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The Veneziano amplitude in $\mathrm{AdS}_5\times\mathrm{S}^3$ from an 8-dimensional effective action

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Abstract: We study four-point functions of arbitrary half-BPS operators in a 4-dimensional $\mathcal{N}=2$ SCFT with flavour group SO(8) at genus-zero and strong 't Hooft coupling, corresponding — via AdS/CFT — to the (α' expansion of the) Veneziano amplitude on an $\mathrm{AdS}_5\times\mathrm{S}^3$ background. We adapt a procedure first proposed by Abl, Heslop and Lipstein in the context of $\mathrm{AdS}_5\times\mathrm{S}^5$, and postulate the existence of an effective action in terms of an 8-dimensional scalar field valued in the adjoint of the flavour group. The various Kaluza-Klein correlators can then be computed by uplifting the standard AdS/CFT prescription to the full product geometry with AdS bulk-to-boundary propagators and Witten diagrams replaced by suitable $\mathrm{AdS}_5\times\mathrm{S}^3$ versions. After elucidating the main features of the procedure, valid at all orders in α', we show explicit results up to order α'^5. The results provide further evidence of a novel relation between AdS×S and flat amplitudes — which made its first appearance in $\mathcal{N}=4$ SYM — that is perhaps the most natural extension of the well known flat-space limit proposed by Penedones to cases where AdS and S have the same radius.

Keywords: AdS-CFT Correspondence, Scale and Conformal Symmetries, Scattering Amplitudes

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1 Introduction and summary of results

In recent years major progress has been made in the computation of holographic correlators, i.e. correlators in CFT's that admit a gravity dual. It is reasonable to hope that understanding the structure of these observables will shed light on the properties of quantum gravity and its UV completion.

In the well studied case of $\mathcal{N} = 4$ SYM, dual to string theory on $\mathrm{AdS}_5\times\mathrm{S}^5$ [1], a number of different bootstrap approaches have led to a systematic understanding of the structure of four-point functions of half-BPS operators at various order in $1/N$ and $1/\lambda$ [2–23].[1] Despite the apparent complexity of the intermediate steps, the end results are found to be tremendously simple. This is no coincidence: amazingly, it turns out that $\mathcal{N} = 4$ SYM enjoys a hidden 10*d* conformal symmetry in the supergravity limit [26].[2] This symmetry

[1] See [24, 25] for recent reviews on the subject.

[2] The symmetry has also been observed at weak coupling in $\mathcal{N} = 4$ SYM [27] and, recently, extended to higher components of the stress-tensor multiplet [28].

allows a repackaging of all four-point correlators into a single $10d$ object transforming as a four-point function of weight 4 scalars.

Away from supergravity, when α' corrections are switched on, the hidden conformal symmetry is broken; however, string corrections are still found to obey a $10d$ principle [13, 14, 29]. Specifically, the authors of [29] postulate the existence of a 10-dimensional effective action in terms of a single scalar such that, the correlators for *arbitrary* Kaluza-Klein modes computed out of the action via Witten diagrams, perfectly agree with all known results in the literature [10, 12–14]. The underlying idea of the computation is to mimic the standard AdS/CFT procedure, with the difference that standard AdS Witten diagrams are replaced by generalised AdS$\times$S versions. The method leaves a handful of free ambiguities at each order in α' which can be interpreted as an effect of the curvature of the background.

A natural question to ask is whether this approach can be useful in other physical theories of interest.[3] A promising candidate in this respect is a recently studied 4-dimensional $\mathcal{N}=2$ theory with global group SO(8). This is the theory we consider in this paper. This particular SCFT arises as the worldvolume of D3-branes moving near F-theory singularities [34–36] and is dual to an orentifold projection of IIB string theory. In this model, four-point functions of chiral primary operators correspond to the scattering of four super gluons in the bulk with the higher derivative corrections representing the string completion of the field-theory amplitude, which is an $\mathrm{AdS}_5\times\mathrm{S}^3$ version of the Veneziano amplitude. These correlators have already been studied in the field-theory limit in a series of papers both at tree-level, one and two loops [37–40] but little is known for the α' corrections. However, given the many similarities shared by $\mathcal{N}=4$ SYM and the SCFT considered in this paper, in particular the existence of a *reduced* correlator and the fact that this can be re-organized into a single 8-dimensional object [37], we expect the methods of [29] to naturally generalise to this background.

This motivates us to conjecture the existence of an $8d$ action written in terms of a scalar field valued in the adjoint of SO(8), which encodes tree-level correlators of super gluons with *arbitrary* Kaluza-Klein levels up to a small number of ambiguities. The main result of this paper is to spell out the effectiveness of the method and compute explicitly these correlators at the first few orders in α'.

The starting point is to conceive the (α' expansion of the) Veneziano amplitude as arising from an effective potential

$$\begin{aligned} V_{\text{open}} = {}& \frac{1}{8}\frac{\pi^2}{6}\,\mathrm{Tr}[\phi^4]\alpha'^2 + \frac{1}{2}\zeta_3\,\mathrm{Tr}\left[(\partial_\mu\phi)(\partial_\mu\phi)\phi\phi\right]\alpha'^3 + \\ & + \frac{1}{2}\frac{\pi^4}{720}\,\mathrm{Tr}\left[14(\partial_\mu\partial_\nu\phi)(\partial_\mu\partial_\nu\phi)\phi^2 + (\partial_\mu\partial_\nu\phi)\phi(\partial_\mu\partial_\nu\phi)\phi\right]\alpha'^4 - \\ & + \frac{1}{3}\,\mathrm{Tr}\left[2\left(\frac{1}{6}\zeta_3\pi^2+\zeta_5\right)(\partial_\mu\partial_\nu\partial_\rho\phi)(\partial_\mu\partial_\nu\partial_\rho\phi)\phi^2 + \right. \\ & \left. + \left(\frac{1}{6}\zeta_3\pi^2 - 2\zeta_5\right)(\partial_\mu\partial_\nu\partial_\rho\phi)\phi(\partial_\mu\partial_\nu\partial_\rho\phi)\phi\right]\alpha'^5 + \cdots\,, \end{aligned} \tag{1.1}$$

[3] See also [30], where this approach has been used to compute higher-derivative corrections in $\mathrm{AdS}_2\times\mathrm{S}^2$. Another theory where we expect these methods to be effective are the $\mathrm{AdS}_3\times\mathrm{S}^3$ correlators of [31–33].

where the constants are fixed by requiring that the four-point amplitude in momentum space at a given order in α' matches the corresponding term in the low-energy expansion of the flat-space Veneziano amplitude. The idea is then to uplift this potential to $\mathrm{AdS}_5\times\mathrm{S}^3$ by replacing flat derivatives with $\mathrm{AdS}_5\times\mathrm{S}^3$ covariant versions. The various Kaluza-Klein correlators can eventually be computed by using a generalisation of contact Witten diagrams which takes into account the compact space. Nicely, we find that the end results are simple functions of AdS and S variables. In particular, we notice that they can be written in a very compact form in terms of a pre-amplitude which made its first appearance in $\mathcal{N}=4$ SYM [14]. Interestingly, this said pre-amplitude shares a strong similarity with its flat-space counterpart and it is related to the latter via a double integral transform, which is perhaps the most natural generalisation of the integral transform defining the flat-space limit conjectured by Penedones [41] and later proved in [42].

To give a flavour for the remarkable simplicity of the results, the order α'^3 correlator for *arbitrary* Kaluza-Klein correlators reads

$$\tilde{\mathcal{M}}_{p_1p_2p_3p_4}(1234)\big|_{\alpha'^3} = \zeta_3\,(S+T+a_1) \tag{1.2}$$

where $\tilde{\mathcal{M}}_{p_1p_2p_3p_4}(1234)$ is a colour-ordered Mellin *pre-amplitude*, S,T are suitable $\mathrm{AdS}_5\times\mathrm{S}^3$ variables, and a_1 is the only ambiguity left at this order. While the notation will be explained in full in the main body, we can already appreciate the manifest similarity with the flat-space Veneziano amplitude which at this order reads

$$\mathcal{V}_{\text{open}}(1234)\big|_{\alpha'^3} = \zeta_3\,(s+t)\,. \tag{1.3}$$

where s,t are the Mandelstam variables. The compactness of these results seems to point out the existence of an underlying structure yet to be discovered.

The remainder of the paper is organised as follows. In section 2 we describe the general set-up for the object of study here: four-point correlation functions of half-BPS operators dual to supergluon amplitudes on $\mathrm{AdS}_5\times\mathrm{S}^3$. In section 3 we review the structure of the Veneziano amplitude in flat space as well as the procedure outlined in [29] for uplifting to AdS via viewing the correlator as arising from an $8d$ scalar effective action. In section 4 we give explicit results for the α'-corrected correlators up to order α'^5. In the last part of this section we clarify the relation between the flat-space limit known in literature and some possible generalisations. Finally, in section 5 we comment on possible future directions.

Note added. Whilst completing our paper, we were informed by the authors of [43] of their work on a similar topic. We thank them for coordinating the release on the arXiv.

2 Generalities

In this paper we are interested in scattering of super gluons in an $\mathrm{AdS}_5\times\mathrm{S}^3$ background. As mentioned in the introduction, the dual SCFT is a four-dimensional $\mathcal{N}=2$ theory with a certain global group, which plays the role of a gauge group in the bulk. This SCFT arises as the worldvolume theory of D3-branes moving near F-theory 7-branes singularities [34–36]. In the case we are interested in, the 7-branes correspond to a $\mathbb{Z}_2$ orientifold point, with the

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low-energy dynamics of N D3-branes described by a USp($2N$) $\mathcal{N}=2$ gauge theory with an additional SO(8) global symmetry group, due to gauge symmetry enhancement in the D7-brane worldvolume. Let us clarify that it is in principle possible to consider different orientifolds, which give rise to world-volume theories with different global symmetries [35, 36]. Although, as we will see, the computations considered in this paper could be performed without making any reference to the gauge group, all other orbifold constructions do not have a perturbative formulation.[4] In other words, the bulk theory does not have an α' parameter (equivalently, from a CFT perspective, there is no marginal coupling) and an α' expansion would evidently be meaningless.[5]

The presence of the 7-branes breaks the SO(6) isometry group of S^5 to $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_R$. From the point of view of the dual $\mathcal{N}=2$ SCFT, $\mathrm{SU}(2)_R \times \mathrm{U}(1)_R$ becomes the R-symmetry group and $\mathrm{SU}(2)_L$ is an additional global group.

We are interested in the (scalar component of) the $\mathcal{N}=1$ vector multiplet and its Kaluza-Klein tower which are dual to half-BPS scalar operators of the form $\mathcal{O}_p^{I a_1\ldots a_p;\bar{a}_1\ldots\bar{a}_{p-2}}$. These operators are chargeless under $\mathrm{U}(1)_R$, they transform under the spin $\frac{p}{2}$ of $\mathrm{SU}(2)_R$, spin $\frac{p}{2}-1$ of $\mathrm{SU}(2)_L$ and in the adjoint of SO(8). Here I is the colour index, p is the scaling dimension of the operator, $a_1,\ldots,a_p$ are symmetrised $\mathrm{SU}(2)_R$ R-symmetry indices and similarly $\bar{a}_i$ are indices of an additional $\mathrm{SU}(2)_L$ flavour group; these last two groups realise the isometry group of the sphere S^3. A convenient way to deal with the $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_L$ indices is by contracting them with auxiliary bosonic two-component vectors η and $\bar{\eta}$:

$$\mathcal{O}_p^I \equiv \mathcal{O}_p^{I;a_1\ldots a_p;\bar{a}_1\ldots\bar{a}_{p-2}}\eta_{a_1}\ldots\eta_{a_p}\bar{\eta}_{\bar{a}_1}\ldots\bar{\eta}_{\bar{a}_{p-2}}\,. \tag{2.1}$$

We will denote the four-point function of half-BPS operators by

$$G_{\vec{p}}^{I_1I_2I_3I_4}(x_i,\eta_i,\bar{\eta}_i) \equiv \langle \mathcal{O}_{p_1}^{I_1}\mathcal{O}_{p_2}^{I_2}\mathcal{O}_{p_3}^{I_3}\mathcal{O}_{p_4}^{I_4}\rangle. \tag{2.2}$$

Note that the correlator is a function of $x_i,\eta_i,\bar{\eta}_i$ and the charges p_i. In particular, due to the definition (2.1), it is a polynomial in the variables $\eta_i,\bar{\eta}_i$, whose degree is dictated by the external charges p_i. The variables $\eta_i,\bar{\eta}_i$ are contracted via $\langle\eta_i\eta_j\rangle=\eta_{ia}\eta_{jb}\epsilon^{ab}$ and $\langle\bar{\eta}_i\bar{\eta}_j\rangle=\bar{\eta}_{i\bar{a}}\bar{\eta}_{j\bar{b}}\epsilon^{\bar{a}\bar{b}}$.

