

Measuring the entanglement of bipartite pure states

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(July 30, 2007)

Abstract

The problem of the experimental determination of the amount of entanglement of a bipartite pure state is addressed. We show that measuring a single observable does not suffice to determine the entanglement of a given unknown pure state of two particles. Possible minimal local measuring strategies are discussed and a comparison is made on the basis of their best achievable precision.

PACS-numbers: 03.67.-a, 03.65.Bz

arXiv:quant-ph/9910041v1 9 Oct 1999

I. INTRODUCTION

Quantum mechanical states of multiparticle systems can be entangled, a fact well known since the early days of the formulation of the quantum theory [1]. However, the status of this property has changed substantially in recent years. Entanglement can also be viewed as a resource and as such features in different processes of potential practical importance, for example quantum teleportation [2], quantum cryptography [3] or even high precision measurements [4]. From a theoretical point of view, entanglement of bipartite pure states, and its properties under local quantum operations, are reasonably well understood. There is a unique measure of entanglement for these systems, provided by the von Neumann entropy [5], and optimal ways for entanglement manipulation are known [6]. However, there is a remaining practical question which has not been addressed so far: How can one optimally measure the amount of entanglement of an unknown bipartite pure state?

At a first sight, this question may seem obvious. Reconstructing the reduced density operator of any of the two subsystems will do the job. However, the essential point is that we require the determination to be optimal and the reconstruction of the reduced density matrix may provide redundant information, given that we are asking for just a feature of the composite state, its entanglement. This is a single number and the first question to be answered is whether there exists a single operator whose experimental measure may provide us with just the amount of entanglement of the state. Note that further details of the state itself are not of interest in the problem we are posing here [7]. We will prove in the following that such an operator does not exist, a conclusion that confirms what it could initially be thought as an educated guess. Knowing the impossibility of a test using a repeated measurements of a single observable, we will discuss possible strategies aimed to be minimal, in the sense of involving the smallest number of observables. In order to avoid any ambiguity when counting the number of observables involved in a given measurement protocol, we will define such number as the different number of *meters* each observer has to read out. Among different minimal strategies, that is, strategies involving the same number of meters, we will call optimal the one providing the best accuracy when supplied with the same resources. We will show that, in fact, measuring the reduced density matrix turns out to be a minimal way to proceed. Moreover, the protocol can be made optimal when involving projections along mutually orthogonal directions. We have organized the paper as follows. In section II we state formally the problem. Section III shows the impossibility of finding a single observable whose measure may allow the experimental determination of the amount of entanglement. Minimal local strategies are discussed in Section IV. When supplied with the same number of identically prepared bipartite pure states, we discuss the performance of two classes of minimal measurements from the point of view of the achievable precision in determining the amount of entanglement. Section V is devoted to conclusions.

II. AN EXPERIMENTAL SCENARIO

Let us imagine the following situation. We are provided with a state preparator which creates pairs of two-level particles (qubits) in an unknown entangled state. These entangled pairs are distributed to two remote locations where two observers, Alice and Bob, may

perform local measurements as well as interchange classical communication. The internal dynamics of the device is not specified and the only thing Alice and Bob know is that, with high accuracy, the state they share is pure. Therefore, the two-qubit state can be written as

$$\rho = |\psi_{AB}\rangle\langle\psi_{AB}|, \quad (1)$$

where

$$|\psi\rangle_{AB} = a_0|00\rangle + a_1|01\rangle + a_2|10\rangle + a_3|11\rangle. \quad (2)$$

In this expression ($|0\rangle, |1\rangle$) refer to the eigenvectors of the operators σ_z , the complex coefficients a_i , ($i = 0, \dots, 3$), being completely unknown. In addition, we assume that the machine may supply a large number of identical pairs. The aim is to use the resulting pairs for a quantum information task and, therefore, the only property we are interested in is its amount of entanglement. Moreover, we require the measurement aimed to determine the amount of entanglement to be optimal in the following sense. First, the protocol should involve the smallest possible number of observables. Such tests will be called minimal. And secondly, among minimal tests, we will define as optimal the class of protocols that yield the best resolution when supplied with the same resources, i.e., the same number of identically prepared two-level systems.

