

Classically integrable boundary conditions for symmetric-space sigma models

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ABSTRACT

We investigate boundary conditions for the nonlinear sigma model on the compact symmetric space G/H . The Poisson brackets and the classical local conserved charges necessary for integrability are preserved by boundary conditions which correspond to involutions which commute with the involution defining H . Applied to $SO(3)/SO(2)$, the nonlinear sigma model on S^2 , these yield the great circles as boundary submanifolds. Applied to $G \times G/G$, they reproduce known results for the principal chiral model.

1 Introduction

Over the last ten years there has been much investigation of the boundary conditions on the half-line which preserve the integrability of certain 1+1-dimensional field theories. However, relatively little of this has focused on nonlinear sigma models. There has been some work on the $O(3)$ model (the nonlinear sigma model on $SO(3)/SO(2)$) [1], and the results of [2] on the $O(N)$ model have been extended [3]. More recently, general boundary conditions for the principal chiral model have been written down [4].

In this letter we study the classical integrability of the sigma model on a general compact symmetric space G/H , and find that the boundary conditions which preserve integrability (via the conserved bulk charges of [5]) on the half-line are in correspondence with the involutions which commute with that defining H . We work in detail through the specialization of our results to the $O(3)$ model and to the principal chiral model.

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2 The bulk model

We first review briefly the gauged construction of the bulk G/H sigma model, and its canonical structure: for the details see [5, 6]. Take G to be a compact Lie group, and let σ be an involutive automorphism of G whose fixed point set is a subgroup H . Then G/H is symmetric and at the level of the Lie algebras $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. Let $g(t, x)$ be a field taking values in G , and write $j_\mu = g^{-1} \partial_\mu g$. The G/H model should possess the local symmetry $g(t, x) \mapsto g(t, x)h(t, x)$: to achieve this we introduce a gauge field $A_\mu(t, x) \in \mathfrak{h}$ and covariant derivative $D_\mu g \equiv \partial_\mu g - gA_\mu$, so that the current

$$J_R^\mu \equiv g^{-1} D^\mu g = j^\mu - A^\mu \quad (1)$$

is gauge-covariant and we can construct the gauge-invariant Lagrangian

$$\mathcal{L} = -\frac{1}{2} \langle J_{R\mu} J_R^\mu \rangle, \quad (2)$$

where $\langle \rangle$ is a negative-definite invariant inner product on \mathfrak{g} .

The equations of motion from varying g are then $D_\mu J_R^\mu = 0$, while the A_μ equations of motion impose the constraint $J_R^\mu = 0$ on \mathfrak{h} . These equations of motion allow the construction of local conserved charges whose densities are polynomial in J_R^μ [5].

The current J_R^μ is associated with the local right H -symmetry. The model also has a global left G -symmetry, with corresponding gauge-invariant Noether current

$$J_{L\mu} \equiv -(D_\mu g)g^{-1}. \quad (3)$$

We note that there exists a description of the model purely in terms of gauge-invariant objects: setting $q = \sigma(g)g^{-1}$, so that $q(t, x)$ is a field valued in the ‘Cartan immersion’³ of G/H in G , $\{\sigma(g)g^{-1} | g \in G\} = G/H \hookrightarrow G$, one may re-write the Lagrangian as

$$\mathcal{L} = -\frac{1}{2} \langle J_{L\mu} J_L^\mu \rangle = \frac{1}{8} \langle \partial_\mu q^{-1} \partial^\mu q \rangle, \quad (4)$$

since $2J_L^\mu = q^{-1} \partial^\mu q$. The local charges may also be straightforwardly constructed from J_L^μ , but their Poisson brackets [8] are then harder to handle than in the gauged form. We therefore work with the gauged description in this letter.

Let $\{t^a\}$ be a basis of anti-hermitian generators for \mathfrak{g} , obeying

$$[t^a, t^b] = f^{abc} t^c \quad \text{and} \quad \langle t^a t^b \rangle = -\delta^{ab}.$$

³Strictly, the Cartan immersion is defined as $\{g \in G | \sigma(g) = g^{-1}\}$, which may be a finite cover of the set defined here: see the erratum to [6]. The distinction will not be important for us in this letter.

We may choose this basis to be the disjoint union $\{t^{\hat{\alpha}}\} \cup \{t^{\alpha}\}$ of a basis $\{t^{\hat{\alpha}}\}$ for \mathfrak{h} and a basis $\{t^{\alpha}\}$ for \mathfrak{m} . Any $X \in \mathfrak{g}$ can then be decomposed as $X = X^{\hat{\alpha}}t^{\hat{\alpha}} + X^{\alpha}t^{\alpha}$.