$G_{\vec{p}}^{I_1I_2I_3I_4}$ is subject to the superconformal Ward identities [45]

$$G_{\vec{p}}^{I_1I_2I_3I_4} = G_{\mathrm{free},\vec{p}}^{I_1I_2I_3I_4} + I\, G_{\mathrm{int},\vec{p}}^{I_1I_2I_3I_4}\,, \tag{2.3}$$

where the kinematic factor I takes the following form:

$$I = x_{13}^2x_{24}^2\langle\eta_1\eta_3\rangle^2\langle\eta_2\eta_4\rangle^2(x-y)(\bar{x}-y), \tag{2.4}$$

and we have defined

$$x\bar{x}=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2},\qquad y=\frac{\langle\eta_1\eta_2\rangle\langle\eta_3\eta_4\rangle}{\langle\eta_1\eta_3\rangle\langle\eta_1\eta_3\rangle}. \tag{2.5}$$

[4]Another possibility would be to consider a small number of flavour D7-branes in a D3 background, following the construction in [44]. In the limit $N_F \ll N$, this gives rise to a $\mathcal{N}=2$ SCFT with flavour group SU(N_F), dual to SYM on $\mathrm{AdS}_5\times\mathrm{S}^3$ background. As mentioned earlier, our approach is agnostic about the flavour group, thus it can in principle be also applied to this theory. We thank an anonymous referee for pointing this out.

[5]We thank Pietro Ferrero for discussion on this point.

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The non-trivial function left over, denoted by $G^{I_1I_2I_3I_4}_{\mathrm{int},\vec{p}}$, has two units of conformal/internal weights less than $G^{I_1I_2I_3I_4}_{\vec{p}}$. As a consequence, $G^{I_1I_2I_3I_4}_{\mathrm{int},\vec{p}}$ is a polynomial of the *same* degree in $\eta, \bar{\eta}$.

The function $G^{I_1I_2I_3I_4}_{\mathrm{int},\vec{p}}$ admits a genus $(\frac{1}{N})$ expansion. In this paper we are interested in the $\mathcal{O}(\frac{1}{N})$ order, which corresponds to a bulk tree-level amplitude. This can be further expanded in the (square of the) string length α'.[6] Our notation for the α' expansion at tree-level reads

$$G^{I_1I_2I_3I_4}_{\mathrm{tree},\vec{p}} = G^{I_1I_2I_3I_4}_{\mathrm{YM},\vec{p}} + G^{I_1I_2I_3I_4}_{0,\vec{p}}\alpha'^2 + G^{I_1I_2I_3I_4}_{1,\vec{p}}\alpha'^3 + G^{I_1I_2I_3I_4}_{2,\vec{p}}\alpha'^4 + G^{I_1I_2I_3I_4}_{3,\vec{p}}\alpha'^5 + \cdots \quad (2.6)$$

where $G^{I_1I_2I_3I_4}_{\mathrm{YM},\vec{p}}$ is the field-theory amplitude, first computed in [37]. Note that, as explained in [37], the graviton exchange is $1/N$-suppressed and can be neglected at this order. Thus, the only massless fields exchanged at tree-level are the gluons themselves. The goal of the next sections is to outline a procedure to compute the various $G^{I_1I_2I_3I_4}_{i,\vec{p}}$ order by order in α'. We will give explicit results for the first four string corrections, i.e. up to order α'^5.

The function $G^{I_1I_2I_3I_4}_{\mathrm{int},\vec{p}}$ in general depends on the conformal variables x^2_{ij}, the R-symmetry variables η, the internal variables $\bar{\eta}$, and the charges p_i . However, it turns out that $G^{I_1I_2I_3I_4}_{\mathrm{tree},\vec{p}}$ is symmetric under $\eta \leftrightarrow \bar{\eta}$ exchange [37]. As a consequence, the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ variables can be reorganized into SO(4) variables. In other words, $G^{I_1I_2I_3I_4}_{\mathrm{tree},\vec{p}}$ ultimately depends on the charges p_i, spacetime distances x^2_{ij}, and the SO(4) distances

$$y^2_{ij} = \langle \eta_i \eta_j \rangle \langle \bar{\eta}_i \bar{\eta}_j \rangle. \quad (2.7)$$

We will make use of this non-trivial fact[7] in the next section to express $G^{I_1I_2I_3I_4}_{\mathrm{tree},\vec{p}}$ in embedding $\mathrm{SO}(2,4) \times \mathrm{SO}(4)$ coordinates. Note that, unlike $\mathcal{N} = 4$ SYM, the factor I — which is entirely due to superconformal symmetry, and therefore is only a function of x_i, η_i — cannot be written in terms of embedding coordinates.

Finally, let us conclude this section by defining the *generator* of all correlators

$$\langle \mathcal{O}^{I_1}\mathcal{O}^{I_2}\mathcal{O}^{I_3}\mathcal{O}^{I_4} \rangle \equiv \sum_{\{p_i\}} G^{I_1I_2I_3I_4}_{\mathrm{tree},\vec{p}} \quad (2.8)$$

where the sum is performed over all charges $p_i = 2, \ldots, \infty$. This is a rather natural object to consider in this formalism. In fact, as we will see, a very convenient aspect of this method is that it automatically collects all KK correlators into a single function.

3 An 8*d* effective action for the Veneziano amplitude

The purpose of this section is to outline a procedure for computing the α' expansion of the Veneziano amplitude in $\mathrm{AdS}_5 \times \mathrm{S}^3$. As already mentioned, this method was first proposed in the context of $\mathrm{AdS}_5 \times \mathrm{S}^5$ [29], and later applied to higher derivative corrections

[6] This a strong-coupling expansion in the CFT, with the Regge slope related to the Yang-Mills coupling via $\frac{R^4}{\alpha'^2} = g^2_{\mathrm{YM}} N$.

[7] A priori, this is *not* guaranteed. For example, the disconnected correlator, as computed by Wick contractions, it is not symmetric under $\eta \leftrightarrow \bar{\eta}$ exchange [38, 40].

in $\mathrm{AdS}_2\times\mathrm{S}^2$ [30]. Perhaps not surprisingly, we find that a straightforward generalisation of their results leads to a very concrete proposal for tree-level correlators in this background at all orders in α'.

We should perhaps stress again that we are not considering the actual supersymmetric $\mathrm{AdS}_5\times\mathrm{S}^3$ action dual to the CFT we are interested in, but a bosonic version of it, thus *a priori* they do not need to be related. Intuitively, the reason why we expect this simplified bosonic action to capture the correct supersymmetric 8-dimensional Veneziano amplitude is that the *reduced* correlator behaves like a bosonic object. Moreover, since string corrections have the form of contact interactions, it is reasonable to think that they are somewhat insensitive to superpartners.

Beyond these naive (and arguable) arguments, we will see more concretely that the correlators so computed have the correct flat-space limit [41, 42] and large p limit [17], providing a first check of the correctness of the results. In the conclusions we will comment on other possible independent approaches which could help to prove the effectiveness of the method.

3.1 The flat-space Veneziano amplitude

Let us start by recalling the form of the supersymmetric Veneziano amplitude in flat space and some of its properties. With a slight abuse of language, with Veneziano amplitude, we will refer, here and after, to the amplitude obtained by stripping off a kinematic factor from the Veneziano amplitude, where the latter contains information about the polarization of external states.[8] This is best given in terms of colour-ordered amplitudes and, in our conventions, takes the form

$$\begin{aligned}\mathcal{V}^{I_1I_2I_3I_4}_{\text{open}} &= \frac{1}{2}\sum_{\mathcal{P}(2,3,4)} \mathrm{Tr}[T^{I_1}T^{I_2}T^{I_3}T^{I_4}]\mathcal{V}_{\text{open}}(1234) \\ &= \mathrm{Tr}[T^{I_1}T^{I_2}T^{I_3}T^{I_4}]\mathcal{V}_{\text{open}}(1234) + \mathrm{Tr}[T^{I_1}T^{I_4}T^{I_2}T^{I_3}]\mathcal{V}_{\text{open}}(1423) \\ &\quad + \mathrm{Tr}[T^{I_1}T^{I_3}T^{I_4}T^{I_2}]\mathcal{V}_{\text{open}}(1342),\end{aligned} \tag{3.1}$$

where $\mathcal{P}(2,3,4)$ are permutations of points $(2,3,4)$, and in the second equality we exploited the antisymmetry of the $\mathrm{SO}(N)$ generators to reduce the number of independent color traces from 6 to 3. The colour-ordered amplitude $\mathcal{V}_{\text{open}}(1234)$ takes the form (see e.g. [46]):

$$\mathcal{V}_{\text{open}}(1234) = P\exp\left(\sum_{m\geq 1}\zeta_{2m+1}M_{2m+1}\right)\mathcal{V}_{\text{YM}}(1234), \quad \mathcal{V}_{\text{YM}}(1234) = -\frac{1}{s\,t} \tag{3.2}$$

where

$$\begin{aligned}P &= \exp\left(\sum_{m\geq 1}\frac{\zeta_{2m}}{2m}\alpha'^{2m}\left(s^{2m}+t^{2m}-u^{2m}\right)\right), \\ M_{2m+1} &= \frac{1}{2m+1}\alpha'^{2m+1}\left(s^{2m+1}+t^{2m+1}+u^{2m+1}\right),\end{aligned} \tag{3.3}$$

[8] The polarisation information is roughly identified with the factor I in (2.4).

$\mathcal{V}_{\rm YM}(1234)$ is the colour-ordered Yang-Mills amplitude and s,t,u are 8-dimensional Mandelstam variables satisfying the on-shell constraint $s+t+u=0$. Another equivalent way of writing the Veneziano amplitude is in terms of Γ functions:

$$\mathcal{V}_{\rm open}(1234)=\mathcal{V}_{\rm YM}(1234)\frac{\Gamma(1-\alpha' s)\Gamma(1-\alpha' t)}{\Gamma(1+\alpha' u)}. \tag{3.4}$$

The Veneziano amplitude satisfies many interesting properties. For example, there is a disentanglement of Riemann zeta functions with even and odd arguments, as it is most obvious from the form in equation (3.2). This property is closely related to the well known fact that open and closed string amplitude amplitudes are related by a kernel:

$$\mathcal{V}_{\rm closed}=\left(\mathcal{V}_{\rm open}(1234)\right)^2 S \tag{3.5}$$

where $\mathcal{V}_{\rm closed}$ is the Virasoro-Shapiro amplitude, i.e. the amplitude of four closed strings,[9] and the kernel S is defined by

$$S=\frac{1}{\pi\alpha'}\frac{\sin(\pi\alpha' s)\sin(\pi\alpha' t)}{\sin(\pi\alpha' u)}=\frac{st}{u}P^{-2}. \tag{3.6}$$

Notice that S provides a cancellation of all even zetas, as it should be, since the Virasoro-Shapiro amplitude only contains odd zetas. Relations like the one above are also known as Kawai-Lewellen-Tye (KLT) relations [47]. At low-energy, they yield a relation between gluon and graviton amplitudes.

For completeness, let us also recall that colour-ordered amplitudes are related each other by further relations, for example,

$$\mathcal{V}_{\rm open}(1342)=\frac{\sin(\pi\alpha' t)}{\sin(\pi\alpha' u)}\mathcal{V}_{\rm open}(1234), \tag{3.7}$$

which can be derived from monodromy properties of the string world-sheet [48, 49]. In the field-theory limit, i.e. $\alpha'\to 0$, they reduce to the well known Bern-Carrasco-Johansson (BCJ) relations [50] between colour-ordered amplitudes.

