Measuring Time with Minimal Clocks

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Abstract

Being able to measure time, whether directly or indirectly, is a significant advantage for an organism. It allows for the timely reaction to regular or predicted events, reducing the pressure for fast processing of sensory input. Thus, clocks are ubiquitous in biology. In the present paper, we consider minimal abstract pure clocks in different configurations and investigate their characteristic dynamics. We are especially interested in optimally time-resolving clocks. Among these, we find fundamentally diametral clock characteristics, such as oscillatory behavior for purely local time measurement or decay-based clocks measuring time periods of a

scale global to the problem. We include also sets of independent clocks ("clock bags"), sequential cascades of clocks and composite clocks with controlled dependency. Clock cascades show a "condensation effect" and the composite clock shows various regimes of markedly different dynamics.

Keywords— timekeeping, information theory, clocks, oscillation, cycle, transfer entropy

1 Introduction

> Time in itself, absolutely, does not exist; it is always relative to some observer or some object. Without a clock I say 'I do not know the time'.
>
> John Fowles, Áristos

In the present paper, we study how time measurement could look in the most minimal cases which we believe relevant to simple organisms.

Klyubin et al. [13] look at maximizing information flow for a simple navigation task in minimal agents as proxy for other more specific and/or complex achievements, such as homing. One striking phenomenon that appears there is that when they consider information processed by the agent, they find that, via a direct factorization analysis, the internal processing of information by the agent can be roughly decomposed[1] into an essentially spatial and a temporal component. Specifically, the memory of the informationally optimal minimal agent typically contains information about the respective lifetime step of the agent without time ever having been optimized for. In that work, the ability to measure time emerged as a pure side effect.

Here we turn the measurement of time into our primary objective: we consider measuring time as the most intrinsic, least environmentally affected processual quantity, with the following assumptions: time has a well-defined beginning; its tick is discrete, global and accessible to the agent.

We will discuss the motivation behind some of these assumptions and their implications in more detail in the following sections. For now, it will suffice to say that the first assumption is quite natural, as many biological processes have a natural beginning, e.g. some trigger event, which might even be just the beginning of the organism's morphogenetic process. The discreteness of the tick constitutes a more serious conceptual problem, as it poses the question concerning the origin of the natural timescale. We will discuss this point later in more detail in connection with external rhythms (or Central Pattern Generators, CPGs). The final issue is the most subtle one, namely how an external tick becomes accessible to the organism (assuming it is

[1] Other factorizations could be considered such as the multivariate information bottleneck [31] or decompositions based on unique information [3] or complexity-based decomposition [1].

not a self-generated CPG oscillation). This major issue does not fall into the scope of the present paper and will be considered separately in future study.

2 Motivation

2.1 Why Do We Study Clocks in Minimal Systems?

As mentioned above, Klyubin et al. [13] showed that (imperfect) time measurement can emerge as consequence of an agent optimally extracting information about its environment. This information that the agent acquires in its memory is partially determined by the environment and partially intrinsical (via the discrete time tick of the model). That work, however, shows an incomplete (yet more plausible) case of time acquisition.

We, however, believe that it is studying the extreme cases of a phenomenon that is essential to understand better the constraints and conditions that are operating on a potential artificial life (ALife) system and what fundamental effects they elicit. In this sense, studying pure time measurement is, in a way, studying an extreme case, namely the "most intrinsically acquirable" information, requiring only a tick and an initial state to compare with and no further environmental structure/input. This extreme minimal setting is what we set out to study here — to ask the question what time measurement could possibly look like in an informationally limited system.

Is this relevant for survival? We know from Klyubin et al. [13] that some timekeeping at least emerges as a side effect for a more-or-less navigational task — and a hypothesis hovering in the background is that, once discovered, good timekeeping might then be used by a species for exaptation to novel niches where it can complement other information which otherwise were not of sufficient quality.

While the argument for this is too long to be made in full in the present paper, we will sketch it briefly here: an agent needs to acquire a certain amount of *relevant Shannon information* from the environment to carry out a given task at a given performance level [28, 25, 34, 33]. However, in some quite generic ALife-type scenarios it can occur that a sometimes significantly larger amount of Shannon information needs

to be acquired by an agent from the environment than corresponds to the actually relevant information. This can be seen as the information-theoretic analogy to thermal efficiency from physics; or, in the information-theoretic case, its complement, namely the informational inefficiency and closely related to the ratio between excess entropy and statistical complexity [30]. As such, this is quite a generic phenomenon and its occurrence is not the exception, but the rule.

It turns out that this excess or "piggyback" information, while not relevant to the original goal, can sometimes be repurposed for another goal [35]. It was hypothesized earlier that such a phenomenon might be pervasive in evolution and in fact offers explanations for some interesting evolutionary phenomena such as the ability for exaptation or very rapid refinement and specialization of sensors.

Other, subtle reasons why time measurement is essential to understand the nature of cognitive control by an agent is the computational complexity required to keep track of time [37]; these require further studies and are, partly, still open questions themselves. For now, we choose to concentrate on as much simplicity and minimality of the model as we can muster.

2.2 The Discrete, External Tick

The clocks in this paper are modeled with a discrete and finite (and very small) state space with a discrete axis of time. Pairing a discrete space world with a discrete axis of time means that all clocks have regular ticks. In a model where multiple clocks are simulated together, this also implies a synchronized global tick for all subclocks in the system.

A model with an implied regular tick given to all clocks is a considerable assumption. By making this assumption, we are choosing to not study how a synchronized and regular ticking mechanism could emerge and be sustained in a world with continuous time. Instead, we focus entirely on which configurations of minimal clocks we can possibly find once a regular, overall synchronized tick is assumed to exist. Notably, we completely ignore the route towards how such a tick would come about in the first place. We note that emergence and mechanisms of CPGs (see also the remarks in the

section on continuous time further below) are thoroughly studied topics. In our context, the analogous question would consist of the information-theoretic characterization of the process of how synchronization, adaptation and reliability would emerge in minimal oscillators (ultimately with biological plausibility in mind). However, this is a topic that requires a substantial separate study and will be studied in future work.

2.3 Sensing Time vs. Measuring It

Before we proceed, we wish to discuss why we do not include into our considerations the simpler possibility of utilizing external cyclic behavior as a possible paradigm for time measurement.

Generally, having an external "timer" helping an agent to synchronize with its surroundings is a useful approach and probably altogether unavoidable once one moves into continuous time. However, in this paper, we restrict ourselves to considering the extreme case of an almost solipsistic system where the only environmental influence is the defined starting state and the global discrete tick. Emphatically, this model generalizes and morphs naturally into any system with an explicit external clock. The latter, however, constitutes an informationally more intricate case: while it makes time measurement easier for the organism, it renders the analysis of the "pure" time measurement considerably more difficult and will have to be left to future study.

2.4 Continuous Memory

The study of clocks could also be imagined to be conducted in a continuous-space world (i.e. the state space of the clocks is considered to be continuous). We choose to model clocks with discrete state spaces because, in a model with continuous state spaces (and lacking any other constraints such as imposed noise), clocks optimized for time resolution would end up being unbounded in their memory capacity (and also complexity). There, an efficient optimizer could potentially find arbitrarily complex clocks when maximizing information about time.

One possibility to limit this complexity while using continuous state spaces would be

to introduce noise which would naturally limit the achievable resolution. However, this would imply the introduction of additional assumptions. Since our focus is primarily the study of minimalistic clocks, we choose arguably the most direct way to restrict the complexity of the clocks, by limiting their state spaces to be finite (and small).

2.5 Continuous Time

Central Pattern Generators (CPGs) are biological neural circuits that produce oscillatory signals. Marder & Bucher [17] define them as:

" [...] neuronal circuits that when activated can produce rhythmic motor patterns such as walking, breathing, flying, and swimming in the absence of sensory or descending inputs that carry specific timing information. "

CPGs are common in biology. They implement "soft" clock ticks in the form of oscillations, in continuous time and having a continuous state space. They reduce the burden on other units in a biological system of generating ticks or defining phases.

CPGs are examples in nature of how the task of generating regular ticks can be a specialized one, segmented from other processes. And, similar to the way CPGs commonly operate in biological systems to provide ticking to other mechanisms, we hypothesize that in general, clocks can also be seen as being composed of a tick producer and their (possibly probabilistic) tick counter. Of the two parts, we focus our study in this paper on what are essentially probabilistic tick counters, instead of studying the tick generating mechanisms. By modeling the axis of time as made up of discrete moments, the simulated clocks implicitly receive a regular clock tick. Their stochastic behavior is applied once at every time transition.

Since in the present paper we consider probabilistic clocks, these would permit meaningful models for continuous time even with a finite number of states. Now, in the present paper we are explicitly interested in the edge cases of alternators (discrete oscillators) and the possible emergence of counters on top of these.

However, Owen et al. [23] find that displaying an oscillatory behavior with finite states in continuous time at equilibrium is impossible without a hidden state. In their paper, they show that under the conditions described therein, to implement oscillations

in continuous time, one needs a state where the distribution can move into while between transitions. Using a Markov chain as model, that Markov chain requires at least a triad of states in a discrete system in continuous time. While we do not reiterate the precise (and intricate) argument behind the work by Owen et al. [23], one intuition behind this is that in continuous time one essentially cannot move probability masses from one state to another arbitrarily fast and they have to be moved continuously. This necessitates at least a third state to store it. The analogy in a continuous state space would be that a Markovian implementation of an oscillator would require a minimum of two dimensions to realize an oscillation (state and rate of change, corresponding to the second order differential equation for oscillatory dynamics).

Since in the present paper we do not wish to model either hidden or external states, but wish to encompass the whole system, we will again postpone this possible generalization of our scenario to a future study and limit ourselves to modeling discrete-time dynamics of discrete state systems which will already exhibit a richly structured set of phenomena.

2.6 General Comments on Time Measurement

More specifically, in the context of biologically plausible models, we are interested in how a Markovian agent can keep track of the flow of time under strong limitations on memory capacity. Going forward with the earlier assumption of the pre-existence of ticks, measuring time becomes effectively equivalent to counting. We will find that, to make the best out of limited resources, in some circumstances we will need to count probabilistically. This is what we will study in the present paper.

In its conceptually simplest incarnation, the measurement of time would consist essentially of two components: having a reliable generator of periodic behavior (which we pre-assume in the present context); and being able to count the periods. To measure larger time intervals precisely, one needs full-fledged counters, which, in turn, require a comparatively complex logical make-up (and more complexity in temporal structure — such as with multiple distinct event types, multiple kinds of 'clock ticks' or 'events' — opens the way to the general structure of discrete dynamical systems and their algebraic

understanding via semigroup theory, the general study of models of time — see Sec. 3), while not impossible in principle, one would not expect this to appear generically in biologically relevant scenarios and most certainly not in very simple organisms.

