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Generalized sine-Gordon models and quantum braided groups

F. Delduc,[a] M. Magro[a] and B. Vicedo[b]

[a] *Laboratoire de Physique, ENS Lyon et CNRS UMR 5672, Université de Lyon 46, allée d'Italie, 69364 LYON Cedex 07, France*
[b] *School of Physics, Astronomy and Mathematics, University of Hertfordshire, College Lane, Hatfield AL10 9AB, U.K.*

E-mail: Francois.Delduc@ens-lyon.fr, Marc.Magro@ens-lyon.fr, Benoit.Vicedo@gmail.com

Abstract: We determine the quantized function algebras associated with various examples of generalized sine-Gordon models. These are quadratic algebras of the general Freidel-Maillet type, the classical limits of which reproduce the lattice Poisson algebra recently obtained for these models defined by a gauged Wess-Zumino-Witten action plus an integrable potential. More specifically, we argue based on these examples that the natural framework for constructing quantum lattice integrable versions of generalized sine-Gordon models is that of affine quantum braided groups.

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Contents

1 Introduction

The (semi-)symmetric space sine-Gordon models constitute a broad class of generalizations of the sine-Gordon model. They may be obtained through Pohlmeyer reduction of (semi-)symmetric space σ-models [1, 2] (see [3] for a review). Their Lagrangian formulation corresponds to (a fermionic extension of) a gauged WZW model with an integrable potential [2, 4]. Like the sine-Gordon model itself, all these models are classically integrable. However, a key difference between them and the sine-Gordon model is that the Poisson algebra satisfied by their Lax matrix is non-ultralocal. Yet a remarkable feature of this particular non-ultralocal Poisson algebra, recently computed in [5–7], is that it admits an integrable lattice discretization. This promising result suggests that one may be able to define quantum integrable lattice versions of generalized sine-Gordon models.

This hope of being able to construct quantum lattice models for a whole class of non-ultralocal integrable models is an entirely new prospect in the study of non-ultralocality. As indicated above, a first step towards this goal came from the determination of the lattice Poisson algebra for generalized sine-Gordon models. In the present article we take a further step in this direction by quantizing the lattice Poisson algebra obtained in [5–7]. More precisely, we determine the quantized function algebra associated with different examples of generalized sine-Gordon models.

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Before indicating the plan of this article, let us recall that the simplest generalization of the sine-Gordon model, which is also taken as the first example in the present work, corresponds to the complex sine-Gordon model [8–10]. In the continuum theory, the Poisson algebra satisfied by its Lax matrix was computed in [11]. It does not satisfy the criteria which enable the construction of a corresponding lattice Poisson algebra. However, as recalled above, the situation is quite different if one views the complex sine-Gordon model as defined by a SU(2)/U(1) gauged WZW action plus an integrable potential. This is the standpoint taken in this article. Note that there are also indications [12, 13] within factorized scattering theory that the proper definition of the quantum complex sine-Gordon model is at the level of a gauged WZW model.

The content of this article is divided in two parts. The first one, which corresponds to sections 2 and 3, deals with general results. Examples are then presented in the second part, comprised of sections 4 and 5.

The first part begins with a brief review of the results obtained in [5–7]. The Poisson algebra satisfied by the continuum Lax matrix of (semi-)symmetric space sine-Gordon models is recalled in section 2.1. The corresponding lattice Poisson algebra is then given in section 2.2. It forms the starting point of the analysis carried out in the rest of the article. This lattice Poisson algebra is of the quadratic *abcd*-type [14, 15] and depends on four matrices a, b, c and d. These satisfy a number of properties which include those required to ensure antisymmetry of the corresponding Poisson bracket, the Jacobi identity and finally the existence of infinitely many commuting quantities.

The general analysis for the quantum case is performed in section 3. To quantize the lattice Poisson algebra from section 2, we search for a quantum lattice algebra of the general quadratic $\mathcal{ABCD}$-type [14, 15]. As usual, the matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ should tend to the identity in the classical limit $\hbar \to 0$ and reproduce the matrices a, b, c and d, respectively, at the next order. We give a list of natural conditions on $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ which reduce in the classical limit to those satisfied by a, b, c and d in section 2.2. Among these are the conditions required in the general construction of [14, 15]. Taken altogether, these properties lead to the more refined structure of an affine quantum braided group [16, 17], as explained in section 3.2.

Concerning the second part, section 4 is devoted to examples of symmetric space sine-Gordon models. The first model considered is the complex sine-Gordon model. We then go on to consider models related to the affine Lie algebras $A_2^{(1)}$ and $A_2^{(2)}$. They correspond to the Pohlmeyer reduction of the $\mathbb{C}P^2$ and SU(3)/SO(3) σ-models, respectively. In section 5, we initiate the analysis for the $AdS_5 \times S^5$ semi-symmetric space sine-Gordon model [2, 18] by considering the case of the twisted affine loop algebra of $\mathfrak{gl}(4|4)$.

2 Quadratic Poisson algebra

2.1 Poisson algebra in the continuum

As mentioned in the introduction, symmetric space sine-Gordon models are obtained by Pohlmeyer reduction of σ-models on symmetric spaces F/G. We start this section by recalling the classical integrable structure of the resulting gauged WZW models with an

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integrable potential. We then indicate the generalization to semi-symmetric space sine-Gordon models. This section is based on the results in [5–7], to which the reader is referred for more details.

Let $\mathfrak{f} = \text{Lie}(F)$ be a Lie algebra equipped with a $\mathbb{Z}_2$-automorphism $\sigma : \mathfrak{f} \to \mathfrak{f}$, namely such that $\sigma^2 = \text{id}$, and let $\mathfrak{g} = \mathfrak{f}^{(0)} = \{x \in \mathfrak{f} \,|\, \sigma(x) = x\}$ and $\mathfrak{f}^{(1)}$ denote the eigenspaces of σ with eigenvalue ± 1. The phase space of the theory is parametrized by a pair of fields g and A taking values in G and $\mathfrak{g} = \text{Lie}(G)$ respectively, with Poisson brackets [5, 19]

$$\{g_{\underline{\mathbf{1}}}(\sigma), g_{\underline{\mathbf{2}}}(\sigma')\} = 0, \tag{2.1a}$$

$$\{g_{\underline{\mathbf{1}}}(\sigma), A_{\underline{\mathbf{2}}}(\sigma')\} = -2 g_{\underline{\mathbf{1}}}(\sigma) C^{(00)}_{\underline{\mathbf{12}}} \delta_{\sigma\sigma'}, \tag{2.1b}$$

$$\{A_{\underline{\mathbf{1}}}(\sigma), A_{\underline{\mathbf{2}}}(\sigma')\} = -2\big[C^{(00)}_{\underline{\mathbf{12}}}, A_{\underline{\mathbf{2}}}(\sigma)\big]\delta_{\sigma\sigma'} + 2C^{(00)}_{\underline{\mathbf{12}}} \partial_\sigma \delta_{\sigma\sigma'}. \tag{2.1c}$$

We denote by $C^{(00)} + C^{(11)}$ the decomposition of the tensor Casimir C of $\mathfrak{f}$ with respect to the $\mathbb{Z}_2$-grading induced by the involution σ. The Lax matrix is given by

$$\mathcal{L}(\sigma, \lambda) = A(\sigma) + \frac{1}{2}\lambda^{-1}\mu_- g^{-1}(\sigma) T_- g(\sigma) - \frac{1}{2}\lambda \mu_+ T_+, \tag{2.2}$$

where $T_\pm \in \mathfrak{f}^{(1)}$ and $\mu_\pm \in \mathbb{R}$ are constants. It takes values in the twisted polynomial loop algebra $\widehat{\mathfrak{f}}^\sigma \subset \mathfrak{f}[\lambda, \lambda^{-1}]$ of $\mathfrak{f}$. The Lax matrix (2.2) satisfies the non-ultralocal Poisson algebra

$$\begin{aligned}\{\mathcal{L}_{\underline{\mathbf{1}}}(\sigma, \lambda), \mathcal{L}_{\underline{\mathbf{2}}}(\sigma', \mu)\} &= \big[r_{\underline{\mathbf{12}}}(\lambda/\mu), \mathcal{L}_{\underline{\mathbf{1}}}(\sigma, \lambda) + \mathcal{L}_{\underline{\mathbf{2}}}(\sigma, \mu)\big]\delta_{\sigma\sigma'} \\ &\quad + \big[s_{\underline{\mathbf{12}}}, \mathcal{L}_{\underline{\mathbf{1}}}(\sigma, \lambda) - \mathcal{L}_{\underline{\mathbf{2}}}(\sigma, \mu)\big]\delta_{\sigma\sigma'} + 2 s_{\underline{\mathbf{12}}} \partial_\sigma \delta_{\sigma\sigma'},\end{aligned} \tag{2.3}$$

where the matrices r and s explicitly read

$$r_{\underline{\mathbf{12}}}(\lambda) = \frac{1+\lambda^2}{1-\lambda^2} C^{(00)}_{\underline{\mathbf{12}}} + \frac{2\lambda}{1-\lambda^2} C^{(11)}_{\underline{\mathbf{12}}}, \qquad s_{\underline{\mathbf{12}}} = C^{(00)}_{\underline{\mathbf{12}}}. \tag{2.4}$$

The sum $r + s$ of the matrices in (2.4) is a non-skew-symmetric solution of the modified classical Yang-Baxter equation (mCYBE) on $\widehat{\mathfrak{f}}^\sigma$, which underlies the integrable structure of the model [20].

In the case of a semi-symmetric space sine-Gordon model such as the one associated with $AdS_5 \times S^5$ [2, 18], the involutive automorphism σ is replaced by a $\mathbb{Z}_4$-automorphism with respect to which the Casimir decomposes as $C = C^{(00)} + C^{(13)} + C^{(22)} + C^{(31)}$. The Poisson brackets (2.1) have to be supplemented with the Poisson brackets of the fermionic fields. The corresponding Lax matrix, whose expression may be found in [6], also satisfies the algebra (2.3) but where now

$$r_{\underline{\mathbf{12}}}(\lambda) = \frac{1+\lambda^4}{1-\lambda^4} C^{(00)}_{\underline{\mathbf{12}}} + \frac{2\lambda}{1-\lambda^4} C^{(13)}_{\underline{\mathbf{12}}} + \frac{2\lambda^2}{1-\lambda^4} C^{(22)}_{\underline{\mathbf{12}}} + \frac{2\lambda^3}{1-\lambda^4} C^{(31)}_{\underline{\mathbf{12}}}, \qquad s_{\underline{\mathbf{12}}} = C^{(00)}_{\underline{\mathbf{12}}}. \tag{2.5}$$

The fact that the matrix $s_{\underline{\mathbf{12}}}$ associated with the (semi-)symmetric space sine-Gordon models is simply the projection onto the subalgebra $\mathfrak{g}$ of constant loops in $\widehat{\mathfrak{f}}^\sigma$ is crucial. Indeed, it enables to define a lattice discretization of the Poisson algebra (2.3). Furthermore, as we will see, this has important consequences for the quantum case.