The problem is still rather general and, for simplicity, three further assumptions will be made:

1. The experimental situation is such that it only allows to act on one pair at a time. In other words, we restrict ourselves to incoherent measurements. Alice and Bob are not allowed to store a given number of particles and perform a joint measurement on them [8].
2. No ancillary systems are available and the only allowed incoherent measurements are projective ones.
3. The adopted protocol is rigid, in the sense that we will not accumulate information from a given set of initial measurements and re-adjust our strategy afterwards.

In these conditions, we will show that no single operator measurement allows to determine the amount of entanglement of an unknown bipartite pure state.

III. IMPOSSIBILITY OF A SINGLE-OBSERVABLE MEASURING STRATEGY

The amount of entanglement of a bipartite pure state is given by its von Neumann entropy,

$$E(\psi_{AB}) = -tr(\rho_A \log_2 \rho_A) = -tr(\rho_B \log_2 \rho_B) \quad (3)$$

where $\rho_{A(B)} = tr_{B(A)}\rho$ is the reduced density matrix of each subsystem and ρ is given by Eq. (1). In terms of the concurrence C [9], defined as

$$\begin{aligned}
C^2(\psi_{AB}) &= |\langle \psi | \sigma_y \otimes \sigma_y | \psi^* \rangle|^2 \\
&= |a_0 a_3 - a_1 a_2|^2 \\
&= 4 \det \rho_A = 4 \det \rho_B,
\end{aligned} \tag{4}$$

the amount of entanglement can be expressed in a compact form as follows

$$\begin{aligned}
E(\psi_{AB}) &= - \left(\frac{1 + \sqrt{1 - C^2}}{2} \right) \log_2 \left(\frac{1 + \sqrt{1 - C^2}}{2} \right) \\
&\quad - \left(\frac{1 - \sqrt{1 - C^2}}{2} \right) \log_2 \left(\frac{1 - \sqrt{1 - C^2}}{2} \right).
\end{aligned} \tag{5}$$

It should be noted that if all coefficients a_i were real, the concurrence could be obtained via the repeated measurement of a single observable, $\sigma_y \otimes \sigma_y$. We will now prove that in general, i.e. where no a priori information is provided about the state of the bipartite system, it is not possible to evaluate $C^2(\psi_{AB})$ by means of measuring a set of orthogonal projectors $\mathbb{P}_i = |O_i\rangle\langle O_i|$, $\sum_{i=0}^3 \mathbb{P}_i = \mathbb{1}$, where the $|O_i\rangle$'s form an orthonormal basis of certain operator \hat{O} . This measurement would allow us to compute the four probabilities $p_i = |\langle O_i | \psi \rangle|^2$ and therefore it provides three independent real numbers. It is obvious that this will not be enough to fully reconstruct the pure state $|\psi\rangle$ but one may still ask whether the resulting information may be enough to compute a property of the state, its amount of entanglement. In order to check this, let us first re-write the concurrence C in a more convenient form. For that, we will express the state $|\psi_{AB}\rangle$ in terms of the eigenbasis of the operator \hat{O} as

$$|\psi_{AB}\rangle = \sum_{i=0}^3 \langle O_i | \psi \rangle |O_i\rangle \equiv \sum_{i=0}^3 m_i e^{i\phi_i} |O_i\rangle, \tag{6}$$

where the coefficients m_i 's are purely real and $\psi_i \in [0, 2\pi)$. Then, C^2 can be written as

$$\begin{aligned}
C^2(\psi_{AB}) &= \left| \sum_{i=0}^3 \sum_{j=0}^3 \langle O_i | \psi \rangle^* \langle O_i | \sigma_y \otimes \sigma_y | O_j^* \rangle \langle O_j | \psi \rangle^* \right|^2 \\
&= \left| \sum_{i=0}^3 \sum_{j=0}^3 m_i m_j e^{-i\phi_i} e^{-i\phi_j} \langle O_i | \sigma_y \otimes \sigma_y | O_j^* \rangle \right|^2.
\end{aligned} \tag{7}$$

