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POPULATIONS OF SOLUTIONS TO CYCLOTOMIC BETHE EQUATIONS

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ABSTRACT. We study solutions of the Bethe Ansatz equations for the cyclotomic Gaudin model of [VY14a]. We give two interpretations of such solutions: as critical points of a cyclotomic master function, and as critical points with cyclotomic symmetry of a certain "extended" master function. In finite types, this yields a correspondence between the Bethe eigenvectors and eigenvalues of the cyclotomic Gaudin model and those of an "extended" non-cyclotomic Gaudin model.

We proceed to define *populations* of solutions to the cyclotomic Bethe equations, in the sense of [MV04], for diagram automorphisms of Kac-Moody Lie algebras.

In the case of type A with the diagram automorphism, we associate to each population a vector space of quasi-polynomials with specified ramification conditions. This vector space is equipped with a $\mathbb{Z}_2$-gradation and a non-degenerate bilinear form which is (skew-)symmetric on the even (resp. odd) graded subspace. We show that the population of cyclotomic critical points is isomorphic to the variety of isotropic full flags in this space.

CONTENTS

1. INTRODUCTION

Let $\mathfrak{g}$ be a complex Kac-Moody Lie algebra and $\sigma : \mathfrak{g} \to \mathfrak{g}$ an automorphism of order $M \in \mathbb{Z}_{\geq 1}$. Let $\omega \in \mathbb{C}^\times$ be a primitive Mth root of unity. We may choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $\sigma(\mathfrak{h}) = \mathfrak{h}$. We have the canonical pairing $\langle \cdot, \cdot \rangle : \mathfrak{h}^* \otimes \mathfrak{h} \to \mathbb{C}$, and the simple roots $\alpha_i \in \mathfrak{h}^*$ and coroots $\alpha_i^\vee \in \mathfrak{h}$, where i runs over the set I of nodes of the Dynkin diagram.

Consider the following system of equations in $m \in \mathbb{Z}_{\geq 0}$ variables $\boldsymbol{t} = (t_1, \dots, t_m) \in \mathbb{C}^m$ and labels $\mathbf{c} = (\mathsf{c}(1), \dots, \mathsf{c}(m)) \in I^m$:

$$0 = \sum_{k=0}^{M-1} \sum_{i=1}^{N} \frac{\langle \sigma^k \Lambda_i, \alpha^\vee_{\mathsf{c}(j)} \rangle}{t_j - \omega^k z_i} - \sum_{k=0}^{M-1} \sum_{\substack{i=1 \\ i \neq j}}^{m} \frac{\langle \sigma^k \alpha_{\mathsf{c}(i)}, \alpha^\vee_{\mathsf{c}(j)} \rangle}{t_j - \omega^r t_i} + \frac{1}{t_j} \left(- \sum_{k=1}^{M-1} \frac{\langle \sigma^k \alpha_{\mathsf{c}(j)}, \alpha^\vee_{\mathsf{c}(j)}, \rangle}{1 - \omega^k} + \langle \Lambda_0, \alpha_{\mathsf{c}(j)} \rangle \right),$$

$$j = 1, \dots, m, \quad (1.1)$$

where $\Lambda_0, \Lambda_1, \dots, \Lambda_N \in \mathfrak{h}^*$ are weights (with $\sigma\Lambda_0 = \Lambda_0$) and $z_1, \dots, z_N$ are non-zero points in the complex plane whose orbits, under the action of the cyclic group $\omega^{\mathbb{Z}}$, are pairwise disjoint.

When $\sigma = \mathrm{id}$, $\omega = 1$ and $\Lambda_0 = 0$, these equations reduce to the following well-known set of equations in mathematical physics:

$$0 = \sum_{i=0}^{N} \frac{\langle \Lambda_i, \alpha^\vee_{\mathsf{c}(j)} \rangle}{t_j - z_i} - \sum_{\substack{i=1 \\ i \neq j}}^{m} \frac{\langle \alpha_{\mathsf{c}(i)}, \alpha^\vee_{\mathsf{c}(j)} \rangle}{t_j - t_i}, \quad j = 1, \dots, m. \tag{1.2}$$

These are the equations for critical points of the *master functions* [SV91] which appear in the integral expressions for hypergeometric solutions to the Knizhnik-Zamolodchikov (KZ) equations. They are also (at least for simple $\mathfrak{g}$) the *Bethe equations* of the quantum Gaudin model [RV95, FFR94, BF94].

The equations (1.1) were introduced (for simple $\mathfrak{g}$) in the study of cyclotomic generalizations of the Gaudin model [VY14a, VY14b] – see also [Skr06, CY07, Skr13] – as we recall in §3 below. Let us call them the *cyclotomic Bethe equations.* (Cyclotomic generalizations of the KZ equations were studied in [Enr08, Bro12], and appear in, in particular, the representation theory of cyclotomic Hecke algebras [VV10].)

It is natural to ask whether the cyclotomic Bethe equations (1.1) can be interpreted as the equations for critical points of some master function. In the present paper we begin by giving two different such interpretations. First, they are indeed the critical point equations for a *cyclotomic master function*, which we write down in (2.5). But they are also the equations for critical points with cyclotomic – more precisely $S_m \ltimes (\mathbb{Z}/M\mathbb{Z})^m$ – symmetry of what we call an *extended master function*, (2.11).

Master functions correspond to weighted configurations of hyperplanes. The cyclotomic master function corresponds to a hyperplane arrangement in $\mathbb{C}^m$ whose hyperplanes include $t_i = \omega^k t_j$, $1 \le i < j \le m$, for each $k \in \mathbb{Z}/M\mathbb{Z}$. By contrast, the extended master function corresponds to a hyperplane arrangement in $\mathbb{C}^{mM}$, but has only those hyperplanes corresponding to the type A root system, *i.e.* $t_i = t_j$, $1 \le i < j \le mM$, *etc.* Because the extended master function is a master function of this standard form, its critical point equations are the Bethe equations for a certain standard (*i.e.* non-cyclotomic) Gaudin model, which we call the *extended Gaudin model.* This observation leads to our first result: a correspondence between the spectrum of the cyclotomic Gaudin model and a "cyclotomic" part of the spectrum of the extended Gaudin model. See Theorem 3.4.

Solutions to the Bethe equations (1.2) form families called *populations.* Populations were first introduced in [ScV03, MV04], where a *generation procedure* was given which produces families of new solutions to the Bethe equations starting from a given solution. A population is then defined to be the Zariski closure of the set of all solutions to the Bethe equations obtained by repeated application of this generation procedure, starting from a given solution. It is known that if $\mathfrak{g}$ is simple then every population is isomorphic to the flag variety of the Langlands dual Lie algebra ${}^L\mathfrak{g}$. This was shown in [MV04] for types A, B, C and in all finite types in [MV05, Fre05]. (A population can also be understood as the variety of Miura opers with a given underlying oper; see [MV05, Fre05].)

In the present work our main goal is to initiate the study of *cyclotomic populations*: populations of solutions to the equations (1.1).

We formulate in §4 a definition of cyclotomic populations for $\mathfrak{g}$ a general Kac-Moody Lie algebra and σ any diagram automorphism of $\mathfrak{g}$ satisfying the *linking condition.* (We also place certain restrictions on the weight Λ_0; see 4.1.) The linking condition [FFS96] states that, for every node $i \in I$, the restriction of the Dynkin diagram to the orbit $\sigma^{\mathbb{Z}}(i)$ consists either of disconnected nodes (in which case i has *linking number* $L_i = 1$), or of a number of disconnected copies of the A_2 Dynkin diagram (in which case i has *linking number* $L_i = 2$). What the linking condition ensures is that it is possible to "fold" the Dynkin diagram by the automorphism σ. See §2.3 and [FFS96].

In §4 we define the cyclotomic population to be the Zariski closure of the set of all cyclotomic critical points obtained by repeated application of a certain "cyclotomic generation procedure", starting from a given cyclotomic critical point. So the key ingredient is this generation procedure. Let us describe it, in outline. There is an "elementary cyclotomic generation" step associated to each orbit $\sigma^{\mathbb{Z}}(i)$. There are two cases: $L_i = 1$ and $L_i = 2$.

First, suppose $i \in I$ is a node with linking number $L_i = 1$. A critical point $(\boldsymbol{t}, \mathsf{c})$ is represented by a tuple of polynomials, $\boldsymbol{y} = (y_i(x))_{i \in I}$, where the roots of the polynomial $y_i(x)$, $i \in I$, are the Bethe variables t_s of "colour" i, *i.e.* those such that $\mathsf{c}(s) = i$. Following [MV04], one defines a function of x,

$$y_i^{(i)}(x; c) := y_i(x) \int^x \xi^{\left\langle \Lambda_0, \alpha_i^\vee \right\rangle} T_i(\xi) \prod_{j \in I} y_j(\xi)^{-\left\langle \alpha_j, \alpha_i^\vee \right\rangle} d\xi + c y_i(x), \tag{1.3}$$

depending on a parameter $c \in \mathbb{C}$. Here $T_i(x)$, $i \in I$, are certain functions encoding the "frame" data *i.e.* the points $z_1, \ldots, z_N$ and the weights $\Lambda_1, \ldots, \Lambda_N$; see (4.5). The Bethe equations ensure that $y_i^{(i)}(x; c)$ is in fact a polynomial, and moreover that if we consider the new tuple $\boldsymbol{y}^{(i)}(c)$ in which $y_i(x)$ is replaced by $y_i^{(i)}(x; c)$, then for almost all values of c this new tuple again represents a solution to the Bethe equations. Call the replacement $\boldsymbol{y} \mapsto \boldsymbol{y}^{(i)}(c)$ *elementary generation in direction* i. Now suppose the initial tuple $\boldsymbol{y}$ represents a cyclotomic point. That means

$$y_{\sigma j}(\omega x) \simeq y_j(x), \qquad j \in I;$$

see Lemma 4.5. Since the orbit $\sigma^{\mathbb{Z}}(i)$ consists of disconnected nodes of the Dynkin diagram, the operations of elementary generation in the directions $\sigma^{\mathbb{Z}}(i)$ commute. By performing each of them once, in any order, we can arrange to arrive at a new cyclotomic point. See Theorem 4.6.

Next, suppose $i \in I$ is a node with linking number $L_i = 2$. Then for every copy of the A_2 diagram, with nodes say j and $\bar{\jmath}$, one must perform the sequence of generation steps $j, \bar{\jmath}, j$. Doing this for each copy of A_2 in turn, in any order, we can arrange to arrive at a new cyclotomic point. See Theorem 4.20.

When $L_i = 2$ there is a subtlety coming from our assumptions about the weight at the origin, Λ_0. Throughout §4, motivated by [VY14a], we assume that $\left\langle \Lambda_0, \alpha_i^\vee \right\rangle$ is non-integral when $L_i = 2$. That means that the expression (1.3) develops a branch point at the origin. The upshot is that at certain intermediate steps, the weight at the origin is shifted to $\mathsf{s}_i \cdot \Lambda_0$, before eventually being shifted back to Λ_0. See Proposition 4.10 and compare [MV08].

In either case, $L_i = 1$ or $L_i = 2$, we write $\boldsymbol{y}^{(i,\sigma)}(c)$ for the tuple of polynomials representing the new cyclotomic critical point. It depends on a single parameter c. The replacement $\boldsymbol{y} \mapsto \boldsymbol{y}^{(i,\sigma)}(c)$ is the elementary cyclotomic generation, in the direction of the orbit $\sigma^{\mathbb{Z}}(i)$.

To a critical point $(\boldsymbol{t}, \mathsf{c})$ represented by a tuple of polynomials $\boldsymbol{y}$ one can associate a weight Λ_∞. See (2.7) and (4.10). For fixed $\Lambda_0, \Lambda_1, \ldots, \Lambda_N$, we may regard Λ_∞ as encoding the number of roots t_s of each "colour" $i \in i$, *i.e.* the degrees of the polynomials $y_i(x)$. It is known that $\Lambda_\infty(\boldsymbol{y}^{(i)}(c))$ is equal either to $\Lambda_\infty(\boldsymbol{y})$ or to $\mathsf{s}_i \cdot \Lambda_\infty(\boldsymbol{y})$, where $\mathsf{s}_i \cdot$ denotes the shifted action of the Weyl reflection in root α_i. See [MV04]. We have an analogous statement in the cyclotomic case. Namely, there is a "folded" Weyl group W^σ with generators s_i^σ. See §2.3. And we show that $\Lambda_\infty(\boldsymbol{y}^{(i,\sigma)}(c))$ is equal either to $\Lambda_\infty(\boldsymbol{y})$ or to $\mathsf{s}_i^\sigma \cdot \Lambda_\infty(\boldsymbol{y})$. For the precise statement see Theorems 4.6 and 4.20.

We proceed in §5 to treat in detail the case of type A with the diagram automorphism.

Recall first from [MV04] the structure of populations in type A_R, $R \in \mathbb{Z}_{\geq 1}$, for the master functions associated to marked points $z_1, \ldots, z_N$ and integral dominant weights $\Lambda_1, \ldots, \Lambda_N$. In that setting, every population of critical points is isomorphic to a variety of full flags in a certain $R+1$-dimensional vector space $\mathcal{K}$ of polynomials. The ramification points of $\mathcal{K}$ are $z_1, \dots, z_N$ and ∞, and the ramification data at these points are specified by the weights $\Lambda_1, \dots, \Lambda_N$ and an integral dominant weight $\tilde{\Lambda}_\infty$. Given a full flag $\mathcal{F} = \{0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{R+1} = \mathcal{K}\}$ in $\mathcal{K}$, pick any basis $(u_i(x))_{i=1}^{R+1}$ of polynomials adjusted to this flag, *i.e.* such that $F_k = \operatorname{span}_{\mathbb{C}}(u_1(x), \dots, u_k(x))$. Then define a tuple of functions $\boldsymbol{y}^{\mathcal{F}} = (y_k^{\mathcal{F}}(x))_{k=1}^R$ by

$$y_k^{\mathcal{F}}(x) = \operatorname{Wr}(u_1(x), \dots, u_k(x))/(T_1^{k-1}(x)T_2^{k-2}(x)\dots T_{k-1}(x))$$

where – as in (1.3) above – the $(T_i(x))_{i=1}^R$ are functions encoding the "frame" data $z_1, \dots, z_N$ and $\Lambda_1, \dots, \Lambda_N$, and where $\operatorname{Wr}(u_1(x), \dots, u_k(x))$ denotes the Wronskian determinant. The ramification properties of $\mathcal{K}$ ensure that the $y_k^{\mathcal{F}}(x)$ are in fact polynomials. Moreover the map $\mathcal{F} \mapsto \boldsymbol{y}^{\mathcal{F}}$ is an isomorphism of varieties from the variety of full flags in $\mathcal{K}$ to the population associated with $\mathcal{K}$. The space $\mathcal{K}$ is the kernel of a certain linear differential operator $\mathcal{D}$ of order $R+1$ (essentially a type A oper). This operator $\mathcal{D}$ can be defined in terms of the $(T_i(x))_{i=1}^R$ together with the polynomials $(y_i(x))_{i=1}^R$ of (any) point in the population. (See §5.4.)

Now let us discuss how the picture changes in our present setting. For us, the weight at the origin Λ_0 need not be integer dominant. We assume it satisfies weaker assumptions given in (5.3). These assumptions mean that we are led to consider vector spaces $\mathcal{K}$ of *quasi*-polynomials: that is, polynomials in $x^{\frac{1}{2}}$. The local behaviour of these quasi-polynomials near the origin is encoded in Λ_0. The remaining ramification points are $z_1, \dots, z_N$, $-z_1, \dots, -z_N$, and ∞. See Definition 5.2.

The space of quasi-polynomials $\mathcal{K}$ admits a natural $\mathbb{Z}_2$ gradation $\mathcal{K} = \mathcal{K}_{\mathrm{O}} \oplus \mathcal{K}_{\mathrm{Sp}}$. We call flags which respect this gradation *decomposable*. Decomposable full flags are classified by their *type*; see §5.3. In particular the flags $\mathcal{F} \in FL_S(\mathcal{K})$ of a certain preferred type S, (5.14), are sent to polynomials under the map $\mathcal{F} \mapsto \boldsymbol{y}^{\mathcal{F}}$. This map of varieties $FL_S(\mathcal{K}) \to \mathbb{P}(\mathbb{C}[x])^R$ is an isomorphism onto its image. The *cyclotomic population* is then the set of cyclotomic tuples in this image, *i.e.* the set of tuples $\boldsymbol{y}^{\mathcal{F}}$, $\mathcal{F} \in FL_S(\mathcal{K})$, such that $y_i(x) \simeq y_{R+1-i}(-x)$, $i = 1, \dots, R$. The question is: which flags in $FL_S(\mathcal{K})$ map to cyclotomic tuples?

To answer this question we introduce the notion of a *cyclotomically self-dual space of quasi-polynomials*. The space $\mathcal{K}$ has a natural dual space $\mathcal{K}^\dagger$ of quasi-polynomials – see §5.5 – and we say $\mathcal{K}$ is cyclotomically self-dual if for all $v(x) \in \mathcal{K}$, $v(-x) \in \mathcal{K}^\dagger$. (Compare the very similar notion of a self-dual space of polynomials in [MV04].) We show that a sufficient condition for $\mathcal{K}$ to be cyclotomically self-dual is that there exists at least one full flag $\mathcal{F}$ in $\mathcal{K}$ such that $\boldsymbol{y}^{\mathcal{F}}$ is cyclotomic (Theorem 5.14). If $\mathcal{K}$ is cyclotomically self-dual then it admits a canonical non-degenerate bilinear form B. We show that, for all full flags $\mathcal{F}$ in $\mathcal{K}$, the tuple $\boldsymbol{y}^{\mathcal{F}}$ is cyclotomic if and only if $\mathcal{F}$ is isotropic with respect to B (Theorem 5.17).

Therefore the cyclotomic population is isomorphic to the variety $FL_S^\perp(\mathcal{K})$ of isotropic flags of type S in $\mathcal{K}$. The bilinear form B is symmetric on $\mathcal{K}_{\mathrm{O}}$ and skew-symmetric on $\mathcal{K}_{\mathrm{Sp}}$, and these subspaces are mutually orthogonal with respect to B (Theorem 5.23). Hence this variety $FL_S^\perp(\mathcal{K})$ is isomorphic to the direct product of spaces of isotropic flags $FL^\perp(\mathcal{K}_{\mathrm{Sp}}) \times FL^\perp(\mathcal{K}_{\mathrm{O}})$.

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2. Master functions and cyclotomic symmetry

2.1. **Kac-Moody algebras.** Let I be a finite set of indices and $A = (a_{i,j})_{i,j\in I}$ a generalized Cartan matrix, *i.e.* $a_{i,i} = 2$ and $a_{i,j} \in \mathbb{Z}_{\leq 0}$ whenever $i \neq j$, with $a_{i,j} = 0$ if and only if $a_{j,i} = 0$. Let $\mathfrak{g} := \mathfrak{g}(A)$ be the corresponding complex Kac-Moody Lie algebra [Kac83, §1], $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, and

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

a triangular decomposition. Let $\alpha_i \in \mathfrak{h}^*$, $\alpha_i^\vee \in \mathfrak{h}$, $i \in I$ be collections of simple roots and coroots respectively. We have $\dim \mathfrak{h} = |I| + \dim \ker A = 2|I| - \operatorname{rank} A$. By definition,

$$\langle \alpha_i, \alpha_j^\vee \rangle = a_{j,i},$$

where $\langle \cdot, \cdot \rangle : \mathfrak{h}^* \otimes \mathfrak{h} \to \mathbb{C}$ is the canonical pairing.

We assume that A is symmetrizable, *i.e.* there exists a diagonal matrix $D = \operatorname{diag}(d_i)_{i \in I}$, whose entries are coprime positive integers, such that the matrix $B = DA$ is symmetric. Let $(\cdot, \cdot)$ be the associated symmetric bilinear form on $\mathfrak{h}^*$. We have $(\alpha_i, \alpha_j) = d_i a_{i,j}$ and

$$\langle \lambda, \alpha_i^\vee \rangle = 2(\lambda, \alpha_i)/(\alpha_i, \alpha_i) \quad \text{for all } \lambda \in \mathfrak{h}^*.$$

The form $(\cdot, \cdot)$ is non-degenerate. Therefore it gives an identification $\mathfrak{h} \cong_{\mathbb{C}} \mathfrak{h}^*$ and hence a non-degenerate symmetric bilinear form on $\mathfrak{h}$ which we also write as $(\cdot, \cdot)$.

Let $\mathcal{P} := \{\lambda \in \mathfrak{h}^* : \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}\}$ be the integral weight lattice and $\mathcal{P}_+ := \{\lambda \in \mathfrak{h}^* : \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}\}$ the set of dominant integral weights.

Let $W \subset \operatorname{End}(\mathfrak{h}^*)$ be the Weyl group. It is generated by the reflections s_i, $i \in I$, given by

$$\mathsf{s}_i(\lambda) := \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i, \quad \lambda \in \mathfrak{h}^*.$$

Let $\rho \in \mathfrak{h}^*$ be a vector such that $\langle \rho, \alpha_i^\vee \rangle = 1$ for $i \in I$. We use $\cdot$ to denote the *shifted action* of the Weyl group, *i.e.*

$$\mathsf{s} \cdot \lambda := w(\lambda + \rho) - \rho, \quad \mathsf{s} \in W,\ \lambda \in \mathfrak{h}^*.$$

2.2. **Diagram automorphism.** Suppose σ is an automorphism of the Dynkin diagram [Kac83, §4.7] of A . That is, σ is a permutation of the index set I such that

$$a_{\sigma i, \sigma j} = a_{i,j} \,.$$

Let M be the order of σ and let $\omega \in \mathbb{C}^\times$ be a primitive Mth root of unity.

To such a permutation is associated a *diagram automorphism* $\mathfrak{g} \to \mathfrak{g}$ of the Kac-Moody Lie algebra [FFS96], which we shall also write as σ. We have

$$\sigma E_i = E_{\sigma i}, \quad \sigma F_i = F_{\sigma i}, \quad \sigma \alpha_i^\vee = \alpha_{\sigma i}^\vee, \quad i \in I$$

where $E_i \in \mathfrak{n}, F_i \in \mathfrak{n}^-$, $i \in I$, are a set of Chevalley generators of $[\mathfrak{g}, \mathfrak{g}]$. This defines σ on the derived subalgebra $[\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$. For the action of σ on the derivations *i.e.* on a complement of $[\mathfrak{g}, \mathfrak{g}]$ in $\mathfrak{g}$, see [FFS96, §3.2]. This action may be chosen to ensure that $\sigma : \mathfrak{g} \to \mathfrak{g}$ has order M and respects the bilinear form $(\cdot, \cdot)$ on $\mathfrak{h}$:

$$(\sigma X, \sigma Y) = (X, Y) \quad \text{for all} \quad X, Y \in \mathfrak{h}.$$

The action of σ on $\mathfrak{h}^*$ is defined by $\sigma\lambda := \lambda \circ \sigma^{-1}$ so that $\langle \sigma\lambda, \sigma X \rangle = \langle \lambda, X \rangle$ for all $\lambda \in \mathfrak{h}^*, X \in \mathfrak{h}$. Note that then $\sigma \alpha_i = \alpha_{\sigma i}$ for all $i \in I$.

Let $\mathfrak{g}^\sigma \subset \mathfrak{g}$ be the Lie subalgebra of elements invariant under σ. We have

$$\mathfrak{g}^\sigma = \mathfrak{n}_-^\sigma \oplus \mathfrak{h}^\sigma \oplus \mathfrak{n}_+^\sigma$$

with $\mathfrak{n}_\pm^\sigma = \mathfrak{g}^\sigma \cap \mathfrak{n}_\pm$ and $\mathfrak{h}^\sigma = \mathfrak{g}^\sigma \cap \mathfrak{h}$.