2.2 Lattice Poisson algebra

The lattice Poisson algebra corresponding to (2.3) in the continuum limit is

$$\{\mathcal{L}^n_{\underline{\mathbf{1}}}(\lambda), \mathcal{L}^n_{\underline{\mathbf{2}}}(\mu)\} = a_{\underline{\mathbf{12}}}(\lambda/\mu)\mathcal{L}^n_{\underline{\mathbf{1}}}(\lambda)\mathcal{L}^n_{\underline{\mathbf{2}}}(\mu) - \mathcal{L}^n_{\underline{\mathbf{1}}}(\lambda)\mathcal{L}^n_{\underline{\mathbf{2}}}(\mu)d_{\underline{\mathbf{12}}}(\lambda/\mu), \tag{2.6a}$$

$$\{\mathcal{L}^n_{\underline{\mathbf{1}}}(\lambda), \mathcal{L}^{n+1}_{\underline{\mathbf{2}}}(\mu)\} = -\mathcal{L}^{n+1}_{\underline{\mathbf{2}}}(\mu)c_{\underline{\mathbf{12}}}\mathcal{L}^n_{\underline{\mathbf{1}}}(\lambda), \tag{2.6b}$$

$$\{\mathcal{L}^{n+1}_{\underline{\mathbf{1}}}(\lambda), \mathcal{L}^n_{\underline{\mathbf{2}}}(\mu)\} = \mathcal{L}^{n+1}_{\underline{\mathbf{1}}}(\lambda)b_{\underline{\mathbf{12}}}\mathcal{L}^n_{\underline{\mathbf{2}}}(\mu), \tag{2.6c}$$

$$\{\mathcal{L}^n_{\underline{\mathbf{1}}}(\lambda), \mathcal{L}^m_{\underline{\mathbf{2}}}(\mu)\} = 0, \qquad |n-m| \geq 2, \tag{2.6d}$$

where the lattice Lax matrix $\mathcal{L}^n$ encodes the physical degrees of freedom at the n^{th} site of the lattice. On the lattice, the property of non-ultralocality is encoded in the Poisson brackets (2.6b) and (2.6c) which express the fact that Lax matrices at adjacent sites n and $n+1$ do not Poisson commute. The Poisson algebra (2.6) fits into the general scheme of quadratic $abcd$-algebras considered in [14, 15]. However, in the present case, the four matrices a, b, c and d are expressed in terms of the matrices r and s given in (2.4) or (2.5) together with [21] a skew-symmetric solution α of the mCYBE on $\mathfrak{g}$ as follows

$$a(\lambda) = r(\lambda) + \alpha, \qquad b = -s - \alpha, \qquad c = -s + \alpha, \qquad d(\lambda) = r(\lambda) - \alpha. \tag{2.7}$$

In particular, b and c do not depend on the spectral parameter. By virtue of their explicit expressions (2.7), the matrices a, b, c and d satisfy the following properties:

- The first set of properties ensures that equations (2.6) define a Poisson bracket. They are

$$a_{\underline{\mathbf{12}}}(\lambda) = -a_{\underline{\mathbf{21}}}(\lambda^{-1}), \qquad d_{\underline{\mathbf{12}}}(\lambda) = -d_{\underline{\mathbf{21}}}(\lambda^{-1}), \qquad b_{\underline{\mathbf{12}}} = c_{\underline{\mathbf{21}}} \tag{2.8}$$

for the antisymmetry and

$$[a_{\underline{\mathbf{12}}}(\lambda/\mu), a_{\underline{\mathbf{13}}}(\lambda)] + [a_{\underline{\mathbf{12}}}(\lambda/\mu), a_{\underline{\mathbf{23}}}(\mu)] + [a_{\underline{\mathbf{13}}}(\lambda), a_{\underline{\mathbf{23}}}(\mu)] = 0, \tag{2.9a}$$

$$[d_{\underline{\mathbf{12}}}(\lambda/\mu), d_{\underline{\mathbf{13}}}(\lambda)] + [d_{\underline{\mathbf{12}}}(\lambda/\mu), d_{\underline{\mathbf{23}}}(\mu)] + [d_{\underline{\mathbf{13}}}(\lambda), d_{\underline{\mathbf{23}}}(\mu)] = 0, \tag{2.9b}$$

$$[a_{\underline{\mathbf{12}}}(\lambda), c_{\underline{\mathbf{13}}}] + [a_{\underline{\mathbf{12}}}(\lambda), c_{\underline{\mathbf{23}}}] + [c_{\underline{\mathbf{13}}}, c_{\underline{\mathbf{23}}}] = 0, \tag{2.9c}$$

$$[d_{\underline{\mathbf{12}}}(\lambda), b_{\underline{\mathbf{13}}}] + [d_{\underline{\mathbf{12}}}(\lambda), b_{\underline{\mathbf{23}}}] + [b_{\underline{\mathbf{13}}}, b_{\underline{\mathbf{23}}}] = 0 \tag{2.9d}$$

for the Jacobi identity.

- An additional property of the matrices b and c, which is not required in the general formalism of [14, 15], is that they are themselves solutions of the classical Yang-Baxter equation

$$[b_{\underline{\mathbf{12}}}, b_{\underline{\mathbf{13}}}] + [b_{\underline{\mathbf{12}}}, b_{\underline{\mathbf{23}}}] + [b_{\underline{\mathbf{13}}}, b_{\underline{\mathbf{23}}}] = 0, \tag{2.10a}$$

$$[c_{\underline{\mathbf{12}}}, c_{\underline{\mathbf{13}}}] + [c_{\underline{\mathbf{12}}}, c_{\underline{\mathbf{23}}}] + [c_{\underline{\mathbf{13}}}, c_{\underline{\mathbf{23}}}] = 0. \tag{2.10b}$$

This is a consequence of the facts that α is a solution of the mCYBE on $\mathfrak{g}$ and that s identifies with the Casimir on $\mathfrak{g}$.

- Another important property which ensures that the algebra (2.6) leads to the existence of an infinite family of commuting integrals of motion reads

$$a(\lambda) + b = c + d(\lambda). \tag{2.11}$$

Indeed, introducing the monodromy $T = \mathcal{L}^N \dots \mathcal{L}^1$, its Poisson bracket can be derived from the local lattice Poisson algebra (2.6) using the relation (2.11) and reads

$$\{T_{\underline{1}}(\lambda), T_{\underline{2}}(\mu)\} = a_{\underline{12}}(\lambda/\mu) T_{\underline{1}}(\lambda) T_{\underline{2}}(\mu) + T_{\underline{1}}(\lambda) b_{\underline{12}} T_{\underline{2}}(\mu) \\ - T_{\underline{2}}(\mu) c_{\underline{12}} T_{\underline{1}}(\lambda) - T_{\underline{2}}(\mu) T_{\underline{1}}(\lambda) d_{\underline{12}}(\lambda/\mu).$$

It then immediately follows using (2.11) once more that the quantities $\mathrm{tr}\,\big(T^p(\lambda)\big)$ Poisson commute.

- Finally, the matrices r and s, in (2.4) as well as (2.5), are related by $\lim_{\lambda\to 0} r(\lambda) = s$ and $\lim_{\lambda\to\infty} r(\lambda) = -s$. An immediate consequence of this is that

$$\lim_{\lambda\to 0} a(\lambda) = -b, \qquad \lim_{\lambda\to\infty} a(\lambda) = c, \tag{2.12a}$$

$$\lim_{\lambda\to 0} d(\lambda) = -c, \qquad \lim_{\lambda\to\infty} d(\lambda) = b. \tag{2.12b}$$

3 Quantum lattice algebra

3.1 Quadratic algebra

On general grounds, the quantum lattice algebra, whose classical limit corresponds to the Poisson algebra (2.6), should be of the following form [14, 15]

$$\mathcal{A}_{\underline{12}}(q, \lambda/\mu) \hat{\mathcal{L}}^n_{\underline{1}}(\lambda) \hat{\mathcal{L}}^n_{\underline{2}}(\mu) = \hat{\mathcal{L}}^n_{\underline{2}}(\mu) \hat{\mathcal{L}}^n_{\underline{1}}(\lambda) \mathcal{D}_{\underline{12}}(q, \lambda/\mu), \tag{3.1a}$$

$$\hat{\mathcal{L}}^n_{\underline{1}}(\lambda) \hat{\mathcal{L}}^{n+1}_{\underline{2}}(\mu) = \hat{\mathcal{L}}^{n+1}_{\underline{2}}(\mu) \mathcal{C}_{\underline{12}}(q) \hat{\mathcal{L}}^n_{\underline{1}}(\lambda), \tag{3.1b}$$

$$\hat{\mathcal{L}}^{n+1}_{\underline{1}}(\lambda) \mathcal{B}_{\underline{12}}(q) \hat{\mathcal{L}}^n_{\underline{2}}(\mu) = \hat{\mathcal{L}}^n_{\underline{2}}(\mu) \hat{\mathcal{L}}^{n+1}_{\underline{1}}(\lambda), \tag{3.1c}$$

$$\hat{\mathcal{L}}^n_{\underline{1}}(\lambda) \hat{\mathcal{L}}^m_{\underline{2}}(\mu) = \hat{\mathcal{L}}^m_{\underline{2}}(\mu) \hat{\mathcal{L}}^n_{\underline{1}}(\lambda), \qquad |n-m| \geq 2, \tag{3.1d}$$

where $\hat{\mathcal{L}}^n = \mathcal{L}^n + O(\hbar)$ denotes the quantum lattice Lax matrix which encodes the physical degrees of freedom at the n^{th} site of the lattice. As usual, $q = e^{i\hbar}$ and the classical limit corresponds to $\hbar \to 0$. In particular, one has in this limit

$$\mathcal{A}_{\underline{12}}(e^{i\hbar}, \lambda) = 1 + i\hbar\, a_{\underline{12}}(\lambda) + O(\hbar^2), \qquad \mathcal{B}_{\underline{12}}(e^{i\hbar}) = 1 + i\hbar\, b_{\underline{12}} + O(\hbar^2), \tag{3.2a}$$

$$\mathcal{C}_{\underline{12}}(e^{i\hbar}) = 1 + i\hbar\, c_{\underline{12}} + O(\hbar^2), \qquad \mathcal{D}_{\underline{12}}(e^{i\hbar}, \lambda) = 1 + i\hbar\, d_{\underline{12}}(\lambda) + O(\hbar^2). \tag{3.2b}$$