It is convenient to eliminate the auxiliary gauge field A_{μ} from the Lagrangian using its equation of motion $j_{\mu}^{\hat{\alpha}} = A_{\mu}^{\hat{\alpha}}$ before passing to phase space. The Lagrangian is then

$$\mathcal{L} = \frac{1}{2}j_0^{\alpha}j_0^{\alpha} - \frac{1}{2}j_1^{\alpha}j_1^{\alpha} = \frac{1}{2}E_i^{\alpha}E_j^{\alpha}\partial_0\phi^i\partial_0\phi^j - \frac{1}{2}E_i^{\alpha}E_j^{\alpha}\partial_1\phi^i\partial_1\phi^j \quad (5)$$

where $\{\phi^i\}$ is a chart on G and $E_i^{\alpha} = (g^{-1}\partial_i g)^{\alpha}$ are the vielbeins mapping between Lie algebra and tangent space indices. The momentum conjugate to ϕ^i is

$$\pi_i = \frac{\partial L}{\partial(\partial_0\phi^i)} = E_i^{\alpha}E_j^{\alpha}\partial_0\phi^j, \quad (6)$$

and from this we define

$$\mathcal{J}^a = E^{ai}\pi_i, \quad (7)$$

so that $\mathcal{J}^{\alpha} = j_0^{\alpha}$, and $\mathcal{J}^{\hat{\alpha}} \approx 0$ must be imposed as a constraint. One then finds the Poisson brackets

$$\begin{aligned} \{\mathcal{J}^a(x), \mathcal{J}^b(y)\} &= -f^{abc}\mathcal{J}^c(x)\delta(x-y) \\ \{\mathcal{J}^a(x), j_1^b(y)\} &= -f^{abc}j_1^c(x)\delta(x-y) + \delta^{ab}\delta'(x-y) \\ \{j_1^a(x), j_1^b(y)\} &= 0, \end{aligned} \quad (8)$$

and the Hamiltonian density

$$H \approx \frac{1}{2}\mathcal{J}^{\alpha}\mathcal{J}^{\alpha} + \frac{1}{2}j_1^{\alpha}j_1^{\alpha} = \frac{1}{2}j_0^{\alpha}j_0^{\alpha} + \frac{1}{2}j_1^{\alpha}j_1^{\alpha}. \quad (9)$$

It follows that the constraint $\mathcal{J}^{\hat{\alpha}} \approx 0$ is weakly preserved under time evolution (so there are no secondary constraints), and is first class, generating the gauge symmetry

$$\delta_{\Lambda}X(x) = - \int dy \Lambda^{\hat{\alpha}}(y) \{\mathcal{J}^{\hat{\alpha}}(y), X(x)\} \quad (10)$$

where $\Lambda^{\hat{\alpha}}(x)$ is some \mathfrak{h} -valued parameter which specifies the gauge transformation. Thus the canonical formalism has been consistently completed, and we find that the degrees of freedom of the model in this gauged Hamiltonian description are the two \mathfrak{m} -valued currents $\mathcal{J}^{\alpha} = j_0^{\alpha}$ and j_1^{α} , which transform covariantly, together with the single \mathfrak{h} -valued gauge connection $j_1^{\hat{\alpha}} = A_1^{\hat{\alpha}}$.

The Hamiltonian above is defined only up to addition of an arbitrary function of the constraint. We choose to set this function to be zero, so that the Hamiltonian is well-defined, time-evolution is unique, and the time-dependent part of the gauge freedom is fixed. The remaining components $j_0^{\hat{\alpha}} = A_0^{\hat{\alpha}}$ are then determined by the zero-curvature identity $\partial_0 j_1 + \partial_1 j_0 + [j_0, j_1] \equiv 0$: the equations of motion following from the Hamiltonian and brackets are compatible with this identity if and only if $j_0^{\hat{\alpha}}$ vanishes.

3 Boundary conditions

Consider now the model on the half-line $x \leq 0$. Demanding that the action $S = \int_{-\infty}^0 dx \mathcal{L}$ be stationary produces the bulk equations of motion for $x < 0$, together with the boundary equation

$$j_0^\alpha j_1^\alpha = 0 \quad (11)$$

at $x = 0$.

Let us now impose the boundary condition $g(t, 0) = g_0 l(t)$, with g_0 fixed and $l \in D$, where D is some submanifold of G , chosen (without loss of generality) such that $1 \in D$. This is the Dirichlet part of the boundary condition; the boundary equation of motion will supplement this with a Neumann condition. Our goal is to determine the allowed D .