Let us define a new matrix K with elements given by

$$K_{ij} = \langle O_i | \sigma_y \otimes \sigma_y | O_j^* \rangle. \tag{8}$$

In terms of this quantity, the squared concurrence can be written as

$$C^2(\psi_{AB}) = \sum_{i,j,k,l=0}^3 m_i m_j m_k m_l e^{i(\phi_k + \phi_l - \phi_i - \phi_j)} K_{ij} K_{kl}^*. \tag{9}$$

Looking at this expression one can already formulate the guess that it will not be possible to obtain C^2 from just measuring the probabilities p_i , given that no information about the relative phases ψ_i will be unveiled by the measurement. In what follows we will prove explicitly that C^2 , and therefore the amount of entanglement of the bipartite pure state, cannot be an univaluated function of the probabilities p_i (equivalently, of the coefficients m_i).

A. An useful lemma

Let us define two new auxiliary matrices S and σ with matrix elements given by

$$S_{ij} = \langle O_j^* | O_i \rangle \quad (10)$$

and

$$\sigma_{ij} = \langle O_i | \sigma_y \otimes \sigma_y | O_j \rangle. \quad (11)$$

It is easy to check that the following properties hold.

1. The matrix K of Eq. 8 satisfies $K = K^T$, as it follows immediately from the hermiticity of the operator $\sigma_y \otimes \sigma_y$.
2. If the $|O_i\rangle$'s form an orthonormal basis, the corresponding conjugate vectors $|O_i^*\rangle$ also form an orthonormal basis. Then, the matrix S defined above is just the change of basis matrix between the two representations, i. e.,

$$|O_i\rangle = \sum_j S_{ij} |O_j^*\rangle, \quad (12)$$

and therefore $S^\dagger S = \mathbb{1}$ and $|Det(S)| = 1$. In particular, this implies that $Det(S^\dagger) \neq 0$.

3. $Det(\sigma) = 1$.

We now have all the ingredients for proving the following lemma.

Lemma: $det(K) \neq 0$.

Proof: The matrix elements of K can be written in terms of those of σ and S as follows:

$$\begin{aligned} K_{ij} &= \langle O_i | \sigma_y \otimes \sigma_y | O_j^* \rangle \\ &= \sum_{l=0}^3 \langle O_i | \sigma_y \otimes \sigma_y | O_l \rangle \langle O_l | O_j^* \rangle \\ &= \sum_{l=0}^3 \sigma_{il} S_{jl}^* = \sum_{l=0}^3 \sigma_{il} S_{lj}^\dagger \end{aligned}$$

Therefore, $K = \sigma S^\dagger$ and the lemma follows from properties 2 and 3.

B. Impossibility of a single-observable test

We will now prove that assuming that the measurement of a single observable allows to determine the concurrence of the state, and therefore its amount of entanglement, yields a contradiction with the previous lemma. Given that the state $|\psi\rangle$ is unknown, the test we are seeking must be universal, that is, the hypothetical observable \hat{O} has to provide the amount of entanglement of whatever input state. The idea underlying our proof is to show

that there will always be a particular case yielding to a contradiction. Therefore, if no a priori information is provided, the minimal test will necessarily require measuring more than one observable.