Rather, one would typically expect to find less precise, but simpler and more robust solutions. The present paper investigates how the most minimal of such models could look and which characteristic properties we expect them to exhibit. In our discussion we include both explicit clocks characterized by a distinct apparatus which clearly have a time-measurement function such as the suprachiasmatic nucleus in the mammalian hypothalamus involved in the control of circadian rhythms [12], but also implicit clocks (which exhibit some aspects of clock-like behavior, such as being correlated to time, but without a dedicated mechanism) like the feeding-hunger cycle of an animal. We do not differentiate these classes a priori since our formalism does not discriminate between them. Instead, our study will concentrate on systems which, given their constraints, are maximally able to measure time, not considering any other tasks — thus we study clocks as "pure" as they can be under the circumstances given. We expressly are not studying how an agent could infer time from correlations in the environment (using an — external — sundial for example), but only in how time can be tracked intrinsically. For such a study of the evolvability and robustness of internal time-keeping mechanisms in gene regulatory networks in the context of (possibly intermittent and noisy) external periodic variation, see e.g. [14]. In the same vein as considering Artificial Life as about understanding life-as-we-know-it vs. life-as-it-could-be [15], and in view of how life pervasively makes use of clocks, the present paper studies possible clocks themselves, or, in analogy to Chris Langton's research program: "time-as-it-could-be-measured".

3 Temporal Dynamics: The Algebra of Time

We begin with some general comments on the structure of time. Whether in classical or relativistic physics, or in more general models (e.g., ancient Indian notions of cyclical time, or in modern automata networks), there is a commonality to notions and models of time: From the perspective of a single organism or agent, events in time satisfy a

grammatical constraint attributed to Aristotle in [27]: If α, β and γ are each sequences of events, then: if γ follows β in an agent's experience, and, prior to both, α occurs, that is exactly the same as when β follows α, and γ occurs after both, i.e.

$$\alpha(\beta\gamma) = (\alpha\beta)\gamma.$$

In short, sequences of events in time from the perspective of an individual agent satisfy the *associative law*. A structure with a 'multiplication' operation satisfying this law is thus a model of time, called a *semigroup* in mathematics. Here, concatenation is silently used to denote the binary operation of following one sequence of events by another, which is the associative multiplication operation of the semigroup of event sequences.[2]

Specifically for time which has only a single kind of clock tick t, let t^n denote the occurrence of n clock ticks one after another. One can distinguish classes of time which exhibit only a single type of ticks (or event) t as follows: determine the first $k > 0$ for which there is an $\ell > 0$ such that $t^k = t^{k+\ell}$, and whether such a k exists or not. Choosing ℓ minimal, time cycles every ℓ ticks after an initial transient $t, t^2, \ldots, t^{k-1}$. There are then four qualitatively different kinds of single clock tick-generated time [19]: (a) cyclical time ($k = 1$), where changes repeat in cycles of exactly ℓ steps, (b) purely transient time ($\ell = 1$) — with no change happening after k ticks, (c) a transient followed by a cycle ($k > 1, \ell > 1$), where after k initial transient steps time becomes an ℓ-step cycle, and, finally, (d) infinite non-repeating discrete time indexed by positive integers, in the remaining case when no positive k and ℓ exist with $t^k = t^{k+\ell}$. In this classification, ℓ is the length of the attractor cycle that time eventually must enter, if k exists; and then $k - 1$ is the length of the transient before the attractor is reached. Case (d) can be viewed as having an infinite transient, $k = \infty$, with no attractor, $\ell = 0$.

Note that all these models of time based on a single type of clock tick are necessarily *commutative*: in other words, here the order of event sequences does not matter, i.e., t^a followed by t^b is the same as t^b followed by t^a, since both are comprised of exactly

[2] Equivalently, we can identify events and sequences of events with operators transforming the agent's world, in which case the associative operation is the composition of operators.

$a + b$ clock ticks t. (Every event sequence α is some number of ticks, i.e., $\alpha = t^a$ for some $a \geq 1$, and any other sequence $\beta = t^b$ (for some $b \geq 1$) is too. Thus it follows $\alpha\beta = \beta\alpha$.) This is a special property of semigroups generated by a single event type, which are classified as above into four types. Non-commutativity becomes possible as soon as there is more than one type of event or clock-tick, e.g., when the agent can choose between two or more actions, or undergo at least two different kinds of events, whose order may not be deterministic.

This type of modeling of time as a semigroup is common practice in dynamical systems theory. Note that it is in general not a *group*, which would be a model of time where every sequence of events must be reversible; in particular, there may be an 'earliest time' or multiple 'Garden-of-Eden' times beyond which a process may not be reversed (whenever there is a non-trivial transient, i.e., $k > 1$); furthermore, the process need not be reversible at all. Note that among the above models, cyclic time with $k = 1$ (case (a)), forms a group where enough repetitions of t reverse any sequence, and the infinite counter (case (d)), embeds in a group: time indexed by the integers $..t^{-2}, t^{-1}, t^0, t, t^2, ...$ When $\ell \geq 0$, the cycle $t^k, ..., t^{k+\ell-1}$ comprises a group substructure of the model of time, i.e., it implies a *local pool of reversibility.* Note that with these models of time, one can never reverse events suitably to re-enter the transient part of the dynamics.[3]

The study of (possibly general) structures satisfying these laws becomes the study of models of time. This viewpoint has deep connections with the theory of discrete or continuous dynamical systems [27, 20]. In the finite discrete deterministic case it leads to the Krohn-Rhodes theory, a branch of mathematics (algebraic automata theory) where discrete dynamical systems can be decomposed using (non-unique)

[3]There is a subtle point here for the infinite case (d): if the clock tick t is realized as an operator on agent's world (mapping states to states), the embedding of positive natural numbers $\mathbb{N}^+$ (the infinite transient) into all integers $\mathbb{Z}$ extends to an embedding of operators if and only if the operator $\varphi(t)$ corresponding to t is itself invertible (with its inverse associated to t^{-1}, and so that t^0 gives the identity operator). In the latter case, once the embedding is done, one could assert of the resulting system that $k = 1$ ('there is no [longer any] earliest time') and $\ell = \infty$ ('the apparent transient has become part of an infinite, reversible cycle'). That is, the embedding $\mathbb{N}^+ \hookrightarrow \mathbb{Z}$ induces an corresponding embedding for the powers of the operator $\varphi(t)$, since negative powers of the operator $\varphi(t)$ are well-defined. (This fails when $\varphi(t)$ is not invertible.) See [20] for full details relating events and operators in models of time.

iterative coarse-graining into a cascade of irreducible components. The composite clocks discussed later constitute a special case of such a decomposition.

With more event types than time ticks (e.g. consider an agent which has the choice of different actions and thus "futures"), one has to generalize further. Multi-event semigroups can be interpreted as agents that can have multiple alternative timelines. Here, in particular, the commutative law no longer holds in general: Washing ones hands before eating, or putting a glass on the table and pouring water has not the same outcome as when the events occur in the reverse order, but has a markedly different outcome from the other way round. Thus, in this case, $\alpha\beta$ is not necessarily the same as $\beta\alpha$. In this case, processes in time are not commutative; in these cases a more complete picture of semigroup theory needs to be invoked to fully describe the scenario.

In the remaining sections of the paper, we will not refer to different event types, but instead consider nondeterministic representations of time which emerge through informational optimality criteria. This causes much more complicated and interesting dynamics to arise that partly (but only partly) mirrors some of the aspects of the semigroup decomposition (see Sec. 9). It turns out that, even just considering the simple semigroup of time, the probabilistic setting under informational optimality gives rise to a rich and complex landscape extending and generalizing the discrete semigroup model into the probabilistic continuum.

We would note that the paper is at this stage not going to consider the case of multiple time observers, but only a single observer with a coherent tick. It will be insightful to consider the generalization to multiple observers. Even without considering relativistic scenarios, having non-synchronized observers (such as multiple sub-organelles in an organism, or, on a larger scale, the existence of local times before the introduction of standardized time zones or precise clocks due to the necessity to fit together train schedules) complicates the situation considerably.

4 The Cost of Measuring Time

Recent work by Barato & Seifert [2] highlighted interest in the problem of time measurement by asking if clocks must pay a thermodynamic cost to run [37]. Their stance is grounded in fundamental trade-offs of physics which cannot be subverted. But, at the level of organisms which are far remote from those physical trade-offs, those physical limitations, conservation laws and constraints do not apply in a straightforward manner. Concretely, we work in a near-macroscopic, classical (non-quantum) Markovian universe without presuming the additional structure of microphysics (including microreversibility or Hamiltonian dynamics). In particular, we cannot assume an obvious generalization of the physical concept of energy to a fully general Markovian system. Thus, it is not obvious how to quantify computation cost in terms completely analogous to thermodynamics. The only concepts that carry over are of entropic and information-theoretic nature. Specifically, Shannon information has been shown to be a highly generalizable measure of information processing cost [24]; a universal measure that can be used to compare systems of a different nature, and does not presuppose any structure on the state space of events; it permits to ignore labels and only considers the statistical coincidences.

Henceforth, all costs will therefore be expressed in the language of information theory; they will refer essentially to information storage and communication costs and constraints.

4.1 Small State Spaces

In the beginning of this paper we focus on the most minimal clocks (and later proceed to slightly more complex arrangements) for a number of reasons. For one, this will make it easier to explore the full solution space. The second reason is significantly more subtle, and we will only be able to sketch it here: essentially, it is not clear what measure of complexity to utilize for the cost of running a larger single-component counter and/or the complexity of running the transition itself; natural candidates for such costs might be predictive information [4], statistical complexity [6] or forecasting complexity [7].

However, the possible candidates are not limited to these three measures, and many other plausible information-theoretic alternatives can be conceived[4]. In absence of a canonical measure for the complexity of a single-component clock operation, here we limit ourselves to investigate the most minimal clocks possible, namely a 2-state (1-bit) clock and we will here not further concern ourselves with taking into account the informational cost of actually running this clock.

4.2 Information Flow Between Modules

We said above that we prefer to start out with already minimal clocks by default to avoid dealing with the question of how complex the operation of a clock is. But what if an agent needs a larger clock? In this case, we will build it out of smaller clocks. Again, we avoid the question how to cost this compositional complexity [37].

As we will state more quantitatively below (Sec. 9.1), such a compound clock will perform better if its components "cooperate". For them to be able to cooperate, they must be able to exchange information. Therefore any limitation in this clock's internal information flow will reduce the performance of the clock.

The motivation for limiting the information flow is that such a flow will in general be costly in biology, even outside of the earlier mentioned thermodynamic considerations; information processing in biology is expensive per se (even far away from the Landauer limit) [16]. This will limit how many components can communicate with each other and at which bandwidth.

All in all, in our studies below, when looking for candidates for clocks, we will consider Markov chains with a small number of states and when we move on to larger clocks we prefer to build them out of minimal clocks and use the information flow between the components as cost to consider.

[4]We will investigate this question in a separate paper, but for a discussion of the issue of computational complexity in a Shannon context, see e.g. [37].

5 Other Relevant Work

We mentioned earlier the work by Klyubin et al. [13] who consider the maximization of information flows in a very simple agent/environment system. The resulting agent controllers generate a rich set of behaviors, with a side effect that the controllers' dynamics causes the agents' internal states to partially encode location, but also partially time information. More precisely, when one inspects the agents' memory, it turns out that it provides partial information about the point in time from the beginning of the experiment. Some information about time and space can be extracted from the agent memory separately through a factorization process. Note, however, that, in that experiment, the joint encoding of spatiotemporal state in the agents' memory is merely a side effect of the homing information optimization task and not directly optimized for.