To write the condition in terms of currents, we define

$$\mathfrak{d} \equiv \{l^{-1} \delta l \mid l \in D, \delta l \in T_l D\} \subset \mathfrak{g}; \quad (12)$$

we shall consider only those \mathfrak{d} which are linear subspaces of \mathfrak{g} . We have $j_0 \in \mathfrak{d}$ and, since $j_0^{\hat{\alpha}} \equiv 0$, the only non-trivial part of this condition is

$$j_0|_{\mathfrak{m}} \in \mathfrak{d} \cap \mathfrak{m}. \quad (13)$$

Let $R = R^\top = R^{-1} : \mathfrak{m} \rightarrow \mathfrak{m}$ be the linear map which restricts to $+1$ on $\mathfrak{d} \cap \mathfrak{m}$ and -1 on $\mathfrak{d}^\perp \cap \mathfrak{m}$ – that is, R is the orthogonal reflection through $\mathfrak{d} \cap \mathfrak{m}$. The Dirichlet boundary condition then has the form

$$j_0^\alpha = R^{\alpha\beta} j_0^\beta. \quad (14)$$

The boundary equation of motion (11) then requires

$$j_1^\alpha = -R^{\alpha\beta} j_1^\beta. \quad (15)$$

We have said nothing yet about the conditions on $j_1^{\hat{\alpha}}$, but these will be fixed by demanding consistency with the Poisson brackets. First we consider the extension of the boundary condition to the whole line by

$$j_0^\alpha(x) = R^{\alpha\beta} j_0^\beta(-x), \quad j_1^\alpha(x) = -R^{\alpha\beta} j_1^\beta(-x). \quad (16)$$

(This allows us to handle the nonultralocal term in (8), which is consistent with the boundary condition since R is orthogonal.) Since $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$, the brackets $\{j_0^\alpha, j_1^\beta\}$ and $\{j_0^{\hat{\alpha}}, j_1^{\hat{\beta}}\}$ fix the behaviour of $j_1^{\hat{\alpha}}$:

$$j_1^{\hat{\alpha}}(x) = -S^{\hat{\alpha}\hat{\beta}} j_1^{\hat{\beta}}(-x), \quad (17)$$

where the matrices R and S must obey

$$S^{\hat{\gamma}\hat{\kappa}} f^{\alpha\beta\hat{\kappa}} = R^{\alpha\delta} R^{\beta\epsilon} f^{\delta\epsilon\hat{\gamma}} \Leftrightarrow R^{\alpha\delta} R^{\beta\epsilon} S^{\hat{\gamma}\hat{\kappa}} f^{\delta\epsilon\hat{\kappa}} = f^{\alpha\beta\hat{\gamma}}. \quad (18)$$

In a basis in which $f^{\hat{\alpha}\gamma\delta} f^{\hat{\beta}\gamma\delta} = k\delta^{\hat{\alpha}\hat{\beta}}$, we have explicitly

$$S^{\hat{\alpha}\hat{\beta}} \equiv k^{-1} f^{\hat{\alpha}\gamma\delta} R^{\gamma\epsilon} R^{\delta\lambda} f^{\hat{\beta}\epsilon\lambda} \quad (19)$$

(so S is symmetric) and the requirement (18) implies also that

$$S^2 = 1 \quad \text{and} \quad S^{\hat{\alpha}\hat{\delta}} S^{\hat{\beta}\hat{\epsilon}} S^{\hat{\gamma}\hat{\kappa}} f^{\delta\hat{\epsilon}\hat{\kappa}} = f^{\hat{\alpha}\hat{\beta}\hat{\gamma}}. \quad (20)$$

Thus it emerges that the map $\tau : \mathfrak{g} \rightarrow \mathfrak{g}$ defined by

$$\tau : \begin{cases} \mathfrak{h} & \rightarrow \mathfrak{h} & X & \mapsto SX \\ \mathfrak{m} & \rightarrow \mathfrak{m} & X & \mapsto RX \end{cases} \quad (21)$$

is an involutive automorphism of \mathfrak{g} which, by construction, commutes with σ , the automorphism that defines the symmetric target space G/H . On taking \mathfrak{d} to be the +1-eigenspace of τ (we had not previously specified $\mathfrak{d} \cap \mathfrak{h}$), we have that \mathfrak{d} is a subalgebra of \mathfrak{g} , D is the subgroup $\exp \mathfrak{d} \subset G$, and G/D is itself a symmetric space.