Consider the particular case where $m_0 = m_1 = 1/\sqrt{2}$ and $m_2 = m_3 = 0$. In this case, Eq. 9 takes the form:

$$\begin{aligned} C^2 &= \frac{1}{4}(|K_{00}|^2 + |K_{11}|^2 + 4|K_{01}|^2 \\ &\quad + 2e^{i\phi}(K_{00}K_{01}^* + K_{01}K_{11}^*) + 2e^{-i\phi}(K_{01}K_{00}^* + K_{11}K_{01}^*) \\ &\quad + e^{2i\phi}K_{00}K_{11}^* + e^{-2i\phi}K_{11}K_{00}^*) \end{aligned}$$

where we have called $\phi = \phi_1 - \phi_0$ and had make use of property 1. If we assume that C^2 is only a function of the real numbers m_i , i. e. independent of the relative phase ϕ , the fact that the functions $(1, e^{i\phi}, e^{-i\phi}, e^{2i\phi}, e^{-2i\phi})$ are linearly independent yields the set of equalities

$$C^2 = \frac{1}{4}(|K_{00}|^2 + |K_{11}|^2 + 4|K_{01}|^2) \quad (13)$$

$$K_{00}K_{01}^* + K_{01}K_{11}^* = 0 \quad (14)$$

$$K_{00}K_{11}^* = 0 \quad (15)$$

Eq. 15 implies that either $K_{00} = 0$ and/or $K_{11} = 0$. Taking Eq. 14 into account, this corresponds to the cases where $K_{01} \neq 0$ or $K_{01} = 0$. In other words, we obtain that two among the three complex numbers (K_{00}, K_{01}, K_{11}) must be zero.

If we repeat this argument for all the cases where any two of the coefficients m_i are equal to $1/\sqrt{2}$ and the remaining two equal to zero, we end up with the requirement that in all the following sets of three complex numbers

$$\begin{aligned} &(K_{00}, K_{01}, K_{11}) \quad (K_{00}, K_{02}, K_{22}) \quad (K_{00}, K_{03}, K_{33}) \\ &(K_{11}, K_{12}, K_{22}) \quad (K_{11}, K_{13}, K_{33}) \\ &(K_{22}, K_{23}, K_{33}) \end{aligned}$$

there must be at least two of them equal to zero in any set. This fact imposes a certain symmetry for the allowed K -matrices. Explicitly, K can only be one of the following

$$K_1 = \begin{pmatrix} K_{00} & 0 & 0 & 0 \\ 0 & 0 & K_{12} & K_{13} \\ 0 & K_{12} & 0 & K_{23} \\ 0 & K_{13} & K_{23} & 0 \end{pmatrix} \quad (16)$$

or

$$K_2 = \begin{pmatrix} 0 & K_{01} & K_{02} & 0 \\ K_{01} & 0 & K_{12} & 0 \\ K_{02} & K_{12} & 0 & 0 \\ 0 & 0 & 0 & K_{33} \end{pmatrix}, \quad (17)$$

and analogous forms obtained when interchanging the roles of the indexes, or of the form

$$K_3 = \begin{pmatrix} 0 & K_{01} & K_{02} & K_{03} \\ K_{01} & 0 & K_{12} & K_{13} \\ K_{02} & K_{12} & 0 & K_{23} \\ K_{03} & K_{13} & K_{23} & 0 \end{pmatrix}, \quad (18)$$

It should be noted that many other cases could be obtained if any of the matrix elements written as non-zero were zero, however these additional cases are not of interest here, as will become clear afterwards.

Our proof ends with showing that in any of the allowed forms for K , some of the possibly nonzero coefficients in K turn out to be zero. Therefore, all the allowed forms for K , i.e. all forms compatible with the requirement of being the concurrence a univaluated function of the real numbers p_i , will have determinant equal zero, which contradicts the lemma stated before.

This can be easily shown for matrices of the form K_1 or K_2 just following an argument parallel to one used above and choosing three of the coefficients m_i equal to $1/\sqrt{3}$ and the remaining one equal to zero. Let us analyze here the case of K -matrices of the form K_3 . If we set the m_i 's coefficients to the values $m_0 = 0$ and $m_i = 1/\sqrt{3}$ for $(i = 1, \dots, 3)$, the squared concurrence given by Eq. 9 reads

