]_0 \in \mathfrak{g}\otimes\mathbb{C}[[t]], \tag{A.4}$$

where, recall, $A[n]_{z_i} = A \otimes (t - z_i)^n \in \mathfrak{g} \otimes \mathbb{C}((t - z_i))$. It is regular at ∞ and its expansion there is given by

$$-\iota_{t^{-1}} f(t) = -\partial_u^{(p)} \sum_{k=0}^{T} \sum_{n=0}^{\infty} (\omega^k u)^n \omega^k \sigma^k A[-n-1]_\infty = -T \partial_u^{(p)} \sum_{n=0}^{\infty} u^n (\Pi_{-n-1} A)[-n-1]_\infty, \quad \text{(A.5)}$$

where recall that for the modes at infinity we write $A[n]_\infty = A \otimes t^n$ in our conventions.

Thus, since $[f(t).(x \otimes v_0)] = 0$, we have

$$\begin{aligned}\big[S(u)x \otimes v_0\big] &:= \big[x \otimes \frac{1}{2} I^a[-1] I_a[-1] v_0\big] = \big[I^a(u) x \otimes \tfrac{1}{2} I_a[-1] v_0\big] \\ &+ \sum_{p=1}^{T-1} \frac{\omega^p}{(\omega^p - 1)u} \big[x \otimes \tfrac{1}{2} (\sigma^p I^a)[0] I_a[-1] v_0\big] + \sum_{p=1}^{T-1} \frac{\omega^p}{(\omega^p - 1)^2 u^2} \big[x \otimes \tfrac{1}{2} (\sigma^p I^a)[1] I_a[-1] v_0\big]\end{aligned} \quad \text{(A.6)}$$

where, putting the expansions at $0, z_1, \ldots, z_N, \infty$ together, we shall write, for any $A \in \mathfrak{g}$,

$$A(u) := \left(\sum_{i=1}^{N} \sum_{n=0}^{\infty} \sum_{k=0}^{T-1} \frac{\omega^{-kn} (\sigma^k A)[n]_{z_i}}{(u - \omega^{-k} z_i)^{n+1}} + T \sum_{n=0}^{\infty} \frac{(\Pi_n A)[n]_0}{u^{n+1}} + T \sum_{n=0}^{\infty} u^n (\Pi_{-n-1} A)[-n-1]_\infty \right). \quad \text{(A.7)}$$

The equality in (A.6) is an example of what was called "cyclotomic swapping" in [VY16a]. Then, defining the element $F \in \mathfrak{g}^\sigma$ and number K as in (2.22), we see after one further such "swapping" that

$$S(u) = \frac{1}{2} I_a(u) I^a(u) + \frac{1}{u} F(u) + \frac{1}{u^2} K. \quad \text{(A.8)}$$

Consider first the terms in $\frac{1}{2} I_a(u) I^a(u)$ acting at sites $i, j \in \{1, \ldots, N\}$, $i \neq j$. Let us write

$$z_{ik} := \omega^{-k} z_i \quad \text{and} \quad \partial_{ik}^{(n)} = \frac{1}{n!} \left(\frac{\partial}{\partial z_{ik}} \right)^n. \quad \text{(A.9)}$$

Then $\frac{1}{(u - z_{ik})^{n+1}} = \partial_{ik}^{(n)} \frac{1}{u - z_{ik}}$ and we get the terms

$$\begin{aligned}&\frac{1}{2} \sum_{n,m=0}^{\infty} \sum_{\substack{i,j=1 \\ j \neq i}}^{N} \sum_{k,l=0}^{T-1} \partial_{ik}^{(n)} \partial_{jl}^{(m)} \frac{1}{u - z_{ik}} \frac{1}{u - z_{jl}} \omega^{-kn-lm} (\sigma^k I_a)[n]_{z_i} (\sigma^l I^a)[m]_{z_j} \\ &= \frac{1}{2} \sum_{n,m=0}^{\infty} \sum_{\substack{i,j=1 \\ j \neq i}}^{N} \sum_{k,l=0}^{T-1} \partial_{ik}^{(n)} \partial_{jl}^{(m)} \left(\frac{1}{u - z_{ik}} \frac{1}{z_{ik} - z_{jl}} + \frac{1}{u - z_{jl}} \frac{1}{z_{jl} - z_{ik}} \right) \omega^{-kn-lm} (\sigma^k I_a)[n]_{z_i} (\sigma^l I^a)[m]_{z_j} \\ &= \frac{1}{2} \sum_{n,m=0}^{\infty} \sum_{\substack{i,j=1 \\ j \neq i}}^{N} \sum_{k,l=0}^{T-1} \partial_{ik}^{(n)} \partial_{jl}^{(m)} \frac{1}{u - z_{ik}} \frac{1}{z_{ik} - z_{jl}} \omega^{-kn-lm} \left\{ (\sigma^k I_a)[n]_{z_i}, (\sigma^l I^a)[m]_{z_j} \right\} \\ &= \sum_{n,m=0}^{\infty} \sum_{\substack{i,j=1 \\ j \neq i}}^{N} \sum_{k,l=0}^{T-1} \partial_{ik}^{(n)} \partial_{jl}^{(m)} \frac{1}{u - z_{ik}} \frac{1}{z_{ik} - z_{jl}} \omega^{-kn-lm} (\sigma^k I_a)[n]_{z_i} (\sigma^l I^a)[m]_{z_j}\end{aligned} \quad \text{(A.10)}$$

where $\{a,b\} := ab + ba$ denotes the anti-commutator. Now here

$$\partial_{ik}^{(n)}\partial_{jl}^{(m)}\frac{1}{u-z_{ik}}\frac{1}{z_{ik}-z_{jl}} = \sum_{p=0}^{n}\left(\partial_{ik}^{(p)}\frac{1}{u-z_{ik}}\right)\left(\partial_{ik}^{(n-p)}\partial_{jl}^{(m)}\frac{1}{z_{ik}-z_{jl}}\right)$$
$$= \sum_{p=0}^{n}\frac{1}{(u-z_{ik})^{p+1}}(-1)^{n-p}\begin{bmatrix}n+m-p\\ m\end{bmatrix}\frac{1}{(z_{ik}-z_{jl})^{n-p+m+1}}. \quad \text{(A.11)}$$