Besides having the correct classical limits, the quantum matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ satisfy certain further properties which can be considered as the quantum analogs of those given in the previous section for a, b, c and d. Most of these properties ensure that the algebra (3.1) is well defined and leads to the existence of infinitely many commuting integrals of motion. The remaining conditions are very natural from a mathematical point of view. The full list of properties satisfied by the matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ of sections 4 and 5 is as follows:

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- The first set of properties arises from considerations of the consistency of the algebra (3.1). By exchanging the tensor indices $\underline{\mathbf{1}} \leftrightarrow \underline{\mathbf{2}}$ and the spectral parameters $\lambda \leftrightarrow \mu$ in equations (3.1), they may be rewritten as

$$\begin{aligned}
\mathcal{A}_{\underline{\mathbf{21}}}(q,\mu/\lambda)^{-1}\hat{\mathcal{L}}^n_{\underline{\mathbf{1}}}(\lambda)\hat{\mathcal{L}}^n_{\underline{\mathbf{2}}}(\mu) &= \hat{\mathcal{L}}^n_{\underline{\mathbf{2}}}(\mu)\hat{\mathcal{L}}^n_{\underline{\mathbf{1}}}(\lambda)\mathcal{D}_{\underline{\mathbf{21}}}(q,\mu/\lambda)^{-1},\\
\hat{\mathcal{L}}^n_{\underline{\mathbf{1}}}(\lambda)\hat{\mathcal{L}}^{n+1}_{\underline{\mathbf{2}}}(\mu) &= \hat{\mathcal{L}}^{n+1}_{\underline{\mathbf{2}}}(\mu)\mathcal{B}_{\underline{\mathbf{21}}}(q)\hat{\mathcal{L}}^n_{\underline{\mathbf{1}}}(\lambda),\\
\hat{\mathcal{L}}^{n+1}_{\underline{\mathbf{1}}}(\lambda)\mathcal{C}_{\underline{\mathbf{21}}}(q)\hat{\mathcal{L}}^n_{\underline{\mathbf{2}}}(\mu) &= \hat{\mathcal{L}}^n_{\underline{\mathbf{2}}}(\mu)\hat{\mathcal{L}}^{n+1}_{\underline{\mathbf{1}}}(\lambda),\\
\hat{\mathcal{L}}^n_{\underline{\mathbf{1}}}(\lambda)\hat{\mathcal{L}}^m_{\underline{\mathbf{2}}}(\mu) &= \hat{\mathcal{L}}^m_{\underline{\mathbf{2}}}(\mu)\hat{\mathcal{L}}^n_{\underline{\mathbf{1}}}(\lambda), \qquad |n-m| \geq 2.
\end{aligned}$$

  Therefore, to guarantee that these latter equations do not impose any new relations on the quantum lattice Lax matrix $\hat{\mathcal{L}}^n$, we should require that

$$\mathcal{A}_{\underline{\mathbf{12}}}(q,\lambda)\mathcal{A}_{\underline{\mathbf{21}}}(q,\lambda^{-1}) = \mathcal{D}_{\underline{\mathbf{12}}}(q,\lambda)\mathcal{D}_{\underline{\mathbf{21}}}(q,\lambda^{-1}) \propto 1, \tag{3.3a}$$
$$\mathcal{C}_{\underline{\mathbf{12}}}(q) = \mathcal{B}_{\underline{\mathbf{21}}}(q). \tag{3.3b}$$

  These are the quantum counterparts of the classical properties (2.8).

  On the other hand, sufficient conditions for the consistency of the algebra (3.1) read [14, 15]

$$\mathcal{A}_{\underline{\mathbf{12}}}(q,\lambda/\mu)\mathcal{A}_{\underline{\mathbf{13}}}(q,\lambda)\mathcal{A}_{\underline{\mathbf{23}}}(q,\mu) = \mathcal{A}_{\underline{\mathbf{23}}}(q,\mu)\mathcal{A}_{\underline{\mathbf{13}}}(q,\lambda)\mathcal{A}_{\underline{\mathbf{12}}}(q,\lambda/\mu), \tag{3.4a}$$
$$\mathcal{D}_{\underline{\mathbf{12}}}(q,\lambda/\mu)\mathcal{D}_{\underline{\mathbf{13}}}(q,\lambda)\mathcal{D}_{\underline{\mathbf{23}}}(q,\mu) = \mathcal{D}_{\underline{\mathbf{23}}}(q,\mu)\mathcal{D}_{\underline{\mathbf{13}}}(q,\lambda)\mathcal{D}_{\underline{\mathbf{12}}}(q,\lambda/\mu), \tag{3.4b}$$
$$\mathcal{A}_{\underline{\mathbf{12}}}(q,\lambda)\mathcal{C}_{\underline{\mathbf{13}}}(q)\mathcal{C}_{\underline{\mathbf{23}}}(q) = \mathcal{C}_{\underline{\mathbf{23}}}(q)\mathcal{C}_{\underline{\mathbf{13}}}(q)\mathcal{A}_{\underline{\mathbf{12}}}(q,\lambda), \tag{3.4c}$$
$$\mathcal{D}_{\underline{\mathbf{12}}}(q,\lambda)\mathcal{B}_{\underline{\mathbf{13}}}(q)\mathcal{B}_{\underline{\mathbf{23}}}(q) = \mathcal{B}_{\underline{\mathbf{23}}}(q)\mathcal{B}_{\underline{\mathbf{13}}}(q)\mathcal{D}_{\underline{\mathbf{12}}}(q,\lambda), \tag{3.4d}$$

  which constitute the quantum analogs of equations (2.9).

- Since the classical matrices b and c satisfy the CYBE (2.10), it is natural to seek matrices $\mathcal{B}(q)$ and $\mathcal{C}(q)$ which are themselves solutions of the quantum Yang-Baxter equation (QYBE). We shall therefore impose the following further conditions on these matrices

$$\mathcal{B}_{\underline{\mathbf{12}}}(q)\mathcal{B}_{\underline{\mathbf{13}}}(q)\mathcal{B}_{\underline{\mathbf{23}}}(q) = \mathcal{B}_{\underline{\mathbf{23}}}(q)\mathcal{B}_{\underline{\mathbf{13}}}(q)\mathcal{B}_{\underline{\mathbf{12}}}(q), \tag{3.5a}$$
$$\mathcal{C}_{\underline{\mathbf{12}}}(q)\mathcal{C}_{\underline{\mathbf{13}}}(q)\mathcal{C}_{\underline{\mathbf{23}}}(q) = \mathcal{C}_{\underline{\mathbf{23}}}(q)\mathcal{C}_{\underline{\mathbf{13}}}(q)\mathcal{C}_{\underline{\mathbf{12}}}(q). \tag{3.5b}$$

  Although these properties are not required in the general formalism of [14, 15] for quadratic quantum lattice algebras of the type (3.1), they will play a very important role for us in underpinning the algebraic structure underlying the integrable models considered.

- Another property which plays a central role in the interpretation of the algebra (3.1), to be described shortly, and which we shall require our set of four quantum R-matrices $\mathcal{A}(q,\lambda)$, $\mathcal{B}(q)$, $\mathcal{C}(q)$ and $\mathcal{D}(q,\lambda)$ to satisfy is

$$\mathcal{A}(q,\lambda)\mathcal{B}(q) = \mathcal{C}(q)\mathcal{D}(q,\lambda). \tag{3.6}$$

Even though the classical limit of this equation is equivalent to the classical property (2.11), it is not the appropriate quantum generalization of the latter.

Instead, the correct quantum analog of (2.11) is the existence of a numerical matrix $\gamma(q)$ satisfying the following relation

$$\gamma_{\underline{\mathbf{2}}}(q)\mathcal{B}_{\underline{\mathbf{12}}}(q)\gamma_{\underline{\mathbf{1}}}(q)\mathcal{A}_{\underline{\mathbf{12}}}(q,\lambda) = \mathcal{D}_{\underline{\mathbf{12}}}(q,\lambda)\gamma_{\underline{\mathbf{1}}}(q)\mathcal{C}_{\underline{\mathbf{12}}}(q)\gamma_{\underline{\mathbf{2}}}(q). \tag{3.7}$$

In order for this equation to reduce to (2.11) in the classical limit, the matrix $\gamma(q)$ should be such that it tends to the identity matrix as $q \to 1$. The property (3.7) is essential to ensure the passage from the local commutation relations (3.1) to the global commutation relation [14, 15]

$$\mathcal{A}_{\underline{\mathbf{12}}}(q,\lambda/\mu)\hat{T}_{\underline{\mathbf{1}}}(\lambda)\mathcal{B}_{\underline{\mathbf{12}}}(q)\hat{T}_{\underline{\mathbf{2}}}(\mu) = \hat{T}_{\underline{\mathbf{2}}}(\mu)\mathcal{C}_{\underline{\mathbf{12}}}(q)\hat{T}_{\underline{\mathbf{1}}}(\lambda)\mathcal{D}_{\underline{\mathbf{12}}}(q,\lambda/\mu) \tag{3.8}$$

for the quantum monodromy defined as $\hat{T}(\lambda) = \hat{\mathcal{L}}^N(\lambda)\gamma(q)\hat{\mathcal{L}}^{N-1}(\lambda)\gamma(q)\ldots\gamma(q)\hat{\mathcal{L}}^1(\lambda)$. It is in this sense that the relation (3.7) is the quantum analog of the classical property (2.11).