2.3. **The linking condition and the folded diagram.** For any $i \in I$ let

$$M_i := |\{i, \sigma i, \sigma^2 i, \ldots, \sigma^{M-1} i\}|$$

be the length of the orbit of the node i under the automorphism σ of the Dynkin diagram A. Define

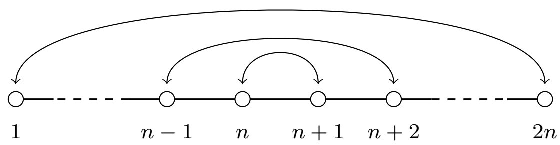
$$L_i := 1 - \sum_{k=1}^{M_i - 1} a_{\sigma^k i, i}. \tag{2.1}$$

Note that $L_i \geq 1$. Following [FFS96], we say that σ obeys the *linking condition* if and only if

$$L_i \leq 2 \quad \text{for all} \quad i \in I. \tag{2.2}$$

To understand the meaning of this condition, consider the restriction of the Dynkin diagram to the orbit of the node i. If $L_i = 1$ then this induced subgraph has no edges at all. If $L_i = 2$ then it consists of $M_i/2$ disconnected copies of the type A_2 Dynkin diagram.

Remark 2.1. If A is of finite type, then all diagram automorphisms obey the linking condition. Moreover, in all finite types except A_{2n}, $n \in \mathbb{Z}_{\geq 1}$, we in fact have $L_i = 1$ for every node i: that is, no two distinct nodes in the same σ-orbit are ever linked by an edge of the Dynkin diagram. In type A_{2n} the non-trivial diagram automorphism gives $L_i = 2$ for $i \in \{n, n+1\}$ and $L_i = 1$ otherwise:



Remark 2.2. If A is of affine type then all diagram automorphisms obey the linking condition with the following exception. In type $\mathrm{A}_n^{(1)}$, $n \in \mathbb{Z}_{\geq 2}$, let R be a generator of the cyclic subgroup C_{n+1} of the full automorphism group of the Dynkin diagram (which is the dihedral group D_{n+1}). Then R does not obey the linking condition. Indeed, the R-orbit of any node i is the whole diagram, and $L_i = 1 + n$.

Given any diagram automorphism satisfying the linking condition it is possible to define a *folded* Dynkin diagram. Let us make a choice of subset

$$I_\sigma \subseteq I \tag{2.3}$$

consisting of exactly one representative of each σ-orbit. Then the Cartan matrix $A^\sigma = (a^\sigma_{i,j})_{i,j \in I_\sigma}$ of the folded diagram is given by

$$a^\sigma_{i,j} = L_i \sum_{k=0}^{M_i - 1} a_{\sigma^k i, j}.$$

Remark 2.3. Compare §3.3 of [FFS96], noting that our convention $a_{j,i} = \langle \alpha_i, \alpha_j^\vee \rangle$ differs from that of [FFS96].

Lemma 2.4 ([FFS96])**.** *If σ obeys the linking condition then A^σ (and its transpose) is a symmetrizable Cartan matrix whose type (finite, affine, or indefinite) is the same as that of A.* ☐

For each $i \in I_\sigma$ let us define also

$$\alpha_i^{\vee,\sigma} := L_i \sum_{k=0}^{M_i - 1} \alpha^\vee_{\sigma^k i}$$

and

$$E_i^\sigma := \sum_{k=0}^{M_i - 1} E_{\sigma i}, \qquad F_i^\sigma := L_i \sum_{k=0}^{M_i - 1} F_{\sigma i}.$$

Then we have

$$\left[E_i^\sigma, F_j^\sigma\right] = \delta_{i,j} \alpha_i^{\vee,\sigma}, \quad \left[\alpha_i^{\vee,\sigma}, E_j^\sigma\right] = E_j^\sigma a^\sigma_{j,i}, \quad \left[\alpha_i^{\vee,\sigma}, F_j^\sigma\right] = -F_j^\sigma a^\sigma_{j,i} \qquad i, j \in I_\sigma. \tag{2.4}$$

Thus $\alpha_i^{\vee,\sigma}, E_i^\sigma, F_i^\sigma$, $i \in I_\sigma$ generate a copy of (the derived subalgebra of) the Kac-Moody Lie algebra $\mathfrak{g}(A^\sigma)$ inside $\mathfrak{g}^\sigma := \{X \in \mathfrak{g} : \sigma X = X\}$. Next, for all $i \in I_\sigma$, if we let

$$\alpha_i^\sigma := \frac{L_i}{M_i} \sum_{k=0}^{M_i - 1} \alpha_{\sigma^k i} \in \mathfrak{h}^*$$

then $\langle \alpha_i^\sigma, \alpha_j^{\vee,\sigma} \rangle = a_{j,i}^\sigma$. Define W^σ to be the group generated by the elements $\mathsf{s}_i^\sigma \in \mathrm{End}(\mathfrak{h}^*)$ given by

$$\mathsf{s}_i^\sigma(\lambda) := \lambda - \langle \lambda, \alpha_i^{\vee,\sigma} \rangle \alpha_i^\sigma, \qquad i \in I_\sigma.$$

Lemma 2.5. *W^σ is a subgroup of W. Indeed, we have*

$$\mathsf{s}_i^\sigma = \begin{cases} \prod_{k=0}^{M_i-1} \mathsf{s}_{\sigma^k i} & L_i = 1 \\ \left(\prod_{k=0}^{M_i/2-1} \mathsf{s}_{\sigma^k i}\right) \left(\prod_{k=0}^{M_i/2-1} \mathsf{s}_{\sigma^{k+M_i/2} i}\right) \left(\prod_{k=0}^{M_i/2-1} \mathsf{s}_{\sigma^k i}\right) & L_i = 2. \end{cases}$$

□

2.4. **The cyclotomic master function.** Let $\mathbf{\Lambda} = (\Lambda_i)_{i=1}^N$ be a collection of $N \in \mathbb{Z}_{\geq 0}$ integral dominant weights $\Lambda_i \in \mathcal{P}_+$. Let $\boldsymbol{z} = (z_i)_{i=1}^N$ be a collection of nonzero points $z_i \in \mathbb{C}^\times$ such that $\omega^{\mathbb{Z}} z_i \cap \omega^{\mathbb{Z}} z_j = \emptyset$ whenever $i \neq j$. We shall call Λ_i *the weight at* z_i.

In addition, we pick a weight $\Lambda_0 \in \mathfrak{h}^{\sigma,*}$. We call Λ_0 *the weight at the origin.*

Let $\mathbf{c} = (\mathsf{c}(j))_{j=1}^m$ be an m-tuple of elements of I, and introduce variables $\boldsymbol{t} = (t_j)_{j=1}^m$. We shall say that t_j is a *variable of colour* $\mathsf{c}(j)$.

We define the *cyclotomic master function* $\Phi = \Phi_{\mathfrak{g},\sigma}(\boldsymbol{t}; \mathbf{c}; \boldsymbol{z}; \mathbf{\Lambda}, \Lambda_0)$ associated to these data to be

$$\begin{aligned}\Phi := \sum_{i=1}^N \left(\frac{1}{2} \sum_{k=1}^{M-1} (\Lambda_i, \sigma^k \Lambda_i) + (\Lambda_i, \Lambda_0) \right) \log z_i + \sum_{k=0}^{M-1} \sum_{1 \leq i < j \leq n} (\Lambda_i, \sigma^k \Lambda_j) \log(z_i - \omega^k z_j) \\ - \sum_{k=0}^{M-1} \sum_{i=1}^N \sum_{j=1}^m (\alpha_{\mathsf{c}(j)}, \sigma^k \Lambda_i) \log(t_j - \omega^k z_i) \\ + \sum_{k=0}^{M-1} \sum_{1 \leq i < j \leq m} (\alpha_{\mathsf{c}(i)}, \sigma^k \alpha_{\mathsf{c}}(j)) \log(t_i - \omega^k t_j) + \sum_{i=1}^m \left(\frac{1}{2} \sum_{k=1}^{M-1} (\alpha_{\mathsf{c}(i)}, \sigma^k \alpha_{\mathsf{c}(i)}) - (\alpha_{\mathsf{c}(i)}, \Lambda_0) \right) \log t_i\end{aligned} \tag{2.5}$$

A point $\boldsymbol{t}$ with complex coordinates is called a *critical point* of the cyclotomic master function if

$$\frac{\partial \Phi}{\partial t_i} = 0, \qquad i = 1, \ldots, m,$$

or equivalently (in view of Lemma 2.6 below) if the following equations are satisfied:

$$0 = \sum_{k=0}^{M-1} \sum_{i=1}^N \frac{(\alpha_{\mathsf{c}(j)}, \sigma^k \Lambda_i)}{t_j - \omega^k z_i} - \sum_{k=0}^{M-1} \sum_{\substack{i=1 \\ i \neq j}}^m \frac{(\alpha_{\mathsf{c}(j)}, \sigma^k \alpha_{\mathsf{c}(i)})}{t_j - \omega^r t_i} + \frac{1}{t_j} \left(- \sum_{k=1}^{M-1} \frac{(\alpha_{\mathsf{c}(j)}, \sigma^k \alpha_{\mathsf{c}(j)})}{1 - \omega^k} + (\alpha_{\mathsf{c}(j)}, \Lambda_0) \right) \tag{2.6}$$

for $j = 1, \ldots, m$. Call this system of equations the *cyclotomic Bethe equations.*

Lemma 2.6. *For any* $\lambda \in \mathfrak{h}^*$, $\sum_{k=1}^{M-1} \frac{(\lambda, \sigma^k \lambda)}{1-\omega^k} = \frac{1}{2} \sum_{k=1}^{M-1} \left(\frac{(\lambda,\sigma^k\lambda)}{1-\omega^k} + \frac{(\lambda,\sigma^k\lambda)}{1-\omega^{-k}} \right) = \frac{1}{2} \sum_{k=1}^{M-1} (\lambda, \sigma^k \lambda)$. □

Define Λ_∞, *the weight at infinity*, to be

$$\Lambda_\infty := \Lambda_0 + \sum_{k=0}^{M-1} \sum_{i=1}^N \Lambda_{\sigma^k(i)} - \sum_{k=0}^{M-1} \sum_{i=1}^m \alpha_{\sigma^k \mathsf{c}(i)}. \tag{2.7}$$

The group S_m acts on pairs of m-tuples $(\boldsymbol{t}, \mathbf{c})$ by permuting indices:

$$\rho.(\boldsymbol{t}, \mathbf{c}) = \Big(\big(t_{\rho^{-1}(1)}, \ldots, t_{\rho^{-1}(m)} \big), \big(\mathsf{c}(\rho^{-1}(1)), \ldots, \mathsf{c}(\rho^{-1}(m)) \big) \Big).$$

The group $\mathbb{Z}/M\mathbb{Z}$ acts on pairs $(t, \mathsf{c}) \in \mathbb{C} \times I$ by $k.(t, \mathsf{c}) = (\omega^k t_i, \sigma^k \mathsf{c})$. This gives rise to an action of the wreath product $S_m \wr (\mathbb{Z}/M\mathbb{Z}) := S_m \ltimes (\mathbb{Z}/M\mathbb{Z})^m$ on pairs of tuples $(\boldsymbol{t}, \mathbf{c}) \in \mathbb{C}^m \times I^m$.

Lemma 2.7. *Up to an additive constant, the cyclotomic master function Φ is invariant under the pull-back of this action of $S_m \wr (\mathbb{Z}/M\mathbb{Z})$. In particular, if $\boldsymbol{t}$ is a critical point of $\Phi(\boldsymbol{t};\mathsf{c})$ then $X.\boldsymbol{t}$ is a critical point of $\Phi(X.\boldsymbol{t};X.\mathsf{c})$, for all $X \in S_m \wr (\mathbb{Z}/M\mathbb{Z})$.* □

2.5. **The extended master function.** The equations (2.6) admit another, closely related, interpretation. Recall the definition of the (usual) master function [SV91]. Namely, let $\tilde{\boldsymbol{\Lambda}} = (\tilde\Lambda_i)_{i=0}^{\tilde N}$ be a collection of $\tilde N + 1 \in \mathbb{Z}_{\geq 0}$ weights $\tilde\Lambda_i \in \mathfrak{h}^*$, and let $\tilde{\boldsymbol{z}} = (\tilde z_i)_{i=0}^{\tilde N}$ be a collection of nonzero points $\tilde z_i \in \mathbb{C}^\times$. Pick $\tilde m \in \mathbb{Z}_{\geq 0}$, let $\mathsf{c} = (\mathsf{c}(j))_{j=1}^{\tilde m}$ be an $\tilde m$-tuple of elements of I and introduce variables $\boldsymbol{t} = (t_j)_{j=1}^{\tilde m}$. The *master function* associated to these data is

$$\widetilde\Phi := \sum_{0\leq i<j\leq \tilde N} (\tilde\Lambda_i,\tilde\Lambda_j)\log(\tilde z_i - \tilde z_j) - \sum_{i=0}^{\tilde N}\sum_{j=1}^{\tilde m}(\alpha_{\mathsf{c}(j)},\tilde\Lambda_i)\log(t_j-\tilde z_i) + \sum_{1\leq i<j\leq \tilde m}(\alpha_{\mathsf{c}(i)},\alpha_{\mathsf{c}(j)})\log(t_i-t_j). \tag{2.8}$$

It is a function of the variables $\boldsymbol{t}$, depending on the parameters c, $\tilde{\boldsymbol{z}}$ and $\tilde{\boldsymbol{\Lambda}}$. The critical points of the master function are those points $\boldsymbol{t}$ with complex coordinates such that $\partial\widetilde\Phi/\partial t_j = 0$ for $j = 1,\dots,\tilde m$, *i.e.* those points such that the following equations are satisfied:

$$0 = \sum_{i=0}^{\tilde N} \frac{(\alpha_{\mathsf{c}(j)},\tilde\Lambda_i)}{t_j - \tilde z_i} - \sum_{\substack{i=1\\ i\neq j}}^{\tilde m}\frac{(\alpha_{\mathsf{c}(j)},\alpha_{\mathsf{c}(i)})}{t_j-t_i}, \quad j=1,\dots,\tilde m. \tag{2.9}$$

In this paper we are concerned with the following special case. Let $\tilde N = NM$, choose $(\tilde z_i)_{i=0}^{NM}$ to be

$$\tilde z_0 = 0, \quad \tilde z_{k+Mi} = \omega^k z_i, \qquad k = 0,1,\dots,M-1,\ i=1,\dots,N, \tag{2.10a}$$

and choose the weights at these points to be

$$\tilde\Lambda_0 = \Lambda_0, \quad \tilde\Lambda_{k+Mi} = \sigma^k\Lambda_i \tag{2.10b}$$

where z_i, Λ_i, $i=1,\dots,N$, and Λ_0 are as in §2.4. We call the master function in this case the *extended master function*, $\widehat\Phi = \widehat\Phi_{\mathfrak{g},\sigma}(\boldsymbol{t};\mathsf{c};\boldsymbol{z};\boldsymbol{\Lambda};\Lambda_0)$. It is given by

$$\begin{aligned}\widehat\Phi := &\sum_{k=0}^{M-1}\sum_{i=1}^{N}(\Lambda_0,\sigma^k\Lambda_i)\log(-\omega^k z_i) + \sum_{k,l=0}^{M-1}\sum_{1\leq i<j\leq n}(\sigma^k\Lambda_i,\sigma^l\Lambda_j)\log(\omega^k z_i - \omega^l z_j)\\ &+ \sum_{0\leq k<l\leq T-1}\sum_{i=1}^{N}(\sigma^k\Lambda_i,\sigma^l\Lambda_i)\log(\omega^k-\omega^l)z_i\\ &-\sum_{j=1}^{\tilde m}(\alpha_{\mathsf{c}(j)},\Lambda_0)\log(t_j) - \sum_{k=0}^{M-1}\sum_{i=1}^{N}\sum_{j=1}^{\tilde m}(\alpha_{\mathsf{c}(j)},\omega^k\Lambda_i)\log(t_j-\omega^k z_i) + \sum_{1\leq i<j\leq\tilde m}(\alpha_{\mathsf{c}(i)},\alpha_{\mathsf{c}(j)})\log(t_i-t_j).\end{aligned} \tag{2.11}$$

and the critical point equations (2.9) take the form

$$0 = \sum_{k=0}^{M-1}\sum_{i=1}^{N}\frac{(\alpha_{\mathsf{c}(j)},\sigma^k\Lambda_i)}{t_j-\omega^k z_i} + \frac{(\alpha_{\mathsf{c}(j)},\Lambda_0)}{t_j} - \sum_{\substack{i=1\\ i\neq j}}^{\tilde m}\frac{(\alpha_{\mathsf{c}(j)},\alpha_{\mathsf{c}(i)})}{t_j-t_i}, \quad j=1,\dots,\tilde m. \tag{2.12}$$

The group $S_{\tilde m}$ acts on pairs of $\tilde m$-tuples $(\boldsymbol{t},\mathsf{c})$ by permuting indices:

$$\rho.(\boldsymbol{t},\mathsf{c}) = \Big(\big(t_{\rho^{-1}(1)},\dots,t_{\rho^{-1}(\tilde m)}\big),\big(\mathsf{c}(\rho^{-1}(1)),\dots,\mathsf{c}(\rho^{-1}(\tilde m))\big)\Big). \tag{2.13}$$

Lemma 2.8. *Any master function of the form (2.8) is invariant under the pull-back of this action of $S_{\tilde m}$. In particular the extended master function (2.11) is invariant.* □

Let us call a point $(\boldsymbol{t}, \mathsf{c}) \in \mathbb{C}^{\tilde{m}} \times I^{\tilde{m}}$ a *cyclotomic point* if we have $\tilde{m} = Mm$ for some $m \in \mathbb{Z}_{\geq 0}$ and, by acting with some permutation in $S_{\tilde{m}}$, we can arrange that

$$t_{i+mk} = \omega^k t_i \qquad \mathsf{c}(i + mk) = \sigma^k \mathsf{c}(i), \qquad i = 1, \dots, m, \; k = 0, \dots, M - 1. \tag{2.14}$$

Lemma 2.9. *This point $(t_i)_{i=1}^{\tilde{m}}$ is a critical point of the extended master function if and only if $(t_i)_{i=1}^{m}$ is a critical point of the cyclotomic master function, i.e. $(t_i)_{i=1}^{m}$ obeys* (2.6)*.*

Proof. Given (2.14), the equation (2.12) for t_j is nothing but the corresponding equation in (2.6) and the equation for t_{j+km}, $k = 1, \dots, M-1$, is actually the same equation up to an overall factor of ω^{-k}. (To see this one must use the compatibility of σ with the inner product: $(\sigma x, y) = (x, \sigma^{-1} y)$.) $\square$

Thus, the cyclotomic Bethe equations (2.6) are also the equations for cyclotomic critical points of the extended master function.

3. Gaudin models and the Bethe ansatz equations

Our first result, Theorem 3.4, concerns the relationship between critical points and the eigenvalues of Gaudin Hamiltonians. Suppose, for this section only, that the Cartan matrix is of finite type, *i.e.* that $\mathfrak{g}$ is semisimple, and that σ is an automorphism of $\mathfrak{g}$ of order $M > 1$. Recall [Gau76, Gau83] that the *quadratic Gaudin Hamiltonians* are the following $\tilde{N} + 1$ elements of $U(\mathfrak{g})^{\otimes(\tilde{N}+1)}$:

$$\tilde{\mathcal{H}}^{(i)} := \sum_{\substack{j=0\\ j\neq i}}^{\tilde{N}} \sum_{a=1}^{\dim \mathfrak{g}} \frac{I^{a(i)} I_a^{(j)}}{\tilde{z}_i - \tilde{z}_j}, \qquad i = 0, 1, \dots, \tilde{N},$$

where I_a, $a = 1, \dots, \dim \mathfrak{g}$, is a basis of $\mathfrak{g}$, I^a is the dual basis with respect to the non-degenerate invariant bilinear form $(\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, and we write $X^{(i)}$ for X acting in the ith tensor factor. (For convenience we number these factors starting from 0.)

For $\Lambda \in \mathfrak{h}^*$, let M_Λ denote the Verma module over $\mathfrak{g}$ with highest weight Λ, $M_\Lambda := \operatorname{Ind}_{\mathfrak{h}\oplus\mathfrak{n}_+}^{\mathfrak{g}} \mathbb{C}\mathsf{v}_\Lambda$. Let us represent the $\tilde{\mathcal{H}}^{(i)}$ as linear maps in $\operatorname{End}\left(\bigotimes_{i=0}^{\tilde{N}} M_{\tilde{\Lambda}_i}\right)$. Then the following can be shown using the techniques of the Bethe Ansatz.

Theorem 3.1 ([BF94, RV95])**.** *To any critical point $\boldsymbol{t}$ of the master function $\widetilde{\Phi}$, i.e. to any solution to the equations* (2.9)*, there corresponds a simultaneous eigenvector $\tilde{\psi}_{\boldsymbol{t}}$ of the linear operators $\tilde{\mathcal{H}}^{(i)} \in \operatorname{End}(\bigotimes_{i=0}^{\tilde{N}} M_{\tilde{\Lambda}_i})$. For each $i = 0, \dots, \tilde{N}$ the eigenvalue of $\tilde{\mathcal{H}}^{(i)}$ on $\tilde{\psi}_{\boldsymbol{t}}$ is*

$$\tilde{E}^{(i)} := \frac{\partial \widetilde{\Phi}}{\partial \tilde{z}_i} = \sum_{\substack{j=0\\ j\neq i}}^{\tilde{N}} \frac{(\tilde{\Lambda}_i, \tilde{\Lambda}_j)}{\tilde{z}_i - \tilde{z}_j} - \sum_{j=1}^{\tilde{m}} \frac{(\Lambda_i, \alpha_{\mathsf{c}(j)})}{\tilde{z}_i - t_j}. \tag{3.1}$$

The eigenvector $\tilde{\psi}_{\boldsymbol{t}}$ is given explicitly by

$$\tilde{\psi}_{\boldsymbol{t}} = \sum_{\boldsymbol{n} \in P_{\tilde{m}, \tilde{N}+1}} \bigotimes_{i=0}^{\tilde{N}} \frac{F_{c(n_1^i)} F_{c(n_2^i)} \dots F_{c(n_{p_i-1}^i)} F_{c(n_{p_i}^i)} \mathsf{v}_{\tilde{\Lambda}_i}}{\left(w_{n_1^i} - w_{n_2^i}\right) \dots \left(w_{n_{p_i-1}^i} - w_{n_{p_i}^i}\right)\left(w_{n_{p_i}^i} - z_i\right)}. \tag{3.2}$$

where the sum $\boldsymbol{n} \in P_{\tilde{m}, \tilde{N}+1}$ is over ordered partitions of the labels $\{1, \dots, \tilde{m}\}$ into $\tilde{N} + 1$ parts. $\square$

The fact that this simultaneous eigenvector is nonzero is proved for $\mathfrak{g} = \mathfrak{sl}_n$ nondegenerate critical points in [MV00], for $\mathfrak{g} = \mathfrak{sl}_n$ isolated critical points in [MTV06], and for semisimple $\mathfrak{g}$ and isolated critical points in [Var11].