Thus we find the following terms in $\frac{1}{2}I_a(u)I^a(u)$:

$$\sum_{n,m=0}^{\infty}\sum_{\substack{i,j=1\\ j\neq i}}^{N}\sum_{k,l=0}^{T-1}\sum_{p=0}^{n}\frac{\omega^{-kn-lm}}{(u-z_{ik})^{p+1}}(-1)^{n-p}\begin{bmatrix}n+m-p\\ m\end{bmatrix}\frac{1}{(z_{ik}-z_{jl})^{n-p+m+1}}(\sigma^k I_a)[n]_{z_i}(\sigma^l I^a)[m]_{z_j}$$
$$= \sum_{i=1}^{N}\sum_{k=0}^{T-1}\sum_{p=0}^{\infty}\frac{1}{(u-\omega^{-k}z_i)^{p+1}}\left(\sum_{\substack{j=1\\ j\neq i}}^{N}\sum_{l=0}^{T-1}\sum_{r,m=0}^{\infty}\frac{\omega^{-k(r+p)-lm}(-1)^r\left[\begin{smallmatrix}r+m\\ m\end{smallmatrix}\right]}{(\omega^{-k}z_i-\omega^{-l}z_j)^{r+m+1}}(\sigma^k I_a)[r+p]_{z_i}(\sigma^l I^a)[m]_{z_j}\right)$$
$$= \sum_{i=1}^{N}\sum_{k=0}^{T-1}\sum_{p=0}^{\infty}\frac{\omega^{-kp+k}\sigma^k}{(u-\omega^{-k}z_i)^{p+1}}\left(\sum_{\substack{j=1\\ j\neq i}}^{N}\sum_{l=0}^{T-1}\sum_{r,m=0}^{\infty}\frac{\omega^{(k-l)m}(-1)^r\left[\begin{smallmatrix}r+m\\ m\end{smallmatrix}\right]}{(z_i-\omega^{k-l}z_j)^{r+m+1}}I_a[r+p]_{z_i}(\sigma^{l-k} I^a)[m]_{z_j}\right)$$
$$= \sum_{i=1}^{N}\sum_{k=0}^{T-1}\sum_{p=0}^{\infty}\frac{\omega^{-kp+k}\sigma^k}{(u-\omega^{-k}z_i)^{p+1}}\left(\sum_{\substack{j=1\\ j\neq i}}^{N}\sum_{l=0}^{T-1}\sum_{r,m=0}^{\infty}\frac{\omega^{-lm}(-1)^r\left[\begin{smallmatrix}r+m\\ m\end{smallmatrix}\right]}{(z_i-\omega^{-l}z_j)^{r+m+1}}I_a[r+p]_{z_i}(\sigma^{l} I^a)[m]_{z_j}\right) \quad \text{(A.12)}$$

In part by a similar calculation we find the terms in $\frac{1}{2}I_a(u)I^a(u)$ acting solely at one site $i \in \{1,\ldots,N\}$, namely

$$\sum_{i=1}^{N}\sum_{k=0}^{T-1}\sum_{p=0}^{\infty}\frac{\omega^{-kp+k}\sigma^k}{(u-\omega^{-k}z_i)^{p+1}}\times$$
$$\left(\sum_{l=1}^{T-1}\sum_{r,m=0}^{\infty}\frac{\omega^{-lm}(-1)^r\left[\begin{smallmatrix}r+m\\ m\end{smallmatrix}\right]}{((1-\omega^{-l})z_i)^{r+m+1}}\frac{1}{2}\left\{I_a[r+p]_{z_i},(\sigma^l I^a)[m]_{z_i}\right\}+\sum_{n=0}^{p-1}\frac{1}{2}I_a[n]_{z_i}I^a[p-n-1]_{z_i}\right), \quad \text{(A.13)}$$

where the second term is an "on-diagonal" term in $\frac{1}{2}I_a(u)I^a(u)$.

The cross terms in $\frac{1}{2}I_a(u)I^a(u)$ acting at sites $i \in \{1, \dots, N\}$ and 0 are given by (here we use an obvious trick to make the calculation resemble the one above)

$$
\begin{aligned}
&T \sum_{n,m=0}^{\infty} \sum_{i=1}^{N} \sum_{k=0}^{T-1} \partial_{ik}^{(n)} \partial_z^{(m)} \frac{1}{u - z_{ik}} \frac{1}{u - z}\bigg|_{z=0} \omega^{-kn} (\sigma^k I_a)[n]_{z_i} (\Pi_m I^a)[m]_0 \\
&\quad = T \sum_{n,m=0}^{\infty} \sum_{i=1}^{N} \sum_{k=0}^{T-1} \partial_{ik}^{(n)} \partial_z^{(m)} \left(\frac{1}{u - z_{ik}} \frac{1}{z_{ik} - z} + \frac{1}{u - z} \frac{1}{z - z_{ik}} \right)\bigg|_{z=0} \omega^{-kn} (\sigma^k I_a)[n]_{z_i} (\Pi_m I^a)[m]_0 \\
&\qquad = T \sum_{n,m=0}^{\infty} \sum_{i=1}^{N} \sum_{k=0}^{T-1} \sum_{p=0}^{n} \frac{1}{(u - z_{ik})^{p+1}} \frac{(-1)^{n-p} \begin{bmatrix} n+m-p \\ m \end{bmatrix}}{(z_{ik} - z)^{n-p+m+1}}\Bigg|_{z=0} \omega^{-kn} (\sigma^k I_a)[n]_{z_i} (\Pi_m I^a)[m]_0 \\
&\qquad\quad + T \sum_{n,m=0}^{\infty} \sum_{i=1}^{N} \sum_{k=0}^{T-1} \sum_{p=0}^{m} \frac{1}{(u - z)^{p+1}} \frac{(-1)^{m-p} \begin{bmatrix} n+m-p \\ n \end{bmatrix}}{(z - z_{ik})^{n-p+m+1}}\Bigg|_{z=0} \omega^{-kn} (\sigma^k I_a)[n]_{z_i} (\Pi_m I^a)[m]_0 \\
&\qquad = T \sum_{n,m=0}^{\infty} \sum_{i=1}^{N} \sum_{k=0}^{T-1} \sum_{p=0}^{n} \frac{1}{(u - \omega^{-k} z_i)^{p+1}} \frac{(-1)^{n-p} \begin{bmatrix} n+m-p \\ m \end{bmatrix}}{z_i^{n-p+m+1}} \omega^{k(-p+m+1)} (\sigma^k I_a)[n]_{z_i} (\Pi_m I^a)[m]_0 \\
&\qquad\quad + T \sum_{n,m=0}^{\infty} \sum_{i=1}^{N} \sum_{k=0}^{T-1} \sum_{p=0}^{m} \frac{1}{u^{p+1}} \frac{(-1)^{n+1} \begin{bmatrix} n+m-p \\ n \end{bmatrix}}{z_i^{n-p+m+1}} \omega^{k(-p+m+1)} (\sigma^k I_a)[n]_{z_i} (\Pi_m I^a)[m]_0
\end{aligned}
\tag{A.14}
$$