With the monodromy matrix so defined and satisfying the quadratic algebra (3.8), the property which ultimately guarantees the existence of an infinite family of commuting operators is the existence of another numerical matrix $\widetilde{\gamma}(q)$ such that

$$\widetilde{\mathcal{A}}_{\underline{\mathbf{12}}}(q,\lambda)\widetilde{\gamma}_{\underline{\mathbf{1}}}(q)\widetilde{\mathcal{B}}_{\underline{\mathbf{12}}}(q)\widetilde{\gamma}_{\underline{\mathbf{2}}}(q) = \widetilde{\gamma}_{\underline{\mathbf{2}}}(q)\widetilde{\mathcal{C}}_{\underline{\mathbf{12}}}(q)\widetilde{\gamma}_{\underline{\mathbf{1}}}(q)\widetilde{\mathcal{D}}_{\underline{\mathbf{12}}}(q,\lambda) \tag{3.9}$$

where the matrices $\widetilde{\mathcal{A}}$, $\widetilde{\mathcal{B}}$, $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{D}}$ are defined as

$$\widetilde{\mathcal{A}} = (\mathcal{A}^{t_1t_2})^{-1}, \quad \widetilde{\mathcal{B}} = [(\mathcal{B}^{t_1})^{-1}]^{t_2}, \quad \widetilde{\mathcal{C}} = [(\mathcal{C}^{t_2})^{-1}]^{t_1}, \quad \widetilde{\mathcal{D}} = (\mathcal{D}^{t_1t_2})^{-1}.$$

Here x^t denotes the (super-)transpose of x. In every example considered in this article, $\widetilde{\gamma}(q)$ is a diagonal matrix tending to the identity in the limit $q \to 1$ and is therefore also consistent in the classical limit with the relation (2.11).

The global monodromy algebra (3.8) together with the property (3.9) ensure [14, 15] that the operators $\operatorname{tr}\big(\widetilde{\gamma}(q)^t\hat{T}(\lambda)\big)$ commute for different values of the spectral parameter.

- In each example we also have the following relations

$$\lim_{\lambda\to 0}\mathcal{A}(q,\lambda) = \mathcal{B}(q)^{-1}, \qquad \lim_{\lambda\to\infty}\mathcal{A}(q,\lambda) = \mathcal{C}(q),$$
$$\lim_{\lambda\to 0}\mathcal{D}(q,\lambda) = \mathcal{C}(q)^{-1}, \qquad \lim_{\lambda\to\infty}\mathcal{D}(q,\lambda) = \mathcal{B}(q)$$

which are natural quantum analogs of (2.12). Using these relations we observe that (3.4c), (3.4d) and (3.5) can all be obtained as appropriate limits of the QYBE (3.4a) and (3.4b).

3.2 Affine quantum braided group

Given a set of four matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ constructed to satisfy all of the above properties, it turns out that the algebraic structure underlying the quantum integrability of the corresponding quantum model is precisely that of an affine quantum braided group [16, 17].

Indeed, using the relation (3.6) together with (3.3b), the four matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ may be expressed in terms of just two matrices $\mathcal{R}$ and $\mathcal{Z}$ as follows

$$\begin{aligned} \mathcal{A}_{\underline{\mathbf{12}}}(q,\lambda) &= \mathcal{Z}_{\underline{\mathbf{21}}}(q)\mathcal{R}_{\underline{\mathbf{12}}}(q,\lambda)\mathcal{Z}_{\underline{\mathbf{12}}}(q)^{-1}, \qquad & \mathcal{B}_{\underline{\mathbf{12}}}(q) &= \mathcal{Z}_{\underline{\mathbf{12}}}(q), \\ \mathcal{D}_{\underline{\mathbf{12}}}(q,\lambda) &= \mathcal{R}_{\underline{\mathbf{12}}}(q,\lambda), & \mathcal{C}_{\underline{\mathbf{12}}}(q) &= \mathcal{Z}_{\underline{\mathbf{21}}}(q), \end{aligned}$$

The relations in (3.3a) then translate into the single unitarity condition

$$\mathcal{R}_{\underline{\mathbf{12}}}(q,\lambda)\mathcal{R}_{\underline{\mathbf{21}}}(q,\lambda^{-1}) \propto 1. \tag{3.10}$$

Note that there is no unitarity condition on the matrix $\mathcal{Z}$. Moreover, the full set of QYB-type relations (3.4) and (3.5) is equivalent to

$$\mathcal{R}_{\underline{\mathbf{12}}}(q,\lambda/\mu)\mathcal{R}_{\underline{\mathbf{13}}}(q,\lambda)\mathcal{R}_{\underline{\mathbf{23}}}(q,\mu) = \mathcal{R}_{\underline{\mathbf{23}}}(q,\mu)\mathcal{R}_{\underline{\mathbf{13}}}(q,\lambda)\mathcal{R}_{\underline{\mathbf{12}}}(q,\lambda/\mu), \tag{3.11a}$$

$$\mathcal{Z}_{\underline{\mathbf{12}}}(q)\mathcal{Z}_{\underline{\mathbf{13}}}(q)\mathcal{Z}_{\underline{\mathbf{23}}}(q) = \mathcal{Z}_{\underline{\mathbf{23}}}(q)\mathcal{Z}_{\underline{\mathbf{13}}}(q)\mathcal{Z}_{\underline{\mathbf{12}}}(q), \tag{3.11b}$$

$$\mathcal{R}_{\underline{\mathbf{12}}}(q,\lambda)\mathcal{Z}_{\underline{\mathbf{13}}}(q)\mathcal{Z}_{\underline{\mathbf{23}}}(q) = \mathcal{Z}_{\underline{\mathbf{23}}}(q)\mathcal{Z}_{\underline{\mathbf{13}}}(q)\mathcal{R}_{\underline{\mathbf{12}}}(q,\lambda), \tag{3.11c}$$

$$\mathcal{Z}_{\underline{\mathbf{12}}}(q)\mathcal{Z}_{\underline{\mathbf{13}}}(q)\mathcal{R}_{\underline{\mathbf{23}}}(q,\lambda) = \mathcal{R}_{\underline{\mathbf{23}}}(q,\lambda)\mathcal{Z}_{\underline{\mathbf{13}}}(q)\mathcal{Z}_{\underline{\mathbf{12}}}(q). \tag{3.11d}$$

In terms of the matrices $\mathcal{R}$ and $\mathcal{Z}$, the quantum monodromy algebra (3.8) then reads

$$\mathcal{R}_{\underline{\mathbf{12}}}(q,\lambda/\mu)\mathcal{Z}_{\underline{\mathbf{12}}}(q)^{-1}\hat{T}_{\underline{\mathbf{1}}}(\lambda)\mathcal{Z}_{\underline{\mathbf{12}}}(q)\hat{T}_{\underline{\mathbf{2}}}(\mu) = \mathcal{Z}_{\underline{\mathbf{21}}}(q)^{-1}\hat{T}_{\underline{\mathbf{2}}}(\mu)\mathcal{Z}_{\underline{\mathbf{21}}}(q)\hat{T}_{\underline{\mathbf{1}}}(\lambda)\mathcal{R}_{\underline{\mathbf{12}}}(q,\lambda/\mu), \tag{3.12}$$

which is exactly the relation defining an affine quantum braided group as introduced in [16, 17].

In the remaining sections we present the quantum R-matrices $\mathcal{A}(q,\lambda)$, $\mathcal{B}(q)$, $\mathcal{C}(q)$ and $\mathcal{D}(q,\lambda)$ entering the affine quantum braided group (3.12) for various models.

4 Symmetric space sine-Gordon models

4.1 Complex sine-Gordon model

Automorphism. In the setup of section 2, consider the case of the Lie algebra $\mathfrak{f} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and define the $\mathbb{Z}_2$-automorphism $\sigma : \mathfrak{f} \to \mathfrak{f}$ as the flip

$$\sigma(x, y) = (y, x),$$

for any $x, y \in \mathfrak{su}(2)$. The corresponding eigenspaces of σ read

$$\mathfrak{g} = \mathfrak{f}^{(0)} = \{(x,x) \,|\, x \in \mathfrak{su}(2)\}, \qquad \mathfrak{f}^{(1)} = \{(x,-x) \,|\, x \in \mathfrak{su}(2)\}. \tag{4.1}$$

Now introduce the standard basis for $\mathfrak{su}(2)$, namely

$$\mathsf{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathsf{E} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathsf{F} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{4.2}$$

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in terms of which a basis of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ reads

$$\mathsf{H}_1 = (\mathsf{H}, 0), \quad \mathsf{E}_1 = (\mathsf{E}, 0), \quad \mathsf{F}_1 = (\mathsf{F}, 0), \qquad \mathsf{H}_2 = (0, \mathsf{H}), \quad \mathsf{E}_2 = (0, \mathsf{E}), \quad \mathsf{F}_2 = (0, \mathsf{F}).$$

Let us also introduce the block matrices $1_1 = (1, 0)$ and $1_2 = (0, 1)$ where 1 is the 2×2 identity matrix. In terms of the above we may write down a basis of the subspaces $\mathfrak{f}^{(0)}$ and $\mathfrak{f}^{(1)}$ as

$$\begin{aligned} \mathsf{h}^{(0)} &= \mathsf{H}_1 + \mathsf{H}_2, & \mathsf{e}^{(0)} &= \mathsf{E}_1 + \mathsf{E}_2, & \mathsf{f}^{(0)} &= \mathsf{F}_1 + \mathsf{F}_2, \\ \mathsf{h}^{(1)} &= \mathsf{H}_1 - \mathsf{H}_2, & \mathsf{e}^{(1)} &= \mathsf{E}_1 - \mathsf{E}_2, & \mathsf{f}^{(1)} &= \mathsf{F}_1 - \mathsf{F}_2 \end{aligned} \tag{4.3}$$

respectively.

