THE BERGMAN-SHELAH PREORDER ON TRANSFORMATION SEMIGROUPS

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Abstract. Let $\mathbb{N}^{\mathbb{N}}$ be the semigroup of all mappings on the natural numbers $\mathbb{N}$, and let U and V be subsets of $\mathbb{N}^{\mathbb{N}}$. We write $U \preccurlyeq V$ if there exists a countable subset C of $\mathbb{N}^{\mathbb{N}}$ such that U is contained in the subsemigroup generated by V and C. We give several results about the structure of the preorder $\preccurlyeq$. In particular, we show that a certain statement about this preorder is equivalent to the Continuum Hypothesis.

The preorder $\preccurlyeq$ is analogous to one introduced by Bergman and Shelah on subgroups of the symmetric group on $\mathbb{N}$. The results in this paper suggest that the preorder on subsemigroups of $\mathbb{N}^{\mathbb{N}}$ is much more complicated than that on subgroups of the symmetric group.

1. Introduction and background

The semigroup of all mappings from $\mathbb{N} = \{0, 1, 2, \ldots\}$ to itself is denoted by $\mathbb{N}^{\mathbb{N}}$. Given subsets U and V of $\mathbb{N}^{\mathbb{N}}$, we write $U \preccurlyeq V$ if there exists a countable subset C of $\mathbb{N}^{\mathbb{N}}$ such that U is contained in the subsemigroup $\langle V, C\rangle$ generated by V and C. It follows from a classical result by Sierpiński [1] that if $U \preccurlyeq V$, then there exist $f, g \in \mathbb{N}^{\mathbb{N}}$ such that $U \subseteq \langle V, f, g\rangle$. So replacing the word 'countable' above by 'finite' or even '2-element' yields an equivalent definition of $\preccurlyeq$. We write $U \approx V$ if $U \preccurlyeq V$ and $V \preccurlyeq U$, and we write $U \prec V$ if $U \preccurlyeq V$ and $U \not\approx V$.

The semigroup $\mathbb{N}^{\mathbb{N}}$ has a natural topology: the product topology arising from the discrete topology on $\mathbb{N}$; see [8, Section 9.B(7)] for further details. Under this topology, composition of functions is continuous, making $\mathbb{N}^{\mathbb{N}}$ a *topological semigroup*. Let S_∞ denote the symmetric group on $\mathbb{N}$, i.e. the group of invertible elements of $\mathbb{N}^{\mathbb{N}}$. As it happens the function $x \mapsto x^{-1}$ on S_∞ is also continuous, and so S_∞ is a *topological group* with the induced topology. We refer to subgroups of S_∞ and subsemigroups of $\mathbb{N}^{\mathbb{N}}$ that are closed in the relevant topologies as *closed subgroups* and *closed subsemigroups*, respectively. It is a well-known fact that the closed subsemigroups of $\mathbb{N}^{\mathbb{N}}$ are precisely the endomorphism semigroups of relational structures on $\mathbb{N}$ and that the closed subgroups of S_∞ are the corresponding automorphisms groups; see, for example, [3, Theorem 5.8].

The preorder $\preccurlyeq$ is analogous to a preorder on the subsets of S_∞ introduced in [2]: if $U, V \subseteq S_\infty$, then U is less than V whenever U is contained in the subgroup generated by $V \cup C$ for some countable $C \subseteq S_\infty$. Once again, insisting that C is finite, or even of size 2, yields an equivalent definition; see [2, Lemma 3(i)]. In [2] it is shown that the closed subgroups of S_∞ fall into four equivalence classes with respect to this preorder. Various classes of subsemigroups of $\mathbb{N}^{\mathbb{N}}$ are classified according to $\approx$ in [10] and [11]. The situation is much more complicated in $\mathbb{N}^{\mathbb{N}}$, as in particular, there are infinitely many distinct $\approx$-classes containing closed subsemigroups. For example, define for each $n \geq 2$

$$\mathfrak{F}_n = \{f \in \mathbb{N}^{\mathbb{N}} : |f(\mathbb{N})| \leq n\}.$$

It is straightforward to show that $\mathfrak{F}_n$ is a closed subsemigroup of $\mathbb{N}^{\mathbb{N}}$ for all $n \geq 2$. Furthermore each $\mathfrak{F}_n$ is an ideal of $\mathbb{N}^{\mathbb{N}}$ and so if $U \subseteq \langle \mathfrak{F}_n, C\rangle$ for some $U, C \subseteq \mathbb{N}^{\mathbb{N}}$, then $U \setminus \mathfrak{F}_n \subseteq \langle C\rangle$. Hence $U \preccurlyeq \mathfrak{F}_n$ if and only if $U \setminus \mathfrak{F}_n$ is countable. But $|\mathfrak{F}_m \setminus \mathfrak{F}_n| = 2^{\aleph_0}$ whenever $m > n$ and so $\mathfrak{F}_2 \prec \mathfrak{F}_3 \prec \cdots$.

We prove five results that exhibit the complicated structure of $\preccurlyeq$ and its sensitivity to set-theoretic assumptions.

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In Theorem 2.1, we show that the Continuum Hypothesis holds if and only if there exists a subsemigroup S of $\mathbb{N}^\mathbb{N}$ such that $S \approx \mathbb{N}^\mathbb{N}$ and for all subsemigroups T of S either $T \approx \mathbb{N}^\mathbb{N}$ or T is equivalent to the trivial semigroup $\{1_\mathbb{N}\}$. We prove that for every closed subsemigroup S of $\mathbb{N}^\mathbb{N}$ with cardinality $2^{\aleph_0}$ there is a closed subsemigroup T of $\mathfrak{F}_2$ of cardinality $2^{\aleph_0}$ such that $T \preccurlyeq S$ (Theorem 3.1). Theorem 3.1 could be viewed as an analogue of the classical theorem that every perfect Polish topological space contains a copy of the Cantor set. To show that T in Theorem 3.1 cannot be replaced by $\mathfrak{F}_2$, we associate a semigroup to each almost disjoint family of subsets of $\mathbb{N}$ with cardinality $2^{\aleph_0}$ and show that any such semigroup is incomparable to $\mathfrak{F}_n$ for all $n \in \mathbb{N}$ (Theorem 4.1). We prove that there are anti-chains of $\approx$-classes containing closed subsemigroups of $\mathbb{N}^\mathbb{N}$ with arbitrary finite length (Theorem 5.1). Finally, we show that there exists a chain of $\approx$-classes with length $\aleph_1$ containing (not necessarily closed) subsemigroups of $\mathfrak{F}_2$ (Theorem 6.1), establishing a new lower bound for the number of $\approx$-classes.