Next, the cross terms in $\frac{1}{2}I_a(u)I^a(u)$ acting at sites $i \in \{1, \dots, N\}$ and ∞ are

$$
\begin{aligned}
&T \sum_{k=0}^{T-1} \sum_{n,m=0}^{\infty} \sum_{i=1}^{N} \frac{\omega^{-kn}}{(u - \omega^{-k} z_i)^{n+1}} u^m (\Pi_{-m-1} I_a)[-m-1]_\infty (\sigma^k I^a)[n]_{z_i} \\
&\quad = T \sum_{k=0}^{T-1} \sum_{n,m=0}^{\infty} \sum_{i=1}^{N} \left(\sum_{p=0}^{m-n-1} \begin{bmatrix} n+p \\ n \end{bmatrix} u^{m-n-1-p} (\omega^{-k} z_i)^p + \sum_{p=0}^{n} \begin{bmatrix} m \\ p \end{bmatrix} \frac{(\omega^{-k} z_i)^{m-p}}{(u - \omega^{-k} z_i)^{n-p+1}} \right) \times \\
&\qquad\qquad \omega^{-kn} (\Pi_{-m-1} I_a)[-m-1]_\infty (\sigma^k I^a)[n]_{z_i}
\end{aligned}
\tag{A.15}
$$

where we used the identity

$$
\frac{u^m}{(u - z)^{n+1}} = \sum_{p=0}^{m-n-1} \begin{bmatrix} n+p \\ n \end{bmatrix} u^{m-n-1-p} z^p + \sum_{p=0}^{n} \begin{bmatrix} m \\ p \end{bmatrix} \frac{z^{m-p}}{(u - z)^{n-p+1}}.
$$

This can be seen by writing the left hand side as

$$
\frac{u^m}{(u - z)^{n+1}} = \frac{1}{n!} \frac{\partial^n}{\partial z^n} \frac{u^m}{u - z} = \frac{1}{n!} \frac{\partial^n}{\partial z^n} \sum_{k=0}^{m-1} u^k z^{m-k-1} + \frac{1}{n!} \frac{\partial^n}{\partial z^n} \frac{z^m}{u - z}.
$$

After taking the n derivatives with respect to z, each of the two terms on the left hand side then evaluate to the two terms on the left hand side of the above identity.

The pole term in $(u-\omega^{-k}z_i)$ in (A.15) is (here $r=n-p$, $n=r+p$)

$$
\begin{aligned}
&T\sum_{i=1}^{N}\sum_{k=0}^{T-1}\sum_{r=0}^{\infty}\frac{1}{(u-\omega^{-k}z_i)^{r+1}}\sum_{p,m=0}^{\infty}\omega^{-kr-km}z_i^{m-p}\begin{bmatrix}m\\ p\end{bmatrix}(\Pi_{-m-1}I_a)[-m-1]_\infty(\sigma^k I^a)[r+p]_{z_i}\\
&\qquad=T\sum_{i=1}^{N}\sum_{k=0}^{T-1}\sum_{r=0}^{\infty}\frac{\omega^{-kr+k}\sigma^k}{(u-\omega^{-k}z_i)^{r+1}}\left(\sum_{p,m=0}^{\infty}z_i^{m-p}\begin{bmatrix}m\\ p\end{bmatrix}(\Pi_{-m-1}I_a)[-m-1]_\infty I^a[r+p]_{z_i}\right)\\
&=T\sum_{i=1}^{N}\sum_{k=0}^{T-1}\sum_{r=0}^{\infty}\frac{\omega^{-kr+k}\sigma^k}{(u-\omega^{-k}z_i)^{r+1}}\left(\sum_{n,p=0}^{\infty}z_i^{n}\begin{bmatrix}n+p\\ p\end{bmatrix}(\Pi_{-n-p-1}I_a)[-n-p-1]_\infty I^a[r+p]_{z_i}\right)
\end{aligned}
\tag{A.16}
$$

where we use the fact that $\sigma^k\Pi_{-m-1}=\omega^{-km-k}\Pi_{-m-1}$.

The powers of u in (A.15) are (here $r=m-n-1-p$, $p=m-n-1-r$)

$$
\begin{aligned}
&T\sum_{i=1}^{N}\sum_{k=0}^{T-1}\sum_{m,n=0}^{\infty}\sum_{r=0}^{m-n-1}u^r\omega^{-km+k+kr}z_i^{m-n-1-r}\begin{bmatrix}m-1-r\\ n\end{bmatrix}(\Pi_{-m-1}I_a)[-m-1]_\infty(\sigma^k I^a)[n]_{z_i}\\
&=T\sum_{r=0}^{\infty}u^r\sum_{i=1}^{N}\sum_{k=0}^{T-1}\sum_{n=0}^{\infty}\sum_{m=r+n+1}^{\infty}\omega^{-km+k+kr}z_i^{m-n-1-r}\begin{bmatrix}m-1-r\\ n\end{bmatrix}(\Pi_{-m-1}I_a)[-m-1]_\infty(\sigma^k I^a)[n]_{z_i}
\end{aligned}
\tag{A.17}
$$

The cross terms in $\frac{1}{2}I_a(u)I^a(u)$ acting at sites 0 and ∞ can be conveniently written as

$$
\begin{aligned}
&T^2\sum_{r=0}^{\infty}\frac{1}{u^{r+1}}\sum_{n=r}^{\infty}(\Pi_nI_a)[n]_0(\Pi_{r-n+1}I^a)[-n+r-1]_\infty\\
&\qquad\qquad+T^2\sum_{r=0}^{\infty}u^r\sum_{n=0}^{\infty}(\Pi_nI_a)[n]_0(\Pi_{-r-n-2}I^a)[-n-r-2]_\infty.
\end{aligned}
\tag{A.18}
$$

Finally the diagonal terms in $\frac{1}{2}I_a(u)I^a(u)$ acting at sites 0 and ∞ respectively are

$$
\frac{T^2}{2}\sum_{r=1}^{\infty}\frac{1}{u^{r+1}}\sum_{n=0}^{r-1}(\Pi_nI_a)[n]_0(\Pi_{r-n-1}I^a)[r-n-1]_0 \tag{A.19}
$$

and

$$
\frac{T^2}{2}\sum_{r=0}^{\infty}u^r\sum_{n=0}^{r}(\Pi_{-n-1}I_a)[-n-1]_\infty(\Pi_{-r+n-1}I^a)[-r+n-1]_\infty. \tag{A.20}
$$