3.2 The $\mathbf{AdS_5\times S^3}$ effective action

The general idea is to write down an effective action starting from the Veneziano amplitude and uplift it to $\mathrm{AdS}_5\times\mathrm{S}^3$. Thus, let us expand the flat-space colour-ordered Veneziano amplitude, at the first few orders in α'

$$\begin{aligned}\mathcal{V}_{\rm open}(1234)=&-\frac{1}{s\,t}+\frac{\pi^2}{6}\alpha'^2+(s+t)\zeta_3\alpha'^3+\frac{\pi^4}{720}(7s^2+7t^2+u^2)\alpha'^4+\\&+\frac{1}{3}\left(\left(\frac{1}{6}\zeta_3\pi^2+\zeta_5\right)(s^3+t^3)+\left(\frac{1}{6}\zeta_3\pi^2-2\zeta_5\right)u^3\right)\alpha'^5+\cdots,\end{aligned} \tag{3.8}$$

[9] As for the Veneziano amplitude, with a slight abuse of language, we will refer to Virasoro-Shapiro amplitude as to the amplitude obtained by stripping off a kinematic factor from the tree-level four-point amplitude in type IIB string theory.

where the even zetas have been evaluated. Excluding the field theory term the remaining expression can be viewed as arising from a scalar effective action, with contact terms containing an increasing number of derivatives. The first few terms take the form

$$\begin{aligned}V_{\text{open}} = &\frac{1}{8}\frac{\pi^2}{6}\,\mathrm{Tr}[\phi^4]\alpha'^2 + \frac{1}{2}\zeta_3\,\mathrm{Tr}\left[(\partial_\mu\phi)(\partial_\mu\phi)\phi\phi\right]\alpha'^3 + \\ &+ \frac{1}{2}\frac{\pi^4}{720}\,\mathrm{Tr}\left[14(\partial_\mu\partial_\nu\phi)(\partial_\mu\partial_\nu\phi)\phi^2 + (\partial_\mu\partial_\nu\phi)\phi(\partial_\mu\partial_\nu\phi)\phi\right]\alpha'^4 + \\ &+ \frac{1}{3}\,\mathrm{Tr}\left[2\left(\frac{1}{6}\zeta_3\pi^2 + \zeta_5\right)(\partial_\mu\partial_\nu\partial_\rho\phi)(\partial_\mu\partial_\nu\partial_\rho\phi)\phi^2 + \right. \\ &\left. + \left(\frac{1}{6}\zeta_3\pi^2 - 2\zeta_5\right)(\partial_\mu\partial_\nu\partial_\rho\phi)\phi(\partial_\mu\partial_\nu\partial_\rho\phi)\phi\right]\alpha'^5 + \cdots\,,\end{aligned} \tag{3.9}$$

where the field $\phi \equiv \phi^I T^I$ is valued in the adjoint of the gauge group SO(8). It is easy to see that after going to momentum space these higher derivative contact terms provide polynomials in the Mandelstam variables and we recover the full colour-dressed amplitude (3.1) with the partial-ordered amplitudes as given in (3.8), order by order in α'.

Following [29], let us now uplift this potential to an $\mathrm{AdS}_5\times\mathrm{S}^3$ background, replacing partial flat derivatives with their $\mathrm{AdS}_5\times\mathrm{S}^3$ covariant[10] counterparts. In doing so, we should note however that the uplift is not unique. This is essentially due to two reasons. First, the covariant derivatives no longer commute and therefore the way of arranging the derivatives in the action is ambiguous. Secondly, at any order in α' there will be terms involving lower numbers of derivatives — that appeared at previous orders — compensated by the AdS (and S) radius R, which would vanish in the flat-space limit, i.e. the limit in which the radius of both the non-compact and the compact space is large. These ambiguities can only be fixed by other methods. This is a very important point which we will better clarify later on, and that is completely analogous to the $\mathrm{AdS}_5\times\mathrm{S}^5$ case[11] [29]: the procedure can intrinsically only access the $\mathrm{AdS}_5\times\mathrm{S}^3$ *completion* of the flat amplitude, while it cannot probe true curvature effects, that manifest themselves in the form of ambiguities. Here, with *completion* we mean *the largest sub-amplitude which directly descends from flat space.* As will become clear after introducing Mellin space, this notion of flat-space limit is intrinsically an extension — which takes into account the full AdS×S geometry — of the more familiar flat-space limit [41, 42] and it automatically reduces to the latter when all but AdS variables are set to zero. From now on we will use the terminology $\mathrm{AdS}_5\times\mathrm{S}^3$ completion as in the sense explained above, i.e. the largest $\mathrm{AdS}_5\times\mathrm{S}^3$ sub-amplitude surviving the large radius limit.

[10] Let us stress that these are $\mathrm{AdS}_5\times\mathrm{S}^3$ covariant derivatives, they are *not* covariant with respect to the gauge/global group SO(8).

[11] See also [14], where similar statements were found from a CFT perspective.

In summary, the $\mathrm{AdS}_5{\times}\mathrm{S}^3$ potential has the following form[12]

$$\begin{aligned}
V^{\text{open}}_{\mathrm{AdS}_5\times\mathrm{S}^3} = &\frac{1}{8}A\,\mathrm{Tr}[\phi^4]\alpha'^2 + \frac{1}{8}B\,\mathrm{Tr}\left[(\nabla_\mu\phi)(\nabla_\mu\phi)\phi\phi\right]\alpha'^3+\\
&+\frac{1}{16}\,\mathrm{Tr}\big[C(\nabla_\mu\nabla_\nu\phi)(\nabla_\mu\nabla_\nu\phi)\phi^2+\frac{1}{2}D(\nabla_\mu\nabla_\nu\phi)\phi(\nabla_\mu\nabla_\nu\phi)\phi\big]\alpha'^4+\\
&+\mathrm{Tr}\left[E\,\nabla^2(\nabla_\mu\phi)(\nabla_\mu\phi)\phi\phi+F\,\nabla^2(\nabla_\mu\phi)\phi(\nabla_\mu\phi)\phi\right]\alpha'^4+\\
&+\frac{1}{32}\,\mathrm{Tr}\big[G(\nabla_\mu\nabla_\nu\nabla_\rho\phi)(\nabla_\mu\nabla_\nu\nabla_\rho\phi)\phi^2+\\
&+\frac{1}{32}H(\nabla_\mu\nabla_\nu\nabla_\rho\phi)\phi(\nabla_\mu\nabla_\nu\nabla_\rho\phi)\phi\big]\alpha'^5+\cdots\,,
\end{aligned} \tag{3.10}$$

with

$$\begin{aligned}
A(\alpha') &= \frac{\pi^2}{6}+a_1\frac{\alpha'}{R^2}+a_2\left(\frac{\alpha'}{R^2}\right)^2+\cdots\\
B(\alpha') &= \zeta_3+b_1\frac{\alpha'}{R^2}+b_2\left(\frac{\alpha'}{R^2}\right)^2+\cdots\\
C(\alpha') &= \frac{\pi^4}{720}7+c_1\frac{\alpha'}{R^2}+c_2\left(\frac{\alpha'}{R^2}\right)^2+\cdots\\
D(\alpha') &= \frac{\pi^4}{720}+d_1\frac{\alpha'}{R^2}+d_2\left(\frac{\alpha'}{R^2}\right)^2+\cdots\\
E(\alpha') &= e_1+e_2\left(\frac{\alpha'}{R^2}\right)+\cdots\\
F(\alpha') &= f_1+f_2\left(\frac{\alpha'}{R^2}\right)+\cdots\\
G(\alpha') &= \frac{1}{3}\left(\frac{1}{6}\zeta_3\pi^2+\zeta_5\right)+g_1\frac{\alpha'}{R^2}+g_2\left(\frac{\alpha'}{R^2}\right)^2+\cdots\\
H(\alpha') &= \frac{1}{3}\left(\frac{1}{6}\zeta_3\pi^2-2\zeta_5\right)+h_1\frac{\alpha'}{R^2}+h_2\left(\frac{\alpha'}{R^2}\right)^2+\cdots\,.
\end{aligned} \tag{3.11}$$

Notice that the only terms which non trivially contribute to the $\mathrm{AdS}_5{\times}\mathrm{S}^3$ completion are A, B, C, D, G, H while E, F are novel AdS terms that vanish upon taking the limit. We will refer to the latter as ambiguities. From now on, we will set $R = 1$.

In the next section we will compute $\mathrm{AdS}_5{\times}\mathrm{S}^3$ Witten diagrams associated to this action which will provide a prediction for the four-point function of arbitrary KK modes order by order in α'.

3.3 AdS×S Witten diagrams in embedding space

In the remaining part of the section, we will generalise the formulae of [29], which are valid for general $\mathrm{AdS}_{\theta+1}{\times}\mathrm{S}^{\theta+1}$ backgrounds, to the most general $\mathrm{AdS}_{\theta_1+1}{\times}\mathrm{S}^{\theta_2+1}$ theory. We will keep the dimensions θ_1, θ_2 generic for most of the discussion. The formulae for the $\mathrm{AdS}_5{\times}\mathrm{S}^3$ case can then be recovered by taking $\theta_1 = 4$, $\theta_2 = 2$.

[12]The various normalisations are chosen so that the normalisation of the associated Mellin amplitudes matches with the corresponding flat amplitude coefficient, as we will see later on.

We will make use of the embedding space formalism. Here, bulk points in $\mathrm{AdS}_{\theta_1+1}$ and S^{θ_2+1} are defined via

$$\hat{X}^2 = -\left(\hat{X}^{-1}\right)^2 - \left(\hat{X}^0\right)^2 + \sum_{i=1}^{\theta_1}\left(\hat{X}^i\right)^2 = -1,$$
$$\hat{Y}^2 = \sum_{i=-1}^{\theta_2}\left(\hat{Y}^i\right)^2 = 1. \tag{3.12}$$

On the other hand, boundary coordinates satisfy $X^2 = Y^2 = 0$. Here we are using a slight abuse of language, since the sphere, being compact, does not have a boundary. However, the condition $Y^2 = 0$ is still meaningful in this context and it corresponds to the statement of tracelessness of R-symmetry indices. The boundary coordinates X, Y are related to spacetime (x_{ij}^2) and R-symmetry variables (y_{ij}^2) via:

$$x_{ij}^2 = -2X_i \cdot X_j, \qquad y_{ij}^2 = -2Y_i \cdot Y_j. \tag{3.13}$$

In embedding coordinates the action of covariant derivatives can be conveniently defined in terms of projectors. These read:

$$\mathcal{P}_A^B = \delta_A^B + \hat{X}_A\hat{X}^B, \qquad \mathcal{P}_I^J = \delta_I^J - \hat{Y}_I\hat{Y}^J. \tag{3.14}$$

Note that the bulk coordinates are in the kernel of the respective projectors:

$$\mathcal{P}_A^B\hat{X}^A = 0, \tag{3.15}$$
$$\mathcal{P}_I^J\hat{Y}^I = 0. \tag{3.16}$$

We will be particularly interested in the covariant derivative of a rank-N tensor defined by [41, 51]

$$\nabla_A T_{A_1A_2\ldots A_N} = \mathcal{P}_A^C\mathcal{P}_{A_1}^{C_1}\ldots\mathcal{P}_{A_N}^{C_N}\partial_C\mathcal{P}_{C_1}^{E_1}\ldots\mathcal{P}_{C_N}^{E_N}T_{E_1\ldots E_N}. \tag{3.17}$$

In this notation, an AdS contact Witten diagram in embedding space reads

$$D^{(\theta_1)}_{\Delta_1\Delta_2\Delta_3\Delta_4}(X_i) = \frac{1}{(-2)^{2\Sigma_\Delta}}\int_{\mathrm{AdS}}\frac{d^{\theta_1+1}\hat{X}}{P_1^{\Delta_1}P_2^{\Delta_2}P_3^{\Delta_3}P_4^{\Delta_4}}, \qquad P_i = \hat{X}\cdot X_i, \tag{3.18}$$

where we have defined $\Sigma_\Delta = (\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4)/2$.

Analogously, one can define sphere contact diagrams as

$$B^{(\theta_2)}_{p_1p_2p_3p_4}(Y_i) = (-2)^{2\Sigma_p}\int_{\mathrm{S}} d^{\theta_2+1}\hat{Y}Q_1^{p_1}Q_2^{p_2}Q_3^{p_3}Q_4^{p_4}, \qquad Q_i = \hat{Y}\cdot Y_i, \tag{3.19}$$

where we defined $\Sigma_p = (p_1 + p_2 + p_3 + p_4)/2$. Note, this last integral can be explicitly evaluated. After some combinatorics one gets [29, 52]:

$$B^{(\theta_2)}_{p_1p_2p_3p_4}(Y_i) = \mathcal{N}_S\sum_{\{d_{ij}\}}\prod_{i<j}\frac{(Y_i\cdot Y_j)^{d_{ij}}}{\Gamma[d_{ij}+1]} \tag{3.20}$$

where the sum runs over the set

$$\left\{(d_{12},d_{13},d_{14},d_{23},d_{24},d_{34}) : 0 \le d_{ij} = d_{ji}, \quad d_{ii} = 0, \quad \sum_{i=1}^{4} d_{ij} = p_j\right\},$$

and the factor $\mathcal{N}_S$ reads

$$\mathcal{N}_S = 2 \cdot 2^{\Sigma_p} \frac{\pi^{\theta_2/2+1} \prod_i \Gamma(p_i+1)}{\Gamma(\Sigma_p + \theta_2/2 + 1)}. \tag{3.21}$$

As we mentioned already, the main idea will be to work directly in the product geometry AdS×S. In this context it is therefore natural to define

$$\mathcal{W}_i \equiv \frac{1}{(P_i + Q_i)}, \tag{3.22}$$

which is related to the *generalised bulk-to-boundary propagator* via [29]

$$G(X_i, \hat{X}; Y_i, \hat{Y}) = \mathcal{C}_{\Delta_i} \frac{1}{(-2)^{\Delta_i}} \mathcal{W}_i^{\Delta_i}, \tag{3.23}$$

with the normalisation given by

$$\mathcal{C}_{\Delta} = \frac{\Gamma(\Delta)}{2\pi^{\frac{\theta_1+\theta_2}{2}} \Gamma(\Delta + \frac{\theta_1+\theta_2}{2} + 1)}. \tag{3.24}$$

Note that when $\Delta = (\theta_1 + \theta_2)/2$ the generalised propagator obeys the following equation

$$\nabla^2 \mathcal{W}_i^{\Delta_i} \equiv \nabla^2_{\hat{X}} \mathcal{W}_i^{\Delta_i} + \nabla^2_{\hat{Y}} \mathcal{W}_i^{\Delta_i} = \Delta_i \frac{\theta_2 - \theta_1}{2} \mathcal{W}_i^{\Delta_i}, \tag{3.25}$$

therefore its "mass" is controlled by the difference between the non-compact and the compact space dimensions $\theta_1 - \theta_2$. In particular, for $\mathcal{N} = 4$ SYM it satisfies a massless equation, while in $\mathrm{AdS}_5 \times \mathrm{S}^3$ it satisfies the equation for a massive scalar field.