$$\begin{aligned} C^2 &= \frac{4}{3\sqrt{3}} (|K_{12}|^2 + |K_{13}|^2 + |K_{23}|^2 \\ &\quad + e^{i\alpha} K_{12} K_{13}^* + e^{-i\alpha} K_{13} K_{12}^* \\ &\quad + e^{i\beta} K_{12} K_{23}^* + e^{-i\beta} K_{23} K_{12}^* \\ &\quad + e^{i\gamma} K_{13} K_{23}^* + e^{-i\gamma} K_{23} K_{13}^*) \end{aligned}$$

where we have introduced the relative phases $\alpha = \phi_3 - \phi_2$, $\beta = \phi_3 - \phi_1$ and $\gamma = \phi_2 - \phi_1$. Using again the argument invoked in proving the allowed forms for the matrix K , we obtain that the following equalities must hold

$$\begin{aligned} K_{12} K_{13}^* &= 0 \\ K_{12} K_{23}^* &= 0 \\ K_{13} K_{23}^* &= 0. \end{aligned}$$

Therefore, within the three numbers (K_{12}, K_{13}, K_{23}) , two must be zero. If we repeat the argument making each time one of the m_i 's zero and the other three equal to $1/\sqrt{3}$, we end up with allowed forms for K_3 with either four or two matrix elements different from zero. For instance, an allowed form of K is given by

$$K_3^1 = \begin{pmatrix} 0 & K_{01} & 0 & 0 \\ K_{01} & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{23} \\ 0 & 0 & K_{23} & 0 \end{pmatrix}. \quad (19)$$

But choosing now all the coefficients m_i equal to $1/2$ suffices to get the constraint $K_{01} K_{23}^* = 0$, which yields a zero determinant for K . The same reasoning applies to the other five

possible cases. Therefore, assuming that C^2 is only a function of the probabilities p_i yields to the condition $\det(K) = 0$, but we have proved in the previous section that the orthonormality of the vectors O_i demands the determinant of K to be non zero. As a result, it is not possible to find a single operator \hat{O} whose measurement allows to determine the amount of entanglement of the pure state $|\psi_{AB}\rangle$.

IV. MINIMAL TESTS

The previous analysis shows that a measuring strategy employing a single-observable does not allow Alice and Bob to know the amount of entanglement of the state they are sharing. Knowing this fact, the natural question to ask is to determine the minimal measuring strategy than may allow them to evaluate $E(\psi_{AB})$. It is clear that if they measure two different observables \hat{O}_{AB} of the form analyzed before, they will fully reconstruct the original pure state and can, therefore, compute its amount of entanglement. It is a remarkable fact that acting on the whole Hilbert space of the two particles, we cannot isolate the information related to the amount of entanglement alone by means of measuring a single observable. If no information is known about the state, determining its amount of entanglement leads to a full reconstruction of the state. However, such a non-local implementation may not be the easiest to implement experimentally and local strategies are preferred. We will discuss in this section possible ways to proceed if one is constrained to act locally and we will determine the expected precision of the protocols.

A. Local actions without exchange of classical communication

When Alice and Bob are constrained to act locally and no classical communication can be exchanged among them, the minimal measuring strategy corresponds to the local reconstruction of the reduced density operators. For instance, Alice may reconstruct the operator $\rho_A = \text{tr}_B \rho_{AB}$ and send at the end of the protocol a final message to Bob whose length will depend on the required resolution.

To achieve this, Alice needs to perform three projective measurements along linearly independent directions. We will show in the following that measuring along three orthogonal directions is in fact optimal, in the sense that choosing this configuration yields the smallest associated uncertainty in the experimental determination of the determinant of the reduced density operator (See Eq. (4)).

Let us write the reduced density matrix in the general form

$$\rho_A = \frac{1}{2}(\mathbb{1} + \sigma \cdot S) = \frac{1}{2} \begin{pmatrix} 1 + S_z & S_x - iS_y \\ S_x + iS_y & 1 - S_z \end{pmatrix}. \quad (20)$$

in terms of the corresponding Bloch's vector. With the above parametrization we have a one to one correspondence between directions in three dimensional space and directions within the Bloch sphere. Note that the determinant of ρ_A only depends on the modulus of Bloch's vector. In other words, it is rotationally invariant. Suppose now we are planning to measure the amount of entanglement projecting the reduced density matrix of a given state along

three linearly independent directions. The uncertainty associated with this measurement will depend on:

1. the modulus of the corresponding Bloch's vector, as the amount of entanglement does.
2. the relative position of Bloch's vector with respect to the three projective directions.