Next we turn to the term $F(u)/u$ in (A.8). Noting that

$$
\frac{1}{u}\frac{1}{(u-z)^{n+1}}=\sum_{p=0}^{n}(-1)^p\frac{1}{z^{p+1}}\frac{1}{(u-z)^{n-p+1}}-\frac{1}{u}\frac{(-1)^n}{z^{n+1}}.
$$

we find that in $F(u)/u$ the terms acting at the sites $i \in \{1, \dots, N\}$ are

$$\sum_{i=1}^{N}\sum_{k=0}^{T-1}\sum_{n=0}^{\infty}\left(\sum_{p=0}^{n}(-1)^p \frac{\omega^{-kn}}{(\omega^{-k}z_i)^{p+1}(u-\omega^{-k}z_i)^{n-p+1}} - \frac{1}{u}\frac{\omega^{-kn}(-1)^n}{(\omega^{-k}z_i)^{n+1}}\right)\sigma^k F[n]_{z_i}$$
$$= \sum_{i=1}^{N}\sum_{k=0}^{T-1}\sum_{r=0}^{\infty}\sum_{n=r}^{\infty}(-1)^{n-r}\frac{\omega^{-kn+k(n-r+1)}}{(u-\omega^{-k}z_i)^{r+1}}\frac{1}{z_i^{n-r+1}}\sigma^k F[n]_{z_i} - \frac{1}{u}\sum_{i=1}^{N}\sum_{n=0}^{\infty}\frac{(-1)^n}{z_i^{n+1}}\sum_{k=0}^{T-1}\omega^k\sigma^k F[n]_{z_i}$$
$$= \sum_{i=1}^{N}\sum_{k=0}^{T-1}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{n}\frac{\omega^{-kr+k}}{(u-\omega^{-k}z_i)^{r+1}}\frac{1}{z_i^{n+1}}\sigma^k F[n+r]_{z_i} - \frac{T}{u}\sum_{i=1}^{N}\sum_{n=0}^{\infty}\frac{(-1)^n}{z_i^{n+1}}\Pi_{-1}F[n]_{z_i}. \quad \text{(A.21)}$$

The terms acting at the sites 0 and ∞ in $F(u)/u$ are clearly

$$T\sum_{n=0}^{\infty}\frac{1}{u^{n+2}}(\Pi_n F)[n]_0 + T\sum_{n=0}^{\infty}u^{n-1}(\Pi_{-n-1}F)[-n-1]_\infty$$

Now we collect terms, and use $\sigma^k\Pi_m = \omega^{km}\Pi_m$, $\sigma^k\mathcal{H}_{i,p} = \mathcal{H}_{i,p}$, and the fact that $\Pi_k F = 0$ for all $k \in \mathbb{Z}/T\mathbb{Z} \setminus \{0\}$, to obtain the result.

Appendix B. Y_W-map

B.1. **Vertex Lie algebras.** In this appendix we use the notion of a vertex Lie algebra and related concepts as defined in [VY16b], to which we refer the reader for the precise definition. In particular, all our vertex Lie algebras are finitely generated as $\mathbb{C}[\mathcal{D}]$-modules, of the form

$$\mathscr{L} := \mathbb{C}[\mathcal{D}] \otimes (L^o \oplus \mathbb{C}\mathsf{c})/\mathcal{D}\mathbb{C}[\mathcal{D}] \otimes \mathbb{C}\mathsf{c} \cong_{\mathbb{C}} \mathbb{C}[\mathcal{D}] \otimes L^o \oplus \mathbb{C}\mathsf{c},$$

where L^o is a finite dimensional vector space and $\mathcal{D}$ is the translation operator of the vertex Lie algebra. Moreover, we will always consider the case when $\mathscr{L}$ is $\mathbb{Z}_{\geq 0}$-graded, in the sense that the underlying finite dimensional vector space L^o is itself $\mathbb{Z}_{\geq 0}$-graded, $\mathcal{D}$ is an operator of degree 1, c is of degree 0 and $\deg(a_{(n)}b) = \deg a + \deg b - n - 1$ for any $a, b \in \mathscr{L}$ where ${}_{(n)} : \mathscr{L} \times \mathscr{L} \to \mathscr{L}$ for $n \in \mathbb{Z}_{\geq 0}$ denote the n^{th}-products of $\mathscr{L}$. We denote by $L(0)$ the degree operator on $\mathscr{L}$, given by $L(0)a = (\deg a)a$ for homogeneous $a \in \mathscr{L}$. We will always assume that $\mathscr{L}$ is equipped with an action $\check{R} : \Gamma \to \operatorname{Aut}\mathscr{L}$, $\alpha \mapsto \check{R}_\alpha$ through automorphisms of $\mathscr{L}$, and consider the corresponding twisted action $R : \Gamma \to \mathsf{GL}(\mathscr{L})$, $\alpha \mapsto R_\alpha := \alpha^{L(0)}\check{R}_\alpha$.

To any such vertex Lie algebra $\mathscr{L}$ we can associate a genuine Lie algebra, denoted $\mathsf{L}(\mathscr{L})$ and defined as $\operatorname{Lie}_{\mathbb{C}((t))}\mathscr{L}$ using [VY16b, Lemma 2.2] with $\mathcal{A} = \mathbb{C}((t))$, which is isomorphic as a vector space to $L^o \otimes \mathbb{C}((t)) \oplus \mathbb{C}\mathsf{c}$. The element in $\mathsf{L}(\mathscr{L})$ corresponding to $a \otimes t^n$ with $a \in L^o$ and $n \in \mathbb{Z}$ is denoted by $a(n) \in \mathsf{L}(\mathscr{L})$. We denote by $\mathsf{c}(-1)$ the copy of the element c belonging to $\mathsf{L}(\mathscr{L})$. The definition of $a(n)$, $n \in \mathbb{Z}$, then extends to all $a \in \mathscr{L}$ by repeatedly applying $(\mathcal{D}a)(n) = -na(n-1)$. In particular then $\mathsf{c}(n) = 0$ unless $n = -1$. For a homogeneous vector $a \in L^o$ we also make use of the notation $a[n] \in \mathsf{L}(\mathscr{L})$ for the element corresponding to $a \otimes t^{n+\deg a-1}$. There is a vector space direct sum decomposition $\mathsf{L}(\mathscr{L}) = \mathsf{L}^-(\mathscr{L}) \dotplus \mathsf{L}^+(\mathscr{L})$ where $\mathsf{L}^\pm(\mathscr{L})$ are Lie subalgebras isomorphic as vector spaces to $L^o \otimes \mathbb{C}[[t]] \oplus \mathbb{C}\mathsf{c}$ and $L^o \otimes t^{-1}\mathbb{C}[t^{-1}]$ respectively.