Casimir decomposition. The tensor Casimir is the sum of the tensor Casimirs for each $\mathfrak{su}(2)$, namely

$$C = \frac{1}{2}\mathsf{H}_1 \otimes \mathsf{H}_1 + \mathsf{E}_1 \otimes \mathsf{F}_1 + \mathsf{F}_1 \otimes \mathsf{E}_1 + \frac{1}{2}\mathsf{H}_2 \otimes \mathsf{H}_2 + \mathsf{E}_2 \otimes \mathsf{F}_2 + \mathsf{F}_2 \otimes \mathsf{E}_2,$$

which can be decomposed as $C = C^{(00)} + C^{(11)}$ relative to $\mathfrak{f} = \mathfrak{f}^{(0)} \oplus \mathfrak{f}^{(1)}$, where

$$\begin{aligned} C^{(00)} &= \frac{1}{4}\mathsf{h}^{(0)} \otimes \mathsf{h}^{(0)} + \frac{1}{2}\mathsf{e}^{(0)} \otimes \mathsf{f}^{(0)} + \frac{1}{2}\mathsf{f}^{(0)} \otimes \mathsf{e}^{(0)}, \\ C^{(11)} &= \frac{1}{4}\mathsf{h}^{(1)} \otimes \mathsf{h}^{(1)} + \frac{1}{2}\mathsf{e}^{(1)} \otimes \mathsf{f}^{(1)} + \frac{1}{2}\mathsf{f}^{(1)} \otimes \mathsf{e}^{(1)}. \end{aligned}$$

Classical r-matrices. We let the matrix α appearing in (2.7) be the standard skew-symmetric solution of the mCYBE on $\mathfrak{g} = \mathfrak{su}(2)$, namely

$$\alpha = \frac{1}{2}\big(\mathsf{e}^{(0)} \otimes \mathsf{f}^{(0)} - \mathsf{f}^{(0)} \otimes \mathsf{e}^{(0)}\big).$$

The corresponding non-skew-symmetric solutions b and c of the CYBE defined by (2.7) then read

$$b = -\frac{1}{4}\mathsf{h}^{(0)} \otimes \mathsf{h}^{(0)} - \mathsf{e}^{(0)} \otimes \mathsf{f}^{(0)}, \qquad c = -\frac{1}{4}\mathsf{h}^{(0)} \otimes \mathsf{h}^{(0)} - \mathsf{f}^{(0)} \otimes \mathsf{e}^{(0)}. \tag{4.4}$$

In terms of these, the spectral parameter dependent classical r-matrices in (2.7) may be written as follows

$$a(\lambda) = -\delta(\lambda) b + \big(1 - \delta(\lambda)\big) c, \qquad d(\lambda) = -\delta(\lambda) c + \big(1 - \delta(\lambda)\big) b,$$

where we have introduced the diagonal matrix

$$\delta(\lambda) = \frac{1}{1-\lambda}\big(1_1 \otimes 1_1 + 1_2 \otimes 1_2\big) + \frac{1}{1+\lambda}\big(1_1 \otimes 1_2 + 1_2 \otimes 1_1\big).$$

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Quantum R-matrices. Next we give quantizations of the above classical r-matrices $a(\lambda)$, b, c and $d(\lambda)$. Specifically, these are solutions $\mathcal{A}(q,\lambda)$, $\mathcal{B}(q)$, $\mathcal{C}(q)$ and $\mathcal{D}(q,\lambda)$ of the QYBE with classical limits (3.2) for $q = e^{i\hbar}$. Quantizations of the constant classical r-matrices b and c read

$$\mathcal{B}(q) = q^{-\frac{1}{4}\mathsf{h}^{(0)}\otimes\mathsf{h}^{(0)}} + q^{-\frac{1}{4}}(1-q)\,\mathsf{e}^{(0)}\otimes\mathsf{f}^{(0)}, \tag{4.5a}$$

$$\mathcal{C}(q) = q^{-\frac{1}{4}\mathsf{h}^{(0)}\otimes\mathsf{h}^{(0)}} + q^{-\frac{1}{4}}(1-q)\,\mathsf{f}^{(0)}\otimes\mathsf{e}^{(0)}. \tag{4.5b}$$

In terms of these, quantizations of $a(\lambda)$ and $d(\lambda)$ are respectively given by

$$\mathcal{A}(q,\lambda) = \delta(q,\lambda)\mathcal{B}(q)^{-1} + \big(1-\delta(q,\lambda)\big)\mathcal{C}(q), \tag{4.6a}$$

$$\mathcal{D}(q,\lambda) = \delta(q,\lambda)\mathcal{C}(q)^{-1} + \big(1-\delta(q,\lambda)\big)\mathcal{B}(q), \tag{4.6b}$$

where we have introduced the following q-deformation of the diagonal matrix $\delta(\lambda)$,

$$\delta(q,\lambda) = \frac{\sqrt{q}}{\sqrt{q}-\lambda}\big(1_1\otimes 1_1 + 1_2\otimes 1_2\big) + \frac{q}{q+\lambda}1_1\otimes 1_2 + \frac{1}{1+\lambda}1_2\otimes 1_1.$$

Properties. One can check that the quantum matrices in (4.5) and (4.6) satisfy all the properties listed in section 3. In particular, relations (3.7) and (3.9) hold with diagonal matrices $\gamma(q) = 1$ and $\widetilde{\gamma}(q) = \mathrm{diag}(q,1,q,1)$.

To describe the property (3.3a) of unitarity, consider the diagonal matrix

$$K(q,\lambda) = \delta(q,\lambda)\delta(q,q\lambda)\delta\big(q,q^{-\frac{1}{2}}\lambda\big)^{-1}\delta\big(q,q^{\frac{3}{2}}\lambda\big)^{-1}.$$

It commutes with all four R-matrices $\mathcal{A}(q,\lambda)$, $\mathcal{B}(q)$, $\mathcal{C}(q)$ and $\mathcal{D}(q,\lambda)$ and tends to the identity in the limits $\lambda\to 0$ and $\lambda\to\infty$. Furthermore, in the limit $\hbar\to 0$, one has $K(e^{i\hbar},\lambda) = 1 + O(\hbar^2)$. Therefore the rescaled matrices $\widehat{\mathcal{A}}(q,\lambda) = K(q,\lambda)^{-\frac{1}{2}}\mathcal{A}(q,\lambda)$ and $\widehat{\mathcal{D}}(q,\lambda) = K(q,\lambda)^{-\frac{1}{2}}\mathcal{D}(q,\lambda)$ together with $\mathcal{B}(q)$ and $\mathcal{C}(q)$ satisfy all properties of section 3 including the unitarity conditions

$$\widehat{\mathcal{A}}_{\underline{\mathbf{12}}}(q,\lambda)\widehat{\mathcal{A}}_{\underline{\mathbf{21}}}(q,\lambda^{-1}) = \widehat{\mathcal{D}}_{\underline{\mathbf{12}}}(q,\lambda)\widehat{\mathcal{D}}_{\underline{\mathbf{21}}}(q,\lambda^{-1}) = 1.$$

4.2 Models related to affine Lie algebras $A_2^{(n)}$

In this section we consider generalized sine-Gordon models associated with both the untwisted and twisted affine Lie algebras $A_2^{(1)}$ and $A_2^{(2)}$.

4.2.1 $\mathbb{C}P^2$ symmetric space sine-Gordon model

We begin by considering the symmetric space sine-Gordon theory resulting from the Pohlmeyer reduction of the $\mathbb{C}P^2$ σ-model. This is a coset σ-model on $\mathrm{SU}(3)/(\mathrm{SU}(2)\times\mathrm{U}(1))$ which means that the Lie algebra $\mathfrak{f}$ is equal to $\mathfrak{su}(3)$ while $\mathfrak{f}^{(0)}\simeq\mathfrak{su}(2)\oplus\mathfrak{u}(1)$.

Automorphism. Consider therefore $\mathfrak{f} = \mathfrak{su}(3)$ with Chevalley generators $\mathsf{H}_i, \mathsf{E}_i, \mathsf{F}_i$ given in the fundamental representation by

$$\mathsf{H}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathsf{E}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathsf{F}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.7a}$$

$$\mathsf{H}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \mathsf{E}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathsf{F}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{4.7b}$$

and let $\mathsf{E}_3 = [\mathsf{E}_1, \mathsf{E}_2]$ and $\mathsf{F}_3 = [\mathsf{F}_2, \mathsf{F}_1]$. The $\mathbb{Z}_2$-automorphism σ of $\mathfrak{f}$ is defined by

$$\begin{aligned} &\sigma(\mathsf{H}_1) = \mathsf{H}_1, && \sigma(\mathsf{H}_2) = \mathsf{H}_2, && \sigma(\mathsf{E}_1) = \mathsf{E}_1, && \sigma(\mathsf{F}_1) = \mathsf{F}_1, \\ &\sigma(\mathsf{E}_2) = -\mathsf{E}_2, && \sigma(\mathsf{F}_2) = -\mathsf{F}_2, && \sigma(\mathsf{E}_3) = -\mathsf{E}_3, && \sigma(\mathsf{F}_3) = -\mathsf{F}_3. \end{aligned}$$

Note that this is an inner automorphism since $\sigma(x) = gxg^{-1}$ where $g = \mathrm{diag}(1, 1, -1)$. We take the bases for the corresponding eigenspaces $\mathfrak{f}^{(0)}, \mathfrak{f}^{(1)} \subset \mathfrak{f}$ of eigenvalue ± 1 to be

$$\mathfrak{f}^{(0)} = \langle \mathsf{H}_1, \mathsf{E}_1, \mathsf{F}_1, \mathsf{H}_2' = \mathsf{H}_2 + \tfrac{1}{2}\mathsf{H}_1 \rangle, \qquad \mathfrak{f}^{(1)} = \langle \mathsf{E}_2, \mathsf{F}_2, \mathsf{E}_3, \mathsf{F}_3 \rangle.$$

Notice that H_2' commutes with $\mathsf{H}_1, \mathsf{E}_1, \mathsf{F}_1 \in \mathfrak{f}^{(0)}$ and hence $\mathfrak{f}^{(0)} \simeq \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ as desired.