Now, as mentioned previously, the bulk model is known [5] to possess an infinite number of local commuting charges, of the form

$$q_{\pm s} = \int dx d_{\alpha_1 \dots \alpha_{s+1}} j_{\pm}^{\alpha_1} \dots j_{\pm}^{\alpha_{s+1}} \quad (22)$$

where $d_{\alpha_1 \dots \alpha_{s+1}}$ is a symmetric tensor on \mathfrak{m} invariant under the action of the group H . (That is, $d_{\gamma(\alpha_1 \dots \alpha_s} f_{\alpha)\hat{\beta}\gamma} = 0$.) By arguments similar to those in [4], it may be verified that, at least for classical G , an infinite subset of charges

$$q_{|s|} = q_s \pm q_{-s} \quad (23)$$

remain conserved and commuting in the model on the half line with boundary conditions as above. (The crucial property needed to show this is that for each tensor d ,

$$d_{\alpha_1 \dots \alpha_{s+1}} R^{\alpha_1 \beta_1} \dots R^{\alpha_{s+1} \beta_{s+1}} = \pm d_{\beta_1 \dots \beta_{s+1}}. \quad (24)$$

For classical groups G , each tensor d is the restriction of a symmetric G -invariant tensor on \mathfrak{g} [5], and so this statement is a consequence of the more general result

$$d_{a_1 \dots a_{s+1}} \tau^{a_1 b_1} \dots \tau^{a_{s+1} b_{s+1}} = \pm d_{b_1 \dots b_{s+1}}. \quad (25)$$

This is obviously true, with positive sign, whenever τ is an inner automorphism of \mathfrak{g} , and is in fact also true when τ is outer, for classical G ; again, see [5]. We do not know how to prove it in general.)

Our result is then that there is one classically integrable boundary condition associated to each gauge-equivalence class of involutive automorphisms which commute with σ .⁴ In gauge-invariant language, the field q is restricted to a Dirichlet submanifold $\tilde{D} = \{\sigma(g_0 l)l^{-1}g_0^{-1} | l \in D\}$ of the Cartan immersion.

The global left G symmetry $g(t, x) \mapsto g_L g(t, x)$ is broken by this boundary condition to the remnant $g_0 D g_0^{-1}$ (which is gauge invariant, as it should be), and with the non-local charges forms the algebra $Y(\mathfrak{g}, \mathfrak{d})$ identified in [7].

We shall conclude with some examples, but we observe first that two choices of boundary condition are present in all cases. If $\tau = 1$ then $D = G$ and the boundary condition is pure Neumann, while if $\tau = \sigma$ then $D = H$ and the boundary condition is pure Dirichlet.

4 Example: $SO(3)/SO(2)$

We first construct $SO(3)/SO(2) = S^2$ as the subspace $\{g \in SO(3) | \sigma(g) = g^{-1}\}$. We choose $H = \{M_{\mathbf{v}}(\theta) | 0 \leq \theta < 2\pi\}$, where we denote by $M_{\mathbf{n}}(\theta)$ the rotation through angle θ about axis \mathbf{n} , here chosen to be some fixed \mathbf{v} . Then $SO(3)/SO(2) = \{U \in SO(3) | MUM = U^{-1}\}$ where $\sigma(U) = MUM$, for $M = M_{\mathbf{v}}(\pi)$. Thus $(MU)^2 = 1$, so that $MU = M_{\mathbf{n}}(\pi)$, and we have

$$SO(3)/SO(2) = \{MM_{\mathbf{n}}(\pi) | \mathbf{n} \cdot \mathbf{n} = 1\}. \quad (27)$$

For a mixed boundary condition we choose D to be any $SO(2) \subset SO(3)$, which may be written

$$D = \{P^{-1}M_{\mathbf{v}}(\theta)P | 0 \leq \theta < 2\pi\},$$

and set $g_0 = QP$, so that both P and Q are arbitrary rotations. Then $\tau(U) = (P^{-1}MP)U(P^{-1}MP)$ and

$$\tilde{D} = \{\sigma(g_0 l)l^{-1}g_0^{-1} | l \in D\} = \{MQM_{\mathbf{v}}(\theta)PMP^{-1}M_{\mathbf{v}}^{-1}(\theta)Q^{-1} | 0 \leq \theta < 2\pi\}.$$

