A lattice Poisson algebra for the Pohlmeyer reduction of the $AdS_5 \times S^5$ superstring

F. Delduc [1], M. Magro [1], B. Vicedo [2]

1) Laboratoire de Physique, ENS Lyon et CNRS UMR 5672, Université de Lyon, 46, allée d'Italie, 69364 LYON Cedex 07, France

2) Department of Mathematics, University of York, Heslington, York, YO10 5DD, United Kingdom

Francois.Delduc@ens-lyon.fr, Marc.Magro@ens-lyon.fr, Benoit.Vicedo@gmail.com

Abstract. The Poisson algebra of the Lax matrix associated with the Pohlmeyer reduction of the $AdS_5 \times S^5$ superstring is computed from first principles. The resulting non-ultralocality is mild, which enables to write down a corresponding lattice Poisson algebra.

1 Introduction

We recently showed in [1] that the Poisson algebra of the Lax matrix associated with symmetric space sine-Gordon models, defined through a gauged Wess-Zumino-Witten action with an integrable potential [2], admits an integrable lattice discretization. In the present letter we compute the r/s-matrix structure [3] associated with the Pohlmeyer reduction of $AdS_5 \times S^5$ superstring theory [4, 5] directly from its representation in terms of a fermionic extension of a gauged WZW action with an integrable potential. We similarly find that it is precisely of the type which, after regularization as in [6], admits an integrable lattice discretization of the general form identified in [7, 8].

2 Canonical analysis and Hamiltonian

To begin with we briefly recall some usual notations. We refer the reader to [4] for more details concerning this setup. The superalgebra $\mathfrak{f} = \mathfrak{psu}(2,2|4)$ admits a $\mathbb{Z}_4$-grading, $\mathfrak{f} = \mathfrak{f}^{(0)} \oplus \mathfrak{f}^{(1)} \oplus \mathfrak{f}^{(2)} \oplus \mathfrak{f}^{(3)}$ where $\mathfrak{g} = \mathfrak{f}^{(0)} = \mathfrak{so}(4,1) \oplus \mathfrak{so}(5)$. Let G denote the corresponding Lie group. The supertrace is compatible with the $\mathbb{Z}_4$-grading, in the sense that $\mathrm{Str}(A^{(m)} B^{(n)}) = 0$ for $m + n \neq 0 \bmod 4$. The reduced theory relies on the element $T = \frac{i}{2}\text{diag}(1,1,-1,-1,1,1,-1,-1) \in \mathfrak{f}^{(2)}$. It defines a $\mathbb{Z}_2$-grading of $\mathfrak{f}$ with $\mathfrak{f}^{[0]} = \text{Ker}(\text{Ad}_T)$ and $\mathfrak{f}^{[1]} = \text{Im}(\text{Ad}_T)$. Elements of $\mathfrak{f}^{[0]}$ commute with T while those of $\mathfrak{f}^{[1]}$ anti-commute with T and we have $\text{Str}(A^{[0]}B^{[1]}) = 0$. Finally, projectors on $\mathfrak{f}^{[0]}$ and $\mathfrak{f}^{[1]}$ are given respectively by $P^{[0]} = -[T,[T,\,\cdot\,]_+]_+$ and $P^{[1]} = -[T,[T,\,\cdot\,]]$. Let $\mathfrak{h} = \mathfrak{g}^{[0]}$ be the subalgebra in $\mathfrak{g}$ of elements commuting with T. The corresponding Lie group H is $[SU(2)]^4$.

Our starting point is the field theory introduced in [4]. It corresponds to a fermionic extension of a G/H gauged WZW with a potential term. The action we start with is, taking $\epsilon^{\tau\sigma\xi} = 1$,