Starting from the generalised bulk-to-boundary propagator, we then define generalised AdS×S contact Witten diagrams via

$$D^{\mathrm{AdS}_{\theta_1+1} \times \mathrm{S}^{\theta_2+1}}_{\Delta_1\Delta_2\Delta_3\Delta_4}(X_i, Y_i) = \frac{1}{(-2)^{2\Sigma_\Delta}} \int_{\mathrm{AdS}_{\theta_1+1} \times \mathrm{S}^{\theta_2+1}} d^{\theta_1+1}\hat{X} d^{\theta_2+1}\hat{Y}\ \mathcal{W}_1^{\Delta_1} \mathcal{W}_2^{\Delta_2} \mathcal{W}_3^{\Delta_3} \mathcal{W}_4^{\Delta_4}. \tag{3.26}$$

Note that AdS×S Witten diagrams reduce to a sum of familiar AdS Witten diagrams after Taylor-expanding the propagators. More precisely, we have:

$$\begin{aligned} &D^{\mathrm{AdS}_{\theta_1+1} \times \mathrm{S}^{\theta_2+1}}_{\Delta_1\Delta_2\Delta_3\Delta_4}(X_i, Y_i) \\ &= \sum_{p_i=0}^{\infty} \prod_{i=1}^{4} (-1)^{p_i} \frac{(p_i+1)_{\Delta_i - 1}}{\Gamma(\Delta_i)} D^{(\theta_1)}_{p_1+\Delta_1 p_2+\Delta_2 p_3+\Delta_3 p_4+\Delta_4}(X_i) B^{(\theta_2)}_{p_1p_2p_3p_4}(Y_i). \end{aligned} \tag{3.27}$$

3.4 Generalised Mellin space

We conclude the section by defining a generalised Mellin space representation. This is a very natural generalisation of the familiar Mellin space formalism for AdS amplitudes [41, 53] to the full AdS×S product space [17, 29]. As we will see, the correlators admit very simple expressions when written in this formalism.

Let us first recall the Mellin transform of standard contact Witten diagrams [41]:

$$D^{(\theta_1)}_{\Delta_1\Delta_2\Delta_3\Delta_4}(X_i) = \frac{\frac{1}{2}\pi^{\theta_1/2}\Gamma(\Sigma_\Delta - \theta_1/2)}{(-2)^{\Sigma_\Delta}\prod_i \Gamma(\Delta_i)} \int \frac{d\delta_{ij}}{(2\pi i)^2} \prod_{i<j} \frac{\Gamma(\delta_{ij})}{(X_i \cdot X_j)^{\delta_{ij}}}, \quad \sum_i \delta_{ij} = \Delta_j. \tag{3.28}$$

As we have seen, the sphere function $B^{(\theta_2)}_{p_1p_2p_3p_4}(Y_i)$ also admits a similar representation that can be obtained by explicitly evaluating the sphere integral. Let us recall it here for convenience:

$$B^{(\theta_2)}_{p_1p_2p_3p_4}(Y_i) = 2\cdot 2^{\Sigma_p} \frac{\pi^{\theta_2/2+1}\prod_i \Gamma(p_i+1)}{\Gamma(\Sigma_p + \theta_2/2+1)} \sum_{\{d_{ij}\}} \prod_{i<j} \frac{(Y_i\cdot Y_j)^{d_{ij}}}{\Gamma(d_{ij}+1)}, \quad \sum_i d_{ij} = p_j. \tag{3.29}$$

Inserting the above representations into (3.27) we get

$$D^{\mathrm{AdS}_{\theta_1+1}\times \mathrm{S}^{\theta_2+1}}_{\Delta_1\Delta_2\Delta_3\Delta_4}(X_i,Y_i) = \frac{\pi^{\frac{\theta_1+\theta_2}{2}+1}}{(-2)^{\Sigma_\Delta}\prod_i\Gamma(\Delta_i)} \tag{3.30}$$

$$\sum_{p_i=0}^{\infty}(-1)^{\Sigma_p}\int\frac{d\delta_{ij}}{(2\pi i)^2}\sum_{\{d_{ij}\}}\prod_{i<j}\left(\frac{(Y_i\cdot Y_j)^{d_{ij}}}{(X_i\cdot X_j)^{\delta_{ij}}}\frac{\Gamma(\delta_{ij})}{\Gamma(d_{ij}+1)}\right)(\Sigma_p+\theta_2/2+1)_{\Sigma_\Delta-\frac{\theta_1+\theta_2}{2}-1}. \tag{3.31}$$

This suggests to define the generalised Mellin transform $\mathcal{M}[f]$ of a generic function $f(X_i,Y_i)$ via

$$f(X_i,Y_i)\equiv$$
$$\frac{\pi^{\frac{\theta_1+\theta_2}{2}+1}}{(-2)^{\Sigma_\Delta}}\left(\prod_i\frac{\mathcal{C}_{\Delta_i}}{\Gamma(\Delta_i)}\right)\sum_{p_i=0}^{\infty}(-1)^{\Sigma_p}\int\frac{d\delta_{ij}}{(2\pi i)^2}\sum_{\{d_{ij}\}}\prod_{i<j}\left(\frac{(Y_i\cdot Y_j)^{d_{ij}}}{(X_i\cdot X_j)^{\delta_{ij}}}\frac{\Gamma(\delta_{ij})}{\Gamma(d_{ij}+1)}\right)\mathcal{M}[f], \tag{3.32}$$

where we have $\sum_{i\neq j}\delta_{ij} = p_j + \Delta_j$ and $\sum_{i\neq j} d_{ij} = p_j$. Using this definition, the Mellin transform of a generalised Witten diagram reads

$$\mathcal{M}\left[\left(\prod_i \mathcal{C}_{\Delta_i}\right) D^{AdS_{\theta_1+1}\times S^{\theta_2+1}}_{\Delta_1\Delta_2\Delta_3\Delta_4}(X_i,Y_i)\right] = (\Sigma_p+\theta_2/2+1)_{\Sigma_\Delta-\frac{\theta_1+\theta_2}{2}-1}. \tag{3.33}$$

Finally, when considering higher derivative corrections it will be necessary to compute a decorated version of (3.26) of the generic form

$$\prod_i \mathcal{C}_{\Delta_i}\frac{\left(\prod_{i<j}(X_i\cdot X_j)^{n^X_{ij}}(Y_i\cdot Y_j)^{n^Y_{ij}}\right)}{(-2)^{2\Sigma_\Delta}}\int_{\mathrm{AdS}\times\mathrm{S}} d^{\theta_1+1}\hat{X}d^{\theta_2+1}\hat{Y}\prod_{i=1}^{4}\frac{P_i^{n_i^P}Q_i^{n_i^Q}(\Delta_i)_{n_i}}{(\mathcal{W})^{\Delta_i+n_i}}. \tag{3.34}$$

In appendix A we show that the generalised Mellin expression for this expression is given by

$$\mathcal{M}[(3.34)] = (-2)^{\Sigma_X}(2)^{\Sigma_Y}(-)^{2\Sigma_Q}\left(\prod_{i=1}^{4}\left(p_i+n_i^X+\Delta_i\right)_{n_i^P}(p_i-n_i^Q-n_i^Y+1)_{n_i^Q}\right)\times \left(\prod_{i<j}(\delta_{ij})_{n_{ij}^X}(d_{ij}-n_{ij}^Y+1)_{n_{ij}^Y}\right)(\Sigma_p-\Sigma_Y+\frac{\theta_2}{2}+1)_{\Sigma_\Delta-\frac{\theta_1+\theta_2}{2}-1+\Sigma_X+\Sigma_Y}, \tag{3.35}$$

with $\sum_{i\neq j}\delta_{ij}=p_j+\Delta_j$ and $\sum_{i\neq j}d_{ij}=p_j$. Here, following the conventions of [29], we introduced the notation Σ_Q for half the sum over the n_i^Q; Σ_X, Σ_Y for the sum over the n_{ij}^X, n_{ij}^Y respectively; $n_i^X=\sum_{i\neq j}n_{ij}^X$, $n_i^Y=\sum_{i\neq j}n_{ij}^Y$ and $n_i=n_i^X+n_i^Y+n_i^P+n_i^Q$. Now that all main ingredients are in place we are ready to show some explicit results.

4 Explicit results at orders $\alpha'^{2,3,4,5}$

Having presented the definitions for arbitrary (θ_1,θ_2) we now specialise to the case at hand, tree-level α' correction in $AdS_5\times S^3$, where we set $\theta_1=4$, $\theta_2=2$, $\Delta_i=3$. We begin by considering the constraints imposed on the (δ_{ij},d_{ij}) i.e.

$$\sum_{i\neq j}\delta_{ij}=p_j+1;\qquad \sum_{i\neq j}d_{ij}=p_j-2, \tag{4.1}$$

where here we have employed the shift $p_i\to p_i-2$ due to the lowest correlator being $p_i=2$. While the correlator may be written in terms of the constrained (δ_{ij},d_{ij}), we will find it useful to partially solve these constraints. Following analogous conventions to those in [19], we can write

$$\begin{aligned}
\delta_{12}&=-s+p_{(12)}, & \delta_{34}&=-s+p_{(34)},\\
\delta_{14}&=-t+p_{(14)}, & \delta_{23}&=-t+p_{(23)},\\
\delta_{13}&=-u+p_{(13)}, & \delta_{24}&=-u+p_{(24)},\\
d_{12}&=\tilde{s}+p_{(12)}, & d_{34}&=\tilde{s}+p_{(34)},\\
d_{14}&=\tilde{t}+p_{(14)}, & d_{23}&=\tilde{t}+p_{(23)},\\
d_{13}&=\tilde{u}+p_{(13)}, & d_{24}&=\tilde{u}+p_{(24)},
\end{aligned} \tag{4.2}$$

where we introduced the notation $p_{(ij)}=\frac{p_i+p_j}{2}$. Having done so, we are left with only 6 variables which satisfy

$$s+t+u=\Sigma_p-1;\qquad \tilde{s}+\tilde{t}+\tilde{u}=-\Sigma_p-2. \tag{4.3}$$

It follows that, at given order in α', the correlator will depend on 8 *unconstrained* variables: $s,t,\tilde{s},\tilde{t}$ as well as the charges p_i.

Now, the large p limit [17] suggests to trade the Mellin variables s,t,u with the bold-face variables defined via:

$$\mathbf{s}=s+\tilde{s},\qquad \mathbf{t}=t+\tilde{t},\qquad \mathbf{s}+\mathbf{t}+\mathbf{u}=-3. \tag{4.4}$$

In fact, as we will see, the various correlators will admit a natural stratification w.r.t. to the bold-face variables $\mathbf{s}, \mathbf{t}$. In particular, as explained in [17], in the limit of large $s, t, \tilde{s}, \tilde{t}, p_i$ variables, the correlator approaches to the flat S-matrix with the Mandelstam replaced by the bold-face variables. In conclusion, we will express the correlators in terms of the set $(\mathbf{s}, \mathbf{t}, \tilde{s}, \tilde{t}, p_1, p_2, p_3, p_4)$.

Finally, our notation for the α' expansion of the Mellin amplitudes will closely follow the analogous position space expression (2.6), i.e.[13]

$$\mathcal{M}_{\text{tree}}^{I_1 I_2 I_3 I_4} = \mathcal{M}_{\text{YM}}^{I_1 I_2 I_3 I_4} + \mathcal{M}_0^{I_1 I_2 I_3 I_4}\alpha'^2 + \mathcal{M}_1^{I_1 I_2 I_3 I_4}\alpha'^3 + \mathcal{M}_2^{I_1 I_2 I_3 I_4}\alpha'^4 + \mathcal{M}_3^{I_1 I_2 I_3 I_4}\alpha'^5 + \cdots, \tag{4.5}$$

where the full colour-dressed amplitude $\mathcal{M}_{\text{tree}}^{I_1 I_2 I_3 I_4}$ admits a decomposition in terms of colour-ordered amplitudes

$$\begin{aligned}\mathcal{M}_{\text{tree}}^{I_1 I_2 I_3 I_4} &= \frac{1}{2}\sum_{\mathcal{P}(2,3,4)} \text{Tr}[T^{I_1}T^{I_2}T^{I_3}T^{I_4}]\mathcal{M}_{\text{tree}}(1234) = \text{Tr}[T^{I_1}T^{I_2}T^{I_3}T^{I_4}]\mathcal{M}_{\text{tree}}(1234)\\ &+ \text{Tr}[T^{I_1}T^{I_4}T^{I_2}T^{I_3}]\mathcal{M}_{\text{tree}}(1423) + \text{Tr}[T^{I_1}T^{I_3}T^{I_4}T^{I_2}]\mathcal{M}_{\text{tree}}(1342),\end{aligned} \tag{4.6}$$

4.1 Field-theory correlator ($\alpha' = 0$)

Before presenting our new results, for completeness let us recall the form of the correlator in the field theory limit, first computed in [37], within this formalism. This cannot be directly recovered from the effective action as it is not a polynomial in the Mandelstam and it would require adding a non-local vertex to the action.