Denote by $\mathbb{V}(\mathscr{L})$ the vacuum Verma module over $\mathsf{L}(\mathscr{L})$, namely the $\mathsf{L}(\mathscr{L})$-module induced from the one-dimensional $\mathsf{L}^+(\mathscr{L})$-module $\mathbb{C}v_0$ on which c acts as 1 and $L^o \otimes \mathbb{C}[[t]]$ acts trivially. It is naturally endowed with the structure of a vertex algebra. Finally, let $\mathsf{L}(\mathscr{L})^\Gamma$ denote the subalgebra of Γ-invariant elements, where the action of Γ on $\mathsf{L}(\mathscr{L})$ is defined by letting $\alpha \in \Gamma$ send any $a(n)$ with $a \in \mathscr{L}$ and $n \in \mathbb{Z}$ to $\alpha^{-n-1}(R_\alpha a)(n)$. Given any $a \in \mathscr{L}$ and $n \in \mathbb{Z}$ we define the corresponding *twisted* n^{th}*-mode*

$$a^\Gamma(n) := \sum_{\alpha\in\Gamma} \alpha^{-n-1}(R_\alpha a)(n) \in \mathsf{L}(\mathscr{L})^\Gamma.$$

The two main examples of vertex Lie algebras $\mathscr{L}$ we shall consider are given in [VY16b, Examples 2.5 & 2.6], whose corresponding Lie algebras $\mathsf{L}(\mathscr{L})$ are centrally extended loop algebras and Heisenberg Lie algebras.

B.2. **'Local' and 'global' Lie algebras.** Fix a set $\boldsymbol{x} := \{x_i\}_{i=1}^p$ of $p \in \mathbb{Z}_{\geq 0}$ non-zero points in the complex plane. We attach to each point $x_i \in \boldsymbol{x}$ a local copy of the Lie algebra $\mathsf{L}(\mathscr{L})$ defined as $\mathrm{Lie}_{\mathbb{C}((t-x_i))}\mathscr{L}$ using [VY16b, Lemma 2.2] with $\mathcal{A} = \mathbb{C}((t-x_i))$, which we denote $\mathsf{L}(\mathscr{L})_{x_i}$. We will also use the index x_i on the formal modes $a(n)$ or $a[n]$ when we wish to emphasise that these belong to $\mathsf{L}(\mathscr{L})_{x_i}$. Simiarly, to the origin we attach a local copy $\mathsf{L}(\mathscr{L})_0^\Gamma$ of the Γ-invariant subalgebra $\mathsf{L}(\mathscr{L})^\Gamma$. To the point at infinity we attach the Lie algebra $\mathsf{L}(\mathscr{L}^{\text{op}})_\infty^\Gamma$ defined as the subalgebra of Γ-invariants in $\mathsf{L}(\mathscr{L}^{\text{op}})_\infty := \mathrm{Lie}_{\mathbb{C}((t^{-1}))}\mathscr{L}^{\text{op}}$, where $\mathscr{L}^{\text{op}}$ is the opposite vertex Lie algebra given by the $\mathbb{C}[\mathcal{D}]$-module $\mathscr{L}$ but with opposite n^{th}-products. For any $n \in \mathbb{Z}$ and homogeneous element $a \in \mathscr{L}$, we denote by $a(n)_\infty$ (resp. $a[n]_\infty$) the class of $a \otimes t^n$ (resp. $a \otimes t^{n+\deg a-1}$) in $\mathsf{L}(\mathscr{L}^{\text{op}})_\infty$.

Consider the direct sum $\mathsf{L}(\mathscr{L}^{\text{op}})_\infty^\Gamma \oplus \bigoplus_{i=1}^p \mathsf{L}(\mathscr{L})_{x_i} \oplus \mathsf{L}(\mathscr{L})_0^\Gamma$. We define the ideal

$$I_{\infty,p,0} := \mathrm{span}_{\mathbb{C}}\big\{\mathsf{c}(-1)_{x_i} - T\mathsf{c}(-1)_\infty\big\}_{i=1}^p \cup \big\{\mathsf{c}(-1)_0 - \mathsf{c}(-1)_\infty\big\}, \tag{B.1}$$

and the corresponding quotient Lie algebra

$$\mathsf{L}(\mathscr{L})_{\infty,\boldsymbol{x},0} := \mathsf{L}(\mathscr{L}^{\text{op}})_\infty^\Gamma \oplus \bigoplus_{i=1}^p \mathsf{L}(\mathscr{L})_{x_i} \oplus \mathsf{L}(\mathscr{L})_0^\Gamma \Big/ I_{\infty,p,0}.$$

Let $\mathbb{C}_{\infty,\Gamma\boldsymbol{x},0}(t)$ be the algebra of global rational functions with poles at most at 0, ∞ and the points in $\Gamma\boldsymbol{x}$. It comes equipped with the derivation ∂_t and an action of Γ defined through pullback by the multiplication map $t \mapsto \alpha^{-1}t$ for $\alpha \in \Gamma$. Consider the associated Lie algebra $\mathrm{Lie}_{\mathbb{C}_{\infty,\Gamma\boldsymbol{x},0}(t)}\mathscr{L}$ defined with the help of [VY16b, Lemma 2.2] for $\mathcal{A} = \mathbb{C}_{\infty,\Gamma\boldsymbol{x},0}(t)$. We will denote by $af := \rho(a \otimes f)$ the class in $\mathrm{Lie}_{\mathbb{C}_{\infty,\Gamma\boldsymbol{x},0}(t)}\mathscr{L}$ of an element $a \otimes f \in \mathscr{L} \otimes \mathbb{C}_{\infty,\Gamma\boldsymbol{x},0}(t)$. The action of Γ on $\mathscr{L} \otimes \mathbb{C}_{\infty,\Gamma\boldsymbol{x},0}(t)$ defined for any $\alpha \in \Gamma$ by

$$\alpha.(a \otimes f(t)) := \alpha^{-1}R_\alpha a \otimes f(\alpha^{-1}t),$$

gives rise to an action on $\mathrm{Lie}_{\mathbb{C}_{\infty,\Gamma\boldsymbol{x},0}(t)}\mathscr{L}$ by Lie algebra automorphisms, *cf.* [VY16b, Lemma 2.13].