$$\begin{aligned}
\mathcal{S} = \tfrac{1}{2}\int d\tau d\sigma\, \text{Str}(g^{-1}\partial_+ g g^{-1}\partial_- g) + \tfrac{1}{3}\int d\tau d\sigma d\xi \epsilon^{\alpha\beta\gamma}\, \text{Str}(g^{-1}\partial_\alpha g g^{-1}\partial_\beta g g^{-1}\partial_\gamma g) \\
- \int d\tau d\sigma\, \text{Str}(A_+\partial_- g g^{-1} - A_- g^{-1}\partial_+ g + g^{-1}A_+ g A_- - A_+ A_-) \\
+ \tfrac{1}{2}\int d\tau d\sigma\, \text{Str}(\psi_L[T, D_+\psi_L] + \psi_R[T, D_-\psi_R]) \\
+ \int d\tau d\sigma \big(\mu^2\, \text{Str}(g^{-1}TgT) + \mu\, \text{Str}(g^{-1}\psi_L g \psi_R)\big).
\end{aligned}$$

The fields g, ψ_R, ψ_L and the gauge fields $A_\pm$ respectively take values in G, $\mathfrak{f}^{(1)[1]}$, $\mathfrak{f}^{(3)[1]}$ and in $\mathfrak{h}$. The covariant derivatives are $D_\pm = \partial_\pm - [A_\pm,]$ with $\partial_\pm = \partial_\tau \pm \partial_\sigma$.

Generalizing the analysis of [9] to the case considered here, one finds that the phase space is spanned by the fields $(g, \mathcal{J}_L, A_\pm, P_\pm, \psi_L, \psi_R)$. The field $\mathcal{J}_L$ corresponds to the left-invariant WZW current. Alternatively, one can use instead the right-invariant current $\mathcal{J}_R$, related to $\mathcal{J}_L$ by

$$\mathcal{J}_R = -2\partial_\sigma g g^{-1} + g\mathcal{J}_L g^{-1}.$$

The fields $P_\pm$ are the canonical momenta of $A_\pm$. The non-vanishing Poisson brackets are

$$\begin{aligned}
\{\mathcal{J}_{L\underline{1}}(\sigma), \mathcal{J}_{L\underline{2}}(\sigma')\} &= [C^{(00)}_{\underline{12}}, \mathcal{J}_{L\underline{2}}]\delta_{\sigma\sigma'} + 2C^{(00)}_{\underline{12}}\partial_\sigma\delta_{\sigma\sigma'}, \\
\{\mathcal{J}_{R\underline{1}}(\sigma), \mathcal{J}_{R\underline{2}}(\sigma')\} &= -[C^{(00)}_{\underline{12}}, \mathcal{J}_{R\underline{2}}]\delta_{\sigma\sigma'} - 2C^{(00)}_{\underline{12}}\partial_\sigma\delta_{\sigma\sigma'}, \\
\{\mathcal{J}_{L\underline{1}}(\sigma), g_{\underline{2}}(\sigma')\} &= -g_{\underline{2}}C^{(00)}_{\underline{12}}\delta_{\sigma\sigma'} \\
\{\mathcal{J}_{R\underline{1}}(\sigma), g_{\underline{2}}(\sigma')\} &= -C^{(00)}_{\underline{12}}g_{\underline{2}}\delta_{\sigma\sigma'} \\
\{A_{\pm\underline{1}}(\sigma), P_{\pm\underline{2}}(\sigma')\} &= C^{(00)[00]}_{\underline{12}}\delta_{\sigma\sigma'}, \\
\{\psi_{R\underline{1}}(\sigma), \psi_{R\underline{2}}(\sigma')\} &= \big[T_{\underline{2}}, C^{(13)}_{\underline{12}}\big]\delta_{\sigma\sigma'}, \\
\{\psi_{L\underline{1}}(\sigma), \psi_{L\underline{2}}(\sigma')\} &= \big[T_{\underline{2}}, C^{(31)}_{\underline{12}}\big]\delta_{\sigma\sigma'}.
\end{aligned}$$

In these expressions $C^{(ij)}_{\underline{12}} \in \mathfrak{f}^{(i)} \otimes \mathfrak{f}^{(j)}$ are the components of the tensor Casimir (see [10] for its properties) in the decomposition $C_{\underline{12}} = C^{(00)}_{\underline{12}} + C^{(13)}_{\underline{12}} + C^{(22)}_{\underline{12}} + C^{(31)}_{\underline{12}}$ with respect to the $\mathbb{Z}_4$-grading. The component $C^{(00)[00]}_{\underline{12}}$ is defined in a similar way relative to the $\mathbb{Z}_2$-grading.