Because of conditions 1 and 2, assuming the initial distribution of states to be isotropic, the average uncertainty after measuring sufficiently many states will only depend on the relative position of the three directions we project along. In particular, this implies we can choose a given direction to be the z -axis. We will call \hat{n} and \hat{m} the other two directions so that the angles they form with the z -axis are θ_n and θ_m . Then we can write the average uncertainty as

$$\delta_{av} = f(\theta_n, \theta_m, \phi_{nm}).$$

Here, ϕ_{nm} is the relative azimuth angle $\phi_m - \phi_n$. Moreover, because of condition 2, the following equalities must hold

$$\begin{aligned} f(\theta_n, \theta_m, \phi_{nm}) &= f(\pi - \theta_n, \theta_m, \pi - \phi_{nm}), \\ f(\theta_n, \theta_m, \phi_{nm}) &= f(\theta_n, \pi - \theta_m, \phi_{nm} - \pi), \\ f(\theta_n, \theta_m, \phi_{nm}) &= f(\pi - \theta_n, \pi - \theta_m, \phi_{nm}). \end{aligned}$$

This is equivalent to say that we could redefine the three positive axis (simultaneously or not) without changing the average uncertainty. Finally, from the previous set of equations one obtains that the function f has to have an extremum at

$$\theta_m = \pi/2, \theta_n = \pi/2, \phi_{nm} = \pi/2.$$

We have numerically calculated the average uncertainty δ_{av} over 10.000 states of the composite system uniformly distributed over the complex four dimensional joint Hilbert space and calculated this uncertainty for a series of different measurements with $\theta_m = \pi/2, \theta_n = \pi/2$ but ϕ_{nm} ranging from 0 to π . Results are shown in Figure 1, where we show the mean uncertainty in determining the amount of entanglement δ_{av} , defined as

$$\delta_{av} = \left\langle \sqrt{\left| \frac{\partial \det(\rho_a)}{\partial P_0} \right|^2 \delta P_0 + \left| \frac{\partial \det(\rho_a)}{\partial P_m} \right|^2 \delta P_m + \left| \frac{\partial \det(\rho_a)}{\partial P_n} \right|^2 \delta P_n} \right\rangle. \quad (21)$$

as a function of the relative difference ϕ_{nm} for a fixed value of $\theta_m = \theta_n$. In the definition above, P_0 , P_m and P_n are the probabilities to obtain a *spin up* when measuring along directions \hat{z} , \hat{m} and \hat{n} respectively and where each δP denotes the squared variance given by $\delta P = \frac{P(1-P)}{N}$. The bracket means the average over the isotropic distribution of states. It is clear that δ_{av} reaches in fact a minimum when $\phi_{nm} = \pi/2$. Similar figures can be plotted, all of them supporting that the minimum uncertainty is indeed achieved when the three directions of projection are chosen to be mutually orthogonal.