Consider the ideal in $\mathrm{Lie}_{\mathbb{C}_{\infty,\Gamma\boldsymbol{x},0}(t)}\mathscr{L}$ defined by

$$I_{\infty,\Gamma\boldsymbol{x},0} := \mathrm{span}_{\mathbb{C}}\bigg\{\sum_{\alpha\in\Gamma}\frac{\mathsf{c}}{t-\alpha x}\bigg\}_{x\in\boldsymbol{x}\cup\{0\}} = \mathrm{span}_{\mathbb{C}}\bigg\{\sum_{\alpha\in\Gamma}\frac{\mathsf{c}}{t-\alpha x_i}\bigg\}_{i=1}^p \cup \bigg\{\frac{T\mathsf{c}}{t}\bigg\}, \tag{B.2}$$

and denote the corresponding quotient Lie algebra by

$$\mathsf{L}_{\infty,\Gamma\boldsymbol{x},0}(\mathscr{L}) := \mathrm{Lie}_{\mathbb{C}_{\infty,\Gamma\boldsymbol{x},0}(t)}\mathscr{L}\Big/I_{\infty,\Gamma\boldsymbol{x},0}.$$

Noting that the ideal (B.2) is invariant under the action of Γ, *i.e.* $\Gamma.I_{\infty,\Gamma\boldsymbol{x},0} = I_{\infty,\Gamma\boldsymbol{x},0}$, we obtain a well-defined action of Γ on the quotient $\mathsf{L}_{\infty,\Gamma\boldsymbol{x},0}(\mathscr{L})$. We denote the corresponding subalgebra of

Γ-invariants by

$$\mathsf{L}^{\Gamma}_{\infty,\boldsymbol{x},0}(\mathscr{L}) := \big(\mathsf{L}_{\infty,\Gamma\boldsymbol{x},0}(\mathscr{L})\big)^{\Gamma}. \tag{B.3}$$

Proposition B.1. *There is an embedding of Lie algebras*

$$\boldsymbol{\iota} : \mathsf{L}^{\Gamma}_{\infty,\boldsymbol{x},0}(\mathscr{L}) \hookrightarrow \mathsf{L}(\mathscr{L})_{\infty,\boldsymbol{x},0}. \tag{B.4}$$

Proof. Taking the Laurent expansion of an element of $\mathrm{Lie}_{\mathbb{C}_{\infty,\Gamma\boldsymbol{x},0}(t)}\,\mathscr{L}$ at 0, ∞ and the points in $\boldsymbol{x}$ yields an embedding of Lie algebras

$$\begin{aligned} \boldsymbol{\iota} : \mathrm{Lie}_{\mathbb{C}_{\infty,\Gamma\boldsymbol{x},0}(t)}\,\mathscr{L} &\hookrightarrow \mathsf{L}(\mathscr{L}^{\mathrm{op}})_{\infty} \oplus \bigoplus_{i=1}^{p} \mathsf{L}(\mathscr{L})_{x_i} \oplus \mathsf{L}(\mathscr{L})_0 \\ af &\longmapsto \big(-\rho(a\otimes \iota_{t^{-1}} f), \rho(a\otimes\iota_{t-x_1} f), \ldots, \rho(a\otimes \iota_{t-x_p} f), \rho(a\otimes\iota_t f)\big). \end{aligned} \tag{B.5}$$

Note that under the embedding (B.5) we have $\boldsymbol{\iota}\big(\sum_{\alpha\in\Gamma}\frac{\mathsf{c}}{t-\alpha x}\big) = \mathsf{c}(-1)_x - T\mathsf{c}(-1)_\infty$ for every $x\in\boldsymbol{x}$ and $\boldsymbol{\iota}\big(\frac{T\mathsf{c}}{t}\big) = T\mathsf{c}(-1)_0 - T\mathsf{c}(-1)_\infty$ from which we deduce that the ideal $I_{\infty,\Gamma\boldsymbol{x},0} \subset \mathrm{Lie}_{\mathbb{C}_{\infty,\Gamma\boldsymbol{x},0}(t)}\,\mathscr{L}$ defined in (B.2) is mapped to the ideal $I_{\infty,p,0} \subset \mathsf{L}(\mathscr{L}^{\mathrm{op}})_\infty \oplus \bigoplus_{i=1}^{p} \mathsf{L}(\mathscr{L})_{x_i} \oplus \mathsf{L}(\mathscr{L})_0$ defined in (B.1). By [VY16b, Lemma 2.9] it follows that (B.5) induces an embedding of quotient Lie algebras

$$\boldsymbol{\iota} : \mathsf{L}_{\infty,\Gamma\boldsymbol{x},0}(\mathscr{L}) \hookrightarrow \mathsf{L}(\mathscr{L}^{\mathrm{op}})_\infty \oplus \bigoplus_{i=1}^{p} \mathsf{L}(\mathscr{L})_{x_i} \oplus \mathsf{L}(\mathscr{L})_0 \Big/ I_{\infty,p,0}.$$

Finally, by restricting the latter to the subalgebra of Γ-invariants (B.3) we obtain the desired Lie algebra embedding (B.4). $\square$

Let $\mathcal{M}_{x_i}$ be a left module over $\mathsf{L}(\mathscr{L})_{x_i}$ for each $i=1,\ldots,p$. Following the convention adopted in [VY16b, Section 3], we will always assume that such modules over $\mathsf{L}(\mathscr{L})$ are of level 1, namely that the central element $\mathsf{c}(-1)$ acts as 1. Let $\mathcal{M}_0$ and $\mathcal{M}_\infty$ be left modules over $\mathsf{L}(\mathscr{L})^{\Gamma}_0$ and $\mathsf{L}(\mathscr{L}^{\mathrm{op}})^{\Gamma}_\infty$ respectively, or alternatively $\mathcal{M}_\infty$ is a *right* module over $\mathsf{L}(\mathscr{L})^{\Gamma}_\infty$. Following also the convention of [VY16b, Section 3], all such modules over $\mathsf{L}(\mathscr{L})^{\Gamma}$ will be assumed to be of level $\frac{1}{T}$. The tensor product $\mathcal{M}_\infty \otimes \bigotimes_{i=1}^{p} \mathcal{M}_{x_i} \otimes \mathcal{M}_0$ is then a left module over the Lie algebra $\mathsf{L}(\mathscr{L})_{\infty,\boldsymbol{x},0}$ so that by virtue of Proposition B.1 it also becomes a left module over the global Lie algebra $\mathsf{L}^{\Gamma}_{\infty,\boldsymbol{x},0}(\mathscr{L})$. In particular, we may form the space of coinvariants

$$\mathcal{M}_\infty \otimes \bigotimes_{i=1}^{p} \mathcal{M}_{x_i} \otimes \mathcal{M}_0 \Big/ \mathsf{L}^{\Gamma}_{\infty,\boldsymbol{x},0}(\mathscr{L}).$$

Many of the statements and proofs of [VY16b] concerning cyclotomic coinvariants which did not include the point at infinity can be seen to carry over with minor modifications to the present case.