In [VY14a][1], B. Vicedo and one of the present authors defined *cyclotomic Gaudin Hamiltonians*. The quadratic cyclotomic Gaudin Hamiltonians are the elements of $U(\mathfrak{g})^{\otimes N}$ given by

$$\mathcal{H}_i := \sum_{p=0}^{M-1} \sum_{\substack{j=1 \\ j\neq i}}^{N} \sum_{a=1}^{\dim\mathfrak{g}} \frac{I^{a(i)}\sigma^p I_a^{(j)}}{z_i - \omega^{-p} z_j} + \frac{1}{z_i} \sum_{p=1}^{M-1} \sum_{a=1}^{\dim\mathfrak{g}} \frac{I^{a(i)}\sigma^p I_a^{(i)}}{(1-\omega^p)}, \qquad i = 1, \dots, n.$$

Let us assign to the point z_i the Verma module M_{Λ_i}, $\Lambda_i \in \mathfrak{h}^*$. In other words, let us represent these Hamiltonians as linear maps

$$\mathcal{H}^{(i)} \in \mathrm{End}\left(\bigotimes_{i=1}^{N} M_{\Lambda_i}\right), \qquad i = 1, \dots, N. \tag{3.3}$$

Let (in this section, §3) $\Lambda_0 \in \mathfrak{h}^{\sigma,*}$ be the weight given by

$$\Lambda_0(h) := \sum_{r=1}^{M-1} \frac{\mathrm{tr}_{\mathfrak{n}}(\sigma^{-r}\mathrm{ad}_h)}{1-\omega^r}. \tag{3.4}$$

Theorem 3.2 ([VY14a])**.** *To any critical point of the cyclotomic master function, i.e. to any solution $\boldsymbol{t}$ to the equations* (2.6)*, there corresponds a simultaneous eigenvector $\psi_{\boldsymbol{t}}$ of the linear operators $\mathcal{H}^{(i)}$, $i = 1, \dots, N$. The eigenvalue of $\mathcal{H}^{(i)}$ on $\psi_{\boldsymbol{t}}$ is*

$$E^{(i)} := \frac{\partial\Phi}{\partial z_i} = \sum_{\substack{j=1 \\ j\neq i}}^{N} \sum_{s=0}^{M-1} \frac{(\Lambda_i, \sigma^s\Lambda_j)}{z_i - \omega^s z_j} - \sum_{j=1}^{m} \sum_{s=0}^{M-1} \frac{(\Lambda_i, \sigma^s\alpha_{\mathsf{c}(j)})}{z_i - \omega^s t_j} + \frac{1}{z_i}\left((\Lambda_i, \Lambda_0) + \sum_{s=1}^{M-1} \frac{(\Lambda_i, \sigma^s\Lambda_i)}{1-\omega^s}\right). \tag{3.5}$$

The explicit form of the eigenvector $\psi_{\boldsymbol{t}}$ is

$$\psi_{\boldsymbol{t}} = \sum_{\substack{\boldsymbol{n}\in P_{m,N} \\ (k_1,\dots,k_m)\in\mathbb{Z}_M^m}} \bigotimes_{i=1}^{N} \frac{\check\sigma^{k_{n^i_1}}(F_{c(n^i_1)})\check\sigma^{k_{n^i_2}}(F_{c(n^i_2)}) \dots \check\sigma^{k_{n^i_{p_i-1}}}(F_{c(n^i_{p_i-1})})\check\sigma^{k_{n^i_{p_i}}}(F_{c(n^i_{p_i})})\mathsf{v}_{\Lambda_i}}{\left(\omega^{k_{n^i_1}} w_{n^i_1} - \omega^{k_{n^i_2}} w_{n^i_2}\right) \dots \left(\omega^{k_{n^i_{p_i-1}}} w_{n^i_{p_i-1}} - \omega^{k_{n^i_{p_i}}} w_{n^i_{p_i}}\right)\left(\omega^{k_{n^i_{p_i}}} w_{n^i_{p_i}} - z_i\right)} \tag{3.6}$$

where $\check\sigma(X) := \omega\sigma(X)$. □

On the other hand, consider the (usual) quadratic Gaudin Hamiltonians in the special case (2.10). We refer to this situation as the *extended Gaudin model*, and write $\tilde{\mathcal{H}}^{(i)}$ as $\mathcal{H}^{(i)}_{ext}$. Note that

$$\mathcal{H}^{(i)}_{ext} \in \mathrm{End}\left(M_{\Lambda_0} \otimes \bigotimes_{k=0}^{M-1} \bigotimes_{i=1}^{N} M_{\sigma^k\Lambda_i}\right), \qquad i = 0, 1, \dots, NM. \tag{3.7}$$

The following is then a corollary of Theorem 3.1.

Corollary 3.3. *To any critical point of the cyclotomic master function, i.e. to any solution $\boldsymbol{t}$ to the equations* (2.6)*, there corresponds a simultaneous eigenvector of the linear operators $\mathcal{H}^{(i)}_{ext}$, $i = 0, 1, \dots, nM$, such that $\mathcal{H}^{(0)}_{ext}$ has eigenvalue zero and, for each $k = 0, \dots, M-1$ and $i = 1, \dots, N$, the eigenvalue of $\mathcal{H}^{(k+Mi)}_{ext}$ is given by $\omega^{-k}E_i$ with E_i as in* (3.5)*.*

[1]In [VY14a] $\sigma : \mathfrak{g} \to \mathfrak{g}$ is allowed to be any automorphism commuting with the Cartan involution, not necessarily a diagram involution. A posteriori the Bethe equations and energy eigenvalues depend on the inner part of σ only through the definition of Λ_0, (3.4).

Proof. Let $\boldsymbol{t}$ be the corresponding (by Lemma 2.9) cyclotomic critical point of the extended master function $\widehat{\Phi}$. Then the result is a special case of Theorem 3.1, by substituting (2.10) and (2.14) into (3.1). (To see that $\mathcal{H}^{(0)}_{ext}$ has eigenvalue zero note that

$$\sum_{i=1}^{N}\sum_{s=0}^{M-1}\frac{(\Lambda_0,\sigma^s\Lambda_i)}{0-\omega^s z_i}-\sum_{j=1}^{m}\sum_{s=0}^{M-1}\frac{(\Lambda_0,\sigma^s\alpha_{\mathsf{c}(j)})}{0-\omega^s t_j}=0$$

because $\sum_{s=0}^{M-1}\omega^{-s}\sigma^{-s}\Lambda_0=\Lambda_0\sum_{s=0}^{M-1}\omega^{-s}=0$ since $\sigma\Lambda_0=\Lambda_0$ and $M>1$.) □

In summary, we have the following observation.

Theorem 3.4. *To any critical point of the cyclotomic master function there corresponds* both *a simultaneous eigenvector* (3.6) *of the Hamiltonians $\mathcal{H}^{(i)}$ of the cyclotomic Gaudin model and a simultaneous eigenvector* (3.2) *of the Hamiltonians $\mathcal{H}^{(i)}_{ext}$ of the extended Gaudin model, $i=1,\dots,n$, with the corresponding eigenvalues equal and in both cases being given by* (3.5). □

Remark 3.5. The operators $\mathcal{H}^{(i)}$ and $\mathcal{H}^{(i)}_{ext}$ are acting in different spaces, (3.3) and (3.7) respectively. It would be interesting to relate these operators by some means independent of the Bethe ansatz.

4. Cyclotomic generation procedure

In [ScV03, MV04] a procedure was introduced which generates new critical points of master functions starting from a given initial critical point. There is an "elementary generation" step associated to each $i\in I$. The Zariski closure of the collection of all critical points obtained by recursively applying elementary generations in all possible ways is called the "population" to which the initial critical point belongs.

The extended master functions, (2.11) above, are master functions of the standard form (unlike the cyclotomic master functions (2.5)). Modulo subtleties coming from the fact that the weight Λ_0 at the origin need not be dominant integral, that means the generation procedure can be applied.

In this section we describe this generation procedure and go on to show how, given a cyclotomic critical point, one can obtain new cyclotomic critical points by applying the elementary generation steps in certain carefully chosen combinations. The resulting collections of cyclotomic critical points will be called "cyclotomic populations".

4.1. **Conditions on** Λ_0.

In the remainder of the paper we assume that σ is a diagram automorphism obeying the linking condition (2.2). That means for each $i\in I$, either $L_i=1$ or $L_i=2$.

In addition, in this section, §4, we place the following conditions on the weight $\Lambda_0\in\mathfrak{h}^{\sigma,*}$.

For each $i\in I$ such that $L_i=1$, we suppose that

$$\langle\Lambda_0,\alpha_i^\vee\rangle\in\mathbb{Z}_{\geq 0} \tag{4.1}$$

and

$$\langle\Lambda_0,\alpha_i^\vee\rangle+1\equiv 0\mod M/M_i. \tag{4.2}$$

For each $i\in I$ such that $L_i=2$, we suppose that

$$2\langle\Lambda_0,\alpha_i^\vee\rangle+1\in\mathbb{Z}_{\geq 0}. \tag{4.3}$$

Remark 4.1. One can verify that these conditions are satisfied by the weight Λ_0 of (3.4) in the case of diagram automorphisms of finite-type Dynkin diagrams. Our assumptions on Λ_0 in the treatment of type A_R in §5 below are weaker.

4.2. **Tuples of polynomials.** To any pair $(\boldsymbol{t};\mathsf{c})$ with $\boldsymbol{t}\in\mathbb{C}^{\tilde m}$ and $\mathsf{c}\in I^{\tilde m}$, we may associate a tuple of polynomials $\boldsymbol{y}=(y_1(x),\dots,y_r(x))$, given by

$$y_i(x) := \prod_{\substack{j=1\\ \mathsf{c}(j)=i}}^{\tilde m} (x-t_j), \qquad i\in I. \tag{4.4}$$

We say that this tuple $\boldsymbol{y}$ *represents the pair* $(\boldsymbol{t};\mathsf{c})$. We consider each coordinate $y_i(x)$ only up to multiplication by a non-zero complex number, since we are only concerned with their zeros. So the tuple $\boldsymbol{y}$ defines a point in the direct product $\mathbb{P}(\mathbb{C}[x])^{|I|}$ of $|I|$ copies of the projective space $\mathbb{P}(\mathbb{C}[x])$, where $\mathbb{C}[x]$ is the vector space of complex polynomials in x.

Conversely, given any $\boldsymbol{y}\in\mathbb{P}(\mathbb{C}[x])^{|I|}$ we may extract the pair $(\boldsymbol{t};\mathsf{c})\in\mathbb{C}^{\tilde m}\times I^{\tilde m}$ such that (4.4) holds. This pair is unique up to permutation by an element of $S_{\tilde m}$; see (2.13).

Define $T_i(x)$, $i\in I$, to be

$$T_i(x) := \prod_{s=1}^{N}\prod_{k=0}^{M-1}(x-\omega^k z_s)^{\left\langle \sigma^k\Lambda_s,\alpha_i^\vee\right\rangle}. \tag{4.5}$$

We say that a tuple of polynomials $\boldsymbol{y}=(y_i(x))_{i\in I}\in\mathbb{P}(\mathbb{C}[x])^{|I|}$ is *generic (with respect to* $(T_i(x))_{i\in I}$*)* if for each $i\in I$, $y_i(x)$ has no root in common with $T_i(x)$, or with any $y_j(x)$, $j\in I\setminus\{i\}$, such that $\left\langle\alpha_j,\alpha_i^\vee\right\rangle\neq 0$.

Note that if $\boldsymbol{y}$ represents a critical point of the extended master function $\widehat{\Phi}(\boldsymbol{t};\mathsf{c};\boldsymbol{z};\boldsymbol{\Lambda})$, (2.11), *i.e.* its roots obey (2.12), then the tuple $\boldsymbol{y}$ must be generic.

4.3. **Elementary generation: the $L_i=1$ case.** Throughout this subsection, we suppose $i\in I$ is such that $L_i=1$. That means that the simple roots $\alpha_{\sigma^k i}$, $i=1,\dots,M_i$, are mutually orthogonal. Equivalently it means that the reflections $\mathsf{s}_{\sigma^k i}\in W$, $i=1,\dots,M_i$, are mutually commuting.

Let $y_i^{(i)}(x)$ be of the form

$$y_i^{(i)}(x) = y_i(x)\int^x \xi^{\left\langle\Lambda_0,\alpha_i^\vee\right\rangle}T_i(\xi)\prod_{j\in I}y_j(\xi)^{-\left\langle\alpha_j,\alpha_i^\vee\right\rangle}d\xi, \tag{4.6}$$

so that $y_i^{(i)}(x)$ is a solution to the equation

$$\mathrm{Wr}(y_i(x),y_i^{(i)}(x)) = x^{\left\langle\Lambda_0,\alpha_i^\vee\right\rangle}T_i(x)\prod_{j\in I\setminus\{i\}}y_j(x)^{-\left\langle\alpha_j,\alpha_i^\vee\right\rangle}, \tag{4.7}$$

where $\mathrm{Wr}(f(x),g(x)) := f(x)g'(x)-f'(x)g(x)$ denotes the Wronskian determinant.

Proposition 4.2. *If $\boldsymbol{y}$ represents a critical point then $y_i^{(i)}(x)$ is a polynomial.*

Proof. We have $\left\langle\Lambda_0,\alpha_i^\vee\right\rangle\in\mathbb{Z}_{\geq 0}$ as in (4.1), and for each $s\in\{1,\dots,N\}$, Λ_s is integral dominant so $\left\langle\Lambda_s,\alpha_i^\vee\right\rangle\in\mathbb{Z}_{\geq0}$. So the integrand is a rational function with poles at most at the points t_p, $p\in\{1,\dots,m\}$, for which $\mathsf{c}(p)=i$. Consider such a point t_p. Note that

$$\begin{aligned}\frac{\partial}{\partial x}\log x^{\left\langle\Lambda_0,\alpha_i^\vee\right\rangle}&T_i(x)(x-t_p)^2\prod_{j\in I}y_j(x)^{-\left\langle\alpha_j,\alpha_i^\vee\right\rangle}\\ &= \sum_{k=0}^{M-1}\sum_{i=1}^{N}\frac{\left\langle\sigma^k\Lambda_i,\alpha^\vee_{\mathsf{c}(p)}\right\rangle}{x-\omega^k z_i}+\frac{\left\langle\Lambda_0,\alpha^\vee_{\mathsf{c}(p)}\right\rangle}{x}-\sum_{\substack{i=1\\ i\neq p}}^{\tilde m}\frac{\left\langle\alpha_{\mathsf{c}(i)},\alpha_{\mathsf{c}(p)}\right\rangle}{x-t_i}.\end{aligned}\tag{4.8}$$

This vanishes at $x=t_p$ by virtue of the critical point equations (2.12). It follows that the residue of the integrand at t_p vanishes: indeed, this residue is $(\frac{\partial}{\partial x}x^{\left\langle\Lambda_0,\alpha_i^\vee\right\rangle}T_i(x)(x-t_p)^2\prod_{j\in I}y_j(x)^{-\left\langle\alpha_j,\alpha_i^\vee\right\rangle})_{x=t_p}$

which vanishes if (4.8) vanishes. This shows that $y_i^{(i)}(x)$ is an entire function. It is of polynomial growth for large x. Therefore it is a polynomial. □

If $y_i^{(i)}(x)$ is any solution to (4.7) then so too is $y_i^{(i)}(x) + cy_i(x)$ for any $c \in \mathbb{C}$.

Thus, given any tuple $\boldsymbol{y}$ representing a critical point we have, for each value of a parameter $c \in \mathbb{C}$, a new tuple of polynomials $\boldsymbol{y}^{(i)}$, obtained from the tuple $\boldsymbol{y}$ by replacing $y_i(x)$ with $y_i^{(i)}(x)+cy_i(x)$. We say $\boldsymbol{y}^{(i)}$ is obtained from $\boldsymbol{y}$ by *generation in the ith direction*, and we call $\boldsymbol{y}^{(i)}$ the *immediate descendant of $\boldsymbol{y}$ in the ith direction.*

Proposition 4.3 ([MV04])**.** *The tuple of polynomials $\boldsymbol{y}^{(i)}$ is generic for almost all c. If $\boldsymbol{y}^{(i)}$ is generic then it represents a critical point.* □

The tuples $\boldsymbol{y}^{(i)}$ describe a projective line in $\mathbb{P}(\mathbb{C}[x])^{|I|}$. It will be useful to have the following specific parameterization of this line. There exists a unique solution $y_i^{(i)}(x)$ to the equation (4.7), call it $y_i^{(i)}(x;0)$, such that the coefficient of $x^{\deg y_i}$ in $y_i^{(i)}(x;0)$ is zero. Let us define

$$y_i^{(i)}(x;c) := y_i^{(i)}(x;0) + cy_i(x), \tag{4.9}$$

and define $\boldsymbol{y}^{(i)}(c) \in \mathbb{P}(\mathbb{C}[x])^{|I|}$ to be the tuple obtained from the tuple $\boldsymbol{y}$ by replacing $y_i(x)$ with $y_i^{(i)}(x;0) + cy_i(x)$.

We say generation in the ith direction is *degree-increasing* if $\deg y_i^{(i)} > \deg y_i$ for almost all c.

Recall that there is a weight at infinity, Λ_∞, associated to any critical point. For the critical point represented by $\boldsymbol{y}$ this weight is, cf. (2.7),

$$\Lambda_\infty(\boldsymbol{y}) = \Lambda_0 + \sum_{s=1}^{N}\sum_{k=0}^{M-1} \sigma^k \Lambda_s - \sum_{j\in I} \alpha_j \deg y_j. \tag{4.10}$$

For fixed $\Lambda_0, \Lambda_1, \dots, \Lambda_N$ we can think of Λ_∞ as encoding the degrees of the polynomials y_j. Note that $\deg y_i^{(i)}(x;0) = \deg y_i + \langle \Lambda_\infty, \alpha_i^\vee \rangle + 1$. It follows that the weight at infinity of $\boldsymbol{y}^{(i)}(0)$ is

$$\Lambda_\infty - \alpha_i \left(\langle \Lambda_\infty, \alpha_i^\vee \rangle + 1 \right) = \Lambda_\infty - \langle \Lambda_\infty + \rho, \alpha_i^\vee \rangle \alpha_i = \mathsf{s}_i \cdot \Lambda_\infty. \tag{4.11}$$

This establishes the following lemma.

Lemma 4.4. *Generation in the ith direction (with $L_i = 1$) is degree-increasing if and only if Λ_∞ is i-dominant, i.e. $\langle \Lambda_\infty, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$.*
If generation in the ith direction is degree-increasing, then the weight at infinity associated with the critical point represented by $\boldsymbol{y}_i^{(i)}(c)$ is $\mathsf{s}_i \cdot \Lambda_\infty$. Otherwise it is Λ_∞ for all $c \neq 0$ (and $\mathsf{s}_i \cdot \Lambda_\infty$ for $c = 0$). □

4.4. **Cyclotomic generation: the $L_i = 1$ case.** We continue to suppose that i is such that $L_i = 1$.

If $\boldsymbol{y}$ represents a cyclotomic point then its immediate descendant $\boldsymbol{y}^{(i)}$ in the ith direction generically does not. However if, starting from a cyclotomic critical point, we successively generate in *each* of the directions $\sigma^k i$, $k = 1, \dots, M_i$, in turn, in any order, then we can arrange to arrive at a (new) cyclotomic critical point. This is the content of Theorem 4.6 below.

Let $\simeq$ denote equality up to a constant (independent of x) nonzero factor. Recall the definition (2.14) of a cyclotomic point.

Lemma 4.5. *A tuple of polynomials $\boldsymbol{y}$ represents a cyclotomic point if and only if*

$$y_{\sigma j}(\omega x) \simeq y_j(x)$$

for all $j \in I$. If $y_j(x)$ and $y_{\sigma j}(x)$ share the same leading coefficient for all $j \in I$, then the tuple $\boldsymbol{y}$ represents a cyclotomic point if and only if

$$y_{\sigma j}(\omega x) = \omega^{\deg y_j} y_j(x)$$

for all $j \in I$. □

For the rest of this subsection, we suppose $\boldsymbol{y}$ represents a cyclotomic critical point. Hence in particular $\sigma\Lambda_\infty = \Lambda_\infty$. Let $y_i^{(i)}(x;c) = y_{i,0}^{(i)}(x) + cy_i(x)$ be as in (4.9). (So $y_i^{(i)}(x;c)$ is a parameterization of the space of solutions to (4.7).) Define $\boldsymbol{y}^{(i,\sigma)}(c)$ to be the tuple of polynomials given by

$$y^{(i,\sigma)}_{\sigma^k i}(\omega^k x; c) := \omega^{k \deg y_i^{(i)}} y_i^{(i)}(x;c), \qquad k = 0,1,\dots,M_i - 1, \tag{4.12}$$

and $y_j^{(i,\sigma)}(x;c) = y_j(x)$ for $j \in I \setminus \sigma^{\mathbb{Z}} i$. Recall s_i^σ from Lemma 2.5.

Theorem 4.6. *For almost all $c \in \mathbb{C}$, the tuple $\boldsymbol{y}^{(i,\sigma)}(c)$ represents a cyclotomic critical point. The exceptional values of c form a finite subset of $\mathbb{C}$.*

The weight at infinity of $\boldsymbol{y}^{(i,\sigma)}(c)$ is $\mathsf{s}_i^\sigma \cdot \Lambda_\infty$ if $\langle \Lambda_\infty, \alpha_i^{\vee,\sigma} \rangle \in \mathbb{Z}_{\geq 0}$. Otherwise it is Λ_∞ for all $c \neq 0$, and $\mathsf{s}_i^\sigma \cdot \Lambda_\infty$ for $c = 0$.

Proof. First let us show that $\boldsymbol{y}^{(i,\sigma)}$ represents a cyclotomic point for all $c \in \mathbb{C}$. Comparing our definition of $\boldsymbol{y}^{(i,\sigma)}$ with the criterion in Lemma 4.5, one sees that it is enough to check that

$$y_i^{(i)}(\omega^{M_i} x; c) = \omega^{M_i \deg y_i^{(i)}} y_i^{(i)}(x;c).$$

Inspecting (4.6), we see that this equality holds for all $c \in \mathbb{C}$ if and only if

$$\omega^{M_i \left\langle \Lambda_\infty + \rho, \alpha_i^\vee \right\rangle} = 1. \tag{4.13}$$

But now, given (4.10) and the assumption that Λ_s, $s = 1,\dots,n$ are integral, the following lemma implies that (4.13) holds if and only if we impose the condition (4.2) on Λ_0.