FIGURES

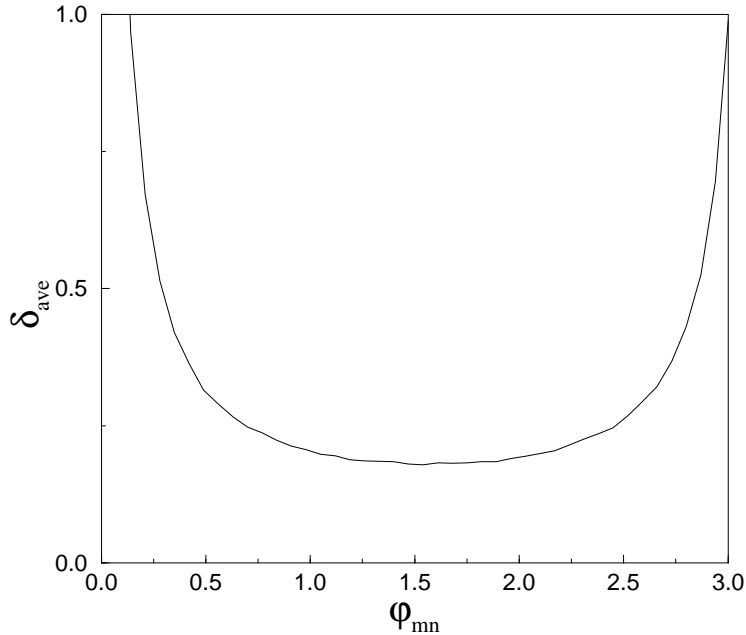


FIG. 1. Average uncertainty as a function of the relative azimuth angle ϕ_{mn} for a fixed value of $\theta_m = \theta_n$. The average uncertainty in determining the amount of entanglement is minimal when the three linearly independent directions of projection are chosen to be orthogonal.

B. Local actions with exchange of classical communication

Let us now assume that Alice and Bob agree to cooperate. Then, the amount of entanglement can be evaluated from the measurement of two Pauli operators in each side. If they agree to measure different operators in each round, they again fully reconstruct the state. However, if one of them always measures the same Pauli operator, for instance they choose to compute the observables $\sigma_z \otimes \mathbf{1}$ and $\mathbf{1} \otimes \sigma_z$ and, in a subsequent round, the observables $\sigma_x \otimes \mathbf{1}$ and $\mathbf{1} \otimes \sigma_x$, for which they should read out three *meters*, they can obtain the amount of entanglement but they will neither get full information about the state itself, nor about the whole reduced density matrix. Indeed, if we denote by P_i ($i=0,1,2,3$) the four probabilities associated to the outcomes $(++,+-,-+$ and $--)$ when measuring $\sigma_z \otimes \mathbf{1}$ and $\mathbf{1} \otimes \sigma_z$ and by P_{++}, P_{+-}, P_{-+} and P_{--} the corresponding probabilities when measuring $\sigma_x \otimes \mathbf{1}$ and $\mathbf{1} \otimes \sigma_x$, it can be easily shown that the probabilities P_{ij} ($i, j = +, -$), can be written in terms of the probabilities P_i as follows:

$$\begin{aligned}
P_{++} &= \frac{1}{2} \left(P_0 + P_1 + 2\sqrt{P_0 P_1} \cos \phi_{01} \right) \\
P_{+-} &= \frac{1}{2} \left(P_0 + P_1 - 2\sqrt{P_0 P_1} \cos \phi_{01} \right) \\
P_{-+} &= \frac{1}{2} \left(1 - P_0 - P_1 + 2\sqrt{P_3 P_2} \cos \phi_{23} \right) \\
P_{--} &= \frac{1}{2} \left(1 - P_0 - P_1 - 2\sqrt{P_3 P_2} \cos \phi_{23} \right),
\end{aligned} \tag{22}$$

where we have rewritten the amplitudes of the initial state as $a_i = m_i e^{i\phi_i}$, ($i=0,1,2,3$), and called $\phi_{ij} = \phi_i - \phi_j$. From the previous set of equations, we see that the functions $\cos(\phi_0 - \phi_1)$ and $\cos(\phi_2 - \phi_3)$ can be expressed in terms of measurable quantities in the form

$$\begin{aligned}
\cos(\phi_0 - \phi_1) &= \frac{2P_{++} - P_0 - P_1}{2\sqrt{P_0 P_1}} \\
\cos(\phi_2 - \phi_3) &= \frac{2P_{-+} + P_0 + P_1}{2\sqrt{P_3 P_2}}.
\end{aligned} \tag{23}$$

Noting that

$$C^2 = 4 \left(P_1 P_2 + P_0 P_3 - 2\sqrt{P_0 P_1 P_2 P_3} \cos(\phi_0 - \phi_1 + \phi_3 - \phi_2) \right). \tag{24}$$

we see that this measuring strategy suffices to determine the squared concurrence and therefore the amount of entanglement of the pure state but it does not allow the full reconstruction of the initial state. As we will show in the next section, this protocol is not optimal, in the sense of providing the best accuracy when measuring locally a minimal set of observables.