However, this is a very special case. In fact, the correlators at this order obey a hidden $8d$ conformal symmetry [37], which implies that the correlator for arbitrary KK modes can be obtained from a generating function which only depends on $8d$ distances. This generating function takes the form of the field-theory correlator with lowest charges:

$$\sum_{\vec{p}} G_{\text{YM},\vec{p}}(x_{ij}^2 + y_{ij}^2) \propto \frac{1}{(x_{12}^2 + y_{12}^2)^3 (x_{34}^2 + y_{34}^2)^3} D_{2321}(x_{ij}^2 + y_{ij}^2). \tag{4.7}$$

Note that this object transforms as the four-point function of weight 3 (i.e. the dimension of scalar in $8d$) operators.

Let us see how to recover this from the point of view of generalised Witten diagrams. Note that the Pochhammer in (3.33) vanishes when

$$\Sigma_\Delta = \frac{\theta_1 + \theta_2}{2} + 1. \tag{4.8}$$

It is immediate to see that, when (4.8) is satisfied — which is the case of the field theory correlator[14] — an AdS×S contact diagram (3.26) becomes proportional to a standard AdS Witten diagram (3.28) with the replacement $X_i \cdot X_j \to X_i \cdot X_j + Y_i \cdot Y_j$. This is nothing but (4.7) written in embedding coordinates.

[13]From now on, we will suppress the subscript $\vec{p}$ for the Mellin amplitudes as we will always be referring to the individual correlators.

[14]In fact, for the field-theory correlator we have $1/2(2+3+2+1) = (4+2)/2+1$.

The generalised Mellin space expression is also very simple and can be written in the form (3.32) with $\Delta = 3$ and the Mellin amplitude:

$$\mathcal{M}_{\mathrm{YM}}(1234) = -\frac{1}{(\delta_{34} - d_{34} - 1)(\delta_{14} - d_{14} - 1)} = -\frac{1}{(\mathbf{s}+1)\,(\mathbf{t}+1)} \tag{4.9}$$

which is in agreement with the expression given in [37]. In the second line we have expressed the amplitude in terms of the bold-face variables $\mathbf{s}, \mathbf{t}$ defined above, as in [40]. Nicely, these colour-ordered amplitudes satisfy BCJ relations for *all* Kaluza-Klein operators completely analogously to flat space [40]. In fact, we have

$$\mathcal{M}_{\mathrm{YM}}(1234) = \frac{(\mathbf{u}+1)}{(\mathbf{t}+1)} \mathcal{M}_{\mathrm{YM}}(1342) \tag{4.10}$$

and similarly for other colour-ordered amplitudes.

4.2 Order α'^2

The first string corrections are at order α'^2. The relevant term in the effective action is

$$S_0 = \frac{1}{8}\zeta_2 \int_{\mathrm{AdS}_5\times \mathrm{S}^3} d^5\hat{X} d^3\hat{Y}\ \mathrm{Tr}\left[\phi(\hat{X},\hat{Y})^4\right]. \tag{4.11}$$

To obtain the correlators we just mimic the standard AdS/CFT procedure, i.e. we take derivatives w.r.t. the boundary data for the bulk field which acts as a source for the scalar operator $\mathcal{O}$. As we stressed already, the actual difference with the common AdS/CFT prescription is just the replacement of AdS bulk-to-boundary propagators with generalised ones. Thus, at this order we get

$$\begin{aligned}\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle|_{\alpha'^2}(1234) &= \zeta_2 \frac{\mathcal{C}_3^4}{(-2)^{12}} \int_{\mathrm{AdS}_5\times \mathrm{S}^3} \frac{d^5\hat{X} d^3\hat{Y}}{\mathcal{W}_1^3\mathcal{W}_2^3\mathcal{W}_3^3\mathcal{W}_4^3} \\ &= \zeta_2\, \mathcal{C}_3^4\, D_{3333}^{\mathrm{AdS}_5\times \mathrm{S}^3}(X_i, Y_i),\end{aligned} \tag{4.12}$$

and analogously for the other colour-ordered correlators. Thus, at this order the correlator is just a single $D_{3333}^{\mathrm{AdS}_5\times \mathrm{S}^3}$ function. The position space expression in terms of standard AdS Witten diagrams (D-functions) for the individual correlators can be directly read off from (3.27). In particular, note that the correlator $\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\rangle$ is proportional to $\bar{D}_{3333}$. As expected, this is the same function showing up as ambiguity in the corresponding field theory one-loop amplitude [39], and can be seen as a one-loop counterterm from the field-theory viewpoint.

We are interested in the Mellin space expression[15] which we can get from (3.33). This is very simple and reads:

$$\mathcal{M}_0(1234) = \zeta_2\, (\Sigma_p - 2)_2\,. \tag{4.13}$$

At this point it is worth taking a break and noticing that the correlators so computed correctly reproduce the flat-space limit as given in [41, 42], which in the following will be referred to as *AdS-type limit.* In fact, Penedones formula states that the Mellin amplitude in

[15] Let us stress again that this is the defined for the individual correlators.

the limit of large Mellin variables approaches the flat amplitude as a function of Mandelstam variables:

$$\mathcal{M}(\delta_{ij}) \xrightarrow[\text{large}\,\delta_{ij}]{} \frac{1}{\Gamma(\Sigma_p-2)} \int_0^\infty d\alpha\, \alpha^{\Sigma_p-1}\, \mathcal{V}_{\text{open}}(\alpha\delta_{ij}), \tag{4.14}$$

where $\mathcal{V}_{\text{open}}$ stands for any of the colour-ordered flat amplitudes. At this order in α' we have $\mathcal{V} = \zeta_2$ is a constant, therefore (4.14) returns the full amplitude and we recover (4.13) straight away.[16]

This fact — which represents a *consistency check* for our results — will be true at all orders as can be seen directly from the Mellin transform of a generic decorated integral (3.35). In fact, note that the decorated integral at a given order produces polynomials in $X_i \cdot X_j$ whose degree is dictated by the number of AdS derivatives hitting the vertex, or, in other words, from the order in α'. Schematically, a vertex of the generic form $\nabla_{\mu_1}\cdots\nabla_{\mu_n}\phi\nabla_{\mu_1}\cdots\nabla_{\mu_n}\phi\phi\phi$ goes like

$$\nabla_{\mu_1}\cdots\nabla_{\mu_n}\phi\nabla_{\mu_1}\cdots\nabla_{\mu_n}\phi\phi\phi \quad \underset{\text{large}\,X}{\sim} \quad (X_i\cdot X_j)^n \quad \underset{\text{Mellin}}{\sim} \quad \delta_{ij}^n. \tag{4.15}$$

Thus, for large Mellin (δ_{ij}) variables, (3.35) approaches[17]

$$\mathcal{M} \xrightarrow[\text{large}\,\delta_{ij}]{} \delta_{ij}^{n_{ij}^X} (\Sigma_p - 2)_{n_{ij}^X+2} \tag{4.16}$$

which is precisely the Pochhammer appearing in (4.14) after integrating (4.14) with $\mathcal{V} \sim \delta_{ij}^{n_{ij}^X}$. At the next orders it will become clear that the AdS-type limit turns out to be a *particular* case of a more general notion of flat-space limit where the *sphere* variables are set to zero.

4.3 Order α'^3

The computation of higher order corrections is conceptually similar, but more computationally involved because it requires evaluating the action of the covariant derivatives on fields. We have used a straightforward generalisation of the algorithm outlined in [29]. At α'^3, the relevant term in the action is:

$$S_1^{\text{main}} = \frac{1}{8}\zeta_3 \int_{\text{AdS}_5\times \text{S}^3} d^5\hat{X} d^3\hat{Y}\, \text{Tr}\left[(\nabla_\mu\phi)(\nabla_\mu\phi)\phi\phi\right]. \tag{4.17}$$

Now, it is easy to see that:

$$(\nabla_\mu \mathcal{W}_1^\Delta)(\nabla_\mu \mathcal{W}_2^\Delta) = \Delta^2 \frac{N_{12}}{\mathcal{W}_1^{\Delta+1}\mathcal{W}_2^{\Delta+1}} \tag{4.18}$$

where we have defined

$$N_{ij} = X_i\cdot X_j + Y_i\cdot Y_j + P_iP_j - Q_iQ_j. \tag{4.19}$$

[16] Note that for $\alpha'=0$ we have $\mathcal{V}_{\text{YM}} = -\frac{1}{s\,t}$ and we correctly reproduce (4.9) in the AdS-type limit.

[17] We remind again that we have shifted the charges p_i by two units, $p_i \to p_i - 2$, and taken $\Delta_i = 3$.

Using the above relation we get, for the colour-ordered correlators,

$$\begin{aligned}\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}^{\text{main}}\rangle|_{\alpha'^3}(1234) = \frac{1}{8}\zeta_3\frac{3^2\,\mathcal{C}_3^4}{(-2)^{12}}\int_{\text{AdS}_5\times\text{S}^3} d^5\hat{X}d^3\hat{Y}\,\frac{1}{(\mathcal{W}_1)^3(\mathcal{W}_2)^3(\mathcal{W}_2)^3(\mathcal{W}_4)^3}\\ \times 2\left(\frac{N_{12}}{\mathcal{W}_1\mathcal{W}_2}+\frac{N_{34}}{\mathcal{W}_3\mathcal{W}_4}+\frac{N_{14}}{\mathcal{W}_1\mathcal{W}_4}+\frac{N_{23}}{\mathcal{W}_2\mathcal{W}_3}\right).\end{aligned} \tag{4.20}$$

This is the generator of all correlators. From (3.34), it is straightforward to get the associated Mellin amplitude for the individual correlators. This is given by

$$\mathcal{M}_1^{\text{main}}(1234) = \zeta_3(\mathcal{M}_1^s + \mathcal{M}_1^t - 3(\Sigma_p-2)_2) \tag{4.21}$$

where we defined the s-type Mellin amplitude

$$\mathcal{M}_1^s = (\Sigma_p-2)_3\mathbf{s} - 3(\Sigma_p-2)_2\tilde{s}, \tag{4.22}$$

with $\mathcal{M}_1^t$, $\mathcal{M}_1^u$ related to $\mathcal{M}_1^s$ by crossing:

$$\begin{aligned}\mathcal{M}_1^t &\equiv \mathcal{M}_1^s[s\to t,\ \tilde{s}\to\tilde{t},\ p_2\leftrightarrow p_4] = (\Sigma_p-2)_3\mathbf{t} - 3(\Sigma_p-2)_2\tilde{t},\\ \mathcal{M}_1^u &\equiv \mathcal{M}_1^s[s\to u,\ \tilde{s}\to\tilde{u},\ p_2\leftrightarrow p_3] = (\Sigma_p-2)_3\mathbf{u} - 3(\Sigma_p-2)_2\tilde{u}.\end{aligned} \tag{4.23}$$

Then, the full colour-ordered amplitude is the sum of the above term (the “main” amplitude) and the ambiguities present at this order. In sum, we have:[18]

$$\begin{aligned}\mathcal{M}_1(1234) &= \zeta_3(\mathcal{M}_1^s+\mathcal{M}_1^t) + a_1(\Sigma_p-2)_2\\ &= \zeta_3\big((\Sigma_p-2)_3(\mathbf{s}+\mathbf{t}) - 3(\Sigma_p-2)_2(\tilde{s}+\tilde{t})\big) + a_1(\Sigma_p-2)_2,\end{aligned} \tag{4.24}$$

and similarly for the other colour-ordered amplitudes. Here a_1 is a free coefficient corresponding to the freedom of adding the α'^2 correction to the amplitude, see (3.11). This is the only ambiguity present at this order. The other possibility would be a term of the form $\sim(\nabla^2\phi)\phi\phi\phi$ but, because of (3.25), it is essentially the same as the ambiguity coming from ϕ^4.

Note that we have the relation

$$\mathcal{M}_1^s + \mathcal{M}_1^t + \mathcal{M}_1^u \propto (\Sigma-2)_2 \tag{4.25}$$

which can be identified as the AdS analogue of the flat on-shell relation $s+t+u=0$.