B.3. Y_W**-map.** A left module M over $\mathsf{L}(\mathscr{L})$ is said to be *smooth* if for all $a\in\mathscr{L}$ and $v\in M$ we have $a(n)v = 0$ for all $n\gg 0$. Likewise, we will say that a right module N over $\mathsf{L}(\mathscr{L})$ is *co-smooth* if for all $a\in\mathscr{L}$ and $\eta\in N$ we have $\eta\, a(n) = 0$ for all $n \ll 0$.

Let $\mathcal{M}_\infty$ be any right co-smooth module over $\mathsf{L}(\mathscr{L})^{\Gamma}$. We define the quasi-module map

$$Y_W(\cdot,u) : \mathbb{V}(\mathscr{L}) \longrightarrow \mathrm{Hom}\big(\mathcal{M}_\infty, \mathcal{M}_\infty((u^{-1}))\big), \qquad A \longmapsto \sum_{n\in\mathbb{Z}} A^{W}_{(n)} u^{-n-1}$$

where u is a formal variable and $A^W_{(n)}$ are endomorphisms of $\mathcal{M}_\infty$ for each $A \in \mathbb{V}(\mathscr{L})$ and $n \in \mathbb{Z}$, by direct analogy with the definition given in [VY16b] for left smooth modules over $\mathsf{L}(\mathscr{L})^\Gamma$ as follows. For any $a \in \mathscr{L}$ we set

$$Y_W(a(-1)v_0, u) := \sum_{\alpha\in\Gamma}\sum_{n\in\mathbb{Z}} (R_\alpha a)(n)(\alpha u)^{-n-1}.$$

Moreover, the map is defined recursively for any other state in $\mathbb{V}(\mathscr{L})$ by letting

$$Y_W(a(-1)B, u) := {:}Y_W(a(-1)v_0, u)Y_W(B, u){:} + \sum_{\alpha\in\Gamma\setminus\{1\}}\sum_{n\geq 0} \frac{1}{((\alpha-1)u)^{n+1}} Y_W\big((R_\alpha a)(n)B, u\big), \tag{B.6}$$

for all $a \in \mathscr{L}$ and $B \in \mathbb{V}(\mathscr{L})$. Here ${:}A(u)B(u){:} = A(u)_+B(u) + B(u)A(u)_-$ denotes the usual normal ordering where for $A(u) = \sum_{n\in\mathbb{Z}} A(n)u^{-n-1}$ we define

$$A(u)_+ = \sum_{n<0} A(n)u^{-n-1} = \sum_{m\geq 0} A(-m-1)u^m, \tag{B.7a}$$

$$A(u)_- = \sum_{n\geq 0} A(n)u^{-n-1}. \tag{B.7b}$$

The following is a direct analogue of [VY16b, Proposition 3.6] for the point at infinity, providing an alternative definition the above Y_W-map on co-smooth modules using co-invariants.

Proposition B.2. *Let $\mathcal{M}_\infty$ be a right co-smooth module over $\mathsf{L}(\mathscr{L})^\Gamma$. We have*

$$-\iota_{u^{-1}}[\underset{\substack{\uparrow\\ u}}{A} \otimes \underset{\substack{\uparrow\\ \infty}}{m_\infty} \otimes \cdots] = \big[\underset{\substack{\uparrow\\ \infty}}{m_\infty} Y_W(A, u) \otimes \cdots\big], \tag{B.8}$$

for all $A \in \mathbb{V}(\mathscr{L})$ and $m_\infty \in \mathcal{M}_\infty$.

Proof. We use induction on the depth of A. When $A = v_0$ the result follows from the analogue of [VY16b, Proposition 3.2] including the point at infinity. For the inductive step, we assume that (B.8) holds for states of depth strictly less than that of A. Without loss of generality we can take the state A to be of the form $A = a(-1)B$ for some $a \in \mathscr{L}$ and $B \in \mathbb{V}(\mathscr{L})$.

Let $\mathcal{M}_{x_i}$, $i = 1, \ldots, p$ be any collection of $p \in \mathbb{Z}_{\geq 0}$ modules attached to the points $x_i \in \mathbb{C}$ for $i = 1, \ldots, p$ which may include the origin. We write $\boldsymbol{x} := \{x_i\}_{i=1}^p$ and let $\boldsymbol{m} \in \bigotimes_{i=1}^p \mathcal{M}_{x_i}$. By definition of the space of coinvariants we have

$$\big[f(t).(\underset{\substack{\uparrow\\ u}}{B} \otimes \underset{\substack{\uparrow\\ \infty}}{m_\infty} \otimes \underset{\substack{\uparrow\\ \boldsymbol{x}}}{\boldsymbol{m}})\big] = 0, \quad \text{where} \quad f(t) = \sum_{\alpha\in\Gamma} \frac{\alpha^{-1}R_\alpha a}{\alpha^{-1}t - u}.$$

In other words,

$$\big[\iota_{t-u}f(t)\underset{\substack{\uparrow\\ u}}{B} \otimes \underset{\substack{\uparrow\\ \infty}}{m_\infty} \otimes \underset{\substack{\uparrow\\ \boldsymbol{x}}}{\boldsymbol{m}}\big] - \big[\underset{\substack{\uparrow\\ u}}{B} \otimes \underset{\substack{\uparrow\\ \infty}}{m_\infty}\big(\iota_{t^{-1}}f(t)\big) \otimes \underset{\substack{\uparrow\\ \boldsymbol{x}}}{\boldsymbol{m}}\big] + \sum_{i=1}^p \big[\underset{\substack{\uparrow\\ u}}{B} \otimes \underset{\substack{\uparrow\\ \infty}}{m_\infty} \otimes \iota_{t-x_i}f(t)\underset{\substack{\uparrow\\ \boldsymbol{x}}}{\boldsymbol{m}}\big] = 0.$$