It seems unlikely that a usable classification of $\approx$-classes and the partial order induced by $\preccurlyeq$ can be found. However, further potentially tractable questions about the structure of $\preccurlyeq$ are, as yet, unanswered. For instance, what is the number of $\approx$-classes? What is the number of $\approx$-classes containing closed subsemigroups? Which preorders can be embedded in $\preccurlyeq$? More specifically, does there exist an infinite anti-chain or an infinite descending chain? Do there exist $U, V \leq \mathbb{N}^\mathbb{N}$ such that $U \prec V$ and whenever $U \preccurlyeq W \preccurlyeq V$ either $W \approx U$ or $W \approx V$?

2. Continuum Hypothesis

The Continuum Hypothesis is the statement: $\aleph_1 = 2^{\aleph_0}$. Gödel [7] and Cohen [4], [5] showed that it is independent of the standard axioms of set theory (ZFC). The Continuum Hypothesis is equivalent to the existence of an uncountable family $\mathcal{F}$ of analytic functions from $\mathbb{C}$ to $\mathbb{C}$ satisfying

$$|\{f(x) : f \in \mathcal{F}\}| \leq \aleph_0$$

for all $x \in \mathbb{C}$, as well as the existence of a function $f = (f_1, f_2)$ from $\mathbb{R}$ onto $\mathbb{R}^2$ such that for all $x \in \mathbb{R}$ either f_1 or f_2 is differentiable at x (see [6] and [12], respectively). For more information on the history of the Continuum Hypothesis see [14] or [15].

In some sense, the above results are analytic versions of the Continuum Hypothesis; in this section we present an algebraic version.

Theorem 2.1. *The following are equivalent:*

(i) *the Continuum Hypothesis;*
(ii) *there exists a subsemigroup S of $\mathbb{N}^\mathbb{N}$ such that $S \approx \mathbb{N}^\mathbb{N}$ and for all subsemigroups T of S either $T \approx \mathbb{N}^\mathbb{N}$ or $T \approx \{1_\mathbb{N}\}$.*

We require two lemmas to prove Theorem 2.1. The proof of the first is essentially Banach's argument [1] for Sierpiński's theorem in [13].

Lemma 2.2. *Let $f \in \mathbb{N}^\mathbb{N}$ be any injective function with $|\mathbb{N} \setminus f(\mathbb{N})| = |\mathbb{N}|$ and let $g_1, g_2, \ldots \in \mathbb{N}^\mathbb{N}$ be arbitrary. Then there exists $h \in \mathbb{N}^\mathbb{N}$ such that $g_1, g_2, \ldots \in \langle f, h\rangle$.*

Proof. Let $X_0 = \mathbb{N} \setminus f(\mathbb{N})$ and let $X_i = f^i(X_0)$ for all $i > 0$. Then clearly $X_0 \cap X_i = \emptyset$ for all $i > 0$. Hence if $k > j$, $X_j \cap X_k = f^j(X_0) \cap f^j(X_{k-j}) = \emptyset$, since f is injective. It follows that $X_0, X_1, \ldots$ are disjoint infinite subsets of $\mathbb{N}$.

Let $X_{0,0}, X_{0,1}, X_{0,2}, \ldots$ be sets partitioning X_0 such that $|X_{0,0}| = |\mathbb{N} \setminus \bigcup_{i=0}^\infty X_i|$ and $|X_{0,i}| = |\mathbb{N}|$ for all $i > 0$. We also let h be any map taking $\mathbb{N} \setminus \bigcup_{i=0}^\infty X_i$ bijectively to $X_{0,0}$ and X_i bijectively to $X_{0,i}$ for all $i > 0$. It is straightforward to verify that hf^ihf maps $\mathbb{N}$ bijectively to $X_{0,i}$ for all $i > 0$. Since h is not yet defined on X_0, we can define it by:

$$h(n) = g_i\big((hf^ihf)^{-1}(n)\big)$$

for all $n \in X_{0,i}$ and for all $i > 0$ and h can be defined arbitrarily on $X_{0,0}$.

It is easy to verify that $g_i = h^2f^ihf$ for all $i > 0$. □

Lemma 2.3. *Let γ be an ordinal and for every $\alpha < \gamma$ let $u_\alpha \in \mathbb{N}^\mathbb{N}$. Then there exist $h, k \in \mathbb{N}^\mathbb{N}$ and for every $\alpha < \gamma$ there is a mapping $g_\alpha \in \mathbb{N}^\mathbb{N}$ such that:*

(i) $g_\alpha g_\beta$ *is the constant function with value* 0 *for all* $\beta < \gamma$*;*
(ii) $u_\alpha = kg_\alpha h$.

Proof. Let X be any infinite coinfinite subset of $\mathbb{N}$ such that $0 \notin X$, let $h : \mathbb{N} \longrightarrow X$ be any bijection, and let $k \in \mathbb{N}^\mathbb{N}$ be any function mapping $\mathbb{N} \setminus X$ bijectively to $\mathbb{N}$. Then for all $\alpha < \gamma$ define $g_\alpha \in \mathbb{N}^\mathbb{N}$ by

$$g_\alpha(n) = \begin{cases} (k|_{\mathbb{N}\setminus X})^{-1} u_\alpha h^{-1}(n) & \text{if } n \in X \\ 0 & \text{if } n \notin X, \end{cases}$$

where $k|_{\mathbb{N}\setminus X}$ denotes the restriction of k to $\mathbb{N} \setminus X$. The mappings h, k, and g_α $(\alpha < \gamma)$ have the required properties. □

Proof of Theorem 2.1. (i) ⇒ (ii). Write $\mathbb{N}^\mathbb{N} = \{f_\alpha : \alpha < \aleph_1\}$ and let $f \in \mathbb{N}^\mathbb{N}$ be an injection such that $\mathbb{N} \setminus f(\mathbb{N})$ is infinite. We define a subset $U = \{u_\alpha : \alpha < \aleph_1\}$ of $\mathbb{N}^\mathbb{N}$ such that every uncountable subset V of U satisfies $V \approx \mathbb{N}^\mathbb{N}$. Set $u_0 = f_0$. If $\alpha < \aleph_1$ and u_β is defined for all $\beta < \alpha$, then, by Lemma 2.2, there exists $u_\alpha \in \mathbb{N}^\mathbb{N}$ such that

$$\{f_\beta : \beta < \alpha\} \subseteq \langle f, u_\alpha \rangle.$$

If V is any uncountable subset of U, then for all $\beta < \aleph_1$ there exists $\lambda(\beta)$ such that $\beta < \lambda(\beta) < \aleph_1$ and $u_{\lambda(\beta)} \in V$. It follows that $f_\beta \in \langle f, u_{\lambda(\beta)} \rangle \subseteq \langle f, V \rangle$ for all $\beta < \aleph_1$ and so $\mathbb{N}^\mathbb{N} \subseteq \langle f, V \rangle$. In particular, $V \approx \mathbb{N}^\mathbb{N}$.