Casimir decomposition. The tensor Casimir of $\mathfrak{su}(3)$ reads

$$C = \frac{1}{6}\big(\mathsf{H}_1 \otimes \mathsf{H}_2 + \mathsf{H}_2 \otimes \mathsf{H}_1\big) + \frac{1}{3}\big(\mathsf{H}_1 \otimes \mathsf{H}_1 + \mathsf{H}_2 \otimes \mathsf{H}_2\big) + \frac{1}{2}\sum_{i=1}^{3}\big(\mathsf{E}_i \otimes \mathsf{F}_i + \mathsf{F}_i \otimes \mathsf{E}_i\big), \tag{4.8}$$

and decomposes as $C = C^{(00)} + C^{(11)}$ with respect to the subspaces $\mathfrak{f} = \mathfrak{f}^{(0)} \oplus \mathfrak{f}^{(1)}$ where

$$\begin{aligned} C^{(00)} &= \frac{1}{3}\mathsf{H}_2' \otimes \mathsf{H}_2' + \frac{1}{4}\mathsf{H}_1 \otimes \mathsf{H}_1 + \frac{1}{2}\big(\mathsf{E}_1 \otimes \mathsf{F}_1 + \mathsf{F}_1 \otimes \mathsf{E}_1\big), \\ C^{(11)} &= \frac{1}{2}\big(\mathsf{E}_2 \otimes \mathsf{F}_2 + \mathsf{F}_2 \otimes \mathsf{E}_2 + \mathsf{E}_3 \otimes \mathsf{F}_3 + \mathsf{F}_3 \otimes \mathsf{E}_3\big). \end{aligned}$$

Classical r-matrices. For the skew-symmetric solution α of the mCYBE on $\mathfrak{g} = \mathfrak{f}^{(0)}$ we shall take the standard solution on its $\mathfrak{su}(2)$ part, namely

$$\alpha = \frac{1}{2}\big(\mathsf{E}_1 \otimes \mathsf{F}_1 - \mathsf{F}_1 \otimes \mathsf{E}_1\big).$$

The corresponding non-skew-symmetric solutions b and c of the CYBE read

$$b = -\frac{1}{3}\mathsf{H}_2' \otimes \mathsf{H}_2' - \frac{1}{4}\mathsf{H}_1 \otimes \mathsf{H}_1 - \mathsf{E}_1 \otimes \mathsf{F}_1, \tag{4.9a}$$

$$c = -\frac{1}{3}\mathsf{H}_2' \otimes \mathsf{H}_2' - \frac{1}{4}\mathsf{H}_1 \otimes \mathsf{H}_1 - \mathsf{F}_1 \otimes \mathsf{E}_1. \tag{4.9b}$$

The spectral parameter dependent r-matrices $a(\lambda)$ and $d(\lambda)$ may then be written as

$$a(\lambda) = -\frac{1}{1-\lambda^2}\, b - \frac{\lambda^2}{1-\lambda^2}\, c + \frac{2\lambda}{1-\lambda^2} C^{(11)}, \tag{4.10a}$$

$$d(\lambda) = -\frac{1}{1-\lambda^2}\, c - \frac{\lambda^2}{1-\lambda^2}\, b + \frac{2\lambda}{1-\lambda^2} C^{(11)}. \tag{4.10b}$$

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Quantum R-matrices. One can check that solutions of the QYBE with classical limits (4.9) as $\hbar \to 0$ with $q = e^{i\hbar}$, are given respectively by

$$\mathcal{B}(q) = q^{-\frac{1}{3}\mathsf{H}_2'\otimes\mathsf{H}_2' - \frac{1}{4}\mathsf{H}_1\otimes\mathsf{H}_1} + q^{-\frac{1}{3}}(1-q)\mathsf{E}_1\otimes\mathsf{F}_1,$$
$$\mathcal{C}(q) = q^{-\frac{1}{3}\mathsf{H}_2'\otimes\mathsf{H}_2' - \frac{1}{4}\mathsf{H}_1\otimes\mathsf{H}_1} + q^{-\frac{1}{3}}(1-q)\mathsf{F}_1\otimes\mathsf{E}_1.$$

Quantizations of the matrices (4.10) then take the following form

$$\mathcal{A}(q,\lambda) = \frac{q^{\frac{1}{3}}}{q^{\frac{1}{3}}-\lambda^2}\,\mathcal{B}(q)^{-1} - \frac{\lambda^2}{q^{\frac{1}{3}}-\lambda^2}\,\mathcal{C}(q) + \frac{2q^{-\frac{1}{3}}(q-1)\lambda}{q^{\frac{1}{3}}-\lambda^2}\,C^{(11)}, \tag{4.11a}$$
$$\mathcal{D}(q,\lambda) = \frac{q^{\frac{1}{3}}}{q^{\frac{1}{3}}-\lambda^2}\,\mathcal{C}(q)^{-1} - \frac{\lambda^2}{q^{\frac{1}{3}}-\lambda^2}\,\mathcal{B}(q) + \frac{2q^{-\frac{1}{3}}(q-1)\lambda}{q^{\frac{1}{3}}-\lambda^2}\,C^{(11)}. \tag{4.11b}$$

Properties. Aside from the general properties listed in section 3, the quantum R-matrices just defined also satisfy $\mathcal{D}_{\underline{\mathbf{12}}}(q,\lambda) = \mathcal{A}_{\underline{\mathbf{21}}}(q,\lambda)$. The unitarity property (3.3a) explicitly reads

$$\mathcal{A}_{\underline{\mathbf{12}}}(q,\lambda)\mathcal{A}_{\underline{\mathbf{21}}}(q,\lambda^{-1}) = \frac{(q-\lambda^2)(q^{-1}-\lambda^2)}{(q^{\frac{1}{3}}-\lambda^2)(q^{-\frac{1}{3}}-\lambda^2)}\mathbf{1},$$

while the relations (3.7) and (3.9) hold for $\gamma(q) = 1$ and $\widetilde{\gamma}(q) = \mathrm{diag}(q,1,1)$.

Connection with universal *R*-matrix. The R-matrix in (4.11a) turns out to be related to the untwisted affine Lie algebra $A_2^{(1)}$, the R-matrix of which in the fundamental representation was given in [22]. In order to see this connection explicitly, it is convenient to use the results of [23], where the universal R-matrix obtained by Khoroshkin and Tolstoy in [24–26] for $A_2^{(1)}$ was evaluated in the fundamental representation. One can directly check that the R-matrix (4.11a) obtained here is proportional to $R^{(2,0,1)}$ in the notation of [23] with the replacement $q \to q^{\frac{1}{2}}$. This connection with the untwisted affine Lie algebra $A_2^{(1)}$ stems from the automorphism σ being inner. In fact, the twisting by the inner automorphism σ can be undone at the quantum level by considering

$$\widehat{\mathcal{A}}_{\underline{\mathbf{12}}}(q,\lambda/\mu) = g_{\underline{\mathbf{1}}}(\lambda)g_{\underline{\mathbf{2}}}(\mu)\mathcal{A}_{\underline{\mathbf{12}}}(q,\lambda/\mu)g_{\underline{\mathbf{1}}}(\lambda)^{-1}g_{\underline{\mathbf{2}}}(\mu)^{-1}$$

where $g(\lambda)$ is the diagonal matrix defined as $g(\lambda) = \mathrm{diag}(1,1,\lambda)$. Up to some overall scalar factor and the replacement $q \to q^{\frac{1}{2}}$, this is precisely the R-matrix $R^{(2,0,0)}$ in the notation of [23].

4.2.2 SU(3)/SO(3) symmetric space sine-Gordon model

Automorphism. We use the same notations (4.7) as in section 4.2.1 for the generators in the fundamental representation of $\mathfrak{f} = \mathfrak{su}(3)$. In the case at hand, the $\mathbb{Z}_2$-automorphism acts on an element x of $\mathfrak{f}$ as

$$\sigma(x) = -\eta x^t \eta^{-1}, \tag{4.12}$$

where the pseudo-metric η is defined by

$$\eta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{4.13}$$

The corresponding eigenspaces $\mathfrak{f}^{(0)}$ and $\mathfrak{f}^{(1)}$ of σ are generated by

$$\mathfrak{f}^{(0)} = \langle \mathsf{h}^{(0)}, \mathsf{e}^{(0)}, \mathsf{f}^{(0)} \rangle, \qquad \mathfrak{f}^{(1)} = \langle \mathsf{h}^{(1)}, \mathsf{e}^{(1)}, \mathsf{f}^{(1)}, \mathsf{E}_3, \mathsf{F}_3 \rangle,$$

where we have introduced the following linear combinations of the generators

$$\begin{aligned} \mathsf{h}^{(0)} &= \mathsf{H}_1 + \mathsf{H}_2, \qquad & \mathsf{e}^{(0)} &= \mathsf{E}_1 + \mathsf{E}_2, \qquad & \mathsf{f}^{(0)} &= \mathsf{F}_1 + \mathsf{F}_2, \\ \mathsf{h}^{(1)} &= \mathsf{H}_1 - \mathsf{H}_2, \qquad & \mathsf{e}^{(1)} &= \mathsf{E}_1 - \mathsf{E}_2, \qquad & \mathsf{f}^{(1)} &= \mathsf{F}_1 - \mathsf{F}_2. \end{aligned}$$

Note in particular that $\mathfrak{f}^{(0)} \simeq \mathfrak{so}(3)$.

Casimir decomposition. The tensor Casimir (4.8) of $\mathfrak{su}(3)$ decomposes with respect to the above $\mathbb{Z}_2$-grading as

$$\begin{aligned} C^{(00)} &= \frac{1}{4}\mathsf{h}^{(0)} \otimes \mathsf{h}^{(0)} + \frac{1}{4}\mathsf{e}^{(0)} \otimes \mathsf{f}^{(0)} + \frac{1}{4}\mathsf{f}^{(0)} \otimes \mathsf{e}^{(0)}, \\ C^{(11)} &= \frac{1}{12}\mathsf{h}^{(1)} \otimes \mathsf{h}^{(1)} + \frac{1}{4}\mathsf{e}^{(1)} \otimes \mathsf{f}^{(1)} + \frac{1}{4}\mathsf{f}^{(1)} \otimes \mathsf{e}^{(1)} + \frac{1}{2}\left(\mathsf{E}_3 \otimes \mathsf{F}_3 + \mathsf{F}_3 \otimes \mathsf{E}_3\right). \end{aligned}$$

Classical r-matrices. Our choice for α is again the standard skew-symmetric solution of the mCYBE on $\mathfrak{f}^{(0)}$, namely

$$\alpha = \frac{1}{4}\big(\mathsf{e}^{(0)} \otimes \mathsf{f}^{(0)} - \mathsf{f}^{(0)} \otimes \mathsf{e}^{(0)}\big).$$

The corresponding non-skew-symmetric solutions b and c of the CYBE take the following form

$$b = -\frac{1}{4}\mathsf{h}^{(0)} \otimes \mathsf{h}^{(0)} - \frac{1}{2}\mathsf{e}^{(0)} \otimes \mathsf{f}^{(0)}, \tag{4.14a}$$

$$c = -\frac{1}{4}\mathsf{h}^{(0)} \otimes \mathsf{h}^{(0)} - \frac{1}{2}\mathsf{f}^{(0)} \otimes \mathsf{e}^{(0)}. \tag{4.14b}$$

In terms of these, the r-matrices $a(\lambda)$ and $d(\lambda)$ are given by the same expressions as in (4.10).