C. Which strategy yields the best resolution?

The measuring strategies described above are both minimal, in the sense of involving the smallest number of meters to be read out. However, it is not obvious whether the precision achieved following these two strategies will be the same. In fact, we will show in the following that it is not. When provided with the same resources, that is, using the same number of identically prepared entangled pairs, we can get the amount of entanglement with higher precision by means of the local reconstruction of the reduced density operator. If we denote by N the number of entangled pairs, N being large in the statistical sense, a numerical simulation with 10^6 states from an isotropic initial distribution yields the following results.

1. The measurement procedure by means of the local reconstruction of the reduced density operator has an associated uncertainty which scales with N as

$$\delta_{loc} = \frac{0.3}{\sqrt{N}}.$$

2. The associated uncertainty with a local measurement of the form described in Sec. IV.B is substantially much larger [10]. More precisely

$$\delta_{loc+cc} = \frac{2.3}{\sqrt{N}}.$$

Note that, once N is given, the resulting number of measurements in each measuring protocol is different. While in the first case each single probability will scale as $P \approx 1/\sqrt{N/3}$, the larger number of measurements yields each probability in the second procedure to scale as $P \approx 1/\sqrt{N/2}$.

From these results one may be lead to the conclusion that the best resolution will always be achieved by means of reconstructing the reduced density operator. However, this may not be true. Imagine that, in the context of the second protocol, Bob measures a different Pauli operator. If the direction of projection is orthogonal to the z -axis, this procedure will also allow to reconstruct the initial state. Will now the associated uncertainty be reduced with respect to the case analyzed above? In the light of the results obtained when measuring the reduced density operator, this should be the case. It should be noted, however, that the number of observables required in this measurement protocol is no longer minimal, as it would require the observer Bob to read out an additional meter.

V. CONCLUSIONS

We have analyzed the problem of determining experimentally the amount of entanglement of bipartite pure states when one has a large supply of identically prepared systems on which one is restricted to act by means of projective measurements. We showed that, provided that the entangled state is totally unknown, no measuring strategy involving a single operator exists. Therefore, acting on the Hilbert space of the composite system does not allow to single out the amount of entanglement without allowing to determine the state completely. When local actions are considered, the minimal protocol for determining the amount of entanglement involves measuring three different observables. We have analyzed here two classes of minimal tests. In the first one, no exchange of classical information is required and entails the local reconstruction of the reduced density operator. The procedure is optimal, in the sense of having the smallest associated uncertainty, when measuring along three mutually orthogonal directions. The second class of protocols requires the use of classical information. Here we have analyzed a possible strategy and showed that it suffices to determine the amount of entanglement of the pure state but not its full reconstruction. The associated resolution turns out to be worse than the one corresponding to the measurement of the reduced density operator. The analyzed protocol is not necessarily the most precise among the whole class of measuring strategies by means of local actions with the exchange of classical information. However, an increase in the resolution would be done at the prize of increasing the number of meters to be read out and the protocol would no longer be minimal. Establishing how the best accuracy achievable within a protocol of this type, and which allows the full reconstruction of the state, compares with the one associated to the reconstruction of the the reduced density operator is an interesting open problem.

Acknowledgements: The authors thank M.B. Plenio, P. Hayden, D. Jonathan, G. Vidal, A.K. Ekert, D.P DiVincenzo and J.I. Cirac for discussions on the subject of this paper. J.M.G.S. also acknowledges M. Ferrero for continuous encouragement and useful discussions. This work has been supported by The Leverhulme Trust, the European Science Foundation, The Engineering and Physical Sciences Research Council (EPSRC) and DGICYT Project No. PB-95-0594 (Spain) and has benefited from the participation in the ESF-QIT workshop in Cambridge.

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