The importance of measuring specifically time (as opposed to having this measurement emerge via side effect) can be seen in other scenarios, for instance in 13- or 17-year cicadas [10, 32]. Also, closer to the level of fundamental physical limits, Chen & Luo [5] find relations that can be obtained by applying Fisher information to the problem of measuring time with quantum clocks and discuss the problem of clock synchronization in the quantum realm.

On the issue of considering global ticks, we note that Karmarkar & Buonomano [11] mention mounting evidence against the view that the brain keeps track of time by counting ticks and instead study the behavior of simulated SDNs (state-dependent neural networks), showing how neural networks can keep track of time implicitly in their states without counting [note the parallelism to the indirect identification of time in 13]. We also note [36] which discusses not only the possible mechanisms of time perception in the brain but also the difficulty of validating them.

Since we hypothesize that our abstract considerations find functional correspondences in nature, the simplest examples would be expected to be found in microorganisms. We would predict that, if our general hypothesis is correct, the characteristics of our results for minimal clocks will be reflected in very simple organisms. We will

preempt here one result detailed in the results section. Under constraints on the memory available to the clock, only two types of clocks will be found — local, short-term clocks measuring the time within a cycle (essentially its phase); and, long-term clocks which distinguish large-scale phases within an overall "lifetime" interval of interest. Thus, we get a very clear dichotomy between local time measurement and global time measurement. We will call the first type cyclic clocks or oscillators, and the second type "drop clocks" (essentially one-off decay-type time measurements). Indeed, it turns out that bacteria show examples of both cyclic and drop clocks. Examples are in the following paragraph.

Hut & Beersma [8] discuss the reasons why day and night require different behaviors, these forming the selective pressures of evolution for the circadian rhythm of bacteria — the periodic cycle of activity in organisms that allows them to adapt to the time of day. This is one central example demonstrating the importance of time-keeping to organisms. They note that the circadian rhythm of cyanobacteria has been reproduced and studied in a test tube. Therefore this is an example of a relatively simple, clearly understood explicit cyclic or *oscillator* (see below) clock. Nutsch et al. [21] study prokaryotic taxis for halobacteria, more specifically signal transduction pathway starting from sensing light and responsible for controlling the switching of the flagellum. This pathway implements a one-off drop clock[5].

They note that while the molecules (the chemistry) involved in this pathway are well studied, the way these components behave together dynamically is not well understood and only speculated on. Building upon an earlier model by Marwan & Oesterhelt [18], they suggest dynamical models to fit experimental findings. These models are synthetic and not based on first principles. Thus, the question about the actual structure and size of the fundamental clocks of bacteria remains currently unanswered.

Remarkably, measuring time can also help bacteria in spatial tasks. Some bacteria use differences along the length of their body to measure gradients [22], however, others instead measure time differences between intensities while moving [21]. This is demonstrating how measurement of space can be converted into measurement of

[5]Which is reset only by an external trigger; some mechanisms can be considered drop clocks triggered at conception and never reset until death, such as telomeres.

time, emphasizing once more the intimate relation between time and space in even the minimal navigational capabilities of bacteria.

6 Models of Clocks

In the real world, time is always encoded in physical systems; in a broad sense, any evolving system with nontrivial dynamics can be regarded as a clock. To read the time, we must perform measurements and estimate the parameter that encodes the time. How precisely we can estimate this parameter depends on the distinguishability of the involved states, and this distinguishability characterizes the quality of a clock. — Chen & Luo [5]

Because our clocks are modeled as Markov chains, they are completely described via their current state and their probabilistic dynamics. We consider only discrete state spaces and also advance in time only in the discrete steps given by the global tick. We reiterate that this means that the clocks receive time ticks for free and that they need not be concerned with the challenge of getting accurate ticks. Apart from these global ticks, the clocks do not receive any information from the environment. We furthermore consider first only the most minimal clock designs. As we proceed, we will consider more complex (composite) clocks.

6.1 Notation

Random variables are written as capital, their values as lowercase letters. The probability of a random variable X to adopt value x will be written as $P(X = x)$ or, where not ambiguous, as $p(x)$ by abuse of notation. The state of a clock which can take on values u and d ("up" and "down") will be denoted by random variables S (possibly subscripted by time t, because in general the probability distribution over the states of the clock changes in time). To model the uncertainty that the agent has about the current time, we treat true time (which the clocks attempt to measure), similarly to [13], also as random variable T which a priori assumes all possible (integer) time values between $t = 0$ and $t = T_{\max}$ with equal probability.

Typical quantities discussed would be, for instance, $P(S = u|T = t) \equiv p(u|t)$, the probability that the clock state is u ("up") at time t. Or else, $P(T = t|S = d) \equiv p(t|d)$ would be the probability that the time is t, given that the current state of S is d, etc. When we optimize clocks, we quantify our criterium of performance as mutual information $I(S;T)$. This mutual information tells how much information a previously uninformed agent would receive about time after looking at its clock (averaged over probabilities of the possible states observed in the clock, "up" or "down"). The quantity $I(S;T)$ is directly related to the probability of guessing time correctly given one observation of the current state of the clock. With this notation, we are ready to define our clock model. In all our experiments, the clocks will be initialized at time $t = 0$ in a fixed known state, specifically u, i.e. $P(S_0 = u) \equiv P(S = u|T = 0) := 1$ and they will advance by one time step per tick according to their respective dynamics.

6.2 Oscillator

We first define the *oscillator* as the 2-state Markov chain with a symmetric probability to change state.

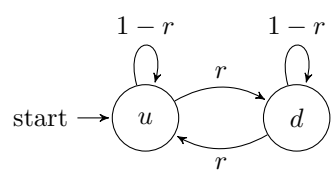


Figure 1: The oscillator clock. It switches one state to the other with probability r.

The plot of $p(u \mid t)$ in Figure 2, shows the distinction between different regimes in which the clock operates, depending on r. We notice the analogy with Damped Harmonic Oscillation.

Stuck For $r = 0$, the clock is stuck in state u (not shown in diagram).

Overdamped For $0 < r < 0.5$, the probability distribution behaves like an overdamped oscillator. Note that, in this regime, the probability distribution of the

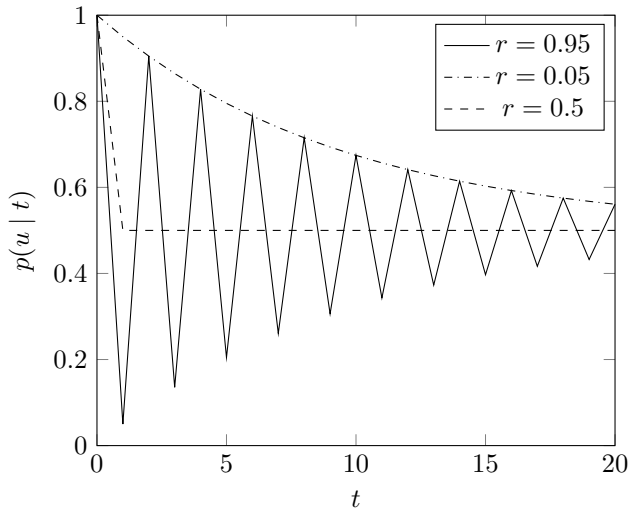


Figure 2: The oscillator clock for different probabilities of switching state.

oscillator clock has the same envelope as the drop clock which we discuss below.

Critically Damped For $r = 0.5$, the clock acts analogous to a critically damped oscillator: it reaches the equilibrium in the shortest possible amount of time — one time step in our discrete time case.

Underdamped For $0.5 < r < 1$, the probability distribution behaves like an underdamped oscillator. This can be interpreted as the clock being initially "synchronized" with time like a clock, but that the synchronization gets lost as the clock's state changes in time, until the correlation between t and s disappears.

Undamped For $r = 1$, the clock state alternates between u and d. Not shown in figure.

These behaviors are shown for the symmetric oscillator, but the asymmetric one (where the transition probabilities for $u \to d$ and $u \leftarrow d$ differ) has the same qualitative behavior, just with a different equilibrium point. All 2-state Markov chains belong to one of these 5 classes. In other words, the class of 2-state clocks can exhibit only this small number of different behaviors. We will refer to this insight in the results section again.

6.3 Drop Clock

Consider the thought experiment of an insect that leaves its nest to forage or explore. Even if it does not find anything, the insect should still initiate a return to its nest at some point in time, otherwise it risks getting lost. Such an insect would profit from a clock telling it if "it's been awhile" since it left its nest.

An oscillator is not well suited for this type of task; it is able to measure a local phase (an odd or even step), but it does not provide much "larger timescale" information, unless it is overdamped. Instead of the latter, it turns out that a *drop clock* is a more natural model for this task.

More complicated models will exist in nature. More complex clocks, such as clocks with time-dependent transition laws, however, require more memory under the Markovian constraint. Another example is that of gene regulatory networks which extends beyond discrete time into continuous time. Here, however, our question strictly focuses on the simplest possible ones.

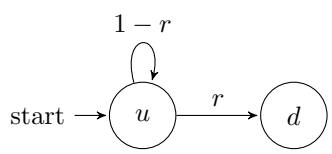


Figure 3: The drop clock, a Markov chain with a probability r to permanently transition to state d and remain there.

The drop clock starts (as all our clocks) in a well defined state, namely u, i.e. with $P(S = u|T = 0) := 1$. After each time step, there is a probability r that the clock "decays" (transitions from state u to state d) and the counterprobability $1 - r$ that it does not. Once the clock has decayed, it remains in state d forever. The behavior that this rule generates is an exponential decay $(1 - r)^t$ in time. An agent using a drop clock infers probable time only from the state of this clock without having any other knowledge of what the time is.

7 Experiments with 1-bit Clocks

Our first experiments are dedicated to the 1-bit (i.e. 2-state) clocks. As discussed earlier, there are only few distinct classes of such clocks. Most notable are the 2-state oscillator and the drop clock. The oscillator offers the maximum of 1 bit of information about time, namely whether one is in an odd or even time step. However, this information is purely local and cannot distinguish whether one is in an early or a later section of a run. However, even with only 1 bit of state, the drop clock can provide this distinction, albeit quite imperfectly, typically at significantly less than 1 bit resolution. For it to provide best results, the probabilistic decay rate of the drop clock (unlike the oscillator) must be attuned to the length of the total time interval of interest; this rate will be in general acquired by evolution or some learning mechanism — however, here, we will directly obtain it by optimization of informational costs. This gives us a handle on what the most effective possible drop clock could possibly be.

We study this next. Note that our axis of time is almost featureless except for two features: length of time and the grain. The oscillator matches the grain (local information) and the drop clock matches the length (global information). As discussed earlier, we do not impose any other features on the axis of time (such as months or seasons, which constitute purely external drivers) because we only study pure time, based on the fundamental tick only. For a meaningful link to external events, as for a consistent treatment of continuous time, one would need to adopt an approach more closely resembling the study of Klyubin et al. [13].