Lemma 4.7. *Suppose $\Lambda \in \mathfrak{h}^*$ is an integral weight. Then, for any $j \in I$,*

$$\sum_{k=0}^{M-1} \langle \sigma^k \Lambda, \alpha_j^\vee \rangle M_j \equiv 0 \mod M.$$

Proof. We have

$$\Big\langle \sum_{k=0}^{M-1} \sigma^k \Lambda, \alpha_j^\vee \Big\rangle M_j = \Big\langle \Lambda, \sum_{k=0}^{M-1} \sigma^{-k} \alpha_j^\vee \Big\rangle M_j = \Big\langle \Lambda, \frac{M}{M_j} \sum_{k=0}^{M_j - 1} \alpha_j^\vee \Big\rangle M_j = M \Big\langle \Lambda, \sum_{k=0}^{M_j-1} \alpha_j^\vee \Big\rangle \in M\mathbb{Z}.$$

□

Now we show that $\boldsymbol{y}^{(i,\sigma)}$ represents a critical point for all but finitely many $c \in \mathbb{C}$. Note that from definition (4.5) we have

$$T_{\sigma j}(\omega x) = \omega^{\left\langle \sum_{s=1}^N \sum_{k=0}^{M-1} \sigma^k \Lambda_s, \alpha_j^\vee \right\rangle} T_j(x), \qquad j \in I. \tag{4.14}$$

Hence, in view of (4.10),

$$\begin{aligned} x^{\left\langle \Lambda_0, \alpha_{\sigma i}^\vee \right\rangle} T_{\sigma i}(\omega x) \prod_{j \in I} y_j(\omega x)^{-\left\langle \alpha_j, \alpha_{\sigma i}^\vee \right\rangle} &= x^{\left\langle \Lambda_0, \alpha_i^\vee \right\rangle} T_{\sigma i}(\omega x) \prod_{j \in I} y_{\sigma j}(\omega x)^{-\left\langle \alpha_{\sigma j}, \alpha_{\sigma i}^\vee \right\rangle} \\ &= \omega^{\left\langle \Lambda_\infty, \alpha_i^\vee \right\rangle} \left(x^{\left\langle \Lambda_0, \alpha_i^\vee \right\rangle} T_i(x) \prod_{j \in I} y_j(x)^{-\left\langle \alpha_j, \alpha_i^\vee \right\rangle} \right). \end{aligned} \tag{4.15}$$

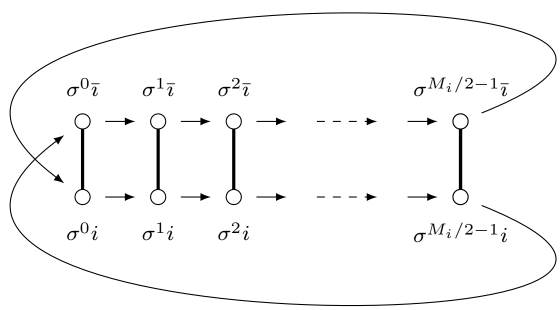
Note also that since $L_i = 1$, no node j in the orbit of i is linked by an edge of the Dynkin diagram to i. That is, no y_j for j in the orbit of i appears on the right of (4.7). Hence, for $k = 1, \dots, M_i - 1$, $y^{(i,\sigma)}_{\sigma^k i}(x;c)$ obeys the equation

$$\mathrm{Wr}(y_{\sigma^k i}(x), y^{(i,\sigma)}_{\sigma^k i}(x;c)) = x^{\left\langle \Lambda_0, \alpha^\vee_{\sigma^k i}\right\rangle} T_{\sigma^k i}(x) \prod_{j\in I\setminus\{\sigma^k i\}} y_j(x)^{-\left\langle \alpha_j, \alpha^\vee_{\sigma^k i}\right\rangle}. \tag{4.16}$$

and the tuple $\boldsymbol{y}^{(i,\sigma)}$ is indeed the result of generating in each of the directions $i, \sigma i, \dots, \sigma^{M_i-1} i$ (in any order). It follows from Proposition 4.3 that $\boldsymbol{y}^{(i,\sigma)}$ is generic for almost all c, and represents a critical point whenever it is generic.

The statements about the weight at infinity follow from Lemma 4.4 and §2.3. This completes the proof of Theorem 4.6. □

4.5. **Elementary generation: the $L_i = 2$ case.** For this subsection we suppose that $i \in I$ is such that $L_i = 2$. That implies M_i is even and the restriction of the Dynkin diagram to the nodes $\sigma^{\mathbb{Z}} i$ consists of $\frac{M_i}{2} \in \mathbb{Z}_{\geq 1}$ disconnected copies of the Dynkin diagram of type A_2, as sketched below.



(4.17)

Here, for brevity, we write $\bar\imath := \sigma^{M_i/2} i$.

Remark 4.8. Among finite and affine types, only the case $M_i/2 = 1$ occurs.

We define $y^{(i)}_i(x)$ by

$$y^{(i)}_i(x) := y_i(x)\, x^{-\left\langle \Lambda_0, \alpha^\vee_i\right\rangle - 1} \int_0^x \xi^{\left\langle \Lambda_0, \alpha^\vee_i\right\rangle} T_i(\xi) \prod_{j\in I} y_j(\xi)^{-\left\langle \alpha_j, \alpha^\vee_i\right\rangle} d\xi.$$

Here the limits $\int_0^x$ mean that $y^{(i)}_i(x)$ is holomorphic at $x = 0$. This condition defines the integral uniquely, since $\left\langle \Lambda_0, \alpha^\vee_i\right\rangle \notin \mathbb{Z}$ by our assumption (4.3).

Proposition 4.9. *If $\boldsymbol{y}$ represents a critical point then $y^{(i)}_i(x)$ is a polynomial. It has degree $\deg y^{(i)}_i = \deg y_i + \left\langle \Lambda_\infty - \Lambda_0, \alpha^\vee_i\right\rangle$.*

Proof. The proof is as for Proposition 4.2. □

Let $\boldsymbol{y}^{(i)} = (y^{(i)}_j(x))_{j\in I}$ be the tuple of polynomials whose ith component $y^{(i)}_i(x)$ is as above whose remaining components are the same as those of $\boldsymbol{y}$, *i.e.*

$$y^{(i)}_j(x) = y_j(x) \quad \text{for all} \quad j \in I \setminus \{i\}.$$

Let $(\boldsymbol{t}^{(i)}; \mathbf{c}^{(i)})$ denote the pair represented by this tuple in the sense of §4.2. It turns out that $\boldsymbol{t}^{(i)}$ is not in general a critical point of the extended master function $\widehat{\Phi}(\boldsymbol{t}^{(i)}; \mathbf{c}^{(i)}; \boldsymbol{z}; \boldsymbol{\Lambda})$, *i.e.* it does not in general obey the equations (2.12). Instead, the following result gives the analogous collection of equations that it does obey, provided $\boldsymbol{y}^{(i)}$ is generic.

Proposition 4.10. *If $\boldsymbol{y}$ represents a critical point and $\boldsymbol{y}^{(i)}$ is generic, then*

$$\frac{\left\langle \mathsf{s}_i \cdot \Lambda_0, \alpha^\vee_{\mathsf{c}^{(i)}(p)} \right\rangle}{t^{(i)}_p} + \sum_{s=1}^N \sum_{k=0}^{M-1} \frac{\left\langle \sigma^k \Lambda_s, \alpha^\vee_{\mathsf{c}^{(i)}(p)} \right\rangle}{t^{(i)}_p - \omega^k z_s} - \sum_{r:r\neq p} \frac{\left\langle \alpha_{\mathsf{c}^{(i)}(r)}, \alpha^\vee_{\mathsf{c}^{(i)}(p)} \right\rangle}{t^{(i)}_p - t^{(i)}_r} = 0$$

for each p.

Proof. By (2.12) for each root t_p in the tuple $\boldsymbol{t}$ we have

$$\frac{\left\langle \Lambda_0, \alpha^\vee_{\mathsf{c}(p)} \right\rangle}{t_p} + \sum_{s=1}^N \sum_{k=0}^{M-1} \frac{\left\langle \sigma^k \Lambda_s, \alpha^\vee_{\mathsf{c}(p)} \right\rangle}{t_p - \omega^k z_s} - \sum_{r:r\neq p} \frac{\left\langle \alpha_{\mathsf{c}(r)}, \alpha^\vee_{\mathsf{c}(p)} \right\rangle}{t_p - t_r} = 0. \tag{4.18}$$

For all roots of colours $j \in I$ such that $\left\langle \alpha_j, \alpha^\vee_i \right\rangle = 0$ this is immediately equivalent to the required equation. So we must consider roots of colour i, and roots of colours $j \in I$ such that $\left\langle \alpha_j, \alpha^\vee_i \right\rangle < 0$.

By definition of $y^{(i)}_i(x)$ we have

$$\mathrm{Wr}(y_i(x), x^{\left\langle \Lambda_0, \alpha^\vee_i \right\rangle + 1} y^{(i)}_i(x)) = x^{\left\langle \Lambda_0, \alpha^\vee_i \right\rangle} T_i(x) \prod_{j\neq i} y_j(x)^{-\left\langle \alpha_j, \alpha^\vee_i \right\rangle} \tag{4.19}$$

or equivalently

$$\frac{y'_i(x)}{y_i(x)} - \frac{y^{(i)\prime}_i(x)}{y^{(i)}_i(x)} - \frac{1 + \left\langle \Lambda_0, \alpha^\vee_i \right\rangle}{x} = \frac{T_i(x) \prod_{j\neq i} y_j(x)^{-\left\langle \alpha_j, \alpha^\vee_i \right\rangle}}{x y_i(x) y^{(i)}_i(x)}. \tag{4.20}$$

By definition of $(\boldsymbol{t}^{(i)}, \mathsf{c}^{(i)})$, the left-hand side of (4.20) is

$$\sum_{r:\mathsf{c}(r)=i} \frac{1}{x - t_r} - \sum_{r:\mathsf{c}^{(i)}(r)=i} \frac{1}{x - t^{(i)}_r} - \frac{1 + \left\langle \Lambda_0, \alpha^\vee_i \right\rangle}{x}. \tag{4.21}$$

Now suppose $j \in I$ is such that $\left\langle \alpha_j, \alpha^\vee_i \right\rangle \in \mathbb{Z}_{<0}$. By definition $y^{(i)}_j(x) = y_j(x)$. Suppose t_p is a root of $y_j(x)$, *i.e.* suppose $\mathsf{c}(p) = j$. Since $\boldsymbol{y}$ represents a critical point, $\boldsymbol{y}$ must be generic, and hence t_p is not a root of $y_i(x)$. By our assumption that $\boldsymbol{y}^{(i)}$ is generic, t_p is not a root of $y^{(i)}_i(x)$ either. Hence the right-hand side of (4.20) is zero at $x = t_p$ and so, in view of (4.21), we have

$$\sum_{r:\mathsf{c}(r)=i} \frac{1}{t_p - t_r} - \sum_{r:\mathsf{c}^{(i)}(r)=i} \frac{1}{t_p - t^{(i)}_r} - \frac{1 + \left\langle \Lambda_0, \alpha^\vee_i \right\rangle}{t_p} = 0.$$

On adding this equation multiplied by $\left\langle \alpha_i, \alpha^\vee_j \right\rangle$ to the equation (4.18), we arrive at

$$\frac{\left\langle \Lambda_0, \alpha^\vee_j \right\rangle - \left\langle \alpha_i, \alpha^\vee_j \right\rangle \left\langle \Lambda_0 + \rho, \alpha^\vee_i \right\rangle}{t^{(i)}_p} + \sum_{s=1}^N \sum_{k=0}^{M-1} \frac{\left\langle \sigma^k \Lambda_s, \alpha^\vee_j \right\rangle}{t^{(i)}_p - \omega^k z_s} - \sum_{r:r\neq p} \frac{\left\langle \alpha_{\mathsf{c}^{(i)}(r)}, \alpha^\vee_j \right\rangle}{t^{(i)}_p - t^{(i)}_r} = 0,$$

which is the required equality (since $\mathsf{s}_i \cdot \Lambda_0 = \Lambda_0 - \left\langle \Lambda_0 + \rho, \alpha^\vee_i \right\rangle \alpha_i$).

It remains to consider roots of colour i. First note that $y_i(x)$ and $y^{(i)}_i(x)$ have no common roots. Indeed, if t were a common root of $y_i(x)$ and $y^{(i)}_i(x)$ then the right-hand side of (4.19) would have to vanish at $x = t$. In other words $y_i(x)$ would have a root in common with the right-hand side of (4.19). But by our definition of what it means for $\boldsymbol{y}$ to be generic, §4.2, this is impossible.

Suppose $t^{(i)}_p$ is any root of $y^{(i)}_i(x)$. By our assumption that $\boldsymbol{y}^{(i)}$ is generic, it follows from (4.19) and Lemmas 4.12 and 4.13 below that

$$\frac{2(1 + \left\langle \Lambda_0, \alpha^\vee_i \right\rangle)}{t^{(i)}_p} - \frac{\left\langle \Lambda_0, \alpha^\vee_i \right\rangle}{t^{(i)}_p} - \sum_{s=1}^N \sum_{k=0}^{M-1} \frac{\left\langle \sigma^k \Lambda_s, \alpha^\vee_i \right\rangle}{t^{(i)}_p - \omega^k z_s} + \sum_{r:r\neq p} \frac{\left\langle \alpha_{\mathsf{c}^{(i)}(r)}, \alpha^\vee_i \right\rangle}{t^{(i)}_p - t^{(i)}_r} = 0, \tag{4.22}$$

which is the required equality. ☐

Remark 4.11. Propositions 4.9 and 4.10 also follow from Theorem 3.5 in [MV08].

Lemma 4.12. *For any $\alpha \in \mathbb{C}$, if $g(x) = x^\alpha \prod_{j=1}^J (x - s_j)$ where $(s_j)_{j=1}^J$ are all distinct and non-zero, then*

$$\left.\frac{g''(x)}{g'(x)}\right|_{x=s_k} = \frac{2\alpha}{s_k} + \sum_{\substack{j=1\\ j\neq k}}^{J} \frac{2}{s_k - s_j}.$$

☐

Lemma 4.13. *If $\mathrm{Wr}(f(x), g(x)) = W(x)$ then*

$$\frac{g''(x)}{g'(x)} - \frac{W'(x)}{W(x)} = \frac{g(x)\big(W(x)f''(x) - W'(x)f'(x)\big)}{f(x)g'(x)W(x)}.$$

Proof. We have $W\,\mathrm{Wr}(f,g)' = W'\,\mathrm{Wr}(f,g)$. Hence

$$W(x)f(x)g''(x) - W'(x)f(x)g'(x) = W(x)f''(x)g(x) - W'(x)f'(x)g(x)$$

and hence the result. ☐

To deal with the case in which $\boldsymbol{y}^{(i)}$ fails to be generic, we shall also need the following observation, which follows from (4.19).

Lemma 4.14. *For any $j \in I$ such that $\langle \alpha_j, \alpha_i^\vee \rangle < 0$, if t is a root of both $y_j(x)$ and $y_i^{(i)}(x)$ then it is a root of $y_i^{(i)}(x)$ with multiplicity 2. In particular, if t is a root of both $y_{\bar\imath}(x)$ and $y_i^{(i)}(x)$ then it is a root of $y_i^{(i)}(x)$ with multiplicity 2.* ☐

Now we define

$$\begin{aligned} y_{\bar\imath}^{(\bar\imath,i)}(x) &:= y_{\bar\imath}(x) \int \xi^{\langle \Lambda_0, \alpha_i^\vee\rangle} T_{\bar\imath}(\xi) \frac{\xi^{1+\langle \Lambda_0,\alpha_i^\vee\rangle} y_i^{(i)}(\xi) \prod_{j\neq i,\bar\imath} y_j(\xi)^{-\langle \alpha_j, \alpha_{\bar\imath}^\vee\rangle}}{y_{\bar\imath}(\xi)^2} d\xi \\ &= y_{\bar\imath}(x) \int \xi^{1+2\langle \Lambda_0, \alpha_i^\vee\rangle} T_{\bar\imath}(\xi) \frac{y_i^{(i)}(\xi) \prod_{j\neq i,\bar\imath} y_j(\xi)^{-\langle \alpha_j, \alpha_{\bar\imath}^\vee\rangle}}{y_{\bar\imath}(\xi)^2} d\xi. \end{aligned} \tag{4.23}$$

Proposition 4.15. *If $\boldsymbol{y}$ represents a critical point then $y_{\bar\imath}^{(\bar\imath,i)}(x)$ is a polynomial.*

Proof. By our assumption (4.3) that $2\langle \Lambda_0, \alpha_i^\vee\rangle + 1 \in \mathbb{Z}_{\geq 0}$, the integrand is regular at $x = 0$. Hence, by Lemma 4.14, it is a rational function with poles at most at those roots of $y_{\bar\imath}(x)$ that are not also roots of $y_i^{(i)}(x)$. Let t_p be any such root. The residue of the integrand at $\xi = t_p$ is

$$\left.\frac{\partial}{\partial x}(x-t)^2 x^{\langle \Lambda_0,\alpha_i^\vee\rangle} T_i(x) \frac{y_i^{(i)}(x) \prod_{j\neq i,\bar\imath} y_j^{(i)}(x)^{-\langle \alpha_j,\alpha_{\bar\imath}^\vee\rangle}}{y_{\bar\imath}(x)^2}\right|_{x=t_p},$$

which must vanish, because according to Proposition 4.10 the following vanishes:

$$\begin{aligned} &\left.\frac{\partial}{\partial x}\log (x-t)^2 x^{\langle \Lambda_0,\alpha_i^\vee\rangle} T_i(x) \frac{y_i^{(i)}(x) \prod_{j\neq i,\bar\imath} y_j^{(i)}(x)^{-\langle \alpha_j,\alpha_{\bar\imath}^\vee\rangle}}{y_{\bar\imath}(x)^2}\right|_{x=t_p} \\ &\qquad = \frac{1+2\langle \Lambda_0, \alpha_i^\vee\rangle}{t_p} + \sum_{s=1}^{N}\sum_{k=0}^{M-1} \frac{\langle \sigma^k \Lambda_s, \alpha_{\bar\imath}^\vee\rangle}{t_p - \omega^k z_s} - \sum_{r: r\neq p} \frac{\langle \alpha_{\mathsf{c}^{(i)}(r)}, \alpha_{\bar\imath}^\vee\rangle}{t_p - t_r^{(i)}}. \end{aligned}$$

(Note $\langle \Lambda_0, \alpha_i^\vee\rangle = \langle \Lambda_0, \alpha_{\bar\imath}^\vee\rangle$ since $\sigma\Lambda_0 = \Lambda_0$). ☐

The polynomial $y_{\bar\imath}^{(\bar\imath,i)}(x)$ is defined up a to the addition of a constant multiple of $y_{\bar\imath}(x)$, coming from the constant of integration in (4.23).

We say generation in the ith direction from $\boldsymbol{y}$ is *degree-increasing* if $\deg y_{\bar\imath}^{(\bar\imath,i)}(x) > \deg y_{\bar\imath}(x)$. Generation in the ith direction is degree-increasing if and only if

$$\left\langle \Lambda_\infty + \rho, \alpha_i^\vee + \alpha_{\bar\imath}^\vee \right\rangle > 0. \tag{4.24}$$

Indeed, if (4.24) holds then

$$\deg y_{\bar\imath}^{(\bar\imath,i)}(x) = \deg y_{\bar\imath}(x) + \left\langle \Lambda_\infty + \rho, \alpha_i^\vee + \alpha_{\bar\imath}^\vee \right\rangle > \deg y_{\bar\imath}(x) \tag{4.25}$$

for all values of the constant of integration. If (4.24) does not hold then $\deg y_{\bar\imath}^{(\bar\imath,i)}(x) \leq \deg y_{\bar\imath}(x)$, with equality for all but one value of the constant of integration in (4.23).

Let $y_{\bar\imath}^{(\bar\imath,i)}(x;0)$ be the unique solution to (4.23) whose coefficient of $x^{\deg y_{\bar\imath}}$ is zero. The degree of $y_{\bar\imath}^{(\bar\imath,i)}(x;0)$ is always given by

$$\deg y_{\bar\imath}^{(\bar\imath,i)}(x;0) = \deg y_{\bar\imath}(x) + \left\langle \Lambda_\infty + \rho, \alpha_i^\vee + \alpha_{\bar\imath}^\vee \right\rangle,$$

whether or not generation is degree-increasing. (Note that $\left\langle \Lambda_\infty + \rho, \alpha_i^\vee + \alpha_{\bar\imath}^\vee \right\rangle$ is odd, by our assumption (4.3), and in particular not zero.)

Let then $\boldsymbol{y}^{(\bar\imath,i)}(c) = (y_j^{(\bar\imath,i)}(x;c))_{j\in I}$ be the tuple of polynomials whose $\bar\imath$th component is

$$y_{\bar\imath}^{(\bar\imath,i)}(x;c) := y_{\bar\imath}^{(\bar\imath,i)}(x;0) + c y_{\bar\imath}(x) \tag{4.26}$$

and whose remaining components are the same as those of $\boldsymbol{y}^{(i)}$, *i.e.*

$$y_i^{(\bar\imath,i)}(x;c) = y_i^{(i)}(x;c), \qquad \text{and} \qquad y_j^{(\bar\imath,i)}(x) = y_j^{(i)}(x) = y_j(x) \qquad \text{for all} \qquad j \in I \setminus \{i,\bar\imath\}.$$

Let $(\boldsymbol{t}^{(\bar\imath.i)}; \mathsf{c}^{(\bar\imath,i)})$ denote the pair represented by this tuple in the sense of §4.2.

The following result says that whenever $\boldsymbol{y}^{(\bar\imath,i)}(c)$ is generic, this new pair $(\boldsymbol{t}^{(\bar\imath,i)}(c), \mathsf{c}^{(\bar\imath,i)})$ obeys the same form of equations as did $(\boldsymbol{t}^{(i)}, \mathsf{c}^{(i)})$.

Proposition 4.16. *If $\boldsymbol{y}$ represents a critical point then, for all $c \in \mathbb{C}$ such that $\boldsymbol{y}^{(\bar\imath,i)}(c)$ is generic, we have*

$$\frac{\left\langle \mathsf{s}_i \cdot \Lambda_0, \alpha^\vee_{\mathsf{c}^{(\bar\imath,i)}(p)} \right\rangle}{t_p^{(\bar\imath,i)}(c)} + \sum_{s=1}^N \sum_{k=0}^{M-1} \frac{\left\langle \sigma^k \Lambda_s, \alpha^\vee_{\mathsf{c}^{(\bar\imath,i)}(p)} \right\rangle}{t_p^{(\bar\imath,i)}(c) - \omega^k z_s} - \sum_{r: r\neq p} \frac{\left\langle \alpha_{\mathsf{c}^{(\bar\imath,i)}(r)}, \alpha^\vee_{\mathsf{c}^{(\bar\imath,i)}(p)} \right\rangle}{t_p^{(\bar\imath,i)}(c) - t_r^{(\bar\imath,i)}(c)} = 0$$

for each p.

Proof. The proof is analogous to that of Proposition 4.10. □

Finally, we define $y_i^{(i,\bar\imath,i)}(x;c)$ by

$$\begin{aligned} y_i^{(i,\bar\imath,i)}(x;c) &= y_i^{(i)}(x)\, x^{\left\langle \Lambda_0, \alpha_i^\vee \right\rangle + 1} \int_0^x \xi^{\left\langle \Lambda_0, \alpha_i^\vee \right\rangle} T_i(\xi) \frac{y_{\bar\imath}^{(\bar\imath,i)}(\xi;c) \prod_{j\in I\setminus\{i,\bar\imath\}} y_j(\xi)^{-\left\langle \alpha_j, \alpha_i^\vee \right\rangle}}{\left(\xi^{\left\langle \Lambda_0, \alpha_i^\vee \right\rangle + 1} y_i^{(i)}(\xi)\right)^2} d\xi \\ &= y_i^{(i)}(x)\, x^{\left\langle \Lambda_0, \alpha_i^\vee \right\rangle + 1} \int_0^x \xi^{-\left\langle \Lambda_0, \alpha_i^\vee \right\rangle - 2} T_i(\xi) \frac{y_{\bar\imath}^{(\bar\imath,i)}(\xi;c) \prod_{j\in I\setminus\{i,\bar\imath\}} y_j(\xi)^{-\left\langle \alpha_j, \alpha_i^\vee \right\rangle}}{y_i^{(i)}(\xi)^2} d\xi. \end{aligned} \tag{4.27}$$

Here the limits $\int_0^x$ mean that $y_i^{(i,\bar\imath,i)}(x;c)$ is holomorphic at $x = 0$. This condition defines the integral uniquely.