The standard analysis shows that there is a total of four constraints,

$$\chi_1 = P_+, \qquad \chi_2 = P_-, \tag{2.2a}$$
$$\chi_3 = \mathcal{J}_R^{[0]} + A_+ - A_- - \tfrac{1}{2}[\psi_L, [T, \psi_L]], \qquad \chi_4 = \mathcal{J}_L^{[0]} + A_+ - A_- + \tfrac{1}{2}[\psi_R, [T, \psi_R]]. \tag{2.2b}$$

The extended Hamiltonian, which has weakly vanishing Poisson brackets with the constraints (2.2), is

$$\begin{aligned} H = \int d\sigma \Big(&\tfrac{1}{4} \operatorname{Str}\big(\mathcal{J}_L{}^2 + \mathcal{J}_R{}^2\big) + \operatorname{Str}\big(\mathcal{J}_R^{[0]} A_+ - \mathcal{J}_L^{[0]} A_-\big) + \tfrac{1}{2} \operatorname{Str}\big[(A_+ - A_-)^2\big] \\ &- \tfrac{1}{2} \operatorname{Str}\big(\psi_L [T, \partial_\sigma \psi_L - [A_+, \psi_L]]\big) - \tfrac{1}{2} \operatorname{Str}\big(\psi_R [T, -\partial_\sigma \psi_R - [A_-, \psi_R]]\big) \\ &- \mu^2 \operatorname{Str}(g^{-1} T g T) - \mu \operatorname{Str}(g^{-1} \psi_L g \psi_R) + v_+ P_+ + v_- P_- + \lambda(\chi_3 - \chi_4)\Big) \end{aligned} \tag{2.3}$$

with $v_+ - v_- = \partial_\sigma(A_+ + A_-) - [A_+, A_-]$. The combination $\chi_3 - \chi_4$ of the constraints generates a gauge invariance.

3 Continuum and lattice Poisson algebras

Up to a gauge transformation, the equations of motion for the fields $(\mathcal{J}_L, g, \psi_L, \psi_R)$ under the Hamiltonian (2.3) are equivalent to the zero curvature equation $\{\mathcal{L}, H\} = \partial_\sigma \mathcal{M} + [\mathcal{M}, \mathcal{L}]$ for the following Lax connection [4]

$$\mathcal{L}(z) = -\tfrac{1}{2}\mathcal{J}_L - \tfrac{1}{2} z \sqrt{\mu} \psi_R - \tfrac{1}{2} z^2 \mu T + \tfrac{1}{2} z^{-1} \sqrt{\mu} g^{-1} \psi_L g + \tfrac{1}{2} z^{-2} \mu g^{-1} T g, \tag{3.1a}$$
$$\mathcal{M}(z) = -\tfrac{1}{2}\mathcal{J}_L + A_- - \tfrac{1}{2} z \sqrt{\mu} \psi_R - \tfrac{1}{2} z^2 \mu T - \tfrac{1}{2} z^{-1} \sqrt{\mu} g^{-1} \psi_L g - \tfrac{1}{2} z^{-2} \mu g^{-1} T g. \tag{3.1b}$$

The field A_+ entering the equations appears as an arbitrary element of $\mathfrak{h}$. We now have all the ingredients needed to compute the Poisson bracket of the Lax matrix (3.1a). The result reads

$$\begin{aligned} 4\{\mathcal{L}_{\underline{\mathbf{1}}}(z_1), \mathcal{L}_{\underline{\mathbf{2}}}(z_2)\} = {}& [r_{\underline{\mathbf{12}}}(z_1, z_2), \mathcal{L}_{\underline{\mathbf{1}}}(z_1) + \mathcal{L}_{\underline{\mathbf{2}}}(z_2)] \delta_{\sigma\sigma'} \\ &+ [s_{\underline{\mathbf{12}}}(z_1, z_2), \mathcal{L}_{\underline{\mathbf{1}}}(z_1) - \mathcal{L}_{\underline{\mathbf{2}}}(z_2)] \delta_{\sigma\sigma'} + 2 s_{\underline{\mathbf{12}}}(z_1, z_2) \partial_\sigma \delta_{\sigma\sigma'}, \end{aligned} \tag{3.2}$$