7.1 Measuring Large Time Scales

We now investigate time measurement by drop clocks at different timescales. We tune the clock by finding the drop probability r maximizing $I(S;T)$ for the particular timescale of the experiment. Optimizing this with a parameter sweep (of values $0 \leq r \leq 1$ at increments of 0.01) gives Fig. 4.

The first result is that the decay rate r that best resolves time with a drop clock clearly depends on the time interval. Such a clock, therefore, must be adapted to the

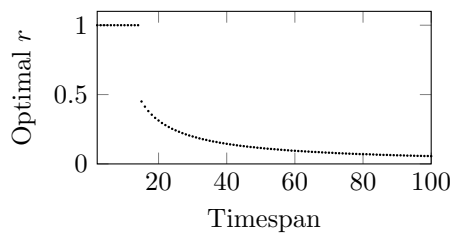


Figure 4: The optimal drop probability r for different timescales. Note that the curve for the drop clock has a discontinuity at $|T| \approx 15$.

particular time interval to resolve. Strikingly, we find two regimes of solutions, namely one with one fixed decay rate (up to $T \approx 15$), and then a time interval-dependent decay rate.

A closer inspection of Fig. 5 shows a relatively complex landscape where a global maximum of time information at the maximal decay rate $r = 1$ is superseded at larger times by maxima at lower decay rates.

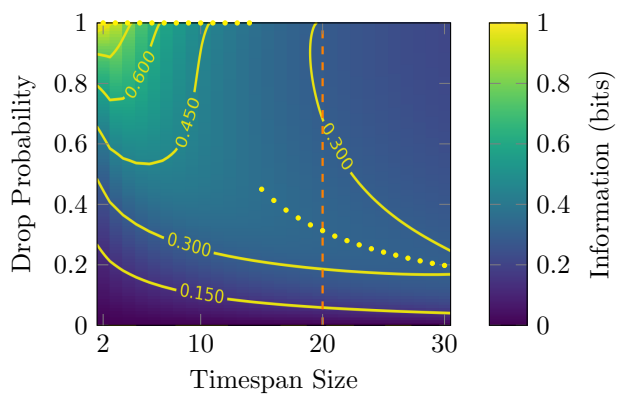


Figure 5: Time information for different drop probabilities and time spans. Taking a slice in this plot at the dashed orange line creates the next plot (Fig 6). The yellow dots show the best decay parameter for that time span (essentially the same data in Fig. 4). The yellow contour lines are drawn for additional clarity.

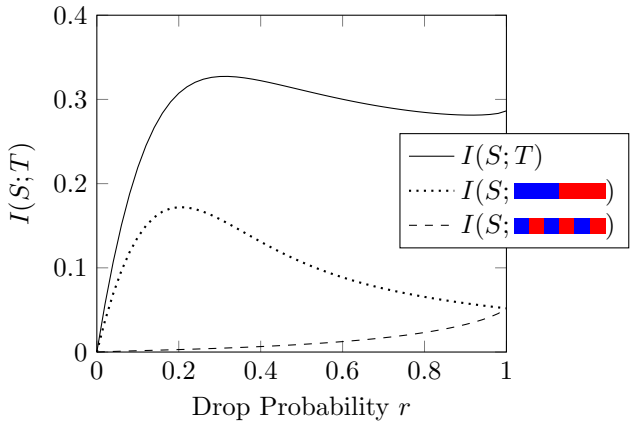


Figure 6: Time information for different drop probabilities at $T = 20$. Apart from total time information, it shows information of clock state S about being in an odd vs. even time step or early vs. late phase of the measured period.

Figure 6 shows (solid line) a vertical slice from Figure 5 (dashed orange line) at $T = 20$. This information curve does not have a unique maximum, but an inflection. This inflection stems from two separate contributions.

We can identify the two-fold origin of the two local maxima by partitioning time into different features: one way to look at time is to distinguish an "earlier" from a "later" part and the other way is to distinguish "odd" vs "even" times. Our interpretation is that the different maxima result from information the clock has about different partitionings of time. The maximum around $r \approx 0.3$ appears to come from the global picture the clock has (the "earlier" vs "later" partitioning) while the second maximum at $r = 1$ comes from the "odd" vs "even" partitioning ( ).

Smaller (slower) decay rates prove better at resolving global timing (early vs. late), and, while the drop clock generally resolves odd/even times only weakly, it still achieves this level resolution best for a hard decay rate for r, i.e. for decaying right away at the beginning of the interval. This explains the phase transition in Fig. 4 between the regime of short time spans to that of large time spans. This transition occurs because of the inflection in the information curve of the sharply initialized drop clock, when one

maximum dips down below the other: different parts of the curve derive from knowing different aspects about the current time and because in some regimes one dominates the other.

We note that these different regimes make the drop clock hard to optimize — short time spans require a different strategy than long ones: not only must the clock be attuned to a particular time scale to best measure global time, but also does the information curve of the clock show two maxima where one overtakes the other. We note that this transition from one maximum to the other corresponds to a first order phase transition, since they keep a finite distance from each other when the transition occurs.

7.2 Bag of Clocks

Having studied the 2-state clocks, we now design a larger experiment. We would like to keep the individual clocks simple, but be able to measure time more accurately. For this purpose, we consider "bags" of independent non-communicating clocks to measure time. Notice that even though the clocks in the bag cannot communicate with each other, the total amount of information that the bag of clocks holds about time will in general still be higher than that of the individual clock. This is because each clock can hold information about a different part of the axis of time.

We start the experiment with an empty bag. The collection is then built up incrementally, one clock at a time. The clock that is currently being added is optimized and after it is optimized so that the current bag maximizes time information, the latest added clock is frozen and added to the collection. We use this incremental process to capture the intuition that, in evolution, existing features tend to be more-or-less "frozen" because they are intertwined with the rest of the organism while it is the new features which are mostly optimized in relation to the existing frozen features.

Of course, it is quite possible that, at a later stage, a frozen feature could soften again; especially feedback from the environment or continuous time might bring about such relaxation and a subsequent co-evolution of clocks. However, the analysis of this case becomes significantly more intricate and we will revisit this in the future.

We will now state our model more precisely. We assume that while a clock is being considered, some n clocks are already in the collection. Given the state of the clock collection $\mathbf{S}^n = (S_1, \ldots, S_n)$, with $n = 0$ indicating the empty collection, the dynamics of clock n is optimized as to maximize $I(\mathbf{S}^n; T)$, always for a fixed total duration of $T_{\max}$; the dynamic parameters of all clocks $k = 1, \ldots, n-1$ are kept fixed during the optimization. Once the optimization is complete, clock n is frozen and a new clock $n+1$ is added and the procedure repeated. In our experiments, we stopped the process once we reached 10 clocks. The parameter space of this optimization is the set of probabilities of the transitions $u \to u$, $u \to d$, $d \to u$ and $d \to d$ for each of the clocks. The clocks that the optimization finds turn out to be either a pure oscillator or else, strictly drop clocks. As the collection grows, so does the achieved time information $I(\mathbf{S}^n; T)$ (Fig. 7). The first clock added to the bag is the oscillator, as intuitively expected, as it resolves perfectly 1 full bit of information. All subsequent additions to the bag, however, turn out to be pure drop clocks, with no oscillatory component. The first two clocks together add ~ 1.5 bits, while every subsequent clock adds significantly less.

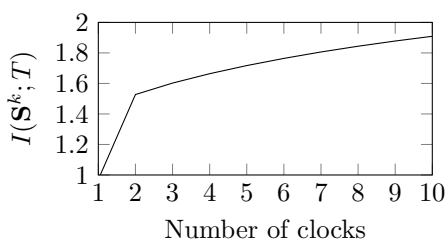


Figure 7: Amount of information as the size of the collection grows for a time interval of length $T_{\max} = 5$.

Analysing the resulting bag of clocks in detail, we notice that all the clocks — except for the first two — strikingly have very similar dynamics (parameters are given in Appendix A.1). In other words, once the first two clocks are added, all further clocks essentially act together as an increasingly refined binomial process, as more clocks are

added[6]. Because the last clocks added to the bag have almost identical parameters, we ran another calculation to explore the behavior of populations of precisely identical drop clocks. We use a bag of pure drop clocks, all with the same $r = 0.1$, starting with one of these clocks and adding more until we reach a total of 10 clocks. We compute $I(\mathbf{S}^n; T)$ for a time scale of size 50 to create Figure 8. The curve in this plot shows same diminishing returns from increasing the size of a clock bag, similar to the curve in the previous plot (Fig. 7 beyond $N = 2$).

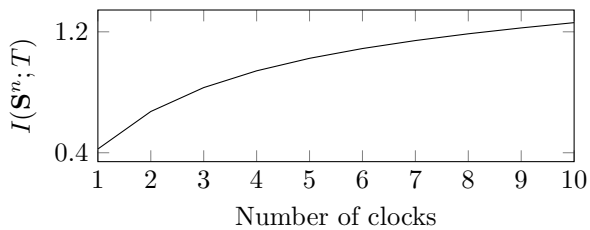


Figure 8: Time information as more identical clocks are added to the set for $T_{\max} = 50$.

8 The Clock Cascade

The clock bag above contained independent clocks which were not permitted to interact with each other. With the clock cascade, however, we introduce a simple interaction scheme.

We arrange the clocks in a queue and begin the simulation by waiting for the first clock to drop. Once the first clock has dropped, the second clock is released and may drop (as all drop clocks do, probabilistically) when time advances after being triggered. The fall of the second clock, in turn, triggers the release of the third clock and so on, creating a "domino" effect. Finally, after the sequence has run through all of the clocks, they lie dormant.

Of course, we could just consider a single decaying counter instead, but we would like here to emphasize that the cascaded counter is constructed from simple individual

[6]Special thanks to Nicola Catenacci-Volpi for this observation.

clocks and the cascade constitutes a biologically relevant architecture [21].

8.1 An Illustrative Example

We take the example of an arrangement of $N = 6$ clocks. We choose this small number here so that the conditional probability (between the state of the arrangement and time) can be plotted easily.

We first used the DIRECT Lipschitzian method for global optimization [9] on the model to maximize $I(\mathbf{S}^n; T)$, the mutual information between the state of the sequence and time. The solution to this optimization; the optimal sequence of decay parameters for these circumstances, is shown in Figure 9.