Proposition 4.17. *For all $c \in \mathbb{C}$, if $\boldsymbol{y}$ represents a critical point then $y_i^{(i,\bar\imath,i)}(x;c)$ is a polynomial.*

Proof. Pick any root $t_p^{(\bar\imath,i)}$ of $y_i^{(i)}(x) = y_i^{(\bar\imath,i)}(x)$. The residue of the integrand at $\xi = t^{(\bar\imath,i)}$ is zero. Indeed, we have

$$\frac{\partial}{\partial x}\log(x-t_p^{(i)})^2 x^{-\langle\Lambda_0,\alpha_i^\vee\rangle-2}T_i(x)\frac{y_{\bar\imath}^{(\bar\imath,i)}(x;c)\prod_{j\in I\setminus\{i,\bar\imath\}}y_j(x)^{-\langle\alpha_j,\alpha_i^\vee\rangle}}{y_i^{(i)}(x)^2}\Bigg|_{x=t_p^{(\bar\imath,i)}}$$
$$=\frac{-\langle\Lambda_0,\alpha_i^\vee\rangle-2}{t_p^{(\bar\imath,i)}}+\sum_{s=1}^N\sum_{k=0}^{M-1}\frac{\langle\sigma^k\Lambda_s,\alpha_{\bar\imath}^\vee\rangle}{t_p^{(\bar\imath,i)}-\omega^k z_s}-\sum_{r:r\neq p}\frac{\langle\alpha_{\mathsf{c}^{(\bar\imath,i)}(r)},\alpha_i^\vee\rangle}{t_p^{(\bar\imath,i)}-t_r^{(\bar\imath,i)}},$$

and this vanishes by Proposition 4.16. □

Let $\boldsymbol{y}^{(i,\bar\imath,i)}(c) = (y_j^{(i,\bar\imath,i)}(x;c))_{j\in I}$ be the tuple of polynomials whose ith component is $y^{(i,\bar\imath,i)}(x;c)$ as above and whose remaining components are those of $\boldsymbol{y}^{(\bar\imath,i)}(c)$, *i.e.*

$$y_{\bar\imath}^{(i,\bar\imath,i)}(x;c) = y_{\bar\imath}^{(\bar\imath,i)}(x;c), \quad \text{and} \quad y_j^{(i,\bar\imath,i)}(x) = y_j(x) \quad \text{for all} \quad j\in I\setminus\{i,\bar\imath\}.$$

Let $(\boldsymbol{t}^{(i,\bar\imath,i)};\mathsf{c}^{(i,\bar\imath,i)})$ denote the pair represented by this tuple in the sense of §4.2.

Proposition 4.18. *If $\boldsymbol{y}$ represents a critical point and $\boldsymbol{y}^{(i,\bar\imath,i)}(c)$ is generic, then $\boldsymbol{y}^{(i,\bar\imath,i)}(c)$ represents a critical point. That is, the pair $(\boldsymbol{t}^{i,\bar\imath,i}(c),\mathsf{c}^{i,\bar\imath,i})$ obeys the equations*

$$\frac{\langle\Lambda_0,\alpha_{\mathsf{c}(p)}^\vee\rangle}{t_p^{(i,\bar\imath,i)}(c)}+\sum_{s=1}^N\sum_{k=0}^{M-1}\frac{\langle\sigma^k\Lambda_s,\alpha_{\mathsf{c}(p)}^\vee\rangle}{t_p^{(i,\bar\imath,i)}(c)-\omega^k z_s}-\sum_{r:r\neq p}\frac{\langle\alpha_{\mathsf{c}(r)},\alpha_{\mathsf{c}(p)}^\vee\rangle}{t_p^{(i,\bar\imath,i)}(c)-t_r^{(i,\bar\imath,i)}(c)}=0 \tag{4.28}$$

for each p.

Proof. The proof is analogous to that of Proposition 4.10. □

We say $\boldsymbol{y}^{(i,\bar\imath,i)}(c)$ is obtained from $\boldsymbol{y}$ by *generation in the ith direction*, and we call $\boldsymbol{y}^{(i,\bar\imath,i)}(c)$ the *immediate descendant of $\boldsymbol{y}$ in the ith direction.* We have the following; cf. Lemma 4.4.

Lemma 4.19. *Generation in the ith direction (with $L_i = 2$) is degree-increasing if and only if $\langle\Lambda_\infty+\rho,\alpha_i^\vee+\alpha_{\bar\imath}^\vee\rangle\in\mathbb{Z}_{>0}$.*
If generation in the ith direction is degree-increasing, then the weight at infinity associated with the critical point represented by $\boldsymbol{y}_i^{(i,\bar\imath,i)}(c)$ is $(\mathsf{s}_i\mathsf{s}_{\bar\imath}\mathsf{s}_i)\cdot\Lambda_\infty$. Otherwise it is Λ_∞ for all $c\neq 0$ (and $(\mathsf{s}_i\mathsf{s}_{\bar\imath}\mathsf{s}_i)\cdot\Lambda_\infty$ for $c=0$). □

Proof. Recall that (4.25) holds if and only if (4.24) holds. Note also that

$$\deg y_i^{(i,\bar\imath,i)} = \deg y_i + \langle\Lambda_\infty+\rho,\alpha_i^\vee+\alpha_{\bar\imath}^\vee\rangle. \tag{4.29}$$

By direct calculation, one verifies that

$$(\mathsf{s}_i\mathsf{s}_{\bar\imath}\mathsf{s}_i)\cdot\Lambda_\infty = \Lambda_\infty-(\alpha_{\bar\imath}+\alpha_i)\langle\Lambda_\infty+\rho,\alpha_i^\vee+\alpha_{\bar\imath}^\vee\rangle \tag{4.30}$$

so we have the result. □

4.6. **Cyclotomic generation: the $L_i = 2$ case.** We continue to suppose that $i\in I$ is such that $L_i = 2$.

Suppose for the rest of this subsection that $\boldsymbol{y}$ represents a cyclotomic critical point. Define $\boldsymbol{y}^{(i,\sigma)}(c)$ to be the tuple of polynomials given by

$$y_{\sigma^k i}^{(i,\sigma)}(\omega^k x;c) := y_i^{(i,\bar\imath,i)}(x;c), \qquad y_{\sigma^k\bar\imath}^{(i,\sigma)}(\omega^k x;c) := y_{\bar\imath}^{(i,\bar\imath,i)}(x;c), \qquad k=0,1,\dots,M_i/2-1, \tag{4.31}$$

and $y_j^{(i,\sigma)}(x;c) = y_j(x)$ for $j\in I\setminus\sigma^{\mathbb{Z}}i$.

Theorem 4.20. *For almost all $c \in \mathbb{C}$, the tuple $\boldsymbol{y}^{(i,\sigma)}(x;c)$ represents a cyclotomic critical point. The exceptional values of c form a finite subset of $\mathbb{C}$.*

The weight at infinity of $\boldsymbol{y}^{(i,\sigma)}(x;c)$ is $\mathsf{s}_i^\sigma \cdot \Lambda_\infty$ if $\left\langle \Lambda_\infty + \rho, \alpha_i^\sigma + \alpha_{\bar{\imath}}^\sigma \right\rangle \in \mathbb{Z}_{\geq 1}$. Otherwise it is Λ_∞ for all $c \neq 0$, and $\mathsf{s}_i^\sigma \cdot \Lambda_\infty$ for $c = 0$.

Proof. First let us show that $\boldsymbol{y}^{(i,\sigma)}(x;c)$ represents a critical point for all but finitely many $c \in \mathbb{C}$. As in the proof of Theorem 4.6, we first observe that $\boldsymbol{y}^{(i,\sigma)}$ is indeed the result of generating in each of the directions $i, \sigma i, \dots, \sigma^{M_i/2-1} i$ (in any order). By Proposition 4.18 it is enough to check that $\boldsymbol{y}^{(i,\bar{\imath},i)}(c)$ is generic for all but finitely many $c \in \mathbb{C}$. This follows from (4.32) and Lemma 4.22, below.

The statements about the weight at infinity follow from Lemma 4.19 and §2.3.

Finally we must check that $\boldsymbol{y}^{(i,\sigma)}(x;c)$ represents a cyclotomic point. Given Lemma 4.5 and the definition (4.31), it is enough to check that

$$y_{\bar{\imath}}^{(i,\bar{\imath},i)}(-x;c) = (-1)^{\deg y_i^{(i,\bar{\imath},i)}} y_i^{(i,\bar{\imath},i)}(x;c). \tag{4.32}$$

This is effectively a statement about the case of type A_2 and we are in the setting of §5 below, with $R = 2n, n = 1, p = 1$. The statement (4.32) follows from Theorem 5.34 and Lemma 5.36. □

Lemma 4.21. *We have*

$$y_j^{(\bar{\imath},i,\bar{\imath})}(-x;c) = (-1)^{\deg y_{\bar{\jmath}}^{(i,\bar{\imath},i)}} y_{\bar{\jmath}}^{(i,\bar{\imath},i)}(x;-c)$$

for all $j \in I$. □

Proof. Note first that from (4.14) we have

$$T_{\bar{\jmath}}(-x) = (-1)^{\left\langle \sum_{k=0}^{M-1} \sum_{s=1}^{N} \omega^k \Lambda_s, \alpha_j^\vee \right\rangle} T_j(x)$$

for all $j \in I$. It follows that

$$y_{\bar{\imath}}^{(\bar{\imath})}(-x) = (-1)^{\deg y_i + \left\langle \Lambda_\infty - \Lambda_0, \alpha_i^\vee \right\rangle} y_i^{(i)}(x).$$

Then, from the definition of $y_{\bar{\imath}}^{(\bar{\imath},i)}(x;c)$ and (4.25) we have that

$$y_i^{(i,\bar{\imath})}(-x;c) = (-1)^{\deg y_{\bar{\imath}}^{(i,\bar{\imath})}} y_{\bar{\imath}}^{(\bar{\imath},i)}(x;\bar{c})$$

if and only if c and $\bar{c}$ are related by $c = (-1)^{2+\left\langle \Lambda_\infty, \alpha_{\bar{\imath}}^\vee + \alpha_i^\vee \right\rangle} \bar{c}$. Since the Λ_s, $s = 1, \dots, N$, are integral, we have

$$(-1)^{\left\langle \Lambda_\infty, \alpha_i^\vee + \alpha_{\bar{\imath}}^\vee \right\rangle} = (-1)^{\left\langle \Lambda_0, \alpha_i^\vee + \alpha_{\bar{\imath}}^\vee \right\rangle} = (-1)^{2\left\langle \Lambda_0, \alpha_i^\vee \right\rangle} = -1,$$

using $\sigma\Lambda_0 = \Lambda_0$ and the property (4.1). □

Lemma 4.22. *For all but finitely many $c \in \mathbb{C}$, $y_{\bar{\imath}}^{(i,\bar{\imath},i)}(x;c)$ and $y_{\bar{\imath}}^{(i,\bar{\imath},i)}(-x;c)$ have no root in common.*

Proof. Recall $y_{\bar{\imath}}^{(i,\bar{\imath},i)}(x;c) = y_{\bar{\imath}}^{(\bar{\imath},i)}(x;c)$. Consider the leading behaviour in small c. As $c \to 0$, some $\deg y_{\bar{\imath}}$ of the roots of $y_{\bar{\imath}}^{(\bar{\imath},i)}(x;c)$ tend to the $\deg y_{\bar{\imath}}$ roots of $y_{\bar{\imath}}(x)$. By the assumption that $\boldsymbol{y}$ was generic and cyclotomic, none of these are roots of $y_{\bar{\imath}}(-x) \simeq y_i(x)$.

Recall (4.25) and the fact that $\left\langle \Lambda_\infty + \rho, \alpha_i^\vee + \alpha_{\bar{\imath}}^\vee \right\rangle$ is odd, by the assumption (4.3).

If $\left\langle \Lambda_\infty + \rho, \alpha_i^\vee + \alpha_{\bar{\imath}}^\vee \right\rangle < 0$, then these are all the roots of $y_{\bar{\imath}}^{(\bar{\imath},i)}(x;c)$.

If $\left\langle \Lambda_\infty + \rho, \alpha_i^\vee + \alpha_{\bar{\imath}}^\vee \right\rangle \not< 0$ then the remaining $\left\langle \Lambda_\infty + \rho, \alpha_i^\vee + \alpha_{\bar{\imath}}^\vee \right\rangle > 0$ roots of $y_{\bar{\imath}}^{(\bar{\imath},i)}(x;c)$ tend to the roots of the equation $cx^{\left\langle \Lambda_\infty + \rho, \alpha_i^\vee + \alpha_{\bar{\imath}}^\vee \right\rangle} + 1 = 0$. This limiting set of roots multiplied by -1 does not intersect itself. This implies the lemma. □

4.7. **Definition of the cyclotomic population.** Suppose $\boldsymbol{y} \in \mathbb{P}(\mathbb{C}[x])^{|I|}$ is a tuple of polyonomials representing a cyclotomic critical point.

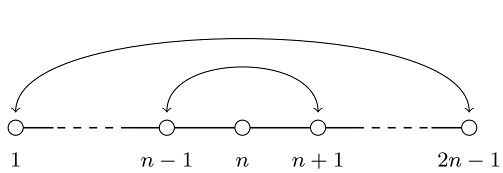
Recall the definition of $\boldsymbol{y}^{(i,\sigma)}(c)$, from §4.5 when $L_i = 1$ and from §4.6 when $L_i = 2$. We say $\boldsymbol{y}^{(i,\sigma)}(c)$ is obtained from $\boldsymbol{y}$ by *cyclotomic generation* in the direction i.

Let us define the *cyclotomic population* originated at $\boldsymbol{y}$ to be the Zariski closure of the set of all tuples of polynomials obtained from $\boldsymbol{y}$ by repeated cyclotomic generation, in all directions $i \in I$.

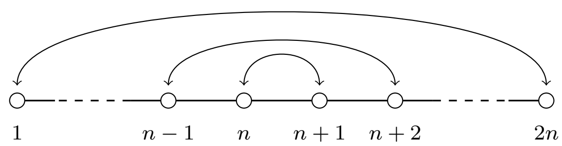
5. The case of type A_R: vector spaces of quasi-polynomials

5.1. **Type A data.** Throughout this section we specialise to $\mathfrak{g} = \mathfrak{sl}_{R+1}$. We shall treat in parallel the cases where $R = 2n-1$ and $R = 2n$, $n \in \mathbb{Z}_{\geq 0}$. We have the usual identification of $\mathfrak{h} \cong \mathfrak{h}^*$ with a subspace of $(R+1)$-dimensional Euclidean space, given by $\alpha_i = \alpha_i^\vee = \epsilon_{i+1} - \epsilon_i$, $i = 1, \dots, R$, where $(\epsilon_i)_{i=1}^{R+1}$ is the standard orthonormal basis.

Let $\sigma : \mathfrak{g} \to \mathfrak{g}$ be the unique non-trivial diagram automorphism, whose order is 2. The nodes I of the Dynkin diagram, and the action of σ on these nodes, are as shown below.



(5.1)



(5.2)

When $R = 2n-1$, then $L_i = 1$ for all $i \in I$, and $M_i = \begin{cases} 1 & i = n \\ 2 & i \neq n. \end{cases}$

When $R = 2n$ then $L_i = \begin{cases} 2 & i = n, n+1 \\ 1 & \text{otherwise} \end{cases}$ and $M_i = 2$ for all $i \in I$.

Let $(z_i)_{i=1}^N$ be nonzero points $z_i \in \mathbb{C}^\times$ such that $z_i \pm z_j \neq 0$ whenever $i \neq j$. Let $\Lambda_1, \dots, \Lambda_N$ be dominant integral weights.

We suppose the weight at the origin, $\Lambda_0 \in \mathfrak{h}^*$, obeys $\sigma\Lambda_0 = \Lambda_0$ (as always). That is,

$$\langle \Lambda_0, \alpha_i^\vee \rangle = \langle \Lambda_0, \alpha_{R+1-i}^\vee \rangle, \qquad i = 1, \dots, R.$$

In addition, we pick and fix an integer $p \in \{0, 1, \dots, n\}$, and suppose that

$$\langle \Lambda_0, \alpha_i^\vee \rangle \in 2\mathbb{Z}_{\geq 0}/M_i \quad \text{if } i \notin \{p, R+1-p\} \tag{5.3a}$$

and

$$\langle \Lambda_0, \alpha_p^\vee \rangle \in \begin{cases} \frac{1}{2}(2\mathbb{Z}_{\geq 0} - 1) = \{-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \dots\} & \text{if } p \leq R/2 \\ 2\mathbb{Z}_{\geq 0} + 1 = \{1, 3, \dots\} & \text{if } p = n \text{ and } R = 2n-1. \end{cases} \tag{5.3b}$$

Note the following particular cases:

- If $R = 2n$ is even and $p = 0$ and then (5.3) just says that Λ_0 is dominant integral.
- If $R = 2n-1$ is odd and $p = 0$ then Λ_0 is dominant integral and $\langle \Lambda_0, \alpha_n^\vee \rangle$ is even.
- If $R = 2n-1$ is odd and $p = n$ then Λ_0 is dominant integral and $\langle \Lambda_0, \alpha_n^\vee \rangle$ is odd.

In the case $p = n$ (and any R) our choice of Λ_0 obeys the assumptions set out in §4.1.

5.2. **Vector spaces of quasi-polynomials.** Let

$$\tilde{T}_i(x) = x^{\left\langle \Lambda_0, \alpha_i^\vee \right\rangle} \prod_{s=1}^{N} (x - z_s)^{\left\langle \Lambda_s, \alpha_i^\vee \right\rangle} (x + z_s)^{\left\langle \Lambda_{R+1-s}, \alpha_i^\vee \right\rangle}, \qquad i \in I. \tag{5.4}$$

Thus $\tilde{T}_i(x) = x^{\left\langle \Lambda_0, \alpha_i^\vee \right\rangle} T_i(x)$ with $T_i(x)$ as in (4.5).

In view of (5.3), $\tilde{T}_i(x) \in \mathbb{C}[x]$ for all $i \notin \{p, R+1-p\}$. If $0 < p < R+1-p < R$ then $\tilde{T}_p(x)$ and $\tilde{T}_{R+1-p}(x)$ belong to $x^{-\frac{1}{2}}\mathbb{C}[x]$. If $p = R+1-p$ then $\tilde{T}_p(x) \in \mathbb{C}[x]$.

We define the *degree*, $\deg p$, of a Laurent polynomial $p(x) \in \mathbb{C}[x^{\pm\frac{1}{2}}]$ to be the leading power of x (for large x) that appears in $p(x)$ with non-zero coefficient.

We will call any polynomial in $x^{\frac{1}{2}}$ a *quasi-polynomial*.

A vector space $V \subset \mathbb{C}[x^{\frac{1}{2}}]$ of quasi-polynomials is *decomposable* if

$$V = V \cap \mathbb{C}[x] \oplus V \cap x^{\frac{1}{2}}\mathbb{C}[x].$$

A tuple of quasi-polynomials is *decomposable* if each element lies in either $\mathbb{C}[x]$ or $x^{\frac{1}{2}}\mathbb{C}[x]$. In particular, a *decomposable basis* of a decomposable vector space $V \subset \mathbb{C}[x^{\frac{1}{2}}]$ is one in which each basis vector lies in either $\mathbb{C}[x]$ or $x^{\frac{1}{2}}\mathbb{C}[x]$.

Define the *divided Wronksian determinant* of quasi-polynomials $u_1, \dots, u_k \in \mathbb{C}[x^{\frac{1}{2}}]$ by

$$\mathrm{Wr}^\dagger(u_1, \dots, u_k) := \frac{\mathrm{Wr}(u_1, \dots, u_k)}{\tilde{T}_1^{k-1}\tilde{T}_2^{k-2} \dots \tilde{T}_{k-1}}, \qquad \mathrm{Wr}(u_1, \dots, u_k) := \det\left(\frac{d^{j-1}u_i}{dx^{j-1}}\right)_{i,j=1}^{k}, \tag{5.5}$$

for $k = 1, \dots, R+1$.

Define

$$\Lambda := \Lambda_0 + \sum_{s=1}^{N} (\Lambda_s + \sigma\Lambda_s), \tag{5.6}$$

and suppose $\tilde{\Lambda}_\infty \in \mathfrak{h}^*$ is a dominant weight such that $\Lambda - \tilde{\Lambda}_\infty = \sum_{i \in I} k_i \alpha_i$ for some $k_i \in \mathbb{Z}_{\geq 0}$. Such a weight defines numbers $d_1, \dots, d_{R+1} \in \mathbb{Z}/2$, $0 \leq d_1 < \dots < d_{R+1}$, by

$$d_1 := \left\langle \Lambda - \tilde{\Lambda}_\infty, \epsilon_1 \right\rangle, \qquad d_k := \left\langle \Lambda - (\mathsf{s}_1 \dots \mathsf{s}_{k-1}) \cdot \tilde{\Lambda}_\infty, \epsilon_1 \right\rangle, \qquad k = 2, \dots R+1. \tag{5.7}$$

Lemma 5.1. *We have*

$$d_k = d_1 + \left\langle \tilde{\Lambda}_\infty + \rho, \alpha_1^\vee + \dots + \alpha_{k-1}^\vee \right\rangle, \qquad k = 2, \dots, R+1. \tag{5.8}$$

Hence, for all $p > 0$, $d_1, \dots, d_p$ and $d_{R+2-p}, \dots, d_{R+1}$ are integers while $d_{p+1}, \dots, d_{R+1-p}$ are half odd integers, i.e. have the form $m + \frac{1}{2}$ for $m \in \mathbb{Z}$. If $p = 0$ then $d_1, \dots, d_{R+1}$ are all integers. □

Proof. We have $d_k - d_1 = \left\langle \tilde{\Lambda}_\infty + \rho - (\mathsf{s}_1 \dots \mathsf{s}_{k-1})(\tilde{\Lambda}_\infty + \rho), \epsilon_1 \right\rangle = \left\langle \tilde{\Lambda}_\infty + \rho, \epsilon_1 - (\mathsf{s}_{k-1} \dots \mathsf{s}_1)\epsilon_1 \right\rangle = \left\langle \tilde{\Lambda}_\infty + \rho, \epsilon_1 - \epsilon_k \right\rangle$ and hence (5.8). □

Definition 5.2. We say a vector space of quasi-polynomials $\mathcal{K} \subset \mathbb{C}[x^{\frac{1}{2}}]$ has *frame* $\tilde{T}_1, \dots, \tilde{T}_R; \tilde{\Lambda}_\infty$ if the following conditions hold:

(i) There is a basis $(u_k(x))_{k=1}^{R+1}$ of $\mathcal{K}$ such that $\deg u_k = d_k$ for each $k = 1, \dots, R+1$.

(ii) For any $z \in \mathbb{C}\backslash\{0\}$ and $v_1, \dots, v_k \in \mathcal{K}$, $k = 1, \dots, R+1$, the divided Wronskian $\mathrm{Wr}^\dagger(v_1, \dots, v_k)$ is regular at z, and moreover, $\mathrm{Wr}^\dagger(v_1, \dots, v_k)$ is nonzero at z for suitable $v_1, \dots, v_k$.

(iii) For all $v_1, \dots, v_k \in \mathcal{K}$, $k = 1, \dots, R+1$, the divided Wronskian $\mathrm{Wr}^\dagger(v_1, \dots, v_k)$ has at $x = 0$ an expansion of the form $\sum_{m \in \mathbb{Z}_{\geq 0}/2} a_m x^m$ and moreover this expansion has nonzero a_0 for suitable $v_1, \dots, v_k$.

In the remainder of this section, $\mathcal{K}$ will denote a decomposable vector space of quasi-polynomials with frame $\tilde{T}_1, \dots, \tilde{T}_R; \tilde{\Lambda}_\infty$.

Conditions (ii) and (iii) specify the *ramification conditions* of $\mathcal{K}$ at every point $z \in \mathbb{C}$. Condition (i) specifies the ramification conditions at ∞. See [MV04, §5.5]. The degrees $0 \le d_1 < d_2 < \dots < d_{R+1}$ will be called the *exponents of $\mathcal{K}$ at infinity.*

Note that conditions (ii) and (iii) together imply in particular that $\mathcal{K}$ has no *base points*. That is, there is no $z \in \mathbb{C}$ such that $u(z) = 0$ for all $u \in \mathcal{K}$. They also imply the following important lemma.

Lemma 5.3. *For all $v_1, \dots, v_k \in \mathcal{K}$, $k = 1, \dots, R+1$, the divided Wronskian $\mathrm{Wr}^\dagger(v_1, \dots, v_k)$ is a quasi-polynomial.* □

Since $\mathcal{K}$ is decomposable it follows from condition (i) that $\mathcal{K}$ admits a decomposable basis $(u_k)_{k=1}^{R+1}$ such that $\deg u_k = d_k$ for each k. We call any such basis a *special* basis.