Applying Lemma 2.3 to U and $\gamma = \aleph_1$ we obtain $g_\alpha \in \mathbb{N}^\mathbb{N}$ for all $\alpha < \aleph_1$ and $h, k \in \mathbb{N}^\mathbb{N}$ with the properties given in the lemma. We set S to be the semigroup consisting of $\{g_\alpha : \alpha < \aleph_1\}$ and the constant mapping with value 0. To verify that S satisfies (ii), let T be any subset of S. If T is uncountable, then $\langle T, h, k \rangle$ contains an uncountable subset of U and so $T \approx \mathbb{N}^\mathbb{N}$ from above. If T is countable, then $T \approx \{1_\mathbb{N}\}$, by definition.

(ii) ⇒ (i). Let T be any subset of S such that $|T| = \aleph_1$. Then, by assumption, $T \approx \mathbb{N}^\mathbb{N}$ and so $2^{\aleph_0} = |\mathbb{N}^\mathbb{N}| = |T| = \aleph_1$, as required. □

3. The structure under $\mathfrak{F}_2$

The following theorem suggests that to understand the structure of $\preccurlyeq$ we should first understand its structure on subsemigroups of $\mathfrak{F}_2$.

Theorem 3.1. *Let* S *be a closed subsemigroup of* $\mathbb{N}^\mathbb{N}$ *of cardinality* $2^{\aleph_0}$*. Then there exists a closed subsemigroup* T *of* $\mathfrak{F}_2$ *such that* $|T| = 2^{\aleph_0}$ *and* $T \preccurlyeq S$*.*

We follow the convention that if $n \in \mathbb{N}$, then $n = \{0, 1, \ldots, n-1\}$. Let $\mathcal{C} = 2^\mathbb{N}$ denote the Cantor set (i.e., all functions from $\mathbb{N}$ to $\{0, 1\}$). Then it is straightforward to prove that $\mathcal{C} \approx \mathfrak{F}_2$.

For a subset A of $\mathbb{N}$, we denote the set of finite sequences of elements of A by $A^{<\mathbb{N}}$ and we write $x = (x(0), x(1), \ldots, x(n-1))$. The length n of x is denoted by $|x|$, and we define

$$x^\frown m = (x(0), x(1), \ldots, x(n-1), m) \quad \text{where } m \in \mathbb{N}.$$

If $f \in \mathbb{N}^\mathbb{N}$, then we denote the restriction $(f(0), f(1), \ldots, f(m-1))$ of f to the set $m = \{0, 1, \ldots, m-1\}$ by $f|_m$. Similarly, if $x \in \mathbb{N}^{<\mathbb{N}}$ and $|x| \geq m$, then $x|_m = (x(0), x(1), \ldots, x(m-1))$.

The proof of the following lemma is similar to that of the fact that every perfect Polish space contains a copy of the Cantor set given in [8, Theorem 6.2].

Lemma 3.2. *Let* S *be a closed subset of* $\mathbb{N}^\mathbb{N}$ *with* $|S| = 2^{\aleph_0}$*. Then there exist* $U \subseteq S$ *and* $f \in \mathcal{C}$ *such that* U *is homeomorphic to* $\mathcal{C}$ *and the map* $\lambda : U \longrightarrow \mathcal{C}$ *defined by* $\lambda(g) = f \circ g$ *for all* $g \in U$ *is a homeomorphism from* U *to* $\lambda(U)$*.*

Proof. By assumption, S is a closed subset of $\mathbb{N}^\mathbb{N}$, and so S is a Polish space. Since $|S| = 2^{\aleph_0}$, the Cantor-Bendixson Theorem [8, Theorem 6.4] implies that there exists a perfect subset of S, i.e. a closed set with no isolated points. Assume without loss of generality that S is perfect. Let

$$\mathcal{S} = \{f|_n \in \mathbb{N}^{<\mathbb{N}} : n \in \mathbb{N},\ f \in S\}$$

(the set of finite restrictions of elements in S), and if $x \in \mathcal{S}$, then define

$$[x]_{\mathcal{S}} = \{y \in \mathcal{S} : y|_{|x|} = x\}$$

(the set of finite extensions of x in $\mathcal{S}$), and

$$E(x) = \{i \in \mathbb{N} : x^\frown i \in \mathcal{S}\}.$$

We begin by showing that there exist $\iota_0, \iota_1 : 2^{<\mathbb{N}} \longrightarrow \mathbb{N}$ and $\sigma : 2^{<\mathbb{N}} \longrightarrow \mathcal{S}$ such that

(i) $\iota_0(2^{<\mathbb{N}}) \cap \iota_1(2^{<\mathbb{N}}) = \emptyset$;
(ii) $\sigma(x^\frown j) \in [\sigma(x)^\frown \iota_j(x)]_{\mathcal{S}}$ for all $j \in \{0, 1\}$ and for all $x \in 2^{<\mathbb{N}}$.

Since S is perfect, for every $x \in \mathcal{S}$ there exists $y \in [x]_{\mathcal{S}}$ such that $|E(y)| \geq 2$. There are two cases to consider.

Case 1. *there exist finite $A \subseteq \mathbb{N}$ and $x \in \mathcal{S}$ such that every $y \in [x]_{\mathcal{S}}$ satisfying $|E(y)| \geq 2$ also has the property that $E(y) \subseteq A$.*

Assume without loss of generality that the set A is minimal with the above property, i.e. for every $B \subsetneq A$ and $z \in \mathcal{S}$ there exists $y \in [z]_{\mathcal{S}}$ such that $|E(y)| \geq 2$, and $E(y) \not\subseteq B$. Let $a \in A$ be arbitrary but fixed. Then for all $x' \in [x]_{\mathcal{S}}$ there exists $y \in [x']_{\mathcal{S}}$ such that $|E(y)| \geq 2$ and $E(y) \not\subseteq A \setminus \{a\}$. But $E(y) \subseteq A$ and so $a \in E(y)$. We have shown that:

($\star$) for all $x' \in [x]_{\mathcal{S}}$ there exists $y \in [x']_{\mathcal{S}}$ such that $|E(y)| \geq 2$ and $a \in E(y)$.