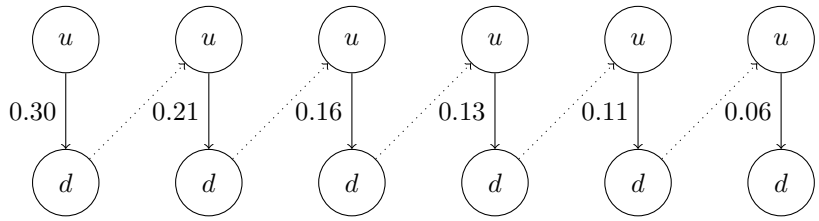
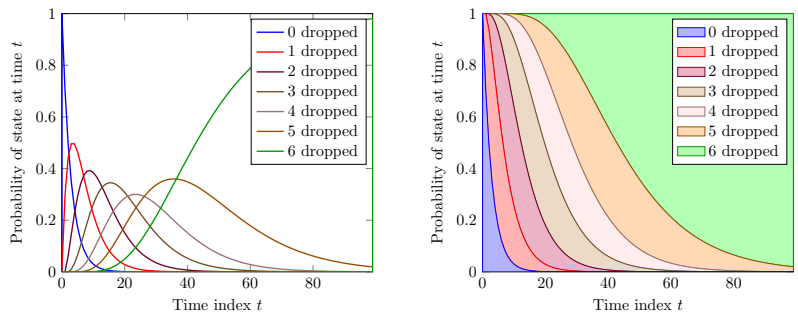


Figure 9: The optimal clock cascade for $N = 6$ and $|T| = 100$. The arrangement contains 6 clocks and they are all in state *up* at the beginning of the simulation. Then, after some time, the first clock drops and by dropping it releases the second clock (notice the dotted arrows). One by one, the sequence runs through all the clocks until the last one decays. The numbers shown in the figure are the (rounded) optimal decay probabilities per tick as found through numerical optimization.

Although we have described the behavior in time of this clock arrangement, we have not described how it could be used as a clock. We do so now. As stated previously, a device is a clock if it correlates to time. But, simply knowing that this correlation exists is not sufficient for an agent that wishes to measure time: it is also necessary to know how the states correspond to time.

An agent would learn (through experience or evolution or in our case, calculation)

what this correspondence is. In the case of the this example, the correspondence produces Figure 10.



(a) *Plot Not stacked.* The dominant region in time of each clock can be seen more clearly in this plot.

(b) *Probabilties stacked.* The gradual shift in time of the probability mass through the sequence of states can be seen in this plot.

Figure 10: The conditional probability distribution $P(\mathbf{S}^n|T = t)$ of the number of clocks that are still up in time, plotted in two different ways (stacked lines and not stacked).

8.2 The Performance of the Cascade in Different Circumstances

We begin our investigations with this model by computing the maximum performance a clock cascade can attain for different circumstances. Namely, we create cascades having different numbers of clocks N and optimize those arrangements for different time windows $|T|$. Taken together, these amounts create the plot in Figure 11.

From the plot, we removed the top left corner (where $N \geq |T|$). We did this because N clocks are already sufficient to optimally track a length of time with N ticks and therefore it is not necessary to study longer cascades. The second reason is that the numerical methods that computed this table returned poor (obviously suboptimal) results when the model was configured with more clocks than time ticks. We hypothesize that the Lipschitz algorithm performs poorly when it is given many

unused parameters as is the case here in the white region on the plot.

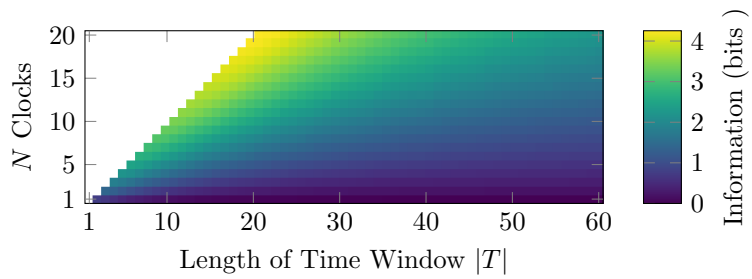


Figure 11: The amount of information that the optimal clock cascade has about time for different time windows and different numbers of clocks. We removed the data from the top left corner of the figure and explain the reason in text.

We observe in Fig. 11 that the performance of the clock cascade varies smoothly when more clocks are added (N is changed) or when the time window T is increased.

8.3 Clock "Condensation"

Although Figure 11 does not reveal different regimes (it shows no abrupt changes in clock performance), we do find for some N and $|T|$ what we consider to be qualitatively different clock cascades. Specifically, we find a "condensation" effect, for example when a cascade of $N = 20$ clocks is optimized for an axis of time of $|T| = 100$ moments (99 time ticks). The way this clock cascade evolves in time is plotted in Figure 22.

To confirm that the condensed clock cascade is a global maximum (and not an artifact of our choice of optimizer) we also run the same optimization with a different algorithm to obtain a very similar result (B.2).

To see where this effect occurs, we show a grid plot of the number of deterministic clocks (that is, the amount of condensation) in the cascade for different N (number of clocks in the cascade in total) and $|T|$ in Figure 13. Mainly the plot shows that the condensation gradually happens as clocks are added until the number of clocks matches the number of ticks and then all clocks in the cascade are deterministic.

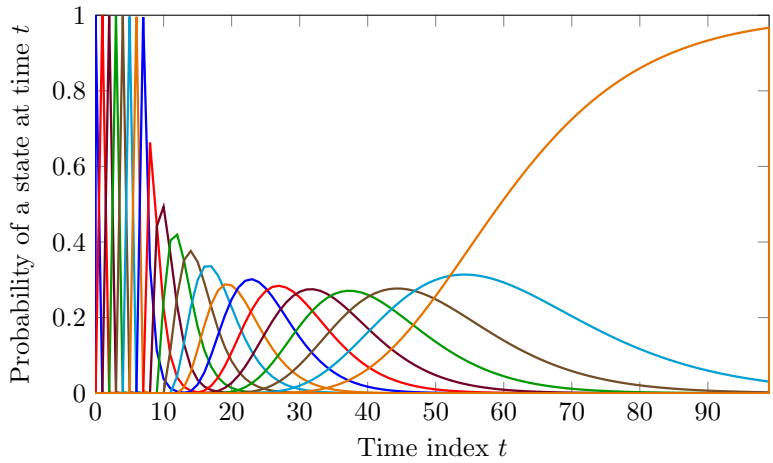


Figure 12: The result of optimizing a cascade of 20 clocks to a time window of 100 ticks. The first 8 clocks in the sequence "condense" (are active only in precisely one time moment each) while the rest are spread along their respective time window. The parameters of each clock are given in Appendix B.1.

8.4 Special Moments in Time

Above, in our calculations with the clock cascade, we have been maximizing for any information about time that the devices can have across the whole time interval of interest. In general, this will be quite an abstract quantity. We expect that a more typical necessity for an organism will be the ability to predict the arrival of a specific moment in time.

We therefore now maximize the amount of information that a clock cascade[7] has about a specific moment in time (which we here arbitrarily picked to be $t = 5$). We repeat the computation for $N = 1, N = 2 \ldots N = 6$ and plot all the clock cascades in this range in Appendix B.3 and we also include the plot for $N = 5$ here in Figure 14.

A cascade of six clocks is (of course) able to predict very well the sixth time

[7]One reason why we choose to predict a specific moment in time with the cascade model is that it is well suited to this task. It gives more information than a bag of drop clocks (we verified this numerically for all $1 \leq T \leq 6$ and $1 \leq N \leq 5$). Still, besides the ability to predict time well, clock models can be relevant for other reasons, such as their robustness or ease of implementation. We leave the comparison of the robustness or cost of implementation of these models for future work.

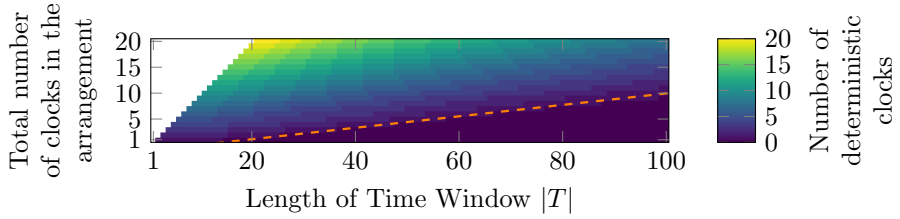
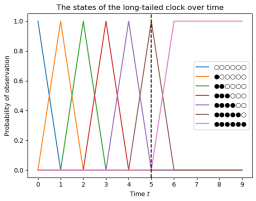


Figure 13: The amount of condensation for different cascade sizes and different time window sizes. Below the orange dotted line, there is no condensation effect.

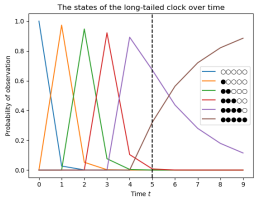
moment (marked on the plot with index $t = 5$) as the arrangement can be made to run deterministically through each clock in the sequence (Figure 14a). But a cascade of five clocks is still able to give some[8] information about this particular moment in time by having probabilistic transitions. Figure 14b shows how, by being probabilistic, the probability of activation of the fifth clock is delayed from the fifth time moment (where it would have occurred, had the cascade been deterministic) to partly cover the sixth moment ($t = 5$).

For a wider picture, it can be seen in Figure 14c how adding clocks to a cascade increases the amount of information it can give about the specific event in time. It increases only gradually as long as the best solution is probabilistic, but then jumps suddenly as the final clock is added which completes the deterministic collection.

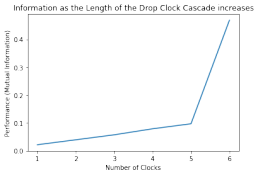
[8]Note that in this paper we optimize for observation of clock *states* and not clock *transitions*. In particular, the observation of the clock is assumed to be made by the agent at some unknown random moment in time. Therefore, a cascade having less than six clocks ($N < 6$) cannot predict $t = 5$ with perfect certainty. Instead, one could imagine a different setup in which the agent monitors (e.g. by continuous polling) the clock and then acts when the last clock drops; or, alternatively, one could consider an event-based detection of the relevant moment. This alternative setup could predict $t = 5$ perfectly with only $N = 5$ clocks (or even $N = 4$ clocks, depending on how observations and actions are causally linked in the model) because it would make use of the mutual information between the *transitions* (rather than the states) of the cascade and time. Strictly spoken, transitions form ordered pairs of states, so being able to identify them would assume a hidden or implicit memory in the agent, contrary to our intention to model an agent without memory outside of the clock state itself. From a formal perspective, using transitions to measure time considers a different state space, namely the space of transitions, i.e. would be translated to our formalism by considering the dual of the transition graph. While in a discrete context, only a minor reduction in clock complexity is gained by considering transitions rather than states, in continuous time, sharp transition events offer a significant advantage, and will require a separate detailed study.



(a) The behavior of the clock cascade for $N = 6$.



(b) And for $N = 5$.



(c) The amount of information as clocks are added to the sequence.

Figure 14: The optimal clock cascade for when maximizing for information about a particular moment in time.

9 The Composite Clock

Beyond the simplest clocks, clock bags and cascades listed above, we consider the next more complex clock, a composite clock consisting of two simple 1-bit clocks each, which, however, are permitted to communicate (therefore making this arrangement more complex), but with a constraint on how much communication is permitted. This constraint is expressed as a measure of information flow between the composite clocks. This information flow constraint is how we penalize modular systems for their complexity. Consistent with the freezing procedure in the bag-of-clocks experiment we freeze the first clock here as well (further comments on "freezing" in Sec. 7.2). In addition to the biological justification, the choice of freezing the first clock also leads to more reliable optimization results. Thus, as before, the first (upper) component of the composite clock becomes an oscillator. This composite structure resembles the hierarchical coarse-graining one finds in the algebraic decomposition theory of semigroups and discrete-event dynamical systems [27, 20].