Lemma 5.4. *Any two special bases $(u_k)_{k=1}^{R+1}$ and $(u'_k)_{k=1}^{R+1}$ are related by a triangular change of basis, $u'_k = \sum_{j \le k} a_{kj} u_j$, such that $a_{kj} = 0$ whenever $d_k - d_j \notin \mathbb{Z}$.* □

Lemma 5.5. *Let $m \in \mathbb{Z}_{\ge 1}$. Let $n_1, \dots, n_m$ be non-negative integers. Then*

$$\mathrm{Wr}(x^{n_1}, \dots, x^{n_m}) = x^{\sum_{i=1}^m n_i - \frac{m(m-1)}{2}} \prod_{1 \le j < i \le m} (n_i - n_j).$$

□

Lemma 5.6. *Let $(u_i(x))_{i=1}^{R+1}$ be a special basis of $\mathcal{K}$. Then*

$$\mathrm{Wr}^\dagger(u_1, \dots, u_{R+1}) = \prod_{1 \le j < i \le R+1} (d_i - d_j).$$

Proof. By Lemma 5.3, $\mathrm{Wr}^\dagger(u_1, \dots, u_{R+1}) \in \mathbb{C}[x^{\frac{1}{2}}]$. We must show that it has degree zero and compute the constant term. From the condition that $\Lambda - \tilde{\Lambda}_\infty \in \mathbb{Z}_{\ge 0}[\alpha_i]_{i \in I}$ it follows that $0 = \langle \Lambda - \tilde{\Lambda}_\infty, -(R+1)\epsilon_1 + R\alpha_1^\vee + (R-1)\alpha_2^\vee + \dots + 2\alpha_{R-1}^\vee + \alpha_R^\vee \rangle$ and therefore

$$(R+1)d_1 = \langle \Lambda - \tilde{\Lambda}_\infty, R\alpha_1^\vee + (R-1)\alpha_2^\vee + \dots + 2\alpha_{R-1}^\vee + \alpha_R^\vee \rangle. \tag{5.9}$$

Then (5.8) implies

$$\sum_{i=1}^{R+1} d_i - \frac{(R+1)R}{2} = \langle \Lambda, R\alpha_1^\vee + (R-1)\alpha_2^\vee + \dots + 2\alpha_{R-1}^\vee + \alpha_R^\vee \rangle. \tag{5.10}$$

The result follows by Lemma 5.5. □

Corollary 5.7. $\mathrm{Wr}^\dagger(v_1, \dots, v_{R+1})$ *is a constant (independent of x) for all $v_1, \dots, v_{R+1} \in \mathcal{K}$.* □

Let $(u_k)_{k=1}^{R+1}$ be a special basis of $\mathcal{K}$. Introduce the subspaces

$$\begin{aligned} \mathcal{K}_{\mathrm{Sp}} &:= \mathrm{span}_{\mathbb{C}}(u_1, \dots, u_p) \oplus \mathrm{span}_{\mathbb{C}}(u_{R+2-p}, \dots, u_{R+1}) \\ \mathcal{K}_{\mathrm{O}} &:= \mathrm{span}_{\mathbb{C}}(u_{p+1}, \dots, u_{R+1-p}), \end{aligned} \tag{5.11}$$

so that

$$\mathcal{K} = \mathcal{K}_{\mathrm{Sp}} \oplus \mathcal{K}_{\mathrm{O}}.$$

By Lemma 5.4, these definitions of do not depend on the choice of special basis $(u_k)_{k=1}^{R+1}$. By Lemma 5.1 we have that, whenever $p > 0$, then

$$\mathcal{K}_{\mathrm{Sp}} = \mathcal{K} \cap \mathbb{C}[x], \qquad \mathcal{K}_{\mathrm{O}} = \mathcal{K} \cap x^{\frac{1}{2}}\mathbb{C}[x]. \tag{5.12}$$

Exceptionally, when $p = 0$, we have $\mathcal{K}_{\mathrm{Sp}} = \{0\}$, $\mathcal{K}_{\mathrm{O}} = \mathcal{K} \subset \mathbb{C}[x]$.

Given a decomposable subspace V, we write $\mathrm{sdim}\, V$ for the pair of numbers

$$\mathrm{sdim}\, V := (\dim V \cap \mathcal{K}_{\mathrm{Sp}} | \dim V \cap \mathcal{K}_{\mathrm{O}}).$$

5.3. **Flags in $\mathcal{K}$.** Let $FL(\mathcal{K})$ denote the space of full (*i.e.* $R+1$-step) flags in $\mathcal{K}$.

We say an r-step flag $\mathcal{F} = \{0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_r = \mathcal{K}\}$ in $\mathcal{K}$ is *decomposable* if each F_k is decomposable.

The space of decomposable full flags in $\mathcal{K}$ has $\binom{R+1}{2p}$ connected components. These connected components are labeled by $2p$-element subsets $Q \subset \{1, \dots, R+1\}$. Define $FL_Q(\mathcal{K})$ to be the subset consisting of the flags $\mathcal{F} = \{0 = F_0 \subset F_1 \subset \dots F_{R+1} = \mathcal{K}\}$ such that for each k,

$$\operatorname{sdim} F_k - \operatorname{sdim} F_{k-1} = \begin{cases} (1|0) & \text{if} \quad k \in Q \\ (0|1) & \text{if} \quad k \notin Q. \end{cases}$$

We call elements of $FL_Q(\mathcal{K})$ *flags of type Q*.

For each Q the variety $FL_Q(\mathcal{K})$ is isomorphic to the direct product of full flag spaces $FL(\mathcal{K}_{\mathrm{Sp}}) \times FL(\mathcal{K}_{\mathrm{O}})$. The isomorphism

$$\eta_Q : FL(\mathcal{K}_{\mathrm{Sp}}) \times FL(\mathcal{K}_{\mathrm{O}}) \to FL_Q(\mathcal{K}) \tag{5.13}$$

sends a pair of flags $F_{1,+} \subset \cdots \subset F_{2p,+}$, $F_{1,-} \subset \cdots \subset F_{R+1-2p,-}$ to the flag $F_1 \subset \cdots \subset F_R$, where $F_k = F_{k_1,+} \oplus F_{k_2,-}$, $k_1 = |Q \cap \{1, ..., k\}|$, $k_2 = k - k_1$.

Call a $2p$-element subset $Q \subset \{1, ..., R+1\}$ *symmetric* if Q is invariant with respect to the involution $k \mapsto R+2-k$. In particular, the following subset S is symmetric

$$S := \{1, ..., p, R+2-p, ..., R+1\}. \tag{5.14}$$

If $(u_k)_{k=1}^{R+1}$ is a special basis of $\mathcal{K}$ then the full flag $\mathcal{F} = \{0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{R+1} = \mathcal{K}\}$ defined by

$$F_k = \operatorname{span}_{\mathbb{C}}(u_1, \dots, u_k), \qquad k = 1, \dots, R+1, \tag{5.15}$$

belongs to $FL_S(\mathcal{K})$. By Lemma 5.4 this flag is independent of the choice of special basis.

To any full flag $\mathcal{F} = \{0 \subset F_1 \subset F_2 \subset \cdots \subset F_{R+1} = \mathcal{K}\}$ in $FL(\mathcal{K})$ one can associate a tuple $\boldsymbol{y}^{\mathcal{F}} = (y_i(x))_{i=1}^R \in \mathbb{P}(\mathbb{C}[x^{\frac{1}{2}}])^R$. Namely, let $(u_k^{\mathcal{F}}(x))_{k=1}^{R+1}$ be any basis of $\mathcal{K}$ such that

$$F_k = \operatorname{span}_{\mathbb{C}}(u_1^{\mathcal{F}}, \dots, u_k^{\mathcal{F}}), \qquad k = 1, \dots, R+1. \tag{5.16}$$

(we say such a basis is *adjusted to* $\mathcal{F}$) and then let

$$y_k^{\mathcal{F}} := \mathrm{Wr}^{\dagger}(u_1^{\mathcal{F}}, \dots, u_k^{\mathcal{F}}), \qquad k = 1, \dots, R. \tag{5.17}$$

By Lemma 5.3, these are quasi-polynomials.

We have the shifted action of the Weyl group of type A_R on weights as in §2.1. The weight at infinity $\Lambda_\infty(\boldsymbol{y}^{\mathcal{F}})$, as in (4.10), belongs to the shifted Weyl orbit of $\tilde{\Lambda}_\infty$ [MV04, §3.6]. It is equal to $\tilde{\Lambda}_\infty$ if and only if $\mathcal{F}$ is the flag given in (5.15).

The map $\mathcal{F} \mapsto \boldsymbol{y}^{\mathcal{F}}$ defines a morphism of varieties,

$$\beta : FL(\mathcal{K}) \to \mathbb{P}(\mathbb{C}[x^{1/2}])^R.$$

This morphism β defines an isomorphism of $FL(\mathcal{K})$ onto its image, as in Lemmas 5.14 – 5.16 of [MV04].

Lemma 5.8. *The image $\beta(FL_S(\mathcal{K}))$ of the variety of flags of type S lies in $\mathbb{P}(\mathbb{C}[x])^R$, i.e. consists of tuples of polynomials.*

Proof. In the exceptional case $p = 0$ no fractional powers are present at all and the result is clear. Suppose $p > 0$. Let $\mathcal{F} \in FL_S(\mathcal{K})$ and let $(u_k^{\mathcal{F}})_{k=1}^{R+1}$ be a basis of $\mathcal{K}$ adjusted to $\mathcal{F}$. By inspection one sees that because $\mathcal{F} \in FL_S(\mathcal{K})$, $\mathrm{Wr}(u_1^{\mathcal{F}}, \dots, u_k^{\mathcal{F}})$ lies in $\mathbb{C}[x]$ (resp. $x^{\frac{1}{2}}\mathbb{C}[x]$) for precisely those k such that the product $\tilde{T}_1^{k-1} \dots \tilde{T}_{k-1}$ lies in $\mathbb{C}[x^{\pm 1}]$ (resp. $x^{\frac{1}{2}}\mathbb{C}[x^{\pm 1}]$). For each k, Lemma 5.3 guarantees that $y_k^{\mathcal{F}} \in \mathbb{C}[x^{\frac{1}{2}}]$. Hence in fact $y_k^{\mathcal{F}} \in \mathbb{C}[x]$. $\square$

Lemma 5.9. *The tuple $\beta(\mathcal{F}) = \boldsymbol{y}^{\mathcal{F}}$ is decomposable if and only if $\mathcal{F}$ is a decomposable flag. If $\mathcal{F}$ is a decomposable flag of type Q then*

$$y_k^{\mathcal{F}} \in \begin{cases} \mathbb{C}[x] & \text{if} \quad |S\triangle Q \cap \{1,\dots,k\}| \in 2\mathbb{Z} \\ x^{\frac{1}{2}}\mathbb{C}[x] & \text{if} \quad |S\triangle Q \cap \{1,\dots,k\}| \in 2\mathbb{Z}+1, \end{cases}$$

where $S\triangle Q := (S\setminus Q)\cup(Q\setminus S)$ denotes the symmetric difference of S and Q. In particular $\boldsymbol{y}^{\mathcal{F}}$ is a tuple of polynomials if and only if $Q = S$. □

5.4. **Fundamental differential operator and the recovery theorem.** To any given a tuple $\boldsymbol{y} = (y_i(x))_{i=1}^R \in \mathbb{P}(\mathbb{C}[x^{\frac{1}{2}}])^R$ of quasi-polynomials, we may associate a differential operator $\mathcal{D}(\boldsymbol{y})$, defined by

$$\begin{aligned} \mathcal{D}(\boldsymbol{y}) &:= \Big(\partial - \log' \frac{\tilde{T}_1\tilde{T}_2\dots\tilde{T}_R}{y_R}\Big)\Big(\partial - \log' \frac{y_R\tilde{T}_1\tilde{T}_2\dots\tilde{T}_{R-1}}{y_{R-1}}\Big)\dots\Big(\partial - \log' \frac{y_2\tilde{T}_1}{y_1}\Big)\Big(\partial - \log' y_1\Big) \\ &= \prod_{i=0}^{\overrightarrow{R}}\Big(\partial - \log' \frac{y_{R+1-i}\prod_{j=1}^{R-i}\tilde{T}_j}{y_{R-i}}\Big), \end{aligned} \tag{5.18}$$

with the understanding that $y_0 = y_{R+1} = 1$. Here $\partial := \partial/\partial x$ and $\log' f := f'(x)/f(x)$.

As in [MV04], we have the following.

Theorem 5.10. *Let $\boldsymbol{y} \in \beta(FL(K))$. Then $\mathcal{K} = \ker\mathcal{D}$.* □

5.5. **The dual space $\mathcal{K}^\dagger$.** Let $\mathcal{K}^\dagger$ be the complex vector space

$$\mathcal{K}^\dagger := \operatorname{span}_{\mathbb{C}}\{\mathrm{Wr}^\dagger(v_1,\dots,v_R) : v_1,\dots,v_R \in \mathcal{K}\} \subset \mathbb{C}[x^{\frac{1}{2}}]. \tag{5.19}$$

The space $\mathcal{K}^\dagger$ is a space of quasi-polynomials by Lemma 5.3. The spaces $\mathcal{K}^\dagger$ and $\mathcal{K}$ are dual with respect to the pairing

$$(\cdot,\cdot) : \mathcal{K}^\dagger \times \mathcal{K} \to \mathbb{C}$$

defined by

$$\big(v_1, \mathrm{Wr}^\dagger(v_2,\dots,v_{R+1})\big) := \mathrm{Wr}^\dagger(v_1,v_2,\dots,v_{R+1}).$$

Given any basis $(u_i(x))_{i=1}^{R+1}$ of $\mathcal{K}$ there is a basis $(W_i(x))_{i=1}^{R+1}$ of $\mathcal{K}^\dagger$ defined by

$$W_i := \mathrm{Wr}^\dagger(u_1,\dots,\widehat{u}_i,\dots,u_{R+1}) \in \mathcal{K}^\dagger, \qquad i = 1,\dots,R+1, \tag{5.20}$$

where $\widehat{u}_i$ denotes omission. We have

$$(u_i, W_j) = 0 \quad \text{if} \quad i \neq j, \qquad (u_i, W_i) \neq 0. \tag{5.21}$$

Let $d_1^\dagger > \dots > d_{R+1}^\dagger$ be the numbers given by

$$d_{R+1}^\dagger := -\big\langle \Lambda - \tilde{\Lambda}_\infty, \epsilon_{R+1}\big\rangle, \qquad d_k^\dagger := -\big\langle \Lambda - (\mathsf{s}_R\dots\mathsf{s}_k)\cdot\tilde{\Lambda}_\infty, \epsilon_{R+1}\big\rangle, \qquad k = 1,\dots,R,$$

cf. (5.7). We have

$$d_k^\dagger = d_{R+1}^\dagger + \big\langle \tilde{\Lambda}_\infty + \rho, \alpha_k^\vee + \dots + \alpha_R^\vee\big\rangle, \qquad k = 1,\dots R, \tag{5.22}$$

by an argument as for Lemma 5.1.

Lemma 5.11. *Let $(u_i(x))_{i=1}^{R+1}$ be a special basis of $\mathcal{K}$. Then $\deg W_k = d_k^\dagger$, $k = 1,\dots,R+1$, and the basis $(W_k)_{k=1}^{R+1}$ is decomposable.*

Proof. From $\Lambda - \tilde\Lambda_\infty \in \mathbb{Z}_{\geq 0}[\alpha_i]_{i\in I}$ we have $0 = \big\langle \Lambda - \tilde\Lambda_\infty, (R+1)\epsilon_{R+1} + \alpha_1^\vee + 2\alpha_2^\vee + \dots + (R-1)\alpha_{R-1}^\vee + R\alpha_R^\vee \big\rangle$, and hence

$$(R+1)d^\dagger_{R+1} = \big\langle \Lambda - \tilde\Lambda_\infty, \alpha_1^\vee + 2\alpha_2^\vee + \dots + R\alpha_R^\vee \big\rangle. \tag{5.23}$$

Now

$$\begin{aligned}
\deg W_{R+1} &= \deg \mathrm{Wr}^\dagger(u_1,\dots,u_R)\\
&= \sum_{i=1}^R d_i - \frac{R(R-1)}{2} - \big\langle \Lambda, (R-1)\alpha_1^\vee + \dots + \alpha_{R-1}^\vee \big\rangle\\
&= Rd_1 + \big\langle \tilde\Lambda_\infty + \rho, (R-1)\alpha_1^\vee + \dots + \alpha_{R-1}^\vee \big\rangle - \frac{R(R-1)}{2} - \big\langle \Lambda, (R-1)\alpha_1^\vee + \dots + \alpha_{R-1}^\vee \big\rangle\\
&= Rd_1 + \big\langle \tilde\Lambda_\infty - \Lambda, (R-1)\alpha_1^\vee + \dots + \alpha_{R-1}^\vee \big\rangle,
\end{aligned} \tag{5.24}$$

where we used (5.8). Hence, using (5.9), we have

$$\begin{aligned}
(R+1)\deg W_{R+1} &= R\big\langle \Lambda - \tilde\Lambda_\infty, R\alpha_1^\vee + (R-1)\alpha_2^\vee + \dots + 2\alpha_{R-1}^\vee + \alpha_R^\vee \big\rangle\\
&\quad - (R+1)\big\langle \Lambda - \tilde\Lambda_\infty, (R-1)\alpha_1^\vee + \dots + \alpha_{R-1}^\vee \big\rangle\\
&= \big\langle \Lambda - \tilde\Lambda_\infty, \alpha_1^\vee + 2\alpha_2^\vee + \dots + R\alpha_R^\vee \big\rangle
\end{aligned} \tag{5.25}$$

since $R(R+1-k) - (R+1)(R-k) = k$. Comparing this with (5.23) we see that $d^\dagger_{R+1} = \deg W_{R+1}$. Then for the remaining W_k, we note that $\deg W_{R+1} - \deg W_k = d_k - d_{R+1}$ for $k = 1,\dots,R$. And by (5.8) and (5.22),

$$d_k - d_{R+1} = -\big\langle \tilde\Lambda_\infty + \rho, \alpha_k^\vee + \dots + \alpha_R^\vee \big\rangle = d^\dagger_{R+1} - d^\dagger_k.$$

Thus $d^\dagger_k = \deg W_k$ for $k = 1,\dots,R+1$. Finally, since the basis $(u_k)_{k=1}^{R+1}$ is decomposable and each $\tilde T_k$ lies in either $\mathbb{C}[x^{\pm 1}]$ or $x^{\frac12}\mathbb{C}[x^{\pm1}]$, it follows that $(W_k)_{k=1}^{R+1}$ is decomposable. □

5.6. **Cyclotomic points and cyclotomic self-duality.** Let us fix $(-1)^m := e^{m\pi i}$ for $m \in \mathbb{Z}/2$. Then given a monomial $q(x) = x^m$, $m \in \mathbb{Z}/2$, we define $q(-x) := (-1)^m x^m$. We extend the transformation $q(x) \mapsto q(-x)$ to Laurent polynomials in $x^{\frac12}$ by linearity.

We say that $\mathcal{K}$ is *cyclotomically self-dual* if

$$u(x) \in \mathcal{K} \Leftrightarrow u(-x) \in \mathcal{K}^\dagger.$$

Lemma 5.12. *If $\mathcal{K}$ is cyclotomically self-dual then*

$$d_k + d_{R+2-k} = R + \big\langle \Lambda, \alpha_1^\vee + \dots + \alpha_R^\vee \big\rangle \qquad k = 1,\dots,R+1.$$

Proof. If $\mathcal{K}$ is cyclotomically self-dual then we must have $d_k = d^\dagger_{R+2-k}$, $k = 1,\dots,R+1$. Comparing (5.8) and (5.22) we see that implies that

$$\big\langle \tilde\Lambda_\infty + \rho, \alpha_1^\vee + \dots \alpha_k^\vee \big\rangle = d_{k+1} - d_1 = d^\dagger_{R+1-k} - d^\dagger_{R+1} = \big\langle \tilde\Lambda_\infty + \rho, \alpha_{R+1-k}^\vee + \dots \alpha_R^\vee \big\rangle, \qquad k = 1,\dots,R$$

and hence

$$\big\langle \tilde\Lambda_\infty, \alpha_k^\vee \big\rangle = \big\langle \tilde\Lambda_\infty, \alpha_{R+1-k}^\vee \big\rangle, \qquad k = 1,\dots,R.$$

Therefore

$$d_k + d_{R+2-k} = 2d_1 + \big\langle \tilde\Lambda_\infty + \rho, \alpha_1^\vee + \dots + \alpha_R^\vee \big\rangle, \qquad k = 1,\dots,R+1,$$

and so, because the right-hand side here does not depend on k,

$$d_k + d_{R+2-k} = \frac{2}{R+1}\sum_{j=1}^{R+1} d_j, \qquad k = 1,\dots,R+1. \tag{5.26}$$

Recall (5.10) and the definition (5.6) of Λ. Using now the fact that $\langle \Lambda, \alpha_i^\vee \rangle = \langle \Lambda, \alpha_{R+1-i}^\vee \rangle$, $i = 1, \dots, R$, we have

$$\sum_{j=1}^{R+1} d_j - \frac{(R+1)R}{2} = \frac{R+1}{2} \langle \Lambda, \alpha_1^\vee + \dots + \alpha_R^\vee \rangle. \tag{5.27}$$

Thus, given (5.26), we have the result. $\square$

If $\mathcal{K}$ is cyclotomically self-dual then there is a non-degenerate bilinear form B on $\mathcal{K}$ defined by

$$B(u(x), v(x)) := (u(x), v(-x)) \tag{5.28}$$

i.e.

$$B(u, v) = \mathrm{Wr}^\dagger(u, v_1, \dots, v_R), \quad \text{where} \quad v(-x) = \mathrm{Wr}^\dagger(v_1, \dots, v_R).$$

Let us call a tuple of quasi-polynomials $\boldsymbol{y} \in \mathbb{P}(\mathbb{C}[x^{\frac{1}{2}}])^R$ *cyclotomic* if

$$y_k(-x) \simeq y_{R+1-k}(x), \qquad k = 1, \dots, R.$$

Proposition 5.13. *Let $\mathcal{F} \in FL(\mathcal{K})$. If the tuple $\beta(\mathcal{F}) \in \mathbb{P}(\mathbb{C}[x^{\frac{1}{2}}])^R$ is cyclotomic then $\mathcal{F}$ is a decomposable flag.*

Proof. Let $\boldsymbol{y}^{\mathcal{F}} = \beta(\mathcal{F})$. To prove that $\mathcal{F}$ is decomposable it is enough to show that each entry $y_k^{\mathcal{F}}$ of this tuple lies in $\mathbb{C}[x]$ or in $x^{1/2}\mathbb{C}[x]$. For each $k = 1, \dots, R$ we have $y_k^{\mathcal{F}}(x) = x^{\frac{1}{2}} a_k(x) + b_k(x)$ for some polynomials $a_k(x)$ and $b_k(x)$ in x. If $\boldsymbol{y}^{\mathcal{F}}$ is cyclotomic then $y_{R+1-k}(x) \simeq y_k^{\mathcal{F}}(-x) = (-1)^{\frac{1}{2}} x^{\frac{1}{2}} a_k(-x) + b_k(-x)$ for each k. That is, $a_{R+1-k}(x) = (-1)^{\frac{1}{2}} c_k a_k(-x)$ and $b_{R+1-k}(x) = c_k b_k(-x)$ for some non-zero constants c_k. But that means

$$\begin{aligned} a_k(x) &= (-1)^{\frac{1}{2}} c_{R+1-k} a_{R+1-k}(-x) &&= -c_{R+1-k} c_k a_k(x) \\ b_k(x) &= c_{R+1-k} b_{R+1-k}(-x) &&= +c_{R+1-k} c_k b_k(x) \end{aligned} \tag{5.29}$$

from which we conclude that at least one of $a_k(x)$ and $b_k(x)$ must vanish. $\square$

Theorem 5.14. *Suppose $\beta(FL(\mathcal{K}))$ contains a cyclotomic tuple. Then $\mathcal{K}$ is cyclotomically self-dual.*

Proof. We shall need the following identity among Wronskian determinants.