Quantum R-matrices. Quantizations of the classical r-matrices (4.14) are given by

$$\mathcal{B}(q) = q^{-\frac{1}{4}\mathsf{h}^{(0)}\otimes\mathsf{h}^{(0)}}\Big(1 - \big(q^{\frac{1}{4}} - q^{-\frac{1}{4}}\big)\mathsf{e}^{(0)} \otimes \mathsf{f}^{(0)} + \big(1 - q^{-\frac{1}{4}}\big)\big(q^{\frac{1}{4}} - q^{-\frac{1}{4}}\big)(\mathsf{e}^{(0)})^2 \otimes (\mathsf{f}^{(0)})^2\Big), \tag{4.15a}$$

$$\mathcal{C}(q) = q^{-\frac{1}{4}\mathsf{h}^{(0)}\otimes\mathsf{h}^{(0)}}\Big(1 - \big(q^{\frac{1}{4}} - q^{-\frac{1}{4}}\big)\mathsf{f}^{(0)} \otimes \mathsf{e}^{(0)} + \big(1 - q^{-\frac{1}{4}}\big)\big(q^{\frac{1}{4}} - q^{-\frac{1}{4}}\big)(\mathsf{f}^{(0)})^2 \otimes (\mathsf{e}^{(0)})^2\Big). \tag{4.15b}$$

As for the R-matrix $\mathcal{A}(q, \lambda)$, based on the established connection of the previous example with the untwisted affine Lie algebra $A_2^{(1)}$, it is natural to expect a similar relation in the present case but this time with the twisted affine Lie algebra $A_2^{(2)}$. The R-matrix of the latter in the fundamental representation was computed in [27]. For our purposes we shall use the results of [28] in which a family of R-matrices in the fundamental representation of

$\mathfrak{su}(3)$ parametrized by two integers s_0 and s_1 was obtained from the universal R-matrix [24–26]. Specifically, we will construct $\mathcal{A}(q,\lambda)$ from the particular solution with $s_0 = 1$ and $s_1 = 0$ which can be rewritten as follows. Introduce a q-deformation η_q of the metric (4.13) as

$$\eta_q = \begin{pmatrix} 0 & 0 & q^{\frac{1}{4}} \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We use this to define the following q-deformation of the automorphism (4.12)

$$\sigma_q(x) = -\eta_q x^t \eta_q^{-1}.$$

Then, up to some overall factor and the replacement $q \to q^{1/4}$, the R-matrix considered in [28] may be rewritten as

$$\widehat{\mathcal{A}}(q,\lambda) = \frac{q^{\frac{1}{4}}}{q^{\frac{1}{2}} - \lambda}\mathcal{B}(q)^{-1} - \frac{q^{\frac{1}{4}}\lambda}{q^{\frac{1}{2}} - \lambda}\mathcal{C}(q) - \frac{\lambda(q^{\frac{1}{2}} - 1)\big(1 + q^{\frac{3}{4}}\big)}{(q^{\frac{1}{2}} - \lambda)\big(\lambda + q^{\frac{3}{4}}\big)} \sum_{i=1}^{3}\sum_{j=1}^{3} E_{ij} \otimes \sigma_q(E_{ji}), \quad (4.16)$$

where E_{ij} denotes the 3×3 matrix whose only non-zero entry is a 1 in the i^{th} row and j^{th} column. The desired quantum R-matrix with the correct classical limit $a(\lambda)$ given in (4.10) may now be obtained by rescaling (4.16) as

$$\mathcal{A}(q,\lambda) = \frac{\big(q^{\frac{1}{2}} - \lambda\big)\big(q^{\frac{3}{4}} + \lambda\big)}{q^{\frac{1}{4}}\big(q^{\frac{1}{6}} - \lambda\big)\big(q^{\frac{7}{12}} + \lambda\big)}\widehat{\mathcal{A}}(q,\lambda). \quad (4.17a)$$

Finally, the matrix $\mathcal{D}(q,\lambda)$ is defined through the relation (3.6). Such a definition automatically satisfies the classical limit (3.2b) and is given explicitly by

$$\mathcal{D}(q,\lambda) = \frac{\big(q^{\frac{1}{2}} - \lambda\big)\big(q^{\frac{3}{4}} + \lambda\big)}{q^{\frac{1}{4}}\big(q^{\frac{1}{6}} - \lambda\big)\big(q^{\frac{7}{12}} + \lambda\big)}\widehat{\mathcal{D}}(q,\lambda), \quad (4.17b)$$

where the quantum R-matrix $\widehat{\mathcal{D}}(q,\lambda)$ admits a similar expression to (4.16), namely

$$\widehat{\mathcal{D}}(q,\lambda) = \frac{q^{\frac{1}{4}}}{q^{\frac{1}{2}} - \lambda}\mathcal{C}(q)^{-1} - \frac{q^{\frac{1}{4}}\lambda}{q^{\frac{1}{2}} - \lambda}\mathcal{B}(q) - \frac{\lambda(q^{\frac{1}{2}} - 1)\big(1 + q^{\frac{3}{4}}\big)}{(q^{\frac{1}{2}} - \lambda)\big(\lambda + q^{\frac{3}{4}}\big)} \sum_{i=1}^{3}\sum_{j=1}^{3} \sigma_q(E_{ij}) \otimes E_{ji}. \quad (4.18)$$

Properties. The matrices (4.15) and (4.17) so defined satisfy all the properties discussed in section 3 as well as the further property $\mathcal{D}_{\underline{\mathbf{12}}}(q,\lambda) = \mathcal{A}_{\underline{\mathbf{21}}}(q,\lambda)$. Furthermore, the rescaled R-matrices (4.16) and (4.18) are both unitary, namely

$$\widehat{\mathcal{A}}_{\underline{\mathbf{12}}}(q,\lambda)\widehat{\mathcal{A}}_{\underline{\mathbf{21}}}(q,\lambda^{-1}) = 1, \qquad \widehat{\mathcal{D}}_{\underline{\mathbf{12}}}(q,\lambda)\widehat{\mathcal{D}}_{\underline{\mathbf{21}}}(q,\lambda^{-1}) = 1.$$

Finally, the relations (3.7) and (3.9) are satisfied with $\gamma(q) = 1$ and $\widetilde{\gamma}(q) = \text{diag}(q^{\frac{1}{4}}, 1, q^{-\frac{1}{4}})$.

5 Affine quantum braided group for $\mathfrak{gl}(4|4)$

Throughout this section we take $\mathfrak{f} = \mathfrak{gl}(4|4)$, a basis of which in the fundamental representation is given by the 8×8 matrices $E_{i,j}$ whose only non-zero entry is a 1 in the i^{th} row and j^{th} column.

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Automorphism. The $\mathbb{Z}_4$-automorphism σ of $\mathfrak{f}$ with the property $\sigma^4 = \mathrm{id}$ is defined by

$$\sigma(x) = -K x^{st} K^{-1},$$

where x^{st} denotes the usual supertranspose of the matrix x and $K = 1_4 \otimes i\sigma_2$. The projection $p^{(k)}$ of $\mathfrak{f}$ onto the corresponding eigenspace $\mathfrak{f}^{(k)}$ of σ is defined for any $x \in \mathfrak{f}$ by

$$p^{(k)}(x) = \frac{1}{4}\big(x + i^{3k}\sigma(x) + i^{2k}\sigma^2(x) + i^k\sigma^3(x)\big). \tag{5.1}$$

The subalgebra $\mathfrak{f}^{(0)}$ corresponds to two copies of the Lie algebra $\mathfrak{so}(5)$ and is spanned by

$$\mathfrak{f}^{(0)} = \big\langle \{\mathsf{h}_i^{(0)}\}_{i=1}^4, \{\mathsf{e}_i^{(0)}\}_{i=1}^8, \{\mathsf{f}_i^{(0)}\}_{i=1}^8 \big\rangle,$$

where the basis vectors are given explicitly in terms of the $E_{i,j}$ as

$$\begin{aligned}
&\mathsf{h}_1^{(0)} = E_{1,1} - E_{2,2}, && \mathsf{h}_2^{(0)} = E_{3,3} - E_{4,4}, && \mathsf{h}_3^{(0)} = E_{5,5} - E_{6,6}, && \mathsf{h}_4^{(0)} = E_{7,7} - E_{8,8},\\
&\mathsf{e}_1^{(0)} = E_{3,4}, && \mathsf{e}_2^{(0)} = \frac{E_{4,2} - E_{1,3}}{\sqrt{2}}, && \mathsf{e}_3^{(0)} = \frac{E_{1,4} + E_{3,2}}{\sqrt{2}}, && \mathsf{e}_4^{(0)} = E_{1,2},\\
&\mathsf{e}_5^{(0)} = E_{8,7}, && \mathsf{e}_6^{(0)} = \frac{E_{7,5} - E_{6,8}}{\sqrt{2}}, && \mathsf{e}_7^{(0)} = \frac{E_{6,7} + E_{8,5}}{\sqrt{2}}, && \mathsf{e}_8^{(0)} = E_{6,5},\\
&\mathsf{f}_i^{(0)} = \big(\mathsf{e}_i^{(0)}\big)^t.
\end{aligned}$$

Casimir decomposition. The tensor Casimir C takes the simple form

$$C = \sum_{i,j=1}^{8} E_{i,j} \otimes W E_{j,i}$$

where we have introduced the diagonal matrix $W = \mathrm{diag}(1_4, -1_4)$. The four components $C^{(00)}$, $C^{(22)}$, $C^{(13)}$ and $C^{(31)}$ of C are obtained by applying the appropriate projections in (5.1). For $C^{(00)}$ we find

$$C^{(00)} = \frac{1}{2}\sum_{i=1}^{4} \mathsf{h}_i^{(0)} \otimes W\mathsf{h}_i^{(0)} + \sum_{i=1}^{8}\big(\mathsf{e}_i^{(0)} \otimes W\mathsf{f}_i^{(0)} + \mathsf{f}_i^{(0)} \otimes W\mathsf{e}_i^{(0)}\big). \tag{5.2}$$