Only the second (lower) component's dynamics will be parametrized and the parameters optimized according to suitable informational criteria. This one-way communication is inspired by the semigroup decomposition of counters [27]. Future work will allow the first clock to be optimized as well and add a feedback channel.

We hypothesize that a more comprehensive optimization of the whole clock system

would begin to optimize the first level of clocks (getting the basic oscillatory tick), which, when converged, will permit the next level to arise. As the oscillation is necessarily optimal for the first level, without further constraints we do not expect any changes at that level, even by further joint evolution.

A more interesting scenario arises when the clock ticks become "soft" (and there might be additional effects once we permit feedback between the levels, which is something we will study in the future). The freezing of a substrate, based on which further, more intricate patterns can evolve, might be an evolutionarily plausible mechanism in itself[9]. For the present paper, this is the basic assumption under which we operate: how can, assuming that the first layer is an optimally converged 1-bit oscillator clock, and assuming its evolution gets henceforth frozen, the following layer evolve to make the total clock as effective as possible (with or without communication).

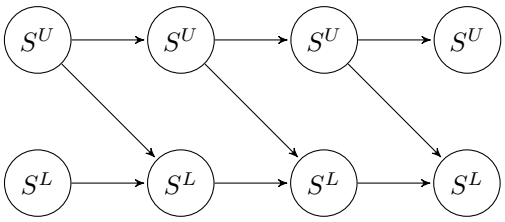


Figure 15: The structure of the composite clock unrolled in time. There is a hierarchy of information flow here. Both clocks send information to themselves but only the upper clock sends information to the lower clock.

To prepare the description of the full system, we write out in detail the first component of the clock (the upper component) as a Markov chain; it is defined by the matrix $A_U = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and initial state distribution[10] $S_0^U = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. In the next step, we design the dynamics of the lower component S^L, but it is not independent anymore (the stochastic kernel of the lower component is not a square matrix). Rather (with a rectangular matrix as a kernel), it can be influenced by the state of the upper component U. The matrix that will drive the behavior of the lower component is a

[9] Alternatively, depending on the configuration, coevolution might also be an option, but for the very fundamental set of studies in the present paper, we will not consider this further.

[10] We use the random variable name as a proxy notation for the whole distribution.

conditional probability distribution, a mapping from the joint state of both components (S_t^U, S_t^L) at time t to the future state of S_{t+1}^L at time $t+1$. There are 4 columns in this matrix because there are four combinations of the input (i.e. condition) states from both components: uu, ud, du, dd (both up, one up and the other down, etc.). In vector notation, the complete probability of the whole system to be in a state is written as:

$$P(S^U, S^L) = \begin{bmatrix} P(S^U = u, S^L = u) \\ P(S^U = u, S^L = d) \\ P(S^U = d, S^L = u) \\ P(S^U = d, S^L = d) \end{bmatrix}, \tag{1}$$

and the transition matrix representing $P(S^L|S^U, S^L)$ for the lower clock (we remind that the upper clock can be directly modeled as an oscillator and that the first coordinate of S^L is the probability for u and the second the probability for d) as the following.

$$A_L = \left[\begin{smallmatrix} \theta_1 & \theta_2 & \theta_3 & \theta_4 \\ 1-\theta_1 & 1-\theta_2 & 1-\theta_3 & 1-\theta_4 \end{smallmatrix} \right]. \tag{2}$$

We make use of the fact that the resulting probabilities add up to 1 and we thus need only one parameter to describe each of the conditionals.

Combine now the matrix for the upper component (A_U) and the matrix for the lower component (A_L) to obtain the complete Markov matrix for the whole clock:

$$A = \begin{bmatrix} 0 & 0 & \theta_3 & \theta_4 \\ 0 & 0 & 1-\theta_3 & 1-\theta_4 \\ \theta_1 & \theta_2 & 0 & 0 \\ 1-\theta_1 & 1-\theta_2 & 0 & 0 \end{bmatrix}.$$

We initialize the lower clock S^L, as always, in state u, i.e. with probability $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. Now all required definitions are complete. Expressed in terms of the joint variable $S = (S^U, S^L)$, and following the conventions from (1), the initial state of the complete clock is $S_0 = \left[\begin{smallmatrix} 1 & 0 & 0 & 0 \end{smallmatrix}\right]'$ (where $'$ is transpose). Using A, the matrix that ticks the clock forward to its next time step, we can simulate the clock starting at time 0 and into

a future time t by repeated matrix multiplication: A^t. For illustration, the first few states look like:

Time dynamics of both parts		
S_0	AS_0	A^2S_0
$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\\theta_1\\1-\theta_1\end{bmatrix}$	$\begin{bmatrix}\theta_1\theta_3+\theta_4\left(1-\theta_1\right)\\\theta_1\left(1-\theta_3\right)+\left(1-\theta_1\right)\left(1-\theta_4\right)\\0\\0\end{bmatrix}$

Figure 16: The probabilistic state of the composite clock for $t = 0$, $t = 1$ and $t = 2$.

9.1 Experiments with the Composite Clock

The current experiments distinguish themselves from the earlier ones by having the participating clocks communicate: we create an information "tap" between the two clocks. Importantly, we will control and limit the amount of information that may flow from the upper to the lower clock. As discussed earlier, all costs/rewards (e.g. flow vs. time resolution) will be expressed exclusively in terms of Shannon information as unique currency in which the quality of time measurement is expressed. The expectation is that as more information is allowed to pass, the overall performance of the composite clock would increase as the components would be more coordinated.

In the same spirit as in the bag-of-clocks experiment, we assume a greedy pre-optimization of the upper clock, which results in an oscillator whose parameters will be frozen. Thus, the experiment will only optimize the remaining parameters of the clock, i.e. the dynamics of the lower clock and the parameters of its dependence on the upper clock (in other words, the parameters θ_1, θ_2, θ_3 and θ_4 of Eq. 2). While we maximize time information as before, a constraint C is imposed on the capacity of communication [*transfer entropy,* 29] from the upper clock to the lower one (T in the index denotes marginalization over all valid times occurring in the experiment). Since

the system is Markovian, the flow only over a single step is considered. Since the upper clock is fixed as the oscillator, no feedback channel is included. We now compute

$$\max_{I(S_T^U;S_{T+1}^L|S_T^L)\leq C} I(S;T) \ .$$

To optimize under the constraint, we use the Lagrangian method, i.e. we maximize $I(S;T) - \lambda I(S_T^U;S_{T+1}^L|S_T^L)$ with Lagrange parameter λ. Scanning through the possible values of λ, we cover the spectrum of clocks that arise throughout all possible constraints. We first used the Lipschitz algorithm to find the red part of the curve in Figure 17. One finds two distinct regimes: the perfect clock at $C = 1$, and distinctly suboptimal clocks in the regime below around $C \approx 0.2$. The curve breaks off at the low end at about $C = 0.04$ due to memory limitations of the Lipschitzian optimizer.

Since the landscape is very complex, in addition to DIRECT, we also applied COBYLA [26] to map additional regions of the Lagrangian landscape and to attain an as complete overview as possible over the solution space. With COBYLA, we used a relaxation method to cover locally optimal parts of the curve which the global Lipschitz optimizer does not access. Concretely, we started optimization at the most permissive information flow bound $C = 1$, optimized, and then tightened the bound slowly, always starting with the clock parameters obtained for the previous constraint C.

The results are shown again in Fig. 17, but they now include the black regions in addition to the red. Since this optimization is not global, it is able to uncover additional structure in the solution space. First of all, when, starting at the optimal clock $C = 1$, C is reduced, the trade-off curve between the constraint C and the time information $I(S;T)$ falls below the globally optimal solutions. The ensuing solutions correspond to "fuzzy" counters (the optimal clock is a binary counter), but do not trade in information flow and achieved time information in a well-balanced way, although part of that portion of the curve is still Pareto-optimal (is not superseded simultaneously by solutions better in terms of both C *and* $I(S;T)$).

In the range $C \approx 0.5 - 0.7$, no solution is found. Between $C \approx 0.25 - 0.5$, a class of locally optimal solutions is found which is not Pareto optimal. Finally, below $C = 0.25$,

one regains the lower C regime which is found with global optimization. The curve continues down to $C = 0$ (continuation of the red to the black curve), which we obtain with COBYLA which is not suffering from the memory problems of the global optimizer. The two clock classes below $C \approx 0.5$ look very similar as a whole clock, but distribute the "counting" differently over their component clocks.

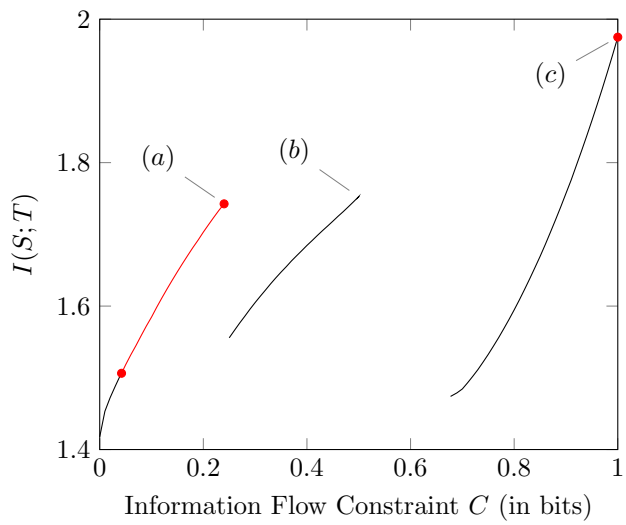
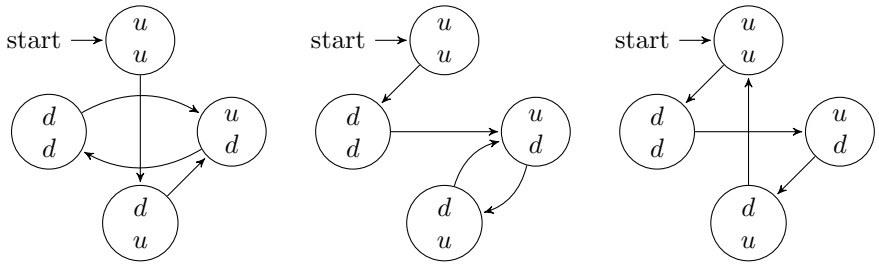


Figure 17: In red, the optimum curve found by the DIRECT global Lipschitz optimizer. In black, the more complete curve found by the COBYLA local optimizer. Diagrams of the clocks at the points of interest a, b and c are given in Figure 18. This plot has a minor error which we discovered after performing the numerical optimizations: see Appendix C for the comparison.

Apart from the discovery of different clock "regimes", a detailed inspection of the space of possible configurations (Fig. 18) demonstrates that, in part, finding parameters which achieve high values of time information is very difficult. Furthermore, despite large permitted flows C, time resolution may still have low values. This makes it clear that measuring time is not an incidental effect that is likely to be found en passant by an evolutionary process; instead, one rather will expect good time measurement abilities to have been explicitly evolved for, either directly or indirectly (via proxy



(a) Where $C \approx 0.25$ (bits). A composite clock with a low (relative to the other two clocks on the right) yet a resonably high performance.