Lemma 5.15 ([MV04])**.** *Given integers $0 \le k \le s$ and functions $f_1, \dots, f_{s+1}$, we have*

$$\begin{aligned} &\mathrm{Wr}(\mathrm{Wr}(f_1, \dots, f_{s-k}, \dots, f_s, \widehat{f}_{s+1}), \\ &\qquad \mathrm{Wr}(f_1, \dots, f_{s-k}, \dots, \widehat{f}_s, f_{s+1}), \dots, \\ &\qquad \mathrm{Wr}(f_1, \dots, f_{s-k}, \widehat{f}_{s-k+1}, \dots, f_{s+1})) = \mathrm{Wr}(f_1, \dots, f_{s-k}) \left(\mathrm{Wr}(f_1, \dots, f_{s+1})\right)^k, \end{aligned}$$

where $\widehat{f}$ denotes omission. $\square$

To prove Theorem 5.14 we argue as for Theorem 6.8 in [MV04]. Let $\mathcal{F} \in FL(\mathcal{K})$ be a full flag in $\mathcal{K}$ and $(u_i(x))_{i=1}^{R+1}$ a basis of $\mathcal{K}$ adjusted to this flag. Let $\boldsymbol{y} = \boldsymbol{y}^{\mathcal{F}}$ be the corresponding tuple of quasi-polynomials as in (5.17), and $(W_i(x))_{i=1}^{R+1}$ the corresponding basis of $\mathcal{K}^\dagger$ as in (5.20). Then Theorem 5.14 follows from the case $k = R+1$ of the following lemma.

Lemma 5.16. *If $\boldsymbol{y}$ is cyclotomic then*

$$\mathrm{span}_{\mathbb{C}}(u_1(-x), \dots, u_k(-x)) = \mathrm{span}_{\mathbb{C}}(W_{R+1}, W_R, \dots, W_{R+2-k}), \qquad k = 1, \dots, R+1.$$

Proof. Let us prove the lemma by induction on k. For $k = 1$ we have

$$u_1(-x) = y_1(-x) \simeq y_R(x) = \mathrm{Wr}^\dagger(u_1, \dots, u_R) = W_{R+1}$$

as required. Assume the statement holds for all values up to some k. For the inductive step it is enough to show that

$$\mathrm{Wr}(u_1(-x), \dots, u_k(-x), W_{R+1-k}) \simeq \mathrm{Wr}(u_1(-x), \dots, u_k(-x), u_{k+1}(-x)). \tag{5.30}$$

Indeed, (5.30) is an inhomogeneous differential equation in $W_{R+1-k}(x)$ and if it holds then it must be that $W_{R+1-k}(x)$ is proportional to $u_{k+1}(-x)$ modulo $\mathrm{span}_{\mathbb{C}}(u_1(-x), \dots, u_k(-x))$, which is sufficient given the inductive assumption.

By the inductive assumption, we have

$$\begin{aligned}
&\mathrm{Wr}(u_1(-x), \dots, u_k(-x), W_{R+1-k}) \\
&\simeq \mathrm{Wr}(W_{R+1}, W_R, \dots, W_{R+1-k+1}, W_{R+1-k}) \\
&= \mathrm{Wr}\,\big(\,\mathrm{Wr}(u_1, \dots, u_{R+1-k-1}, \dots, u_R, \widehat{u}_{R+1}), \\
&\qquad \mathrm{Wr}(u_1, \dots, u_{R+1-k-1}, \dots, \widehat{u}_R, u_{R+1}), \dots, \\
&\qquad \mathrm{Wr}(u_1, \dots, u_{R+1-k-1}, \widehat{u}_{R+1-k}, \dots, u_{R+1})\big)\big/(\tilde{T}_1^{R-1}\tilde{T}_2^{R+1-3}\dots\tilde{T}_{R-1}^1)^{k+1} \\
&= \mathrm{Wr}(u_1, \dots, u_{R-k})\,(\mathrm{Wr}(u_1, \dots, u_{R+1}))^k\big/(\tilde{T}_1^{R-1}\tilde{T}_2^{R+1-3}\dots\tilde{T}_{R-1}^1)^{k+1},
\end{aligned}$$

the final equality by Lemma 5.15. Since $\mathrm{Wr}^\dagger(u_1, \dots, u_{R+1}) = \mathrm{Wr}(u_1, \dots, u_{R+1})/\tilde{T}_1^R\tilde{T}_2^{R-1}\dots\tilde{T}_R^1$ is a nonzero constant by Lemma 5.6 we therefore have

$$\begin{aligned}
\mathrm{Wr}(u_1(-x), \dots, u_k(-x), W_{R+1-k}) &\simeq \mathrm{Wr}(u_1, \dots, u_{R-k})\frac{(\tilde{T}_1^R\dots\tilde{T}_R^1)^k}{(\tilde{T}_1^{R-1}\dots\tilde{T}_{R-1})^{k+1}} \\
&= \frac{\mathrm{Wr}(u_1, \dots, u_{R-k})}{\tilde{T}_1^{R+1-k-2}\dots\tilde{T}_{R+1-k-2}^1}\tilde{T}_R^k\dots\tilde{T}_{R+1-k}^1.
\end{aligned} \tag{5.31}$$

Now we may use again the fact that $\boldsymbol{y}$ is cyclotomic, so $y_k(-x) \simeq y_{R+1-k}(x)$. In view of (5.17) that implies

$$\frac{\mathrm{Wr}(u_1, \dots, u_{R-k})}{\tilde{T}_1^{R+1-k-2}\dots\tilde{T}_{R+1-k-2}^1} \simeq \frac{\mathrm{Wr}(u_1(-x), \dots, u_{k+1}(-x))}{\tilde{T}_1^k(-x)\dots\tilde{T}_k^1(-x)}. \tag{5.32}$$

Recall that $\tilde{T}_{R+1-k}(x) \simeq \tilde{T}_k(-x)$. Hence we have indeed that

$$\mathrm{Wr}(u_1(-x), \dots, u_k(-x), W_{R+1-k}) \simeq \mathrm{Wr}(u_1(-x), \dots, u_{k+1}(-x)), \tag{5.33}$$

as required. □

This completes the proof of Theorem 5.14. □

Given a subspace $U \subset \mathcal{K}$, let

$$U^\perp := \{v \in \mathcal{K} : B(u, v) = 0 \text{ for all } u \in U\}$$

denote its orthogonal complement in $\mathcal{K}$ with respect to the bilinear form B. Recall that a full flag $\mathcal{F} = \{0 = F_0 \subset F_1 \subset F_2 \subset \dots \subset F_R \subset F_{R+1} = \mathcal{K}\} \in FL(\mathcal{K})$ is called *isotropic* with respect to B if $F_k = F_{R+1-k}^\perp$ for $k = 1, \dots, R$.

Theorem 5.17. *Suppose $\mathcal{K}$ is cyclotomically self-dual. A full flag $\mathcal{F} \in FL(\mathcal{K})$ is isotropic if and only if the associated tuple $\boldsymbol{y}^{\mathcal{F}}$ is cyclotomic.*

Proof. Let $(u_i(x))_{i=1}^{R+1}$ be a basis of $\mathcal{K}$ adjusted to $\mathcal{F}$, so that we have (5.17).

For the "only if" direction, suppose $\boldsymbol{y}^{\mathcal{F}}$ is cyclotomic. By Lemma 5.16,

$$F_k = \mathrm{span}_{\mathbb{C}}(u_1, \dots, u_k) = \mathrm{span}_{\mathbb{C}}(W_{R+1}(-x), \dots, W_{R+2-k}(-x)).$$

We also have $F_{R+1-k}^{\perp} = \operatorname{span}_{\mathbb{C}}(u_1, \dots, u_{R+1-k})^{\perp} = \operatorname{span}_{\mathbb{C}}(W_{R+1}(-x), \dots, W_{R+2-k}(-x))$ by (5.21). Therefore $F_k = F_{R+1-k}^{\perp}$.

For the "if" direction, suppose $\mathcal{F} = \{F_k\}$ is isotropic. Since $F_k = F_{R+1-k}^{\perp}$, and given (5.21), we have two bases for F_k, namely $(u_1, \dots, u_k)$ and $(W_{R+1}(-x), \dots, W_{R+2-k}(-x))$. So to prove that $\boldsymbol{y}$ is cyclotomic it suffices to establish the following lemma, which is the converse of Lemma 5.16.

Lemma 5.18. *If*

$$\operatorname{span}_{\mathbb{C}}(u_1(-x), \dots, u_k(-x)) = \operatorname{span}_{\mathbb{C}}(W_{R+1}, W_R, \dots, W_{R+2-k}), \qquad k = 1, \dots, R+1,$$

then $\boldsymbol{y}$ is cyclotomic.

Proof. Examining the induction in the proof of Lemma 5.16, one sees that we also have, by a similar induction, that if $\operatorname{span}_{\mathbb{C}}(u_1(-x), \dots, u_k(-x)) = \operatorname{span}_{\mathbb{C}}(W_{R+1}, W_R, \dots, W_{R+2-k})$ for each k then (5.32) must hold for each k, which says that $\boldsymbol{y}$ is cyclotomic. □

This completes the proof of Theorem 5.17. □

In view of Proposition 5.13 we have the following corollary.

Corollary 5.19. *If $\mathcal{F} \in FL(\mathcal{K})$ is isotropic then $\mathcal{F}$ is decomposable.* □

5.7. **Witt bases and the symmetries of the bilinear form B.** We say that $(r_k)_{k=1}^{R+1}$ is a *Witt basis* of the cyclotomically self-dual space $\mathcal{K}$ if

$$\operatorname{Wr}^{\dagger}(r_1, \dots, \widehat{r}_k, \dots, r_{R+1}) \simeq r_{R+2-k}(-x), \qquad k = 1, \dots, R+1. \tag{5.34}$$

The following lemma gives a useful alternative characterization of Witt bases.

Lemma 5.20. *The basis $(r_k)_{k=1}^{R+1}$ is a Witt basis if and only if*

$$B(r_i, r_j) = 0 \quad \text{whenever} \quad i + j \neq R+2. \tag{5.35}$$

Proof. Suppose $(u_k)_{k=1}^{R+1}$ is a basis of $\mathcal{K}$ and let $(W_k)_{k=1}^{R+1}$ be as in (5.20). Then $(W_i(x))_{i=1}^{R+1}$ and $(u_i(-x))_{i=1}^{R+1}$ are two bases of $\mathcal{K}^{\dagger}$ and so $u_i(-x) = \sum_{j=1}^{R+1} C_{ij} W_j(x)$, for some invertible matrix C_{ij}. We have $B(u_i, u_j) = \sum_{k=1}^{R+1} C_{jk} \operatorname{Wr}^{\dagger}(u_i, u_1, u_2, \dots, \widehat{u}_k, \dots, u_{R+1}) = (-1)^{i-1} C_{ji} \operatorname{Wr}^{\dagger}(u_1, \dots, u_{R+1})$. Hence (5.34) is equivalent to (5.35). □

Theorem 5.21. *Every cyclotomically self-dual space $\mathcal{K}$ has a special basis $(r_k)_{k=1}^{R+1}$ which is also Witt basis, and in which in fact*

$$\operatorname{Wr}^{\dagger}(r_1, \dots, \widehat{r}_k, \dots, r_{R+1}) = (-1)^{-\deg r_{R+2-k}} r_{R+2-k}(-x), \qquad k = 1, \dots, R+1. \tag{5.36}$$

Proof. Let $(u_k(x))_{k=1}^{R+1}$ be a special basis of $\mathcal{K}$. We may suppose that the $u_k(x)$ all have leading coefficient 1. Let $(W_k(x))_{k=1}^{R+1}$ be the basis of $\mathcal{K}^{\dagger}$ as in (5.20). By Lemma 5.11, $\deg W_k = d_k^{\dagger}$. By Lemma 5.5, we have

$$W_k = \operatorname{Wr}^{\dagger}(u_1, \dots, \widehat{u}_k, \dots, u_{R+1}) = D_k x^{d_k^{\dagger}} + \dots,$$

where $\dots$ indicates terms of lower degree in x and where

$$D_k := \prod_{\substack{1 \le j < i \le R+1 \\ i \neq k,\, j \neq k}} (d_i - d_j), \qquad k = 1, \dots, R+1.$$

Since $\mathcal{K}$ is cyclotomically self-dual we must have

$$d_k = d_{R+2-k}^{\dagger}, \qquad k = 1, \dots, R+1.$$

and

$$W_k = \operatorname{Wr}^{\dagger}(u_1, \dots, \hat{u}_k, \dots, u_{R+1}) = D_k (-1)^{-d_{R+2-k}} u_{R+2-k}(-x) + \dots.$$

Now from (5.8) we have

$$d_k - d_l = d_{R+2-l} - d_{R+2-k}, \qquad 1 \le k < l \le R+1,$$

using which one verifies that

$$D_k = D_{R+2-k}, \qquad k = 1, \dots, R+1.$$

Given this equality, if we set

$$q_k := u_k D_k^{\frac{1}{2}} \prod_{j=1}^{R+1} D_j^{-\frac{1}{2R-2}}$$

then we have

$$\mathrm{Wr}^\dagger(q_1, \dots, \widehat{q}_k, \dots, q_{R+1}) = (-1)^{-d_{R+2-k}} q_{R+2-k}(-x) + \dots$$

In this way, we arrive at

$$\mathrm{Wr}^\dagger(q_1, \dots, \widehat{q}_k, \dots, q_{R+1}) = (-1)^{-d_{R+2-k}} q_{R+2-k}(-x) + \sum_{j=2}^{R+1} q_{R+2-j}(-x) c_k^j, \qquad k = 1, \dots, R+1, \tag{5.37}$$

for some constants c_k^j. That is, we have

$$\begin{aligned}
\mathrm{Wr}^\dagger(q_1, \dots, q_{R-1}, q_R, \widehat{q}_{R+1}) &= (-1)^{-d_1} q_1(-x) \\
\mathrm{Wr}^\dagger(q_1, \dots, q_{R-1}, \widehat{q}_R, q_{R+1}) &= (-1)^{-d_2} q_2(-x) + c_R^1 q_1(-x) \\
\mathrm{Wr}^\dagger(q_1, \dots, \widehat{q}_{R-1}, q_R, q_{R+1}) &= (-1)^{-d_3} q_3(-x) + c_{R-1}^2 q_2(-x) + c_{R-1}^1 q_1(-x) \\
&\vdots
\end{aligned}$$

We define

$$\begin{aligned}
(-1)^{-d_1} r_1 &:= (-1)^{-d_1} q_1 \\
(-1)^{-d_2} r_2 &:= (-1)^{-d_2} q_2 + c_R^1 r_1
\end{aligned} \tag{5.38}$$

so that

$$\begin{aligned}
\mathrm{Wr}^\dagger(r_1, r_2, q_3, \dots, q_{R-1}, q_R, \widehat{q}_{R+1}) &= (-1)^{-d_1} r_1(-x) \\
\mathrm{Wr}^\dagger(r_1, r_2, q_3, \dots, q_{R-1}, \widehat{q}_R, q_{R+1}) &= (-1)^{-d_2} r_2(-x) \\
\mathrm{Wr}^\dagger(r_1, r_2, q_3, \dots, \widehat{q}_{R-1}, q_R, q_{R+1}) &= (-1)^{-d_3} q_3(-x) + \tilde{c}_{R-1}^2 r_2(-x) + \tilde{c}_{R-1}^1 r_1(-x) \\
&\vdots
\end{aligned} \tag{5.39}$$

for some new constants $\tilde{c}_k^j$, and we then define

$$(-1)^{-d_3} r_3 := (-1)^{-d_3} q_3 + \tilde{c}_{R-1}^1 r_2 + \tilde{c}_{R-1}^1 r_1,$$

and so on. By an obvious induction, we arrive at a Witt basis $(r_k)_{k=1}^{R+1}$. By construction $\deg r_k = d_k$. Finally, note in (5.37) that c_k^j can be non-zero only when $d_k - d_j \in \mathbb{Z}$ since both sides lie in either $\mathbb{C}[x]$ or $x^{\frac{1}{2}}\mathbb{C}[x]$. Therefore this Witt basis $(r_k)_{k=1}^{R+1}$ is special. □

Lemma 5.22. *Let $(r_k)_{k=1}^{R+1}$ be the Witt basis of Theorem 5.21. Then* $\mathrm{Wr}^\dagger(r_1, \dots, r_{R+1}) = 1$.

Proof. We have

$$\mathrm{Wr}^\dagger(q_1,\dots,q_{R+1}) = \mathrm{Wr}^\dagger(u_1,\dots,u_{R+1}) \prod_{k=1}^{R+1}\left(D_k^{\frac{1}{2}} \prod_{j=1}^{R+1} D_j^{-\frac{1}{2R-2}} \right)$$
$$= \mathrm{Wr}^\dagger(u_1,\dots,u_{R+1}) \prod_{k=1}^{R+1} D_k^{-\frac{1}{R-1}}.$$

But then noting that

$$\prod_{k=1}^{R+1} D_k = \prod_{1\le j<i\le R+1} (d_i-d_j)^{R-1}$$

and recalling Lemma 5.6, one finds

$$\mathrm{Wr}^\dagger(q_1,\dots,q_{R+1}) = 1$$

and hence the result. □

Theorem 5.23. *The subspaces $\mathcal{K}_{\mathrm{Sp}}$ and $\mathcal{K}_{\mathrm{O}}$ are mutually orthogonal with respect to B. The bilinear form B is skew-symmetric on $\mathcal{K}_{\mathrm{Sp}}$ and symmetric on $\mathcal{K}_{\mathrm{O}}$.*

Proof. Let $(r_k)_{k=1}^{R+1}$ be the special Witt basis constructed in Theorem 5.21. From (5.36) and Lemma 5.22, we have

$$B(r_k, r_{R+2-k}) = (-1)^{d_{R+2-k}+k+1}, \qquad k=1,\dots,R+1, \tag{5.40}$$

and $B(r_i,r_j)=0$ if $i+j\neq R+2$. This implies in particular that $\mathcal{K}_{\mathrm{Sp}}$ and $\mathcal{K}_{\mathrm{O}}$ are mutually orthogonal. By Lemma 5.12 it also gives

$$B(r_k, r_{R+2-k})B(r_{R+2-k},r_k) = (-1)^{\left\langle \Lambda, \alpha_1^\vee+\dots+\alpha_R^\vee\right\rangle}, \qquad k=1,\dots R+1.$$

Recall the definition of Λ, (5.6). Now $\left\langle \Lambda_s+\sigma\Lambda_s, \alpha_1^\vee+\dots+\alpha_R^\vee\right\rangle \in 2\mathbb{Z}$ for each $s=1,\dots,N$, since Λ_s is integral. Therefore it follows from (5.3) that

$$\left\langle \Lambda, \alpha_1^\vee+\dots+\alpha_R^\vee\right\rangle \in \begin{cases} 2\mathbb{Z}+1 & p>0 \\ 2\mathbb{Z} & p=0. \end{cases}$$

Consider the case $p>0$. Then we have

$$B(r_k, r_{R+2-k})B(r_{R+2-k},r_k) = -1, \qquad k=1,\dots,R+1. \tag{5.41}$$

Recall from (5.12) that $\deg r_k$ and $\deg r_{R+2-k}$ are both half odd integers if $k=p+1,\dots,R+1-p$, and are integers otherwise. Hence, by (5.40), $B(r_k,r_{R+2-k})$ and $B(r_{R+2-k},r_k)$ lie in $\{(-1)^{\frac12},(-1)^{-\frac12}\}$ if $k=p+1,\dots,R+1-p$ and in $\{1,-1\}$ otherwise. Combining this statement with (5.41) we find

$$B(r_k,r_{R+2-k}) = \begin{cases} -B(r_{R+2-k},r_k) & k=1,\dots,p,R+2-p,\dots,R+1 \\ +B(r_{R+2-k},r_k) & k=p+1,\dots,R+1-p \end{cases}$$

which is the required result.

Finally, consider the case $p=0$. Then

$$B(r_k, r_{R+2-k})B(r_{R+2-k},r_k) = 1, \qquad k=1,\dots,R+1,$$

and since in this case $\deg r_k$ is integral for all k, this implies

$$B(r_k,r_{R+2-k}) = B(r_{R+2-k},r_k), \qquad k=1,\dots,R+1$$

as required. □

The following are corollaries of Theorem 5.23 together with Lemma 5.20.

Corollary 5.24. *Every Witt basis $(r_k)_{k=1}^{R+1}$ of $\mathcal{K}$ is decomposable.* □

A basis $(r_k)_{k=1}^{R+1}$ of K such that

$$B_{ij} := B(r_i, r_j) = \delta_{R+2-i,j} b_i \tag{5.42}$$

with

$$b_k := \begin{cases} (-1)^k & k = 1, \dots, p \\ +1 & k = p+1, \dots, R+1-p \\ (-1)^{R+1-k} & k = R+2-p, \dots, R+1 \end{cases}$$

is called a *reduced Witt basis*. By Lemma 5.20, reduced Witt bases are Witt bases.

Corollary 5.25. *Any Witt basis can be transformed to a reduced Witt basis by a suitable diagonal transformation followed by a suitable permutation of the basis vectors.* □

Corollary 5.26. *For any Witt basis $(r_k)_{k=1}^{R+1}$ of $\mathcal{K}$, the full flag $\mathcal{F} = \{F_1 \subset F_2 \subset \cdots \subset F_{R+1} = \mathcal{K}\}$ given by $F_k = \operatorname{span}_{\mathbb{C}}(r_1, \dots, r_k)$, $k = 1, \dots, R+1$, is isotropic (and hence the corresponding tuple $\boldsymbol{y}^{\mathcal{F}}$ is cyclotomic by Theorem 5.17).*

Conversely, given any isotropic full flag $\mathcal{F} = \{F_1 \subset F_2 \subset \cdots \subset F_{R+1} = \mathcal{K}\}$ there is a Witt basis $(r_k)_{k=1}^{R+1}$ such that $F_k = \operatorname{span}_{\mathbb{C}}(r_1, \dots, r_k)$, $k = 1, \dots, R+1$. If in addition $\mathcal{F}$ is of type S then this basis can be chosen to be a reduced Witt basis. □

Lemma 5.27. *The full flag $\mathcal{F}$ given in* (5.15) *is isotropic and hence the corresponding tuple $\boldsymbol{y}^{\mathcal{F}}$ is cyclotomic.*

Proof. We can choose the special basis $(u_k)_{k=1}^{R+1}$ defining $\mathcal{F}$ to be the Witt basis of Theorem 5.21. Then the result follows from Corollary 5.26. □

5.8. **Isotropic flags.** Recall from §5.3 the notion of a symmetric subset of $\{1, \dots, R+1\}$.

Lemma 5.28. *Let $Q \subset \{1, \dots, R+1\}$ be a $2p$-element subset. The variety $FL_Q(\mathcal{K})$ contains an isotropic flag if and only if Q is symmetric.* □

Lemma 5.29. *If Q is symmetric then the variety $FL_Q^{\perp}(\mathcal{K})$ of isotropic flags is isomorphic to the direct product of spaces of isotropic flags $FL^{\perp}(\mathcal{K}_{\mathrm{Sp}}) \times FL^{\perp}(\mathcal{K}_{\mathrm{O}})$ and the isomorphism of these varieties is given by the map η_Q defined in* (5.13). □

In view of these lemmas and Theorem 5.17, we have the following description of the subspace of all cyclotomic tuples within the image $\beta(FL(\mathcal{K})) \subset \mathbb{P}(\mathbb{C}[x^{\frac{1}{2}}])^R$.