We now use ($\star$) to recursively define $\iota_0, \iota_1 : 2^{<\mathbb{N}} \longrightarrow \mathbb{N}$ and $\sigma : 2^{<\mathbb{N}} \longrightarrow \mathbb{N}^{<\mathbb{N}}$ satisfying (i) and (ii) above. As a first step, let $\sigma(\emptyset) \in [x]_{\mathcal{S}}$ be any element such that $|E(\sigma(\emptyset))| \geq 2$ and $a \in E(\sigma(\emptyset))$.

Assume that $\sigma(z) \in [x]_{\mathcal{S}}$ is defined for some $z \in 2^{<\mathbb{N}}$ such that $|E(\sigma(z))| \geq 2$ and $a \in E(\sigma(z))$. Define $\iota_0(z) = a$ and $\iota_1(z)$ to be any element in $E(\sigma(z)) \setminus \{a\}$. By ($\star$) we can define $\sigma(z^\frown j) \in [\sigma(z)^\frown \iota_j(z)]_{\mathcal{S}}$ such that $|E(\sigma(z^\frown j))| \geq 2$ and $a \in E(\sigma(z^\frown j))$ for $j \in \{0, 1\}$.

Case 2. *for all finite $A \subseteq \mathbb{N}$ and for all $x \in \mathcal{S}$ there exists $y \in [x]_S$ with $|E(y)| \geq 2$ but $E(y) \not\subseteq A$.*

List the elements of $2^{<\mathbb{N}}$ as $x_0, x_1, \ldots$ in any way such that $|x_j| < |x_k|$ implies $j < k$. Let $\sigma(x_0) \in \mathcal{S}$ be such that $|E(\sigma(x_0))| \geq 2$ and let $\iota_0(x_0), \iota_1(x_0) \in E(\sigma(x_0))$ be such that $\iota_0(x_0) \neq \iota_1(x_0)$. Assume that for all $j < k$ we have already defined $\sigma(x_j)$, $\iota_0(x_j)$, and $\iota_1(x_j)$ such that $\sigma(x_j)^\frown \iota_i(x_j) \in \mathcal{S}$ for $i \in \{0, 1\}$. Set $A_k = \{\iota_0(x_l), \iota_1(x_l) : l < k\}$. Write $x_k = x_j{}^\frown r$ for some $r \in \{0, 1\}$ and $j \in \mathbb{N}$. Then $j < k$ from the order on the elements of $2^{<\mathbb{N}}$. Hence by the assumption of this case there exists $\sigma(x_k) \in [\sigma(x_j)^\frown \iota_r(x_j)]_S$ such that $|E(\sigma(x_k))| \geq 2$ and $E(\sigma(x_k)) \not\subseteq A_k$. Let $m, n \in E(\sigma(x_k))$ be such that $m \neq n$ and $m \notin A_k$. If $n \in A_k$, then $n = \iota_l(x_j)$ for some $j < k$ and some $l \in \{0, 1\}$. In this case, set $\iota_l(x_k) = n$ and set $\iota_{l+1 \pmod 2}(x_k) = m$. If $n \notin A_k$, then set $\iota_0(x_k) = m$ and $\iota_1(x_k) = n$.

In either case, the functions ι_0, ι_1 and σ have the required properties.

We will now use ι_0, ι_1 and σ to define U and f. If $x \in \mathcal{C}$, then $\bigcap_{n \in \mathbb{N}} [\sigma(x|_n)]$ is a singleton in S since S is closed and hence complete. Let $\{\Psi(x)\} = \bigcap_{n \in \mathbb{N}} [\sigma(x|_n)]$. Then $\Psi : \mathcal{C} \longrightarrow P$ is a homeomorphism from $\mathcal{C}$ to $\Psi(\mathcal{C})$ and we set $U = \Psi(\mathcal{C})$. Let $f \in \mathcal{C}$ be any mapping such that

$$f(m) = \begin{cases} 0 & \text{if } m \in \iota_0(2^{<\mathbb{N}}) \\ 1 & \text{if } m \in \iota_1(2^{<\mathbb{N}}). \end{cases}$$

Then $\lambda : U \longrightarrow \mathcal{C}$ defined by $\lambda(g) = f \circ g$ is continuous, since $\mathbb{N}^{\mathbb{N}}$ is a topological semigroup. It only remains to prove that λ is injective. Let $\Psi(x), \Psi(y) \in U = \Psi(\mathcal{C})$ such that $\Psi(x) \neq \Psi(y)$. Then, without loss of generality, there exist $m \in \mathbb{N}$ and $z \in 2^{<\mathbb{N}}$ such that $x|_m = z^\frown 0$ and $y|_m = z^\frown 1$. It follows that $\sigma(z)^\frown \iota_0(z)$ is a restriction of $\sigma(x|_m) = \sigma(z^\frown 0)$ and $\sigma(z)^\frown \iota_1(z)$ is a restriction of $\sigma(y|_m) = \sigma(z^\frown 1)$. The number $|\sigma(z)|$ is in the domain of $\sigma(z)^\frown \iota_0(z)$ and hence of $\sigma(x|_m) = \sigma(z^\frown 0)$ and so

$$\Psi(x)\big(|\sigma(z)|\big) = \sigma(x|_m)\big(|\sigma(z)|\big) = \sigma(z^\frown 0)\big(|\sigma(z)|\big) = \big(\sigma(z)^\frown \iota_0(z)\big)\big(|\sigma(z)|\big) = \iota_0(z).$$

Hence

$$\lambda(\Psi(x))(|\sigma(z)|) = (f \circ \Psi(x))(|\sigma(z)|) = f\big(\Psi(x)(|\sigma(z)|)\big) = f(\iota_0(z)) = 0.$$

Likewise, $\Psi(y)\big(|\sigma(z)|\big) = \iota_1(z)$ and so $\lambda(\Psi(y))\big(|\sigma(z)|\big) = f(\iota_1(z)) = 1$. Therefore $\lambda(\Psi(x)) \neq \lambda(\Psi(y))$ and so λ is injective. □

Proof of Theorem 3.1. Let S be a closed subsemigroup of $\mathbb{N}^\mathbb{N}$ with $|S| = 2^{\aleph_0}$. Then, by Lemma 3.2, there exist $U \subseteq S$ and $f \in \mathcal{C}$ such that U is homeomorphic to $\mathcal{C}$ and the map $\lambda : U \longrightarrow \mathcal{C}$ defined by $\lambda(g) = f \circ g$ for all $g \in U$ is a homeomorphism from U to $\lambda(U)$.