Before computing the other α' corrections, let us notice a remarkable simplification occurring for the first two corrections which was already spotted in the $\text{AdS}_5\times\text{S}^5$ case [14]. This will help to write the other α' corrections in a more compact form. The idea is to absorb the various Pochhammers appearing through a double integral transform. Let us thus define the following pre-amplitude

$$\mathcal{M}_n = \frac{i}{2\pi}\int_0^\infty d\alpha\int_{\mathcal{C}} d\beta\ e^{-\alpha-\beta}\alpha^{\Sigma_p-1}(-\beta)^{2-\Sigma_p}\,\tilde{\mathcal{M}}_n(S,T,\tilde{S},\tilde{T}), \tag{4.26}$$

[18] Note that we simplified the result by absorbing the constant term $3(\Sigma_p-2)_2$ into the ambiguity a_1. We will use the freedom to perform such redefinitions at higher orders as well.

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where $\mathcal{C}$ is the Hankel contour. Here $\mathcal{M}_n$ is any of the colour-ordered amplitudes and $\tilde{\mathcal{M}}_n$ is a simplified amplitude, defined in terms of the following variables,

$$S = \alpha s - \beta \tilde{s}, \quad \tilde{S} = \alpha s + \beta \tilde{s}, \tag{4.27}$$

and similarly for t-type and u-type variables. The integral transform (4.26) really just provides the Γ functions needed to reconstruct the various Pochhammer. In fact, it is not difficult to check that in the pre-amplitude $\tilde{\mathcal{M}}_n$ all Pochhammer disappear. For example, the pre-amplitude associated to α'^2 is just

$$\tilde{\mathcal{M}}_0(1234) = \zeta_2. \tag{4.28}$$

On the other hand, the pre-amplitude at order α'^3 reads (4.22) is

$$\tilde{\mathcal{M}}_1(1234) = \zeta_3(S + T + a_1). \tag{4.29}$$

Thus, from these first two examples, we can see that the pre-amplitude is given by the corresponding term in the flat Veneziano amplitude with the Mandelstam variables replaced by S, T plus lower order terms in $S, T, \tilde{S}, \tilde{T}$. This will be true at higher orders as well.

Note, the α integral is nothing but the Penedones integral (4.14), and the above transform reduces to the latter when setting all but the Mellin variables to zero. As mentioned previously, these *non-trivial simplifications* strongly suggest that the usual flat-space limit arises as a particular case of a more general flat-space limit which involves all 8 variables which the Mellin amplitude depends on.

4.4 Order α'^4

Let us now consider α'^4 corrections. The corresponding term in the action is:

$$S_2^{\text{main}} = \frac{1}{16}\frac{\pi^4}{720}\int_{\text{AdS}_5\times\text{S}^3} d^5\hat{X} d^3\hat{Y}\, \text{Tr}\left[7(\nabla_\mu\nabla_\nu\phi)(\nabla_\mu\nabla_\nu\phi)\phi^2 + \frac{1}{2}(\nabla_\mu\nabla_\nu\phi)\phi(\nabla_\mu\nabla_\nu\phi)\phi\right]. \tag{4.30}$$

Let us stress again that this is the only term at this order with a flat-space counterpart. After computing the derivatives we get

$$\begin{aligned}\langle\mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle^{\text{main}}|_{\alpha'^4}(1234) = {}& \frac{1}{16}\frac{\pi^4}{720}\frac{3^2\,\mathcal{C}_3^4}{(-2)^{12}}\int_{\text{AdS}_5\times\text{S}^3} d^5\hat{X} d^3\hat{Y}\frac{1}{(\mathcal{W}_1)^3(\mathcal{W}_2)^3(\mathcal{W}_3)^3(\mathcal{W}_4)^3}\times \\ & 2\bigg[7\left(\frac{L_{12}}{(\mathcal{W}_1)^2(\mathcal{W}_2)^2} + \frac{L_{34}}{(\mathcal{W}_3)^2(\mathcal{W}_4)^2} + \frac{L_{14}}{(\mathcal{W}_1)^2(\mathcal{W}_4)^2} + \frac{L_{23}}{(\mathcal{W}_2)^2(\mathcal{W}_3)^2}\right) \\ & + \frac{L_{13}}{(\mathcal{W}_1)^2(\mathcal{W}_3)^2} + \frac{L_{24}}{(\mathcal{W}_2)^2(\mathcal{W}_4)^2}\bigg],\end{aligned} \tag{4.31}$$

where here we have defined

$$L_{ij} = 16N_{ij}^2 - (P_iP_j - Q_iQ_j)(3P_iP_j - 5Q_iQ_j - P_iQ_j - P_jQ_i). \tag{4.32}$$

From (3.35), we get the associated colour-ordered Mellin amplitude for the individual correlators:

$$\mathcal{M}_2^{\text{main}}(1234) = \frac{\pi^4}{720}(7\mathcal{M}_2^s + 7\mathcal{M}_2^t + \mathcal{M}_2^u), \tag{4.33}$$

where the s-type amplitude reads

$$\mathcal{M}_2^s = (\Sigma_p - 2)_4\, \mathbf{s}^2 - (\Sigma_p - 2)_3\, \mathbf{s}(8\tilde{s} + \Sigma_p + 1) + (\Sigma_p - 2)_2 \left(12\tilde{s}^2 - \frac{3}{8}P + 12\tilde{s} + \frac{3}{2}\Sigma_p\right). \tag{4.34}$$

and analogously for $\mathcal{M}_2^t, \mathcal{M}_2^u$. Here we have defined

$$P \equiv p_1^2 + p_2^2 + p_3^2 + p_4^2. \tag{4.35}$$

As mentioned before, the amplitude automatically stratifies w.r.t. the power of $\mathbf{s}$, $\mathbf{t}$. In particular it respects the large p behaviour [17].

By using (the inverse of) (4.26), we can get the pre-amplitude associated to this expression. This is very simple and reads

$$\tilde{\mathcal{M}}_2^s = S^2 + S\Sigma_p - \frac{5}{2}S + \frac{3}{2}\left(\tilde{S} + \Sigma_p\right) - \frac{3}{8}P. \tag{4.36}$$

Let us now compute the ambiguities. Through explicit calculation we verified that, on the top of the lower-order ambiguities, there are two new independent ambiguities at this order, which we choose to be

$$\mathrm{Tr}\left[(\nabla^2\nabla_\mu\phi)(\nabla^\mu\phi)\phi\phi\right], \qquad \mathrm{Tr}\left[(\nabla^2\nabla_\mu\phi)\phi(\nabla^\mu\phi)\phi\right]. \tag{4.37}$$

After computing the derivatives and using the Mellin space formula (3.35) we obtain, respectively:

$$\mathcal{M}_{2,\text{amb}_1} = \mathcal{M}_{2,\text{amb}}^s + \mathcal{M}_{2,\text{amb}}^t, \tag{4.38}$$

and

$$\mathcal{M}_{2,\text{amb}_2} = \mathcal{M}_{2,\text{amb}}^u, \tag{4.39}$$

with

$$\mathcal{M}_{2,\text{amb}}^s = (\Sigma_p - 2)_3\mathbf{s} + \frac{3}{14}(\Sigma_p - 2)_2(p_1p_2 + p_3p_4 + 2\Sigma_p - 2\Sigma_p^2 - 4\tilde{s}\Sigma_p - 2\tilde{s}), \tag{4.40}$$

and $\mathcal{M}_{2,\text{amb}}^{t,u}$ related to $\mathcal{M}_{2,\text{amb}}^s$ by crossing, cf. (4.23). In terms of the associated pre-amplitude we have

$$\tilde{\mathcal{M}}_{2,\text{amb}}^s = S + \frac{3}{8}\left(2\tilde{S} + 2\Sigma_p - 2\Sigma_p^2 + p_1p_2 + p_3p_4\right). \tag{4.41}$$

In sum, the full colour-ordered (pre-)amplitude at order α'^4 is[19]

$$\begin{aligned}\tilde{\mathcal{M}}_2(1234) =& \frac{\pi^4}{720}(7\tilde{\mathcal{M}}_2^s + 7\tilde{\mathcal{M}}_2^t + \tilde{\mathcal{M}}_2^u) + a_2 + b_1\left(\tilde{\mathcal{M}}_1^s + \tilde{\mathcal{M}}_1^t\right) + \\ &+ e_1(\tilde{\mathcal{M}}_{2,\text{amb}}^s + \tilde{\mathcal{M}}_{2,\text{amb}}^t) + f_1\tilde{\mathcal{M}}_{2,\text{amb}}^u.\end{aligned} \tag{4.42}$$

[19]Here we have used a slight abuse of notation, as the ambiguities here are related to those in (3.11) by some shifts and rescaling.

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Note that this theory will generically contain more ambiguities than $\mathcal{N}=4$ SYM [14, 29], due to the loss of some crossing symmetry.

Moreover, as anticipated, it is easy to check that for large S,T,U the amplitude is given by the corresponding term in the expansion of the Veneziano amplitude with the Mandelstam variables replaced by S,T,U variables:

$$\tilde{\mathcal{M}}_2(1234) \underset{\text{large}\, S,T,U}{\sim} \frac{\pi^4}{720}(7S^2+7T^2+U^2) = \mathcal{V}_{\text{open}}\big|_{\alpha'^4}(1234). \tag{4.43}$$

4.5 Order α'^5

Finally, let us compute the amplitude at order α'^5. The main amplitude at this order reads

$$\begin{aligned} S_3^{\text{main}} =& \frac{1}{3}\frac{1}{32}\int_{\text{AdS}_5\times \text{S}^3} d^5\hat{X} d^3\hat{Y}\, \text{Tr}\left[\left(\frac{1}{6}\zeta_3\pi^2+\zeta_5\right)(\nabla_\mu\nabla_\nu\nabla_\rho\phi)(\nabla_\mu\nabla_\nu\nabla_\rho\phi)\phi^2 + \right. \\ &+ \left.\frac{1}{2}\left(\frac{1}{6}\zeta_3\pi^2-2\zeta_5\right)(\nabla_\mu\nabla_\nu\nabla_\rho\phi)\phi(\nabla_\mu\nabla_\nu\nabla_\rho\phi)\phi\right]. \end{aligned} \tag{4.44}$$

In appendix B we explicitly evaluate the derivatives, along with their Mellin space expression. Instead here we just give the associated Mellin pre-amplitude, which reads

$$\tilde{\mathcal{M}}_3^{\text{main}} = \frac{1}{3}\left(\frac{1}{6}\zeta_3\pi^2+\zeta_5\right)(\tilde{\mathcal{M}}_3^s+\tilde{\mathcal{M}}_3^t) + \frac{1}{3}\left(\frac{1}{6}\zeta_3\pi^2-2\zeta_5\right)\tilde{\mathcal{M}}_3^u \tag{4.45}$$

with[20]

$$\begin{aligned} \tilde{\mathcal{M}}_3^s =& S^3-3S^2\Sigma_p+2S\Sigma_p^2-\frac{11}{8}PS-3S^2+7S\Sigma_p+\frac{9}{2}S\tilde{S}-3\Sigma_p\tilde{S}+\frac{33}{16}P-2\Sigma_p^2 \\ &+p_1p_2+p_3p_4+\frac{45}{2}S-\frac{25}{4}\Sigma_p-\frac{19}{4}\tilde{S}, \end{aligned} \tag{4.46}$$

and $\tilde{\mathcal{M}}_3^t$, $\tilde{\mathcal{M}}_3^u$ defined analogously.

By writing down all possible ambiguities, we find that there are in total 8 independent, and we can choose them to be the 6 associated to previous orders plus two new ones associated to the terms

$$\text{Tr}\left[(\nabla^2\nabla^\mu\nabla^\nu\phi)(\nabla_\nu\nabla_\mu\phi)\phi\phi\right], \qquad \text{Tr}\left[(\nabla^2\nabla^\mu\nabla^\nu\phi)\phi(\nabla_\nu\nabla_\mu\phi)\phi\right] \tag{4.47}$$

The pre-amplitude expressions are $\tilde{\mathcal{M}}^s_{3,\text{amb}}+\tilde{\mathcal{M}}^t_{3,\text{amb}}$ and $\tilde{\mathcal{M}}^u_{3,\text{amb}}$ respectively, with the s-channel ambiguity given by

$$\begin{aligned} \tilde{\mathcal{M}}^s_{3,\text{amb}} =& S^2-\frac{8}{5}S\Sigma^2+\frac{8}{5}\Sigma^3-\frac{2}{5}P\Sigma+\frac{2}{5}(p_1p_2+p_3p_4)(2S-2\Sigma+1)+\frac{8}{5}S\tilde{S} \\ &-\frac{8}{5}\tilde{S}\Sigma+\frac{3}{5}S\Sigma+\frac{39}{10}S-\frac{1}{10}\tilde{S}+\frac{3}{8}P-\frac{4}{5}\Sigma^2-\frac{7}{10}\Sigma, \end{aligned} \tag{4.48}$$

and $\tilde{\mathcal{M}}_3^{t,u}$,$\tilde{\mathcal{M}}_{3,\text{amb}}^{t,u}$ defined accordingly.