Theorem 5.30. *The irreducible components of the space $\beta(FL^{\perp}(\mathcal{K}))$ of all cyclotomic tuples are labeled by symmetric subsets $Q \subset \{1, \dots, R+1\}$. The components do not intersect and each is isomorphic to $FL^{\perp}(\mathcal{K}_{\mathrm{Sp}}) \times FL^{\perp}(\mathcal{K}_{\mathrm{O}})$.* □

5.9. **Infinitesimal deformation of isotropic flags of type S.** The connected Lie group of endomorphisms of $\mathcal{K}$ preserving B acts transitively on the variety of isotropic full flags of type Q, $FL_Q^{\perp}(\mathcal{K})$, for each symmetric subset $Q \subset \{1, \dots, R+1\}$. In particular it acts transitively on $FL_S^{\perp}(\mathcal{K})$, and hence on the cyclotomic tuples of polynomials in the image $\beta(FL_S^{\perp}(\mathcal{K})) \subset \mathbb{P}(\mathbb{C}[x])^R$. We shall describe the infinitesimal action of this group on $\beta(FL_S^{\perp}(\mathcal{K}))$.

The connected Lie group of endomorphisms of $\mathcal{K}$ preserving B preserves each of the subspaces $\mathcal{K}_{\mathrm{Sp}}$ and $\mathcal{K}_{\mathrm{O}}$. Thus this group is the product $\mathrm{Sp}(\mathcal{K}_{\mathrm{Sp}}) \times \mathrm{SO}(\mathcal{K}_{\mathrm{O}})$ of the group of special symplectic transformations in $\mathrm{End}(\mathcal{K}_{\mathrm{Sp}})$ and the group of special orthogonal transformations in $\mathrm{End}(\mathcal{K}_{\mathrm{O}})$. Its Lie algebra $\mathfrak{sp}(\mathcal{K}_{\mathrm{Sp}}) \oplus \mathfrak{so}(\mathcal{K}_{\mathrm{O}})$ consists of all traceless endomorphisms X of $\mathcal{K}$ such that

$$B(Xu, v) + B(u, Xv) = 0$$

for all $u, v \in \mathcal{K}$.

Pick any isotropic full flag $\mathcal{F} = \{F_1 \subset F_2 \subset \cdots \subset F_{R+1} = \mathcal{K}\}$ of type S. Then $\beta(\mathcal{F}) = \boldsymbol{y}^{\mathcal{F}}$ is a cyclotomic tuple of polynomials by Lemma 5.8. Let $(r_k)_{k=1}^{R+1}$ be a reduced Witt basis such that $F_k = \mathrm{span}_{\mathbb{C}}(r_1, \dots, r_k)$, $k = 1, \dots, R+1$. Such a basis exists by Corollary 5.26.

This choice of basis gives identifications $\mathcal{K}_{\mathrm{Sp}} \cong \mathbb{C}^{2p}$ and $\mathcal{K}_{\mathrm{O}} \cong \mathbb{C}^{R+1-2p}$ and hence $\mathfrak{sp}(\mathcal{K}_{\mathrm{Sp}}) \cong \mathfrak{sp}_{2p}$ and $\mathfrak{so}(\mathcal{K}_{\mathrm{O}}) \cong \mathfrak{so}_{R+1-2p}$. The Lie algebra $\mathfrak{sp}_{2p}$ has root system of type C_p. The Lie algebra $\mathfrak{so}_{R+1-2p}$ has root system of type D_{n-p} if $R = 2n-1$ is odd and of type B_{n-p} if $R = 2n$ is even.

Let $(E_{i,j})_{i,j=1}^{R+1}$ be the basis of $\mathrm{End}(\mathcal{K})$ defined by

$$E_{i,j} r_k = \delta_{ik} r_j.$$

The lower-triangular subalgebra of $\mathfrak{sp}(\mathcal{K}_{\mathrm{Sp}}) \cong \mathfrak{sp}_{2p}$ is generated by

$$X_k := E_{k+1,k} + E_{R+2-k,R+1-k}, \qquad k = 1, \dots, p-1,$$

and

$$X_p := E_{R+2-p,p}.$$

When $R = 2n-1$, the lower-triangular subalgebra of $\mathfrak{so}(\mathcal{K}_{\mathrm{O}}) \cong \mathfrak{so}_{2n-2p}$ is generated by

$$Y_k := E_{k+p,k+p-1} - E_{2n-p-k+1,2n-p-k}, \qquad k = 1, \dots, n-p-1,$$

and

$$\tilde{Y}_{n-p-1} := E_{k+p+1,k+p-1} - E_{2n-p-k+1,2n-p-k-1}.$$

When $R = 2n$, the lower-triangular subalgebra of $\mathfrak{so}(\mathcal{K}_{\mathrm{O}}) \cong \mathfrak{so}_{2n-2p+1}$ is generated by

$$Z_k := E_{k+p,k+p-1} - E_{2n-p-k+2,2n-p-k+1}, \qquad k = 1, \dots, n-p.$$

These generators define linear transformations belonging to $\mathrm{End}(\mathcal{K}_{\mathrm{Sp}}) \oplus \mathrm{End}(\mathcal{K}_{\mathrm{O}})$.

Remark 5.31. The Lie algebra $\mathfrak{so}(\mathcal{K}_{\mathrm{O}}) \oplus \mathfrak{sp}(\mathcal{K}_{\mathrm{Sp}})$ is contained in the simple Lie superalgebra $\mathfrak{osp}(\mathcal{K})$ of all orthosymplectic transformations of the space $\mathcal{K}$. See [Kac77] for the definition. It would be interesting to understand the role of this superalgebra here.

For any $k = 1, \dots, p$ and all $c \in \mathbb{C}$, the basis $e^{cX_k}\boldsymbol{r}$ is again a Witt basis of $\mathcal{K}$. Let $e^{cX_k}\mathcal{F}$ denote the corresponding isotropic flag and $\beta(e^{cX_k}\mathcal{F})$ the corresponding tuple representing a cyclotomic point. Let us describe the dependence on c of this tuple.

For $k = 1, \dots, p-1$, we have

$$e^{cX_k}\boldsymbol{r} = (r_1, \dots, r_{k-1}, r_k + c r_{k+1}, r_{k+1}, \dots, r_{R-k}, r_{R+1-k} + c r_{R+2-k}, r_{R+2-k}, \dots, r_{R+1})$$

and hence

$$\beta(e^{cX_k}\mathcal{F}) = (y_1^{\mathcal{F}}, \dots, y_{k-1}^{\mathcal{F}}, y_k(x,c), y_{k+1}^{\mathcal{F}}, \dots, y_{R+1-k}^{\mathcal{F}}, y_{R+1-k}(x,c), y_{R+2-k}^{\mathcal{F}}, \dots, y_{R+1}^{\mathcal{F}})$$

where

$$\begin{aligned} y_k(x,c) &:= \mathrm{Wr}^{\dagger}(r_1, \dots, r_{k-1}, r_k + c r_{k+1}) \\ &= y_k^{\mathcal{F}} + c\,\mathrm{Wr}^{\dagger}(r_1, \dots, r_{k-1}, r_{k+1}) \end{aligned} \tag{5.43a}$$

and

$$\begin{aligned} y_{R+1-k}(x,c) &:= \mathrm{Wr}^{\dagger}(r_1, \dots, r_{R-k}, r_{R+1-k} + c r_{R+2-k}) \\ &= y_{R+1-k}^{\mathcal{F}} + c\,\mathrm{Wr}^{\dagger}(r_1, \dots, r_{R-k}, r_{R+2-k}). \end{aligned} \tag{5.43b}$$

Finally (for $k = p$) we have

$$e^{cX_p}\boldsymbol{r} = (r_1, \dots, r_{p-1}, r_p + c r_{R+2-p}, r_{p+1}, \dots, r_{R+1})$$

and hence

$$\beta(e^{cX_p}\mathcal{F}) = (y_1^{\mathcal{F}}, \dots, y_{p-1}^{\mathcal{F}}, y_p(x,c), y_{p+1}^{\mathcal{F}}, \dots, \dots, y_{R+1}^{\mathcal{F}})$$

$$\begin{aligned} y_p(x,c) &:= \mathrm{Wr}^\dagger(r_1,\dots,r_{p-1},r_p+cr_{R+2-p}) \\ &= y_p^{\mathcal{F}} + c\,\mathrm{Wr}^\dagger(r_1,\dots,r_{p-1},r_{R+2-p}). \end{aligned} \tag{5.44}$$

The flows in $\mathbb{P}(\mathbb{C}[x])^R$ corresponding to the generators of $\mathfrak{so}(\mathcal{K}_{\mathrm{O}})$ can be described similarly.

5.10. **Populations of cyclotomic critical points in type A.** Recall the definition of the extended master function $\widehat{\Phi}$, (2.11). In the setting of the present section (see §5.1) it has the explicit form

$$\begin{aligned} \widehat{\Phi}(\boldsymbol{t};\mathsf{c};\boldsymbol{z};\boldsymbol{\Lambda};\Lambda_0) = \sum_{i=1}^N (\Lambda_0,\Lambda_i)(\log(-z_i)+\log(z_i)) + \sum_{i=1}^N (\Lambda_i,\Lambda_{R+1-i})\log 2z_i \\ + \sum_{1\le i<j\le n}(\Lambda_i,\Lambda_j)\log(z_i-z_j) + \sum_{1\le i<j\le n}(\Lambda_{R+1-i},\Lambda_j)\log(-z_i-z_j) \\ + \sum_{1\le i<j\le n}(\Lambda_i,\Lambda_{R+1-j})\log(z_i+z_j) + \sum_{1\le i<j\le n}(\Lambda_{R+1-i},\Lambda_{R+1-j})\log(-z_i+z_j) \\ - \sum_{j=1}^{\tilde m}(\alpha_{\mathsf{c}(j)},\Lambda_0)\log(t_j) - \sum_{i=1}^N\sum_{j=1}^{\tilde m}(\alpha_{\mathsf{c}(j)},\Lambda_i)\log(t_j-z_i) - \sum_{i=1}^N\sum_{j=1}^{\tilde m}(\alpha_{\mathsf{c}(j)},\Lambda_{R+1-i})\log(t_j+z_i) \\ + \sum_{1\le i<j\le \tilde m}(\alpha_{\mathsf{c}(i)},\alpha_{\mathsf{c}(j)})\log(t_i-t_j) \end{aligned} \tag{5.45}$$

and the critical point equations (2.12) become

$$0 = \sum_{i=1}^N \frac{(\alpha_{\mathsf{c}(j)},\Lambda_i)}{t_j-z_i} + \sum_{i=1}^N \frac{(\alpha_{\mathsf{c}(j)},\Lambda_{R+1-i})}{t_j+z_i} + \frac{(\alpha_{\mathsf{c}(j)},\Lambda_0)}{t_j} - \sum_{\substack{i=1\\ i\neq j}}^{\tilde m}\frac{(\alpha_{\mathsf{c}(j)},\alpha_{\mathsf{c}(i)})}{t_j-t_i}, \quad j=1,\dots,\tilde m. \tag{5.46}$$

Given a tuple of polynomials $\boldsymbol{y}\in\mathbb{P}(\mathbb{C}[x])^R$, we have the pair $(\boldsymbol{t},\mathsf{c})\in\mathbb{C}^{\tilde m}\times I^{\tilde m}$ represented by $\boldsymbol{y}$ in the sense of §4.2. We say the tuple $\boldsymbol{y}$ *represents a critical point of* $\widehat{\Phi}$ if $\boldsymbol{t}$ is a critical point of $\widehat{\Phi}(\boldsymbol{t};\mathsf{c};\boldsymbol{z};\boldsymbol{\Lambda};\Lambda_0)$, *i.e.* if $(\boldsymbol{t},\mathsf{c})$ satisfy the equations (5.46).

The following theorem says that we can go from cyclotomic critical points of the extended master function $\widehat{\Phi}$, (5.45), to decomposable cyclotomically self-dual vector spaces of quasi-polynomials.

Theorem 5.32. *Suppose $\boldsymbol{y}\in\mathbb{P}(\mathbb{C}[x])^R$ represents a cyclotomic critical point of $\widehat{\Phi}$,* (5.45).

The kernel $\ker\mathcal{D}(\boldsymbol{y})$ of the fundamental differential operator $\mathcal{D}(\boldsymbol{y})$, §5.4, is a decomposable cyclotomically self-dual vector space of quasi-polynomials with frame $\tilde T_1,\dots,\tilde T_R;\tilde\Lambda_\infty$, where $\tilde\Lambda_\infty$ is the unique dominant weight in the orbit of $\Lambda_\infty(\boldsymbol{y})$, (4.10), *under the shifted action of the Weyl group of type A_R.*

There exists an isotropic flag $\mathcal{F}\in FL_S^\perp(\ker\mathcal{D}(\boldsymbol{y}))$ such that $\boldsymbol{y}=\beta(\mathcal{F})$.

Proof. Arguing as in [MV04] – see especially Lemma 5.10 – we have that $\ker\mathcal{D}(\boldsymbol{y})$ is a vector space of quasi-polynomials with frame $\tilde T_1,\dots,\tilde T_R;\tilde\Lambda_\infty$, and that the flag $\mathcal{F}\in FL(\ker(\mathcal{D}(\boldsymbol{y})))$ such that $\beta(\mathcal{F})=\boldsymbol{y}$ can be constructed as follows. Define quasi-polynomials $y_i^{(i,i+1,\dots,k)}$, $1\le i\le k\le R$, recursively by

$$\begin{aligned} \mathrm{Wr}(y_k^{(k)},y_k) &= y_{k-1}\tilde T_k y_{k+1}, \\ \mathrm{Wr}(y_i^{(i,i+1,\dots,k)},y_i) &= y_{i-1}\tilde T_i y_{i+1}^{(i+1,\dots,k)}, \qquad i<k \end{aligned}$$

(recall we set $y_0=y_{R+1}=1$ for convenience). Set $u_1=y_1$ and $u_k=y_1^{(1,\dots,k-1)}$ for $k=2,\dots,R+1$. Then $(u_k)_{k=1}^{R+1}$ is a basis of $\ker\mathcal{D}(\boldsymbol{y})$. Moreover $\mathrm{Wr}^\dagger(u_1,\dots,u_k)=y_k$, $k=1,\dots,R$. That is, $\beta(\mathcal{F})=\boldsymbol{y}$ for the flag $\mathcal{F}=\{F_k\}$ given by $F_k=\mathrm{span}_{\mathbb{C}}(u_1,\dots,u_k)$, $k=1,\dots,R+1$. Since $\boldsymbol{y}$ is cyclotomic, Theorem 5.14 states that $\ker \mathcal{D}(\boldsymbol{y})$ is cyclotomically self-dual. By Lemma 5.9, $\mathcal{F}$ is a decomposable flag of type S, and by Theorem 5.17 it is isotropic. □

Conversely, we have the following, arguing as in Lemmas 3.1, 3.2 and 5.15 in [MV04] and using Theorem 5.17.

Theorem 5.33. *Let $\mathcal{K}$ be a decomposable cyclotomically self-dual vector space of quasi-polynomials with frame $\tilde{T}_1, \dots, \tilde{T}_R; \tilde{\Lambda}_\infty$.*

Suppose there exists an isotropic flag $\mathcal{F} \in FL^{\perp}_{S}(\mathcal{K})$ such that the tuple $\boldsymbol{y}^{\mathcal{F}}$ is generic. Then $\boldsymbol{y}^{\mathcal{F}}$ represents a cyclotomic critical point of $\widehat{\Phi}$, (5.45). □

Since being generic is an open condition, the set of generic tuples in the image $\beta(FL^{\perp}_{S}(\mathcal{K}))$ is either empty or it is open and dense in $\beta(FL^{\perp}_{S}(\mathcal{K}))$.

Starting from an initial tuple $\boldsymbol{y}$ that represents a cyclotomic critical point of $\widehat{\Phi}$, (5.45), we may let $\mathcal{K} = \ker \mathcal{D}(\boldsymbol{y})$ as in Theorem 5.32. Then we have the variety

$$\beta(FL^{\perp}_{S}(\mathcal{K})) \cong FL^{\perp}(\mathcal{K}_{\mathrm{Sp}}) \times FL^{\perp}(\mathcal{K}_{\mathrm{O}}), \tag{5.47}$$

where the isomorphism here is by Theorem 5.30. Almost all of the tuples in $\beta(FL^{\perp}_{S}(\mathcal{K}))$ are generic and hence represent cyclotomic critical points of $\widehat{\Phi}$. Call this variety $\beta(FL^{\perp}_{S}(\mathcal{K})) \subset \mathbb{P}(\mathbb{C}[x])^R$ the *cyclotomic population originated at $\boldsymbol{y}$.*

5.11. **The case $p = n$.** Consider the case $p = n$ in (5.3). Namely, suppose that we are either in

- type A_{2n-1} with Λ_0 integral and $\langle \Lambda_0, \alpha_n \rangle$ odd, or
- type A_{2n} with $\langle \Lambda_0, \alpha_i^\vee \rangle \in \mathbb{Z}$ for all $i < n$ and $\langle \Lambda_0, \alpha_n^\vee \rangle \in \frac{1}{2}(2\mathbb{Z}_{\geq 0} - 1) = \{-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \dots\}$.

Then Λ_0 obeys the assumptions from §4.1 and so we are in the setting of §4. That means we have two notions of a cyclotomic population: the one in the previous subsection, and the one in §4.7. Let us show that these two notions coincide.

Theorem 5.34. *Let $p = n$ in* (5.3)*. Let $\boldsymbol{y}$ represent a cyclotomic critical point of the extended master function $\widehat{\Phi}$ of* (5.45)*. Then the variety $\beta(FL^{\perp}_{S}(\mathcal{K}))$ is isomorphic to the variety of isotropic full flags in a complex symplectic vector space of dimension $2n$. The cyclotomic population in $\mathbb{P}(\mathbb{C}[x])^R$ originated at $\boldsymbol{y}$ in the sense of §4.7 coincides with this variety $\beta(FL^{\perp}_{S}(\mathcal{K}))$.*

Proof. When $p = n$ we have either $\mathcal{K}_{\mathrm{O}} = \{0\}$, if $R = 2n - 1$, or $\mathcal{K}_{\mathrm{O}} \cong \mathbb{C}$, if $R = 2n$. In either case $FL^{\perp}(\mathcal{K}_{\mathrm{O}})$ is a point, and (5.47) reduces to

$$\beta(FL^{\perp}_{S}(\mathcal{K})) \cong FL^{\perp}(\mathcal{K}_{\mathrm{Sp}}),$$

i.e. $\beta(FL^{\perp}_{S}(\mathcal{K}))$ is isomorphic to the variety of isotropic full flags in the vector space $\mathcal{K}_{\mathrm{Sp}} \cong \mathbb{C}^{2n}$ endowed with the symplectic form $B|_{\mathcal{K}_{\mathrm{Sp}}}$.

Starting from any such isotropic full flag, $\mathcal{F} \in FL^{\perp}_{S}(\mathcal{K})$, we choose a reduced Witt basis adapted to $\mathcal{F}$ (Corollary 5.26). Then every other flag in $FL^{\perp}_{S}(\mathcal{K})$ can be reached by an element of the lower-triangular (as in §5.9) unipotent subgroup of Sp_{2n}. This subgroup is generated by the one-parameter groups corresponding to negative simple root generators X_k of §5.9. Lemmas 5.35 and 5.36 below show that the flows in $\beta(FL^{\perp}_{S}(\mathcal{K}))$ generated by the X_k coincide with notion of cyclotomic generation from §4. That shows that the set of all tuples of polynomials obtained from $\boldsymbol{y}$ by repeated cyclotomic generation, in all directions $i \in I$, contains a non-empty open subset of $\beta(FL^{\perp}_{S}(\mathcal{K}))$. Therefore it is dense in $\beta(FL^{\perp}_{S}(\mathcal{K}))$. Hence its Zariski closure is $\beta(FL^{\perp}_{S}(\mathcal{K}))$ itself. □

Lemma 5.35. *The image $\beta(e^{cX_k}\mathcal{F}) \in \mathbb{P}(\mathbb{C}[x])^R$ coincides with the tuple $\boldsymbol{y}^{(k,\sigma)}(1/c)$ of Theorem 4.6, for every $k = 1, \dots, n-1$ (and also for $k = n$ when we are in type $A_R = A_{2n-1}$).*

Proof. It is enough to note that, in view of (5.43) and Lemma 5.15, we have

$$\mathrm{Wr}(y_k^{\mathcal{F}}, y_k(x,c)) = c\tilde{T}_k y_{k-1}^{\mathcal{F}} y_{k+1}^{\mathcal{F}}$$
$$\mathrm{Wr}(y_{R+1-k}^{\mathcal{F}}, y_{R+1-k}(x,c)) = c\tilde{T}_{R+1-k} y_{R-k}^{\mathcal{F}} y_{R+2-k}^{\mathcal{F}}.$$

□

It remains to consider the case $k = n$ in type A_{2n}.

Lemma 5.36. *In type A_{2n}, the image $\beta(e^{-cX_n}\mathcal{F}) \in \mathbb{P}(\mathbb{C}[x])^{2n}$ coincides with the tuple $\boldsymbol{y}^{(n,\sigma)}(1/c)$ of Theorem 4.20.*

Proof. We have (5.44) with $p = n$. Namely,

$$e^{cX_n}\boldsymbol{r} = (r_1, \dots, r_{n-1}, r_n + cr_{n+2}, r_{n+1}, r_{n+2}, \dots, r_{2n+1})$$

and hence

$$\beta(e^{cX_n}\mathcal{F}) = (y_1^{\mathcal{F}}, \dots, y_{n-1}^{\mathcal{F}}, y_n(x,c), y_{n+1}(x,c), y_{n+2}^{\mathcal{F}}, \dots, y_{2n+1}^{\mathcal{F}})$$

where

$$\begin{aligned} y_n(x,c) &:= \mathrm{Wr}^\dagger(r_1, \dots, r_{n-1}, r_n + cr_{n+2}) \\ &= y_n^{\mathcal{F}} + c\,\mathrm{Wr}^\dagger(r_1, \dots, r_{n-1}, r_{n+2}) \end{aligned} \tag{5.48}$$

and

$$\begin{aligned} y_{n+1}(x,c) &:= \mathrm{Wr}^\dagger(r_1, \dots, r_{n-1}, r_n + cr_{n+2}, r_{n+1}) \\ &= y_{n+1}^{\mathcal{F}} + c\,\mathrm{Wr}^\dagger(r_1, \dots, r_{n-1}, r_{n+2}, r_{n+1}). \end{aligned} \tag{5.49}$$

Now let

$$y_n^{(n)} := \mathrm{Wr}^\dagger(r_1, \dots, r_{n-1}, r_{n+1}).$$

Then by Lemma 5.15 we have

$$\mathrm{Wr}(y_n^{\mathcal{F}}, y_n^{(n)}) = \tilde{T}_n y_{n-1}^{\mathcal{F}} y_{n+1}^{\mathcal{F}}$$
$$\mathrm{Wr}(y_{n+1}^{\mathcal{F}}, y_{n+1}(x,c)) = -c\tilde{T}_{n+1} y_n^{(n)} y_{n+2}^{\mathcal{F}}$$
$$\mathrm{Wr}(y_n^{(n)}, y_n(x,c)) = \tilde{T}_n y_{n-1} y_{n+1}(x,c).$$

This establishes the lemma. □

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