Thus the left hand side of (B.8) may be written as

$$
\begin{aligned}
-\iota_{u^{-1}}[a(-1)\underset{\overset{\uparrow}{u}}{B} \otimes \underset{\overset{\uparrow}{\infty}}{m_\infty} \otimes \underset{\overset{\uparrow}{\boldsymbol{x}}}{\boldsymbol{m}}] &= -\iota_{u^{-1}}\Bigg[\sum_{\alpha\neq 1}\sum_{n\geq 0}\frac{(R_\alpha a)(n)}{((\alpha-1)u)^{n+1}}\underset{\overset{\uparrow}{u}}{B} \otimes \underset{\overset{\uparrow}{\infty}}{m_\infty} \otimes \underset{\overset{\uparrow}{\boldsymbol{x}}}{\boldsymbol{m}}\Bigg] \\
&- \iota_{u^{-1}}\Bigg[\underset{\overset{\uparrow}{u}}{B} \otimes \underset{\overset{\uparrow}{\infty}}{m_\infty} Y_W(a(-1)v_0,u)_+ \otimes \underset{\overset{\uparrow}{\boldsymbol{x}}}{\boldsymbol{m}}\Bigg] - \iota_{u^{-1}}\Bigg[\underset{\overset{\uparrow}{u}}{B} \otimes \underset{\overset{\uparrow}{\infty}}{m_\infty} \otimes \sum_{i=1}^{p}\sum_{\alpha\in\Gamma}\sum_{n\geq 0}\frac{(R_\alpha a)(n)_{x_i}}{(\alpha u - x_i)^{n+1}}\underset{\overset{\uparrow}{\boldsymbol{x}}}{\boldsymbol{m}}\Bigg].
\end{aligned}
$$

Here we made use of the notation (B.7a). The states at the point u on the right hand side are of lower depth in $\mathbb{V}(\mathscr{L})$, so that we may apply the inductive hypothesis to obtain

$$
\begin{aligned}
-\iota_{u^{-1}}[a(-1)\underset{\overset{\uparrow}{u}}{B} \otimes \underset{\overset{\uparrow}{\infty}}{m_\infty} \otimes \underset{\overset{\uparrow}{\boldsymbol{x}}}{\boldsymbol{m}}] &= \sum_{\alpha\neq 1}\sum_{n\geq 0}\frac{1}{((\alpha-1)u)^{n+1}}\Bigg[\underset{\overset{\uparrow}{\infty}}{m_\infty} Y_W\big((R_\alpha a)(n)B,u\big) \otimes \underset{\overset{\uparrow}{\boldsymbol{x}}}{\boldsymbol{m}}\Bigg] \\
&+ \Bigg[\underset{\overset{\uparrow}{\infty}}{m_\infty} Y_W(a(-1)v_0,u)_+ Y_W(B,u) \otimes \underset{\overset{\uparrow}{\boldsymbol{x}}}{\boldsymbol{m}}\Bigg] + \iota_{u^{-1}}\Bigg[\underset{\overset{\uparrow}{\infty}}{m_\infty} Y_W(B,u) \otimes \sum_{i=1}^{p}\sum_{\alpha\in\Gamma}\sum_{n\geq 0}\frac{(R_\alpha a)(n)_{x_i}}{(\alpha u - x_i)^{n+1}}\underset{\overset{\uparrow}{\boldsymbol{x}}}{\boldsymbol{m}}\Bigg].
\end{aligned}
$$

The first two terms on the right hand side are already in the desired form. Taking the map $\iota_{u^{-1}}$ explicitly in the remaining term we may rewrite it as

$$
\Bigg[\underset{\overset{\uparrow}{\infty}}{m_\infty} Y_W(B,u) \otimes \sum_{i=1}^{p}\sum_{\alpha\in\Gamma}\sum_{m\geq 0}\sum_{n=0}^{m}\begin{bmatrix} m \\ n \end{bmatrix}\frac{x_i^{m-n}}{(\alpha u)^{m+1}}(R_\alpha a)(n)_{x_i}\underset{\overset{\uparrow}{\boldsymbol{x}}}{\boldsymbol{m}}\Bigg]. \tag{B.9}
$$

Now consider the following identity

$$
\sum_{m\geq 0} u^{-m-1}\Big[g_m(t).\big(\underset{\overset{\uparrow}{\infty}}{m_\infty} Y_W(B,u) \otimes \underset{\overset{\uparrow}{\boldsymbol{x}}}{\boldsymbol{m}}\big)\Big] = 0, \quad \text{where} \quad g_m(t) = \sum_{\alpha\in\Gamma}\alpha^{-1}(R_\alpha a)(\alpha^{-1}t)^m.
$$

Using this we may rewrite (B.9) simply as

$$
\sum_{m\geq 0} u^{-m-1}\Big[\underset{\overset{\uparrow}{\infty}}{m_\infty} Y_W(B,u)\big(\iota_{t^{-1}}g_m(t)\big) \otimes \underset{\overset{\uparrow}{\boldsymbol{x}}}{\boldsymbol{m}}\Big] = \Big[\underset{\overset{\uparrow}{\infty}}{m_\infty} Y_W(B,u) Y_W(a(-1)v_0,u)_- \otimes \underset{\overset{\uparrow}{\boldsymbol{x}}}{\boldsymbol{m}}\Big].
$$

Putting the above together and using the recurrence relation (B.6) of the Y_W-map we obtain

$$
\iota_{u^{-1}}[a(-1)\underset{\overset{\uparrow}{u}}{B} \otimes \underset{\overset{\uparrow}{\infty}}{m_\infty} \otimes \underset{\overset{\uparrow}{\boldsymbol{x}}}{\boldsymbol{m}}] = \big[\underset{\overset{\uparrow}{\infty}}{m_\infty} Y_W(a(-1)B,u) \otimes \underset{\overset{\uparrow}{\boldsymbol{x}}}{\boldsymbol{m}}\big],
$$

as required. □

Recall that the vacuum Verma module $\mathbb{V}(\mathscr{L})$ has the structure of a vertex algebra and so is in particular a vertex Lie algebra. We may thus form the associated Lie algebra $\mathsf{L}(\mathbb{V}(\mathscr{L}))$ as well as its subalgebra of Γ-invariants $\mathsf{L}(\mathbb{V}(\mathscr{L}))^\Gamma$. The following is an immediate generalisation of [VY16b, Proposition 5.8].

Proposition B.3. *Let M_∞ be a co-smooth right module over $\mathsf{L}(\mathscr{L})^\Gamma$. There is a well-defined co-smooth right $\mathsf{L}(\mathbb{V}(\mathscr{L}))^\Gamma$-module structure on M_∞ given for all $m_\infty \in M_\infty$ by*

$$
m_\infty A^\Gamma(n) := m_\infty A^W_{(n)}.
$$

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