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[20] We remind that $P \equiv p_1^2+p_2^2+p_3^2+p_4^2$.

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In sum, the full Mellin pre-amplitude reads

$$\begin{aligned}\tilde{\mathcal{M}}_3(1234) =& \frac{1}{3}\left(\frac{1}{6}\zeta_3\pi^2+\zeta_5\right)(\tilde{\mathcal{M}}_3^s+\tilde{\mathcal{M}}_3^t)+\frac{1}{3}\left(\frac{1}{6}\zeta_3\pi^2-2\zeta_5\right)\tilde{\mathcal{M}}_3^u+\\ &+a_3+b_2(\tilde{\mathcal{M}}_1^s+\tilde{\mathcal{M}}_1^t)+c_1(\tilde{\mathcal{M}}_2^s+\tilde{\mathcal{M}}_2^t)+d_1\tilde{\mathcal{M}}_2^u+e_2(\tilde{\mathcal{M}}_{2,\text{amb}}^s+\tilde{\mathcal{M}}_{2,\text{amb}}^t)+\\ &+f_2\tilde{\mathcal{M}}_{2,\text{amb}}^u+l_1(\tilde{\mathcal{M}}_{3,\text{amb}}^s+\tilde{\mathcal{M}}_{3,\text{amb}}^t)+h_1\tilde{\mathcal{M}}_{3,\text{amb}}^u.\end{aligned} \tag{4.49}$$

Finally, note once again that, for large S,T,U, the amplitude reduces to the flat-Veneziano amplitude:

$$\tilde{\mathcal{M}}_3(1234) \underset{\text{large}\,S,T,U}{\sim} \frac{1}{3}\left(\frac{1}{6}\zeta_3\pi^2+\zeta_5\right)(S^3+T^3)+\frac{1}{3}\left(\frac{1}{6}\zeta_3\pi^2-2\zeta_5\right)U^3=\mathcal{V}_{\text{open}}\big|_{\alpha'^5}(1234). \tag{4.50}$$

4.6 Towards a generalised flat-space limit

We conclude this section by commenting on the various flat-space limit we encountered along the way. We would like to address the following question:

$$\boxed{\textit{what is the largest sub-amplitude which directly descends from flat-space?}}$$

A first answer comes from the flat-space limit worked out in [41, 42], which states that the Mellin amplitude and the flat scattering amplitude are related by an integral transform which, in this theory, reads

$$\mathcal{M}(s,t) \xrightarrow[\text{large}\,s,t]{} \frac{1}{\Gamma(\Sigma_p-2)}\int_0^\infty d\alpha\,\alpha^{\Sigma_p-1}\,\mathcal{V}(\alpha s,\alpha t). \tag{4.51}$$

Note that only AdS variables participate in this limit; the sphere variables $\tilde{s},\tilde{t}$ and the charges p_i are just spectators. This does not look completely satisfactory because in this particular theory, AdS and S factors scale in the same way for large radius, thus one would expect a limit where variables are treated in a more symmetric way.

In fact, the authors of [17] point out that in the "large p limit", i.e. in the limit of large $s,t,\tilde{s},\tilde{t},p_i$, the relation above gets upgraded to a more symmetric version with the Mellin amplitude related to the corresponding flat-space scattering process via the integral

$$\mathcal{M}(\mathbf{s},\mathbf{t}) \xrightarrow[\text{large}\,s,t,\tilde{s},\tilde{t},p_i]{} \frac{1}{\Gamma(\Sigma_p-2)}\int_0^\infty d\alpha\,\alpha^{\Sigma_p-1}\,\mathcal{V}(\alpha\mathbf{s},\alpha\mathbf{t}), \tag{4.52}$$

where we recall here for convenience the definition of the bold-face variables

$$\mathbf{s}=s+\tilde{s},\qquad \mathbf{t}=t+\tilde{t},\qquad \mathbf{s}+\mathbf{t}+\mathbf{u}=-3. \tag{4.53}$$

Note that all Mellin amplitudes presented in this work manifestly respect the large p limit, as they should. Note also that when $\tilde{s}=\tilde{t}=0$ one recovers (4.51).

In this paper we have seen that an even more general version of flat-space limit seems to hold, which was noticed already in [14] for $\mathcal{N}=4$ SYM correlators. In particular, all Mellin amplitudes can be written in terms of a pre-amplitude defined via (4.26):

$$\mathcal{M}_n=\frac{i}{2\pi}\int_0^\infty d\alpha\int_{\mathcal{C}}d\beta\;e^{-\alpha-\beta}\alpha^{\Sigma_p-1}(-\beta)^{2-\Sigma_p}\,\tilde{\mathcal{M}}_n(S,T), \tag{4.54}$$

where

$$S = \alpha s - \beta \tilde{s}, \quad \tilde{S} = \alpha s + \beta \tilde{s}. \tag{4.55}$$

Quite nicely, the highest degree terms in S, T are precisely the polynomials appearing in the expansion of the Veneziano amplitude. In other words, the pre-amplitude $\tilde{\mathcal{M}}$ is given by

$$\tilde{\mathcal{M}} = \mathcal{V}_{\text{open}}(S, T) + \text{lower orders in } S, T, \tilde{S}, \tilde{T}. \tag{4.56}$$

Note that the sub-amplitude $\mathcal{V}_{\text{open}}(S, T)$ automatically contains the two flat-space limits just discussed, therefore in this sense it is a generalisation of those.

Finally, the covariantisation of the effective action suggests that the largest sub-amplitude related to flat space is the one obtained with the replacement of the partial derivatives with covariant ones [29], with all other limits just discussed arising as particular cases. An explicit form for this sub-amplitude at all orders in α' is still not known, nor is the analogous expression for the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background. We hope to report on this in the future.

5 Outlook and conclusions

In this paper we initiated the study of tree-level α' corrections to the four-point function of half-BPS operators in a $4d$, $\mathcal{N} = 2$ SCFT with flavour group SO(8), dual to string theory on $\mathrm{AdS}_5 \times \mathrm{S}^3$ [34–36]. In particular, the strong-coupling expansion of these four-point correlators corresponds to the low-energy expansion of an AdS version of the Veneziano amplitude. By generalising a procedure first proposed in [29] in the context of $\mathcal{N} = 4$ SYM, we conjectured that *all* half-BPS four-point correlators can be obtained by evaluating generalised contact Witten diagrams whose vertices come from $8d$ effective potential written in terms of a single scalar field. We then showed explicit results for the first four orders in α'. We found that at each order in α' the various correlators are given by a *main* amplitude, which represents the covariantisation of the flat-space amplitude, plus a certain number of ambiguities which arise as a result of curvature effects of the background and therefore do not have a flat-space counterpart. Nicely, the end results are remarkably simple when written in terms of an integral transform, which is perhaps the most natural generalisation of the integral proposed by Penedones in the flat-space limit [41].

While we believe that the simplicity of these correlators and the fact that they correctly capture the various flat-space limits present in literature are indicative of the validity of the method, the existence of the effective action is still to be proven and it would be interesting to find independent methods to check the conjecture or, more ambitiously, to derive it from first principles. Another consistency check of our results could be provided by an analysis of the spectrum of anomalous dimensions. In $\mathrm{AdS}_5 \times \mathrm{S}^5$ [13, 14], this furnished an independent method that led to the same results as those of [29]. In particular, we expect the α'-corrected anomalous dimensions of the exchanged double-trace operators to induce a splitting of the residual degeneracy left in the field-theory anomalous dimensions computed in [40], as a result of the breaking of the hidden conformal symmetry [54].

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We believe that this work can open a number of interesting directions.

- As mentioned already, these results, together with their $AdS_5 \times S^5$ analogous [14, 29], suggest that there is a more general version of flat-space limit associated to the covariantisation of the derivatives in the effective action. Finding this sub-amplitude might shed light on new relations between flat and AdS amplitudes.

- On the other hand, it is very important to find a systematic way to (list and) compute all the ambiguities, which represent true curvature effects. In $AdS_5 \times S^5$ they can be fixed with various techniques, such as localisation [55–58], symmetry principles [29], dispersive sum rules [22, 23] or also bootstrap approaches [14]. It would be interesting to see whether these methods can also applied to this background [43].

- With these new correlators at hand, it is now possible to extract novel CFT data. As we mentioned, we expect the α'-corrections of the exchanged double-trace operators to drive a splitting of the residual degeneracy left in the field-theory anomalous dimensions. The splitting of this degeneracy is controlled by a characteristic polynomial — an intrinsically non-perturbative object — that enjoys a lot of intriguing features. It would be extremely interesting to find the form of this polynomial, perhaps also deriving the explicit dependence on the dimension θ_1, θ_2.[21]

- It would also be interesting to compute higher genus α' corrections. We believe that similar results to those obtained in [19] for $\mathcal{N} = 4$ SYM will find a natural generalisation to $AdS_5 \times S^3$. In particular, it should be the case that the accidental degeneracy enjoyed by certain classes of tree-level correlators gets broken at higher loops, through a phenomenon known as *sphere splitting* [19].

- Finally, as mentioned already, it is well known that open and closed string amplitudes are related by relations known as KLT relations [47]; in addition, colour-ordered open string amplitudes are related each other through monodromy relations [48, 49]. These are the uplifted "stringy" versions of double-copy [60] of BCJ [50] relations, respectively, and reduce to the latter in the $\alpha' \to 0$ limit. It is our belief that (a suitable generalisation of) these relations will also hold in this set-up, at least at the level of the main amplitude, since this object is directly connected to the flat amplitude. From this point of view, it is promising that the generalised Mellin amplitudes in the field theory limit do satisfy double copy [61] and BCJ relations [40] completely analogous to flat space[22] and moreover the highest degree terms in the pre-amplitude do enjoy these relations, since they literally coincide with the flat Veneziano amplitude written in terms of S, T, U variables. We hope to report on this in the near future.

[21] In fact, it should also be the case that the supergravity anomalous dimensions in $AdS_3 \times S^3$ [59] undergo a similar splitting when string corrections are turned on [54].

[22] For recent developments on double-copy and BCJ relations in different (A)dS backgrounds, see e.g. [62–69] and references therein.

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A Decorated Witten diagrams in Mellin space

In this appendix we fill in the details between (3.33) and (3.35) for the derivation of the Mellin transform of the generalised Witten diagrams with decorations. The details follow closely the original presentation of [29] and provide a slight generalisation to the case of $\mathrm{AdS}_{\theta_1+1}\times \mathrm{S}^{\theta_2+1}$ where the AdS and S dimensions no longer coincide.

Our starting point is the definition of the decorated AdS×S Witten diagram which we repeat here for convenience

$$\prod_i \mathcal{C}_{\Delta_i} \frac{\prod_{i<j}(X_i\cdot X_j)^{n^X_{ij}}(Y_i\cdot Y_j)^{n^Y_{ij}}}{(-2)^{2\Sigma_\Delta}} \int_{\mathrm{AdS}\times \mathrm{S}} d^{\theta_1+1}\hat{X}\, d^{\theta_2+1}\hat{Y} \prod_{i=1}^{4} \frac{P_i^{n_i^P} Q_i^{n_i^Q} (\Delta_i)_{n_i}}{(P_i+Q_i)^{\Delta_i+n_i}}. \tag{A.1}$$

The propogators in the denominator may be Taylor-expanded using

$$\frac{1}{(P_i+Q_i)^{\Delta_i+n_i}} = \sum_{p_i=0}^{\infty} (-1)^{p_i} \frac{(p_i+1)_{\Delta_i+n_i-1}}{\Gamma(\Delta_i+n_i)} \frac{Q_i^{p_i}}{P_i^{p_i+\Delta_i+n_i}}, \tag{A.2}$$

from which we arrive at an expansion for the decorated $AdS\times S$ Witten diagram in terms of regular AdS diagrams and their sphere counterparts given explicitly by

$$\begin{aligned}&\prod_i \mathcal{C}_{\Delta_i} (-2)^{2\Sigma_X+2\Sigma_Y} \left(\prod_{i<j}(X_i\cdot X_j)^{n^X_{ij}}(Y_i\cdot Y_j)^{n^Y_{ij}}\right)\\ &\sum_{p_i=0}^{\infty}\prod_{i=1}^{4}(-)^{p_i}\frac{(p_i+1)_{\Delta_i+n_i-1}}{\Gamma(\Delta_i)} D^{(\theta_1)}_{\Delta_i+p_i+n_i-n_i^P} B^{(\theta_2)}_{p_i+n_i^Q}.\end{aligned} \tag{A.3}$$