Then $\lambda(U)$, being homeomorphic to $\mathcal{C}$, is compact. Hence, since $\mathbb{N}^\mathbb{N}$ is Hausdorff, $\lambda(U) \subseteq \mathcal{C}$ is closed. Let T be the subsemigroup generated by $\lambda(U)$, the transposition $(0\ 1) \in S_\infty$, and the constant function with value 0. Then T is the union of $\lambda(U)$, $\{(0\ 1) \circ \lambda(u) : u \in U\}$, and the constant functions with value 0 and 1. In particular, $T \leq \mathfrak{F}_2$ and T is closed (being the finite union of closed sets). Also $|T| = 2^{\aleph_0}$ and $T \approx \lambda(U)$. Furthermore, $\lambda(U) = \{f \circ g : g \in U\} \subseteq \langle U, f\rangle$ and so $T \approx \lambda(U) \subseteq \langle U, f\rangle \approx U \subseteq S$. In particular, $T \preccurlyeq S$. □

4. Almost disjoint families

If A is a subset of $\mathbb{N}$, then define $s_A \in \mathbb{N}^\mathbb{N}$ by

$$s_A(n) = \begin{cases} n & \text{if } n \in A \\ 0 & \text{if } n \notin A. \end{cases} \tag{1}$$

The *power set* of $A \subseteq \mathbb{N}$ is denoted by $\mathcal{P}(A)$. If $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$, then set

$$S_\mathcal{A} = \{s_A \in \mathbb{N}^\mathbb{N} : A \in \mathcal{A}\}. \tag{2}$$

Note that $S_\mathcal{A}$ is a subsemigroup of $\mathbb{N}^\mathbb{N}$ if and only if $\mathcal{A}$ is closed under taking finite intersections.

A set $\mathcal{A}$ of subsets of $\mathbb{N}$ is called *almost disjoint* if $A \cap B$ is finite for all $A, B \in \mathcal{A}$. It is not hard to show that there exist almost disjoint $\mathcal{A}$ such that $|\mathcal{A}| = 2^{\aleph_0}$; see, for example, [9, Theorem 1.3]. Let

$$\mathfrak{F} = \bigcup_{n\in\mathbb{N}} \mathfrak{F}_n.$$

In this section we prove the following theorem.

Theorem 4.1. *If $\mathcal{A}$ is an almost disjoint family of cardinality $2^{\aleph_0}$, then $S_\mathcal{A}$ is incomparable under $\preccurlyeq$ to $\mathfrak{F}$ and $\mathfrak{F}_n$ for all $n \geq 2$.*

If we identify $\mathcal{P}(\mathbb{N})$ with $2^\mathbb{N}$ equipped with the product topology, then the function $A \mapsto s_A$ is a homeomorphism from $2^\mathbb{N}$ to $S_{\mathcal{P}(\mathbb{N})}$. Thus, since $2^\mathbb{N}$ is compact, $S_{\mathcal{P}(\mathbb{N})}$ is closed in $\mathbb{N}^\mathbb{N}$ and so $S_\mathcal{A}$ is closed in $\mathbb{N}^\mathbb{N}$ if and only if $\mathcal{A}$ is closed in $2^\mathbb{N}$. For example, if $\mathcal{A}$ is the almost disjoint family defined as the infinite paths starting at the root of an infinite binary tree labelled by the natural numbers (without repeats), then $S_\mathcal{A}$ is closed. Hence, by Theorem 3.1, there exists $T \preccurlyeq S_\mathcal{A}$ such that $\{1_\mathbb{N}\} \prec T \preccurlyeq \mathfrak{F}_2$. Note that Theorem 4.1 implies that the semigroup $T \not\approx \mathfrak{F}_2$, and so, in general, T in Theorem 3.1 cannot be replaced by $\mathfrak{F}_2$.

Throughout the remainder of this section we use $\mathcal{A}$ to denote an arbitrary almost disjoint family of cardinality $2^{\aleph_0}$.

Let X and Y be countably infinite sets and let $f, g : X \longrightarrow Y$. Then we say that f is *almost injective* if it is injective on a cofinite subset of X. If all but finitely many elements of X are contained in Y, then we say that X is *almost contained in* Y. If f and g agree on a cofinite subset of X, then we say that f and g are *almost equal*.

Lemma 4.2. *Let $u_0, \ldots, u_r \in \mathbb{N}^\mathbb{N}$ and let N be an infinite subset of $\mathbb{N}$ such that $u_{r-1}\cdots u_0$ is almost injective on N and $u_j \cdots u_0(N)$ is almost contained in some $A(j) \in \mathcal{A}$ for all $j \in \{0, \ldots, r-1\}$. If $B(0), \ldots, B(r-1) \in \mathcal{A}$, and $g = u_r s_{B(r-1)} u_{r-1} \cdots s_{B(0)} u_0$, then $g|_N$ is almost equal to $u_r \cdots u_0|_N$ or a constant function.*

Proof. If $A(i) = B(i)$ for all $i \in \{0, \ldots, r-1\}$, then $g|_N$ is almost equal to $u_r \cdots u_0|_N$ since each $s_{B(i)}$ is the identity of $B(i)$.

If $j \in \{0, \ldots, r-1\}$ is the least value such that $A(j) \neq B(j)$, then $u_j s_{B(j-1)} u_{j-1} \cdots s_{B(0)} u_0|_N$ almost equals $u_j \cdots u_0|_N$ as in the previous case. Since $\mathcal{A}$ is an almost disjoint family and $A(j) \neq B(j)$, it follows that $A(j) \cap B(j)$ is finite. But $u_j \cdots u_1 u_0(N)$ is almost contained in $A(j)$ and so

$$s_{B(j)} u_j s_{B(j-1)} u_{j-1} \cdots s_{B(0)} u_0(n) = s_{B(j)} u_j \cdots u_1 u_0(n) = 0$$

for all but finitely many $n \in N$. Therefore $g|_N$ is almost equal to a constant function. □

Proof of Theorem 4.1. If $\mathcal{B}$ equals the union of $\mathcal{A}$ with the set of all finite subsets of $\mathbb{N}$, then $S_{\mathcal{B}}$ is a semigroup equivalent to $S_{\mathcal{A}}$. Thus we may assume without loss of generality that $\mathcal{A}$ contains all finite sets and $S_{\mathcal{A}}$ is a subsemigroup of $\mathbb{N}^{\mathbb{N}}$.