(b) Where $C \approx 0.4$ (bits). In this clock, the *bottom* clock remains correlated to the *top* clock which is the reason why the information flow is higher for this clock than for the composite clock on the left.

(c) Where $C \approx 1$ (bit). This composite clock together acts as a four-state counter.

Figure 18: Above, the composite clocks of salient points in the tradeoff in Fig. 17. The diagrams show transitions of the composite clock in pairs of states of both the *top* and the *bottom* clock.

criteria). In a hierarchy any more complex than the one discussed here, we expect the clocks to need to be attuned to each other for optimal resolution.

10 Final Comments and Future Work

We have studied the ability of minimal clocks to resolve time information. In particular, even 1-bit clocks can, as drop clocks, provide information about global time if the overall time horizon is known when the clock parameters are set.

Let us summarize one of the central insights: the only features of our axis of time are its total maximal extent and its grain. The oscillator matches the grain and represents local relative time information and the drop clock matches the overall duration, i.e. the global information about the current point in time. Since the clock is Markovian it is per its nature time-local; it is remarkable that it is possible to attune the drop probability to the global time period of interest to effectively extract global time information with a limited, purely local mechanism. In evolution, such a probability can be expected to evolve over many generations for the optimal resolution

of the time intervals of interest.

The relation of measured interval and clock parameters is not straightforward and is marked by an interplay of global and local properties (Fig. 6). When extending to a bag-of-clocks, the first two clocks are an oscillator and a drop clock of a particular time constant, followed by further drop clocks, which are then, however, nearly identical; thus, apart from the first two, the rest of the clocks operate as a nearly binomial process. A cascade of clocks can stretch to reach target moments in time or condense in narrow time windows. Finally, when two clocks are stacked together with limited communication, a rich set of regimes opens up, of which just two, the perfect clock, and a very soft clock, can be Lagrange-optimal. These studies provide a spectrum of candidates for behaviors of minimal clocks which one could try to identify in biological systems.

One central limitation of the present work is the assumption of a fixed, global tick which drives all the clocks. Future work will, therefore, include the consideration of clocks in continuous time, where the dynamics needs to establish and sustain a synchronization between the subclocks in addition to the coordination of their respective resolution regime. Additionally, we conjecture that informationally optimal clocks will exhibit some robustness for the resolution of relevant time regimes. This will form the basis for future studies.

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Appendices

A Drop Clock Bag

A.1 Parameters of Incremental Optimization

Results of optimizing a bag of 10 clocks with incremental freezing. The DIRECT Lipschitz optimizer was used set to 30 iterations.

Clock #	Probability $u \to d$	Probability $d \to u$
1	0.0000084675439042	0.9999915324560957
2	0.0727277345933039	0.0000254026317127
3	0.4945130315500686	0.0002286236854138
4	0.5187471422039323	0.0002286236854138
5	0.5187471422039323	0.0002286236854138
6	0.5187471422039323	0.0002286236854138
7	0.5192043895747599	0.0002286236854138
8	0.5205761316872427	0.0006858710562415
9	0.5205761316872427	0.0006858710562415
10	0.5205761316872427	0.0006858710562415

B Drop Clock Cascade

B.1 Parameters of a Condensed Cascade

Here we show the probability to drop of each step in a cascade of 20 clocks as found by the DIRECT Lipschitz optimizer after 1000 iterations when maximizing for the overall information about time (not specifically about a particular moment). The table shows that the parameters for the first seven clocks are close to one.

Clock #	Drop Probability
1	0.9993141289437585
2	0.9993141289437585
3	0.9993141289437585
4	0.9993141289437585
5	0.9993141289437585
6	0.9993141289437585
7	0.9993141289437585
8	0.6659807956104252
9	0.6659807956104252
10	0.5548696844993141
11	0.4684499314128944
12	0.3984910836762689
13	0.3436213991769547
14	0.3340192043895748
15	0.2572016460905350
16	0.2229080932784636
17	0.1886145404663923
18	0.1570644718792867
19	0.1241426611796982
20	0.0829903978052126

B.2 Alternative Optimization Also Finds Condensation

Here, we run the same experiment of optimizing a cascade of 20 clocks to a time window of 100 moments, but instead of using DIRECT Lipschitz, we optimize using a particle swarm optimizer. Although the results are noisy, they increase our confidence in the validity of the "condensation" solution found by Lipschitz. See plot below (Fig. 19).

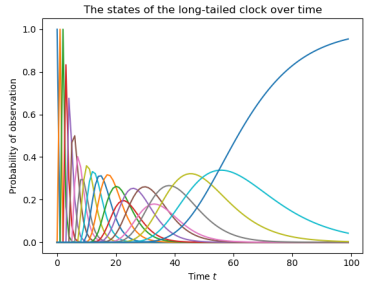


Figure 19: Alternative optimization of the clock cascade also finds the condensation effect.

B.3 Identifying Specific Moments in Time

The behavior of the clock cascade for $N = 1, N = 2 \ldots N = 6$ optimized to give information about a specific moment in time at $t = 5$:

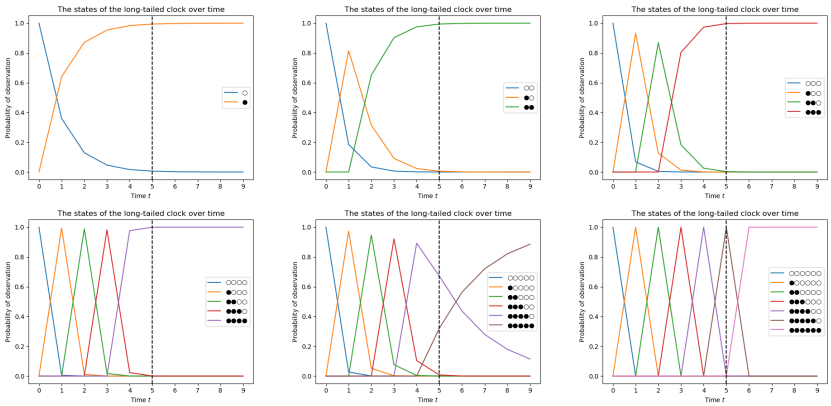


Figure 20: The figure shows in detail how optimal clock cascades stall (lengthen the average amount of time until the decay of their final clock) by decaying at every step probabilistically.

C Composite Clock Computation Error

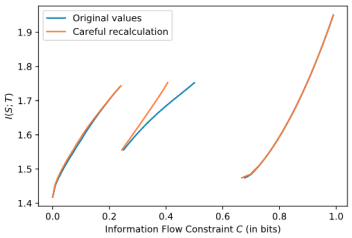
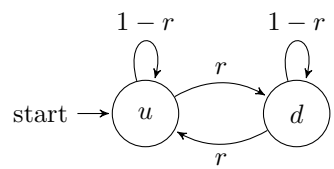
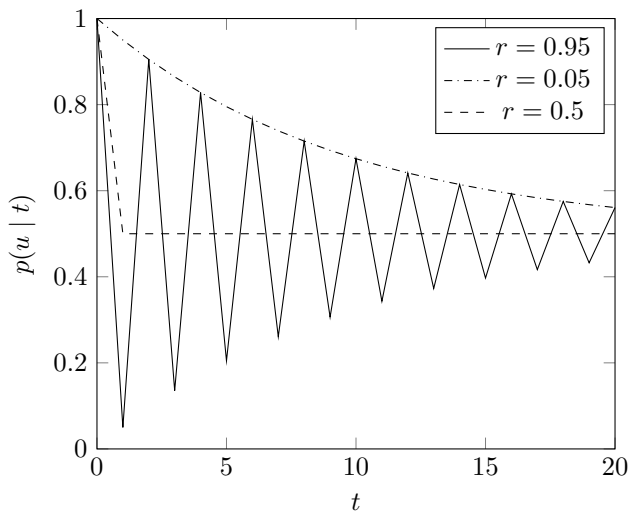


Figure 21: The landscape plotted here is a careful recomputation of Figure 17 showing the same qualitative features but fixing minor errors (in the constraint function, specifically) which do not affect the overall character of the solution.

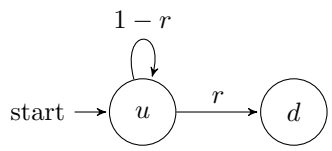
Copies of Figures



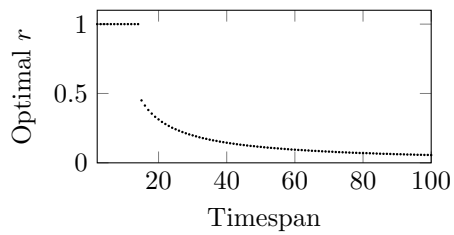
The oscillator clock. It switches one state to the other with probability r.



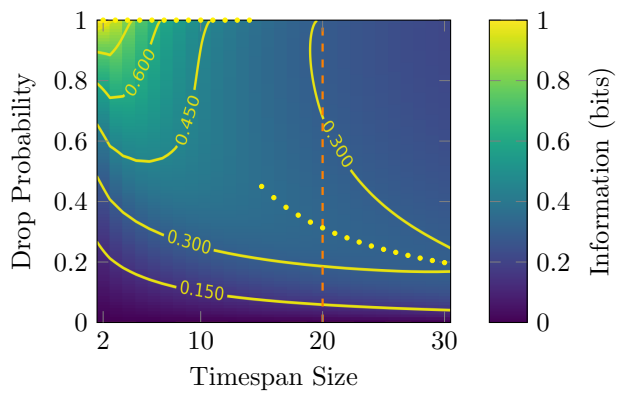
The oscillator clock for different probabilities of switching state.



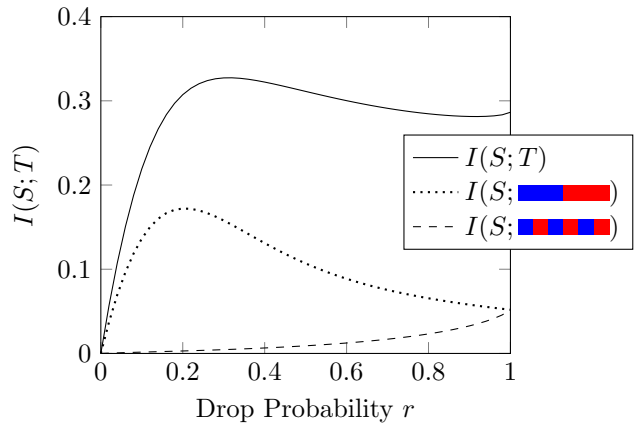
The drop clock, a Markov chain with a probability r to permanently transition to state d and remain there.