Substituting in the following expressions

$$\begin{aligned} D^{(\theta_1)}_{\Delta_i+p_i+n_i-n_i^P} &= \frac{\frac{1}{2}\pi^{\theta_1/2}\Gamma(\Sigma_\Delta+\Sigma_p+\Sigma_Q+\Sigma_X+\Sigma_Y-\theta_1/2)}{(-2)^{\Sigma_\Delta+\Sigma_p+\Sigma_Q+\Sigma_X+\Sigma_Y}\prod_i\Gamma(\Delta_i+p_i+n_i-n_i^P)} \int \frac{d\delta_{ij}}{(2\pi i)^2} \prod_{i<j} \frac{\Gamma(\delta_{ij})}{(X_i\cdot X_j)^{\delta_{ij}}},\\ B^{(\theta_2)}_{p_i+n_i^Q} &= 2\cdot 2^{\Sigma_p+\Sigma_Q} \frac{\pi^{\theta_2/2+1}\prod_i\Gamma(p_i+n_i^Q+1)}{\Gamma(\Sigma_p+\Sigma_Q+\theta_2/2+1)} \sum_{\{d_{ij}\}}\prod_{i<j}\frac{(Y_i\cdot Y_j)^{d_{ij}}}{\Gamma(d_{ij}+1)},\end{aligned} \tag{A.4}$$

for which we have

$$\sum_i \delta_{ij} = \Delta_j + p_j + n_j - n_j^P, \quad \sum_i d_{ij} = p_j + n_j^Q, \tag{A.5}$$

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the decorated Witten diagram (A.1) is found to take the following form

$$\frac{\pi^{\frac{\theta_1+\theta_2}{2}}}{(-2)^{\Sigma_\Delta}}\left(\prod_i \frac{\mathcal{C}_{\Delta_i}}{\Gamma(\Delta_i)}\right)\sum_{p_i=0}^{\infty}(-)^{\Sigma_p}\int\frac{d\delta_{ij}}{(2\pi i)^2}\sum_{\{d_{ij}\}}\prod_{i<j}\left(\frac{(Y_i\cdot Y_j)^{d_{ij}}}{(X_i\cdot X_j)^{\delta_{ij}}}\frac{\Gamma(\delta_{ij})}{\Gamma(d_{ij}+1)}\right)\mathcal{M}_{\Delta_i}[(A.1)]. \tag{A.6}$$

Here we have introduced the notation Σ_Q for half the sum over the n_i^Q and Σ_X, Σ_Y for the sum over the n_{ij}^X, n_{ij}^Y, respectively. Here $\mathcal{M}_{\Delta_i}[(A.1)]$ is the desired representation of (A.1) in generalised Mellin space and is given by

$$\begin{aligned}\mathcal{M}_{\Delta_i}[(A.1)] = {}&(-2)^{\Sigma_X}(2)^{\Sigma_Y}(-)^{2\Sigma_Q}\left(\prod_{i=1}^{4}\left(p_i+n_i^X+\Delta_i\right)_{n_i^P}(p_i-n_i^Q-n_i^Y+1)_{n_i^Q}\right)\times\\ &\left(\prod_{i<j}(\delta_{ij})_{n_{i,j}^X}(d_{ij}-n_{ij}^Y+1)_{n_{ij}^Y}\right)(\Sigma_p-\Sigma_Y+\frac{\theta_2}{2}+1)_{\Sigma_\Delta-\frac{\theta_1+\theta_2}{2}-1+\Sigma_X+\Sigma_Y},\end{aligned} \tag{A.7}$$

where we have the contraints $\sum_i d_{ij}=p_j$ and $\sum_i \delta_{ij}=\Delta_j+p_j$.

B Details at α'^5

In this appendix we collect some explicit expression for the amplitudes at α'^5. First, the computation at this order requires the evaluation of $\nabla_\mu\nabla_\nu\nabla_\rho\mathcal{W}_1^3\;\nabla_\mu\nabla_\nu\nabla_\rho\mathcal{W}_2^3$ and $\nabla^2\nabla_\mu\nabla_\nu\mathcal{W}_1^3\;\nabla_\nu\nabla_\mu\mathcal{W}_2^3$. We find

$$\begin{aligned}\nabla_\mu\nabla_\nu\nabla_\rho\mathcal{W}_1^3\;\nabla_\mu\nabla_\nu\nabla_\rho\mathcal{W}_2^3 = \frac{9}{\mathcal{W}_1^6\mathcal{W}_2^6}\Big(&245P_1^3P_2^3+62P_1^2P_2^3Q_1-3P_1P_2^3Q_1^2+62P_1^3P_2^2Q_2\\ &-575P_1^2P_2^2Q_1Q_2-100P_1P_2^2Q_1^2Q_2-3P_2^2Q_1^3Q_2-3P_1^3P_2Q_2^2-100P_1^2P_2Q_1Q_2^2+457P_1P_2Q_1^2Q_2^2\\ &+14P_2Q_1^3Q_2^2-3P_1^2Q_1Q_2^3+14P_1Q_1^2Q_2^3-163Q_1^3Q_2^3+1045P_1^2P_2^2(X_1\cdot X_2)+62P_1P_2^2Q_1(X_1\cdot X_2)\\ &-3P_2^2Q_1^2(X_1\cdot X_2)+62P_1^2P_2Q_2(X_1\cdot X_2)-2012P_1P_2Q_1Q_2(X_1\cdot X_2)-114P_2Q_1^2Q_2(X_1\cdot X_2)\\ &-3P_1^2Q_2^2(X_1\cdot X_2)-114P_1Q_1Q_2^2(X_1\cdot X_2)+869Q_1^2Q_2^2(X_1\cdot X_2)+1200P_1P_2(X_1\cdot X_2)^2\\ &-1200Q_1Q_2(X_1\cdot X_2)^2+400(X_1\cdot X_2)^3+963P_1^2P_2^2(Y_1\cdot Y_2)-14P_1P_2^2Q_1(Y_1\cdot Y_2)+3P_2^2Q_1^2(Y_1\cdot Y_2)\\ &-14P_1^2P_2Q_2(Y_1\cdot Y_2)-1988P_1P_2Q_1Q_2(Y_1\cdot Y_2)-14P_2Q_1^2Q_2(Y_1\cdot Y_2)+3P_1^2Q_2^2(Y_1\cdot Y_2)\\ &-14P_1Q_1Q_2^2(Y_1\cdot Y_2)+963Q_1^2Q_2^2(Y_1\cdot Y_2)+2400P_1P_2(X_1\cdot X_2)(Y_1\cdot Y_2)-2400Q_1Q_2(X_1\cdot X_2)(Y_1\cdot Y_2)\\ &+1200(X_1\cdot X_2)^2(Y_1\cdot Y_2)+1200P_1P_2(Y_1\cdot Y_2)^2-1200Q_1Q_2(Y_1\cdot Y_2)^2+1200(X_1\cdot X_2)(Y_1\cdot Y_2)^2\\ &+400(Y_1\cdot Y_2)^3\Big)\equiv\frac{9}{\mathcal{W}_1^6\mathcal{W}_2^6}Z_{12}\end{aligned}$$

and

$$\begin{aligned}\nabla^2\nabla_\mu\nabla_\nu\mathcal{W}_1^3\;\nabla_\nu\nabla_\mu\mathcal{W}_2^3 = \frac{9}{\mathcal{W}_1^5\mathcal{W}_2^5}\Big(&31Q_1^2Q_2^2-167P_1^2P_2^2-3P_1P_2^2Q_1-3P_1^2P_2Q_2+3P_1Q_1Q_2^2\\ &+3P_2Q_1^2Q_2+136P_1P_2Q_1Q_2-416P_1P_2(X_1\cdot X_2)+160Q_1Q_2(X_1\cdot X_2)-160(X_1\cdot X_2)(Y_1\cdot Y_2)\\ &-160P_1P_2(Y_1\cdot Y_2)-96Q_1Q_2(Y_1\cdot Y_2)-208(X_1\cdot X_2)^2+48(Y_1\cdot Y_2)^2\Big).\end{aligned} \tag{B.1}$$

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Then, the main amplitude is given by the following integral

$$
\begin{aligned}
&\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle^{\text{main}}|_{\alpha'^5}(1234) = \frac{1}{3}\frac{1}{32}\frac{3^2\,\mathcal{C}_3^4}{(-2)^{12}}\int_{\text{AdS}_5\times\text{S}^3} d^5\hat{X}d^3\hat{Y}\frac{1}{(\mathcal{W}_1)^3(\mathcal{W}_2)^3(\mathcal{W}_3)^3(\mathcal{W}_4)^3}\\
&\left[\left(\frac{1}{6}\zeta_3\pi^2+\zeta_5\right)2\left(\frac{Z_{12}}{(\mathcal{W}_1)^3(\mathcal{W}_2)^3}+\frac{Z_{34}}{(\mathcal{W}_3)^3(\mathcal{W}_4)^3}+\frac{Z_{14}}{(\mathcal{W}_1)^3(\mathcal{W}_4)^3}+\frac{Z_{23}}{(\mathcal{W}_2)^3(\mathcal{W}_3)^3}\right)+\right.\\
&\left.\left(\frac{1}{6}\zeta_3\pi^2-2\zeta_5\right)2\left(\frac{Z_{13}}{(\mathcal{W}_1)^3(\mathcal{W}_3)^3}+\frac{Z_{24}}{(\mathcal{W}_2)^3(\mathcal{W}_4)^3}\right)\right].
\end{aligned}
\tag{B.2}
$$

The associated generalised Mellin amplitude reads

$$
\mathcal{M}_3^{\text{main}} = \frac{1}{3}\left(\frac{1}{6}\zeta_3\pi^2+\zeta_5\right)(\mathcal{M}_3^s+\mathcal{M}_3^t)+\frac{1}{3}\left(\frac{1}{6}\zeta_3\pi^2-2\zeta_5\right)\mathcal{M}_3^u, \tag{B.3}
$$

where

$$
\begin{aligned}
\mathcal{M}_3^s =& (\Sigma_p-2)_5\mathbf{s}^3-\frac{3}{2}\Big[10\tilde{s}+2\Sigma_p-1\Big](\Sigma_p-2)_4\mathbf{s}^2\\
&+\frac{1}{8}\Big[480\tilde{s}^2+120(1+\Sigma_p)\tilde{s}+16\Sigma_p(2+\Sigma_p)-11P+142\Big](\Sigma_p-2)_3\mathbf{s}\\
&-\frac{1}{16}\Big[480(2\tilde{s}+3)\tilde{s}^2-2(33P-164\Sigma_p-654)\tilde{s}\\
&\qquad -16(p_1p_2+p_3p_4)+4\Sigma_p(25+8\Sigma_p)-33P\Big](\Sigma_p-2)_2.
\end{aligned}
\tag{B.4}
$$

Similarly, the s-channel ambiguity is given by

$$
\begin{aligned}
\mathcal{M}_{3,\text{amb}}^s =& (\Sigma_p-2)_4\mathbf{s}^2-\frac{1}{13}\Big[8(7+2\Sigma_p)\tilde{s}+\Sigma_p(5+8\Sigma_p)-4(p_1p_2+p_3p_4)-19\Big](\Sigma_p-2)_3\mathbf{s}\\
&+\frac{1}{104}\Big[32(16\Sigma_p-9)\tilde{s}^2+32(10\Sigma_p^2-8\Sigma-3p_1p_2-3p_3p_4-15)\tilde{s}\\
&\qquad +16(p_1+p_2)(p_3+p_4)(2\Sigma-1)+7P-28\Sigma_p\Big](\Sigma_p-2)_2.
\end{aligned}
\tag{B.5}
$$

The expressions for the related pre-amplitudes are given in the main body, see (4.46) and (4.48) for the Mellin pre-amplitudes $\tilde{\mathcal{M}}_3^s$, $\tilde{\mathcal{M}}_{3,\text{amb}}^s$, respectively.

All in all, the full colour-ordered Mellin amplitude (including all other ambiguities from previous orders) reads

$$
\begin{aligned}
\mathcal{M}_3(1234) =& \frac{1}{3}\left(\frac{1}{6}\zeta_3\pi^2+\zeta_5\right)(\mathcal{M}_3^s+\mathcal{M}_3^t)+\frac{1}{3}\left(\frac{1}{6}\zeta_3\pi^2-2\zeta_5\right)\mathcal{M}_3^u+\\
&+a_3+b_2(\mathcal{M}_1^s+\mathcal{M}_1^t)+c_1(\mathcal{M}_2^s+\mathcal{M}_2^t)+d_1\mathcal{M}_2^u+e_2(\mathcal{M}_{2,\text{amb}}^s+\mathcal{M}_{2,\text{amb}}^t)+\\
&+f_2\mathcal{M}_{2,\text{amb}}^u+l_1(\mathcal{M}_{3,\text{amb}}^s+\mathcal{M}_{3,\text{amb}}^t)+h_1\mathcal{M}_{3,\text{amb}}^u.
\end{aligned}
\tag{B.6}
$$

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