It is clear that:

$$\mathfrak{F}_2 \prec \mathfrak{F}_3 \prec \cdots \prec \mathfrak{F}.$$

So it suffices to show that $\mathfrak{F}_2 \not\preccurlyeq S_{\mathcal{A}}$ and $S_{\mathcal{A}} \not\preccurlyeq \mathfrak{F}$. That $S_{\mathcal{A}} \not\preccurlyeq \mathfrak{F}$ follows since $\mathfrak{F}$ forms an ideal in $\mathbb{N}^{\mathbb{N}}$ and $|S_{\mathcal{A}} \setminus \mathfrak{F}| = 2^{\aleph_0}$.

Let U be any countable subset of $\mathbb{N}^{\mathbb{N}}$. We will show that $\mathfrak{F}_2 \not\subseteq \langle S_{\mathcal{A}}, U\rangle$. Assume without loss of generality that $1_{\mathbb{N}} \in U$. Partition $\mathbb{N}$ into countably many infinite sets $N(u_0, \ldots, u_m)$ indexed by the finite tuples $(u_0, \ldots, u_m) \in U^{m+1}$ for all $m \in \mathbb{N}$. We shall define $f \in \mathfrak{F}_2$ such that

$$f|_{N(u_0,\ldots,u_m)} \neq u_m s_{A(m-1)} u_{m-1} \cdots s_{A(0)} u_0|_{N(u_0,\ldots,u_m)}$$

for any $A(0), \ldots, A(m-1) \in \mathcal{A}$ whereby $f \notin \langle S_{\mathcal{A}}, U\rangle$ and $\mathfrak{F}_2 \not\preccurlyeq S_{\mathcal{A}}$.

Let $u_0, \ldots, u_m \in U$ be arbitrary and let $N := N(u_0, \ldots, u_m)$. Let $r \in \{0, \ldots, m\}$ be the largest value such that $u_{r-1} \cdots u_0$ is almost injective on N and $u_j \cdots u_0(N)$ is almost contained in some element of $\mathcal{A}$ for all $j \in \{0, \ldots, r-1\}$. Such an r exists since the conditions are vacuously satisfied when $r = 0$. We will define $f|_N$ such that no extension of $f|_N$ to an element of $\mathbb{N}^{\mathbb{N}}$ lies in $u_m S_{\mathcal{A}} u_{m-1} \ldots S_{\mathcal{A}} u_0$. If g is any element of $u_m S_{\mathcal{A}} u_{m-1} \ldots S_{\mathcal{A}} u_0$, then, by applying Lemma 4.2 to the string of factors from u_r to u_0 in the expression for g, we see that $g|_N$ is almost equal to either:

(i) $(u_m s_{A(m-1)} u_{m-1} \cdots s_{A(r+1)} u_{r+1} s_{A(r)})(u_r \cdots u_0)$ for some $A(r), \ldots, A(m-1) \in \mathcal{A}$; or
(ii) a constant function.

From the definition of r there are three cases to consider, since one of the following holds:

(a) $u_r \cdots u_0$ is not almost injective on N;
(b) $u_m \cdots u_0$ is almost injective on N and $r = m$;
(c) $u_r \cdots u_0$ is almost injective on N, $r < m$, and $u_r \cdots u_0(N)$ is not almost contained in any set in $\mathcal{A}$.

In each of these cases we shall construct $f|_N$ so that $f|_N$ is constant with value 1 on some infinite coinfinite subset M of N and constant with value 0 on $N \setminus M$. In any of these cases, if $g \in u_m S_{\mathcal{A}} u_{m-1} \ldots S_{\mathcal{A}} u_0$ and (ii) holds, then no matter how M is defined $f|_N \neq g|_N$. Consequently, below we verify that $f|_N \neq g|_N$ for all $g \in u_m S_{\mathcal{A}} u_{m-1} \ldots S_{\mathcal{A}} u_0$ such that (i) holds.

Case (a). Since $u_r \cdots u_0$ is not almost injective on N, there exist infinite disjoint sets $M = \{m_i : i \in \mathbb{N}\} \subseteq N$ and $\{n_i : i \in \mathbb{N}\} \subseteq N$ such that $u_r \cdots u_0(m_i) = u_r \cdots u_0(n_i)$ for all $i \in \mathbb{N}$. In this case, we let $f|_N$ be defined by $f(m_i) = 1$ and $f(n) = 0$ for all $n \in N \setminus M \supseteq \{n_i \in N : i \in \mathbb{N}\}$. If $g \in u_m S_{\mathcal{A}} u_{m-1} \ldots S_{\mathcal{A}} u_0$ and (i) holds, then $g(m_i) = g(n_i)$ for all but finitely many $i \in \mathbb{N}$. Hence $f|_N \neq g|_N$, as required.

Case (b). In this case, we let M be any infinite coinfinite subset of N and define $f|_N$ so that $f(n) = 1$ if $n \in M$ and $f(n) = 0$ if $n \in N \setminus M$. If $g \in u_m S_{\mathcal{A}} u_{m-1} \ldots S_{\mathcal{A}} u_0$ and (i) holds, then $g|_N$ almost equals $u_m \cdots u_0$ and so $g|_N$ is almost injective on N. But $f|_N$ is not almost injective on N and so $g|_N \neq f|_N$.

Case (c). Since $u_r \cdots u_0(N)$ is not almost contained in any set in $\mathcal{A}$, either there exists $A \in \mathcal{A}$ such that $u_r \cdots u_0(N) \cap A$ and $u_r \cdots u_0(N) \setminus A$ are infinite or $u_r \cdots u_0(N) \cap B$ is finite for all $B \in \mathcal{A}$. In the first case, let $M \subseteq N$ be such that both $u_r \cdots u_0(M)$ and $u_r \cdots u_0(N \setminus M)$ contain infinitely many points in A and infinitely many points not in A. Then we define $f|_N$ so that $f(n) = 1$ if $n \in M$ and $f(n) = 0$ if $n \in N \setminus M$. If $g \in u_m S_{\mathcal{A}} u_{m-1} \dots S_{\mathcal{A}} u_0$, (i) holds, and $A = A(r)$, then $g|_{(u_r \cdots u_0)^{-1}(\mathbb{N}\setminus A)\cap N}$ is almost equal to a constant function. If $g \in u_m S_{\mathcal{A}} u_{m-1} \dots S_{\mathcal{A}} u_0$, (i) holds, and $A \neq A(r)$, then $g|_{(u_r \cdots u_0)^{-1}(A)\cap N}$ is almost equal to a constant function. In either case, $g|_N \neq f|_N$.