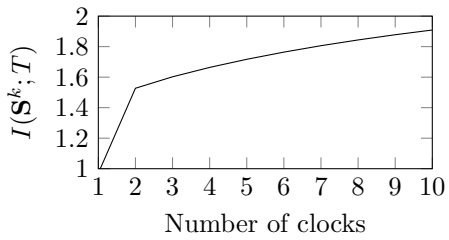
The optimal drop probability r for different timescales. Note that the curve for the drop clock has a discontinuity at $|T| \approx 15$.



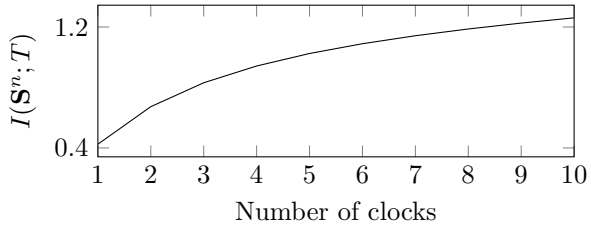
Time information for different drop probabilities and time spans. Taking a slice in this plot at the dashed orange line creates the next plot (Fig 6). The yellow dots show the best decay parameter for that time span (essentially the same data in Fig. 4). The yellow contour lines are drawn for additional clarity.



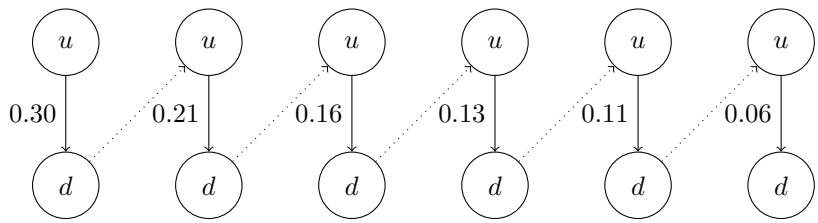
Time information for different drop probabilities at $T = 20$. Apart from total time information, it shows information of clock state S about being in an odd vs. even time step or early vs. late phase of the measured period.



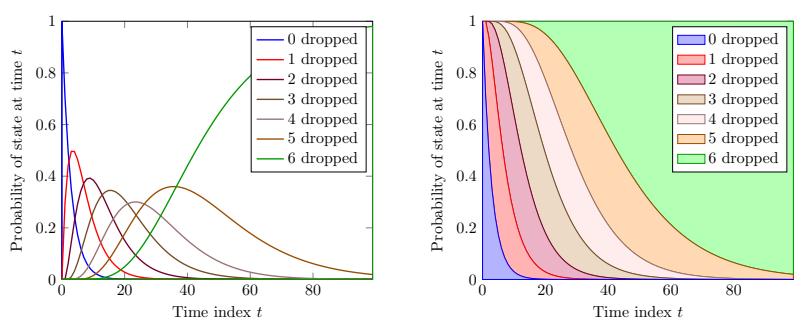
Amount of information as the size of the collection grows for a time interval of length $T_{\max} = 5$.



Time information as more identical clocks are added to the set for $T_{\max} = 50$.



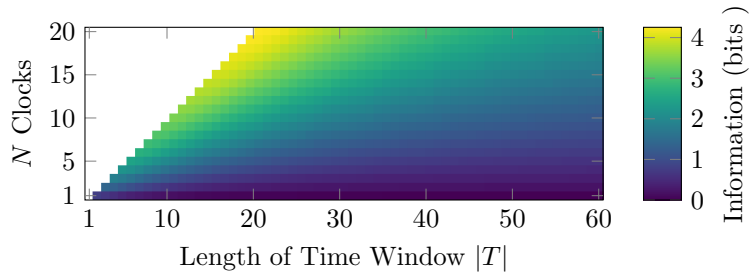
The optimal clock cascade for $N = 6$ and $|T| = 100$. The arrangement contains 6 clocks and they are all in state *up* at the beginning of the simulation. Then, after some time, the first clock drops and by dropping it releases the second clock (notice the dotted arrows). One by one, the sequence runs through all the clocks until the last one decays. The numbers shown in the figure are the (rounded) optimal decay probabilities per tick as found by through numerical optimization.



(a) *Plot Not stacked*. The dominant region in time of each clock can be seen more clearly in this plot.

(b) *Probabilties stacked*. The gradual shift in time of the probability mass through the sequence of states can be seen in this plot.

The conditional probability distribution $P(\mathbf{S}^n|T = t)$ of the number of clocks that are still up in time, plotted in two different ways (stacked lines and not stacked).



The amount of information that the optimal clock cascade has about time for different time windows and different numbers of clocks. We removed the data from the top left corner of the figure and explain the reason in text.

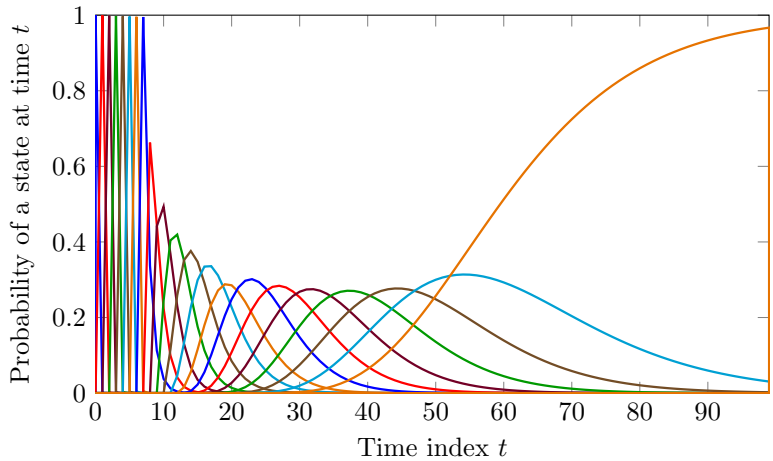
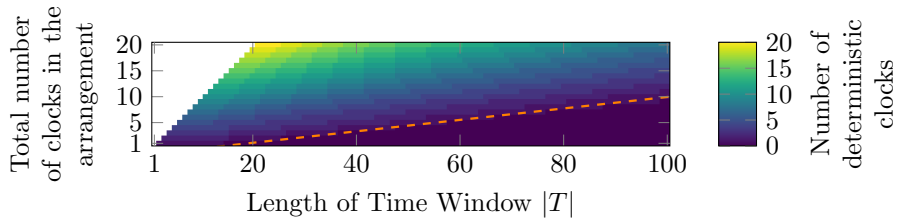
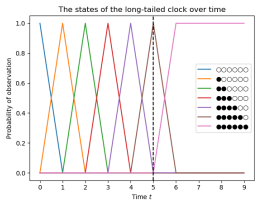


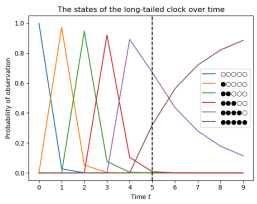
Figure 22: The result of optimizing a cascade of 20 clocks to a time window of 100 ticks. The first 8 clocks in the sequence "condense" (are active only in precisely one time moment each) while the rest are spread along their respective time window. The parameters of each clock are given in Appendix B.1.



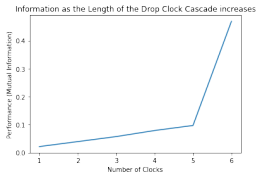
The amount of condensation for different cascade sizes and different time window sizes. Below the orange line, there is no condensation effect.



(a) The behavior of the clock cascade for $N = 6$.

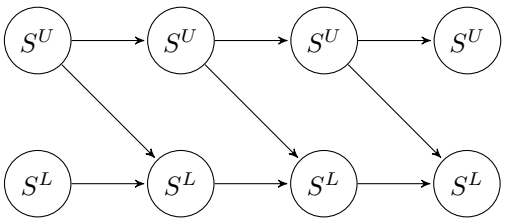


(b) And for $N = 5$.



(c) The amount of information as clocks are added to the sequence.

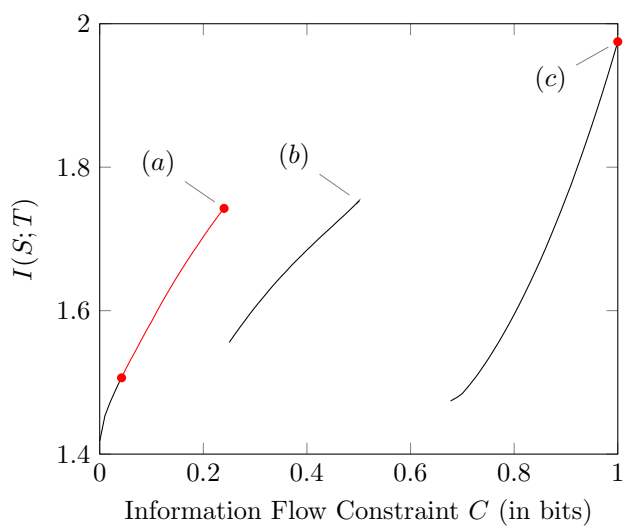
The optimal clock cascade for when maximizing for information about a particular moment in time.



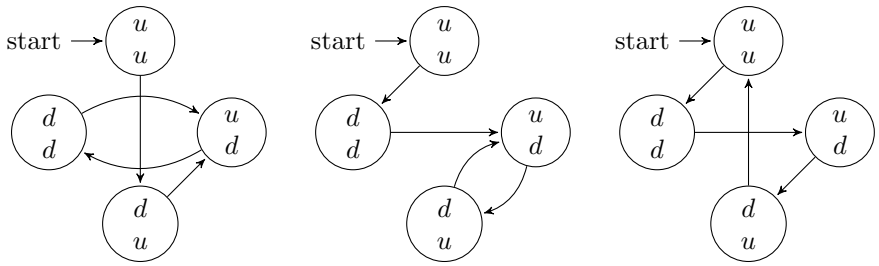
The structure of the composite clock unrolled in time. There is a hierarchy of information flow here. Both clocks send information to themselves but only the upper clock sends information to the lower clock.

Time dynamics of both parts		
S_0	AS_0	A^2S_0
$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\\theta_1\\1-\theta_1\end{bmatrix}$	$\begin{bmatrix}\theta_1\theta_3+\theta_4\left(1-\theta_1\right)\\\theta_1\left(1-\theta_3\right)+\left(1-\theta_1\right)\left(1-\theta_4\right)\\0\\0\end{bmatrix}$

The probabilistic state of the composite clock for $t = 0$, $t = 1$ and $t = 2$.



In red, the optimum curve found by the DIRECT global Lipschitz optimizer. In black, the more complete curve found by the COBYLA local optimizer. Diagrams of the clocks at the points of interest a, b and c are given in Figure 18. This plot has a minor error which we discovered after performing the numerical optimizations: see Appendix C for the comparison.



(a) Where $C \approx 0.25$ (bits). A composite clock with a low (relative to the other two clocks on the right) yet a resonably high performance.

(b) Where $C \approx 0.4$ (bits). In this clock, the *bottom* clock remains correlated to the *top* clock which is the reason why the information flow is higher for this clock than for the composite clock on the left.

(c) Where $C \approx 1$ (bit). This composite clock together acts as a four-state counter.

Above, the composite clocks of salient points in the tradeoff in Fig. 17. The diagrams show transitions of the composite clock in pairs of states of both the *top* and the *bottom* clock.