where the kernels of the r/s-matrices are given by

$$r_{\underline{\mathbf{12}}}(z_1, z_2) = \frac{z_2^4 + z_1^4}{z_2^4 - z_1^4} C^{(00)}_{\underline{\mathbf{12}}} + \frac{2 z_1 z_2^3}{z_2^4 - z_1^4} C^{(13)}_{\underline{\mathbf{12}}} + \frac{2 z_1^2 z_2^2}{z_2^4 - z_1^4} C^{(22)}_{\underline{\mathbf{12}}} + \frac{2 z_1^3 z_2}{z_2^4 - z_1^4} C^{(31)}_{\underline{\mathbf{12}}}, \tag{3.3a}$$
$$s_{\underline{\mathbf{12}}}(z_1, z_2) = C^{(00)}_{\underline{\mathbf{12}}}. \tag{3.3b}$$

One can check explicitly that the kernels (3.3) coincide exactly with the ones that would be obtained from the generalization of the alleviation procedure proposed in [1] to semi-symmetric space σ-models. This is simply a matter of replacing the twisted inner product on the twisted loop algebra considered in [11] by the trigonometric one and to compute the corresponding kernels as explained in [1].

An important property of the above r/s-matrix structure is that s is simply the projection onto the subalgebra $\mathfrak{g}$. In this case, the corresponding Poisson algebra (3.2) can be discretized following [6] by introducing a skew-symmetric solution $\alpha \in \operatorname{End}\mathfrak{g}$ of the modified classical Yang-Baxter equation on $\mathfrak{g}$. Then the matrices

$$a_{\underline{\mathbf{12}}} = (r+\alpha)_{\underline{\mathbf{12}}}, \qquad b_{\underline{\mathbf{12}}} = (-s-\alpha)_{\underline{\mathbf{12}}}, \qquad c_{\underline{\mathbf{12}}} = (-s+\alpha)_{\underline{\mathbf{12}}}, \qquad d_{\underline{\mathbf{12}}} = (r-\alpha)_{\underline{\mathbf{12}}},$$

satisfy all the requirements of [7, 8] in order to define the following consistent lattice algebra,

$$4\{\mathcal{L}^n_{\underline{\mathbf{1}}}, \mathcal{L}^m_{\underline{\mathbf{2}}}\} = a_{\underline{\mathbf{12}}}\mathcal{L}^n_{\underline{\mathbf{1}}}\mathcal{L}^m_{\underline{\mathbf{2}}}\delta_{mn} - \mathcal{L}^n_{\underline{\mathbf{1}}}\mathcal{L}^m_{\underline{\mathbf{2}}}d_{\underline{\mathbf{12}}}\delta_{mn} + \mathcal{L}^n_{\underline{\mathbf{1}}}b_{\underline{\mathbf{12}}}\mathcal{L}^m_{\underline{\mathbf{2}}}\delta_{m+1,n} - \mathcal{L}^m_{\underline{\mathbf{2}}}c_{\underline{\mathbf{12}}}\mathcal{L}^n_{\underline{\mathbf{1}}}\delta_{m,n+1}.$$

This algebra reduces to (3.2) in the continuum limit (see [1]). The corresponding algebra for the monodromy may be found in [1].

4 Conclusion

We have constructed a quadratic lattice Poisson algebra associated with the fermionic extension of the $(SO(4,1)\times SO(5))/[SU(2)]^4$ gauged WZW model with an integrable potential. The fact that one is able to write down such a lattice algebra is quite appealing and in sharp contrast with what happens for the canonical Poisson structure of the $AdS_5 \times S^5$ superstring [10]. Indeed, it brings hope of being able to construct a lattice quantum algebra related to the Pohlmeyer reduction of the $AdS_5 \times S^5$ superstring. The precise link of this Pohlmeyer reduction with the alleviation procedure presented in [1] is under study.

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