In the second case, i.e., $u_r \cdots u_0(N) \cap B$ is finite for all $B \in \mathcal{A}$, we let M be any infinite coinfinite subset of N. Then we define $f|_N$ so that $f(n) = 1$ if $n \in M$ and $f(n) = 0$ if $n \in N \setminus M$. If $g \in u_m S_{\mathcal{A}} u_{m-1} \dots S_{\mathcal{A}} u_0$ and (i) holds, then $g|_N$ is almost equal to a constant function while $f|_N$ maps infinitely many points to both 0 and 1. Hence $f|_N \neq g|_N$. □

5. Anti-chains

In [10] it was proved that $\preccurlyeq$ contains at least two incomparable elements by constructing a subsemigroup S of $\mathbb{N}^\mathbb{N}$ such that $S \not\preccurlyeq \mathfrak{F}_3$ and $\mathfrak{F}_3 \not\preccurlyeq S$. In Section 4 we gave an example of a subsemigroup incomparable to all $\mathfrak{F}_n$. The following theorem shows that there are anti-chains in $\preccurlyeq$ of arbitrary finite length.

Theorem 5.1. *For all $i \in \mathbb{N}$, there exist i distinct closed subsemigroups contained in $\mathfrak{F}$ that are mutually incomparable under $\preccurlyeq$.*

Let $m, k \in \mathbb{N}$ be such that $m \geq 2$ and define $\mathfrak{U}_{k,m}$ to be the semigroup of all $f \in \mathbb{N}^\mathbb{N}$ satisfying

$$f(i) = i \text{ if } i < k \text{ and } f(i) \in \{k, k+1, \dots, k+m-1\} \text{ if } i \geq k.$$

It is easy to see that every $\mathfrak{U}_{k,m}$ is a closed subsemigroup of $\mathbb{N}^\mathbb{N}$. Note that $\mathfrak{U}_{0,m} \approx \mathfrak{F}_m$.

Lemma 5.2. *Let $k, l, m, n \in \mathbb{N}$ be such that $m, n \geq 2$. Then $\mathfrak{U}_{k,m} \preccurlyeq \mathfrak{U}_{l,n}$ if and only if $m \leq n$ and $k + m \leq l + n$.*

Proof. ($\Leftarrow$) We define $g, h \in \mathbb{N}^\mathbb{N}$ such that $\mathfrak{U}_{k,m} \leq \langle \mathfrak{U}_{l,n}, g, h \rangle$. Let $g, h \in \mathbb{N}^\mathbb{N}$ be any mappings such that

$$g(i) = \begin{cases} i & \text{if } 0 \leq i < k \\ i - (k+m) + (l+n) & \text{if } i \geq k \end{cases}$$

$$h(i) = \begin{cases} i & \text{if } 0 \leq i < k \\ i + (k+m) - (l+n) & \text{if } i \geq l+n-m. \end{cases}$$

The mapping h is well-defined since $l + n - m \geq k + m - m = k$. Also, since $g(i) \geq l + n - m$ if $i \geq k$, it follows that $hg = 1_\mathbb{N}$.

Let $f \in \mathfrak{U}_{k,m}$ be arbitrary and let $f' \in \mathbb{N}^\mathbb{N}$ be the map defined by

$$f'(i) = \begin{cases} i & \text{if } i < l+n-m \\ gfh(i) & \text{if } i \geq l+n-m. \end{cases}$$

We prove that $f' \in \mathfrak{U}_{l,n}$. If $i < l + n - m$, then $f'(i) = i$ and, in particular, since $n \geq m$, $f'(j) = j$ for all $j < l$. If $i \geq l + n - m$, then $h(i) = i + (k + m) - (l + n) \geq k$. Hence $k \leq fh(i) \leq k + m - 1$ and so $l \leq l + n - m \leq gfh(i) = f'(i) \leq l + n - 1$. Thus $f' \in \mathfrak{U}_{l,n}$.

To conclude, we show that $f = hf'g$. If $i < k$, then $hf'g(i) = hf'(i) = h(i) = i = f(i)$ since $k \leq l + n - m$. If $i \geq k$, then $g(i) \geq l + n - m$ and so $hf'g(i) = hgfhg(i) = f(i)$. Therefore $f = hf'g$ and so $f \in \langle \mathfrak{U}_{l,n}, g, h \rangle$. Thus $\mathfrak{U}_{k,m} \subseteq \langle \mathfrak{U}_{l,n}, g, h \rangle$ and so $\mathfrak{U}_{k,m} \preccurlyeq \mathfrak{U}_{l,n}$.

($\Rightarrow$) We prove the contrapositive. If $k + m > l + n$, then $\mathfrak{U}_{k,m} \setminus \mathfrak{F}_{l+n}$ is uncountable. Since $\mathfrak{F}_{l+n}$ is an ideal in $\mathbb{N}^\mathbb{N}$, it follows that $\mathfrak{U}_{k,m} \not\preccurlyeq \mathfrak{F}_{l+n}$. But $\mathfrak{U}_{l,n} \subseteq \mathfrak{F}_{l+n}$ and therefore $\mathfrak{U}_{k,m} \not\preccurlyeq \mathfrak{U}_{l,n}$.