Classical r-matrices. Based on the form (5.2) of the Casimir component $C^{(00)}$, we make the following choice for the matrix α,

$$\alpha = \sum_{i=1}^{8}\big(\mathsf{e}_i^{(0)} \otimes W\mathsf{f}_i^{(0)} - \mathsf{f}_i^{(0)} \otimes W\mathsf{e}_i^{(0)}\big).$$

It is straightforward to check that this is a solution of the mCYBE on $\mathfrak{f}^{(0)}$ and is skew-symmetric. The corresponding non-skew-symmetric constant solutions of the CYBE read

$$b = -\frac{1}{2}\sum_{i=1}^{4} \mathsf{h}_i^{(0)} \otimes W\mathsf{h}_i^{(0)} - 2\sum_{i=1}^{8} \mathsf{e}_i^{(0)} \otimes W\mathsf{f}_i^{(0)}, \tag{5.3a}$$

$$c = -\frac{1}{2}\sum_{i=1}^{4} \mathsf{h}_i^{(0)} \otimes W\mathsf{h}_i^{(0)} - 2\sum_{i=1}^{8} \mathsf{f}_i^{(0)} \otimes W\mathsf{e}_i^{(0)}. \tag{5.3b}$$

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The classical r-matrices $a(\lambda)$ and $d(\lambda)$ may then be written in the form

$$a(q,\lambda) = -\frac{1}{1+\lambda^2}b + \frac{\lambda^2}{1+\lambda^2}c + \frac{2\lambda^2}{1-\lambda^4}\big(C^{(00)} + C^{(22)}\big) + \frac{2\lambda}{1-\lambda^4}C^{(13)} + \frac{2\lambda^3}{1-\lambda^4}C^{(31)}, \tag{5.4a}$$

$$d(q,\lambda) = -\frac{1}{1+\lambda^2}c + \frac{\lambda^2}{1+\lambda^2}b + \frac{2\lambda^2}{1-\lambda^4}\big(C^{(00)} + C^{(22)}\big) + \frac{2\lambda}{1-\lambda^4}C^{(13)} + \frac{2\lambda^3}{1-\lambda^4}C^{(31)}. \tag{5.4b}$$

Quantum *R*-matrices Quantizations of (5.3) can be expressed as

$$\mathcal{B}(q) = q^H E_1(q)E_3(q)E_4(q)E_2(q)E_5(q)E_7(q)E_8(q)E_6(q), \tag{5.5a}$$

$$\mathcal{C}_{\underline{\mathbf{12}}}(q) = \mathcal{B}_{\underline{\mathbf{21}}}(q), \tag{5.5b}$$

where the first factor in (5.5a) is the q-exponential of the Cartan part of b which reads

$$H = -\frac{1}{2}\sum_{i=1}^{4} \mathsf{h}_i^{(0)} \otimes W\mathsf{h}_i^{(0)}.$$

The remaining factors in (5.5a) are given by q-exponentials of the q-analogues of each non-Cartan term in the expression (5.3a) for b, namely

$$\begin{aligned}
E_1(q) &= 1\otimes 1 + (q^{-1}-q)\,\mathsf{e}_1^{(0)} \otimes \mathsf{f}_1^{(0)},\\
E_2(q) &= 1\otimes 1 + 2(q^{-\frac12}-q^{\frac12})\,\mathsf{e}_2^{(0)} \otimes \mathsf{f}_2^{(0)},\\
E_3(q) &= 1\otimes 1 - 2(q^{-\frac12}-q^{\frac12})\,[\mathsf{e}_1^{(0)},\mathsf{e}_2^{(0)}]_q \otimes [\mathsf{f}_1^{(0)},\mathsf{f}_2^{(0)}]_q,\\
E_4(q) &= 1\otimes 1 + (q^{-1}-q)\,\big[[\mathsf{e}_1^{(0)},\mathsf{e}_2^{(0)}]_q,\mathsf{e}_2^{(0)}\big]_q \otimes \big[[\mathsf{f}_1^{(0)},\mathsf{f}_2^{(0)}]_q,\mathsf{f}_2^{(0)}\big]_q,\\
E_5(q) &= 1\otimes 1 + (q-q^{-1})\,\mathsf{e}_5^{(0)} \otimes \mathsf{f}_5^{(0)},\\
E_6(q) &= 1\otimes 1 + 2(q^{\frac12}-q^{-\frac12})\,\mathsf{e}_6^{(0)} \otimes \mathsf{f}_6^{(0)},\\
E_7(q) &= 1\otimes 1 - 2(q^{\frac12}-q^{-\frac12})\,[\mathsf{e}_5^{(0)},\mathsf{e}_6^{(0)}]_{q^{-1}} \otimes [\mathsf{f}_5^{(0)},\mathsf{f}_6^{(0)}]_{q^{-1}},\\
E_8(q) &= 1\otimes 1 + (q-q^{-1})\,\big[[\mathsf{e}_5^{(0)},\mathsf{e}_6^{(0)}]_{q^{-1}},\mathsf{e}_6^{(0)}\big]_{q^{-1}} \otimes \big[[\mathsf{f}_5^{(0)},\mathsf{f}_6^{(0)}]_{q^{-1}},\mathsf{f}_6^{(0)}\big]_{q^{-1}}.
\end{aligned}$$

Here $[\mathsf{x},\mathsf{y}]_q = q^{-\frac12}\mathsf{x}\,\mathsf{y} - q^{\frac12}\mathsf{y}\,\mathsf{x}$ denotes the q-commutator of x with y.

Quantizations of (5.4) may now be written in the following form

$$\begin{aligned}
\mathcal{A}(q,\lambda) = \frac{1}{1+\lambda^2}\mathcal{B}(q)^{-1} + \frac{\lambda^2}{1+\lambda^2}\mathcal{C}(q) \qquad\qquad\qquad\qquad\qquad\qquad (5.6a)\\
+ (q^{\frac12}-q^{-\frac12})\left(\frac{2\lambda^2}{1-\lambda^4}\big(C^{(00)}+C^{(22)}\big) + \frac{2\lambda}{1-\lambda^4}C_q^{(13)} + \frac{2\lambda^3}{1-\lambda^4}C_q^{(31)}\right),
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}(q,\lambda) = \frac{1}{1+\lambda^2}\mathcal{C}(q)^{-1} + \frac{\lambda^2}{1+\lambda^2}\mathcal{B}(q) \qquad\qquad\qquad\qquad\qquad\qquad (5.6b)\\
+ (q^{\frac12}-q^{-\frac12})\left(\frac{2\lambda^2}{1-\lambda^4}\big(C^{(00)}+C^{(22)}\big) + \frac{2\lambda}{1-\lambda^4}q^{-10H}C_{q^{-1}}^{(13)} + \frac{2\lambda^3}{1-\lambda^4}q^{-10H}C_{q^{-1}}^{(31)}\right),
\end{aligned}$$

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where $C_q^{(13)}$ and $C_q^{(31)}$ are q-deformations of the respective components $C^{(13)}$ and $C^{(31)}$ of the tensor Casimir. Explicitly, the q-deformation $C_q^{(13)}$ is defined as

$$C_q^{(13)} = -\frac{1}{2} \sum_{m,n=1}^{4} q^{(\epsilon_n - \epsilon_m)\overline{H}} \big(E_{m,n+4} - i\sigma(E_{m,n+4}) \big) \otimes \big(E_{n+4,m} + i\sigma(E_{n+4,m}) \big)$$

with $(\epsilon_n)_{n=1}^4 = (0,4,1,3)$, whereas the q-deformation $C_q^{(31)}$ is obtained from this as $C_{q\underline{\mathbf{12}}}^{(31)} = C_{q^{-1}\underline{\mathbf{21}}}^{(13)}$. Here we have introduced

$$\overline{H} = \frac{1}{2} \sum_{i=1}^{4} \mathsf{h}_i^{(0)} \otimes \mathsf{h}_i^{(0)}.$$

We also have the relation $C_q^{(13)} + C_q^{(31)} = C^{(13)} + C^{(31)}$.

Properties. The matrices (5.5) and (5.6) satisfy all the properties listed in section 3 with

$$\gamma(q) = \mathrm{diag}(1_4, q^5 1_4), \qquad \widetilde{\gamma}(q) = \mathrm{diag}(1, q^{-4}, q^{-1}, q^{-3}, 1, q^{-4}, q^{-1}, q^{-3}). \tag{5.7}$$

Note that the first four and last four entries along the diagonal of $\widetilde{\gamma}$ are just $q^{-\epsilon_n}$. Concerning the unitarity property we have

$$\mathcal{A}_{\underline{\mathbf{12}}}(q,\lambda)\mathcal{A}_{\underline{\mathbf{21}}}(q,\lambda^{-1}) = \frac{(q-\lambda^2)\,(q^{-1}-\lambda^2)}{(1-\lambda^2)^2} 1, \tag{5.8}$$

and similarly for $\mathcal{D}$.

6 Conclusion

We have shown, by way of example, how to quantize the lattice Poisson algebra of (semi-)symmetric space sine-Gordon models previously identified in [5–7]. The quantum lattice algebras obtained for the four models considered each provide new interesting examples of the general formalism laid out in [14, 15]. But moreover, there is a certain uniformity among these examples which hints at a general framework for quantizing (semi-)symmetric space sine-Gordon models.

Indeed, in each of the four models considered, the function algebra can be quantized within the language of affine quantum braided groups. The necessity for the departure from the conventional set-up of affine quantum groups can be seen as a remnant of the non-ultralocality of these models at the classical level. Specifically, the braiding arises as a quantum counterpart of the regularization prescription [21] necessary to unambiguously define the Poisson bracket of the monodromy matrix. This strongly suggests that the general formalism presented in section 3 should be the appropriate language within which to address the quantization of (semi-)symmetric space sine-Gordon models.

Furthermore, the examples discussed in section 4.2 indicate a general procedure for constructing the various R-matrices entering the quantized lattice algebra of these models. Indeed, in the specific cases of the $\mathbb{C}P^2$ and SU(3)/SO(3) symmetric space sine-Gordon

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models, we have shown how these R-matrices can be directly obtained from the R-matrix of, respectively, the untwisted and twisted affine Lie algebras of type A_2 in the fundamental representation through the works of [23, 28].

Finally, in view of ultimately identifying a quantum lattice model for the theories in question, the next challenge is to find explicit quantum lattice Lax operators $\hat{\mathcal{L}}^n$ satisfying the algebra given in section 3.1. This is an important problem which we leave for future work.

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