But $MQM_{\mathbf{v}}(\theta)PMP^{-1}M_{\mathbf{v}}^{-1}(\theta)Q^{-1}$ is conjugate to M , hence squares to one, and hence is a rotation through π ; call it $M_{\mathbf{n}_\theta}(\pi)$, where

$$QM_{\mathbf{v}}(\theta)P \quad : \quad \mathbf{v} \mapsto \mathbf{n}_\theta.$$

⁴Under gauge transformations $g(t, x) \mapsto g(t, x)h(x)$, and in particular $g(t, 0) \mapsto g(t, 0)h(0)$. It is convenient to achieve this by requiring

$$g_0 \mapsto g_0 h(0), \quad l(t) \mapsto h^{-1}(0)l(t)h(0). \quad (26)$$

Then $D \mapsto h(0)^{-1}Dh(0)$ and $\mathfrak{d} \mapsto h(0)^{-1}\mathfrak{d}h(0)$. Note that the requirement that the symmetry conditions (16) and (17) be preserved restricts the allowed gauge transformations, and in particular forces $h^{-1}\partial_1 h|_{x=0}$ to vanish. Thus *at the boundary* the connection $j_1^{\hat{a}}$ does transform covariantly, so there is no inconsistency.

The \mathbf{n}_θ form (any) $S^1 \subset S^2$, but the requirement that τ commutes with σ fixes it to be a great circle – that is, P is a rotation through $\pi/2$.

This result should be compared with the work of Corrigan and Sheng [1]. They find a Lax-pair description (and hence conserved, non-local charges) for the $SO(3)/SO(2)$ model with any S^1 as the boundary Dirichlet submanifold, but explicitly leave open the question of whether these charges are in involution. We have found that amongst the circles only the great circles give boundary conditions compatible with the Poisson bracket structure inherited from the model on the whole line, and then, from the existence of local conserved charges in involution, that these BCs are integrable. This accords with the results of Zhao and He [9], who find that there is no consistent set of Poisson brackets for the model with a general circle as the Dirichlet submanifold (their ‘MD’ condition). The case $G = SO(N)$ was also studied in [2, 3, 9].

5 Example: the Principal Chiral Model

The principal chiral model may be regarded as the sigma model on $G \times G/G$, under the involution

$$\sigma : (g_L, g_R) \mapsto (g_R, g_L),$$

with fixed point $\{(g, g) | g \in G\}$. Then the Cartan immersion is

$$\frac{G \times G}{G} = \{\sigma(n, m)(n, m)^{-1} | n, m \in G\} = \{(mn^{-1}, (mn^{-1})^{-1}) | n, m \in G\} = \{(g, g^{-1}) | g \in G\}.$$

For the boundary conditions of the previous section, we require the involutions τ which commute with σ , and write them in terms of some non-trivial involution α of G , with invariant subgroup H_α . In each case we give the Dirichlet submanifold \tilde{D} in the gauge-invariant formulation, and thereby the submanifold \tilde{D}_G in the usual, ungauged formulation of the principal chiral model, on G .

- (i) $\tau = (1, 1)$: a pure Neumann condition, $\tilde{D} = G \times G/G$, and $\tilde{D}_G = G$.
- (ii) $\tau = \sigma$: a pure Dirichlet condition, $\tilde{D} = \{(e_G, e_G)\}$ and $\tilde{D}_G = \{e_G\}$, realized on the currents $j_\mu^L = \partial_\mu g g^{-1}$ and $j_\mu^R = -g^{-1} \partial_\mu g$ as $j_1^L = -j_1^R$.
- (iii) $\tau = (\alpha, \alpha)$: then $D = H_\alpha \times H_\alpha$, and

$$\tilde{D} = \{\sigma(g_R l_1, g_L l_2)(g_R l_1, g_L l_2)^{-1} | (l_1, l_2) \in H_\tau\} = \{(g_L h g_R^{-1}, (g_L h g_R^{-1})^{-1}) | h = l_2 l_1^{-1} \in H_\alpha\},$$

and so $\tilde{D}_G = g_L H_\alpha g_R^{-1} \subset G$ in the usual formulation.

(iv) $\tau = \sigma(\alpha, \alpha)$: then

$$D = \{(s, t) \in G \times G | (\alpha(t), \alpha(s)) = (s, t)\} = \{(g, \alpha(g)) | g \in G\},$$

while

$$\tilde{D} = \{\sigma(g_R g, g_L \alpha(g))(g_R g, g_L \alpha(g))^{-1} | g \in G\} = \{(g_L k g_R^{-1}, (g_L k g_R^{-1})^{-1}) | k = \alpha(g) g^{-1} \in G/H_\alpha \hookrightarrow G\},$$

so $\tilde{D}_G = g_L \frac{G}{H_\alpha} g_R^{-1} \subset G$. These results agree with those of [4], except that the pure Dirichlet condition was there incorrectly identified as being non-integrable.

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