Now, assume that $m > n$. Let U be an arbitrary countable subset of $\mathbb{N}^\mathbb{N}$. We will show that $\mathfrak{U}_{k,m} \not\subseteq \langle \mathfrak{U}_{l,n}, U \rangle$. We may assume without loss of generality that $1_\mathbb{N} \in U$. Let $\mathcal{K} \subseteq \mathcal{P}(\mathbb{N})$ be the set of finite unions of sets in $\{f^{-1}(i) : i \in \mathbb{N} \text{ and } f \in \langle U \rangle\}$ and let $f \in \langle \mathfrak{U}_{l,n}, U \rangle$ be arbitrary. We will show that there are at most n values i for which $f^{-1}(i) \notin \mathcal{K}$. If $f \in \langle U \rangle$, then $f^{-1}(i) \in \mathcal{K}$ for all $i \in \mathbb{N}$. Otherwise,

$$f = hgu$$

for some $u \in \langle U \rangle$, $g \in \mathfrak{U}_{l,n}$, and $h \in \langle \mathfrak{U}_{l,n}, U \rangle$. If $r \in \{0, 1, \ldots, l-1\}$, then

$$(gu)^{-1}(r) = u^{-1}(r) \in \mathcal{K}.$$

Hence gu has at most n preimages that are not in $\mathcal{K}$, namely the preimages of the elements $l, \ldots, l+n-1$. Every preimage of f is a union of the preimages of gu and, since gu has finite image, it is a finite union. Hence any preimage of f that is not in $\mathcal{K}$ must contain at least one of $(gu)^{-1}(l), \ldots, (gu)^{-1}(l+n-1)$. Thus f has at most n preimages that are not in $\mathcal{K}$.

On the other hand, we show that there exists $f \in \mathfrak{U}_{k,m}$ with $m > n$ preimages that are not in $\mathcal{K}$. Since $\mathcal{K}$ is countable, there exists a partition $A_0, \ldots, A_{m-1}$ of $\mathbb{N} \setminus \{0, 1, \ldots, k-1\}$ such that $A_0, \ldots, A_{m-1} \notin \mathcal{K}$. If f is the element of $\mathfrak{U}_{k,m}$ such that $f^{-1}(k+i) = A_i$ for all $0 \leq i \leq m-1$, then f has the required property. It follows that $f \notin \langle \mathfrak{U}_{l,n}, U \rangle$ and so $\mathfrak{U}_{k,m} \not\preccurlyeq \mathfrak{U}_{l,n}$. □

Proof of Theorem 5.1. Let $i \in \mathbb{N}$ be such that $i \geq 1$. We will show that the i semigroups $\mathfrak{U}_{0,i+1}, \mathfrak{U}_{2,i}, \ldots, \mathfrak{U}_{2i-2,2}$ form an antichain under $\preccurlyeq$. Let $k, l, m, n \in \mathbb{N}$ be such that $k + m = l + n = i + 1$. Then, by Lemma 5.2, $\mathfrak{U}_{2k,m} \preccurlyeq \mathfrak{U}_{2l,n}$ if and only if $m \leq n$ and $2k + m \leq 2l + n$ if and only if $m = n$ and $k = l$ if and only if $\mathfrak{U}_{2k,m} = \mathfrak{U}_{2l,n}$. It follows that the semigroups $\mathfrak{U}_{0,i+1}, \mathfrak{U}_{2,i}, \ldots, \mathfrak{U}_{2i-2,2}$ form an anti-chain in $\preccurlyeq$ of length i. □

6. An uncountable chain

A *chain* inside a partial order is just a totally ordered subset.

Theorem 6.1. *There exists a chain, having length $\aleph_1$, of $\approx$-classes containing (not necessarily closed) subsemigroups of $\mathfrak{F}_2$.*

If $A \subseteq \mathbb{N}$, then we define $f_A \in \mathbb{N}^{\mathbb{N}}$ by

$$f_A(i) = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{if } i \notin A. \end{cases}$$

If $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ containing $\emptyset$ or $\mathbb{N}$, then write

$$F_{\mathcal{A}} = \{f_A \in \mathbb{N}^{\mathbb{N}} : A \in \mathcal{A} \text{ or } \mathbb{N} \setminus A \in \mathcal{A}\}.$$

It is easy to verify that $F_{\mathcal{A}}$ is a subsemigroup of $\mathcal{C} \leq \mathfrak{F}_2$.

Lemma 6.2. *Let $\mathcal{A}$ be a countable union of almost disjoint families $(\mathcal{A}_i)_{i \in \mathbb{N}}$ of subsets of $\mathbb{N}$ where $\mathcal{A}_i$ contains all finite subsets of $\mathbb{N}$ for all $i \in \mathbb{N}$, let A be any infinite subset of $\mathbb{N}$, and let X be any countable subset of $\mathbb{N}^{\mathbb{N}}$. Then there exists $B \subseteq A$ such that $f_B \notin \langle F_{\mathcal{A}}, X \rangle$.*

Proof. Note that if $f_C \in F_{\mathcal{A}_i}$, $g \in \mathbb{N}^{\mathbb{N}}$, and $gf_C \in \mathcal{C}$, then $gf_C \in \{f_C, f_{\mathbb{N} \setminus C}, f_{\mathbb{N}}, f_{\emptyset}\} \subseteq F_{\mathcal{A}_i}$. In particular, $F_{\mathcal{A}} = \bigcup_{i \in \mathbb{N}} F_{\mathcal{A}_i}$ and $\mathcal{C} \cap \langle F_{\mathcal{A}_i}, X \rangle = \mathcal{C} \cap F_{\mathcal{A}_i} \langle X \rangle$. Hence

$$\mathcal{C} \cap \langle F_{\mathcal{A}}, X \rangle = \mathcal{C} \cap \langle \bigcup_{i \in \mathbb{N}} F_{\mathcal{A}_i}, X \rangle = \mathcal{C} \cap \bigcup_{i \in \mathbb{N}} F_{\mathcal{A}_i} \langle X \rangle$$

and so it suffices to find $f \in \mathcal{C}$ such that $B := f^{-1}(1) \subseteq A$ and $f \notin F_{\mathcal{A}_i} \langle X \rangle$ for all $i \in \mathbb{N}$. Let $(U_{i,j})_{i,j \in \mathbb{N}}$ be any infinite sets partitioning A and let $\langle X \rangle = \{x_0, x_1, \ldots\}$. We shall specify a subset $V_{i,j}$ of $U_{i,j}$ for all $i, j \in \mathbb{N}$ such that if $f \in \mathbb{N}^{\mathbb{N}}$ is any mapping such that

$$f(n) = \begin{cases} 1 & \text{if } n \in V_{i,j} \\ 0 & \text{if } n \in U_{i,j} \setminus V_{i,j}, \end{cases}$$

then $f \notin F_{\mathcal{A}_j} x_i$.

If x_i restricted to $U_{i,j}$ is not injective, then there exist distinct $k, l \in U_{i,j}$ with $x_i(k) = x_i(l)$. Thus if $g \in F_{\mathcal{A}_j}$, then $gx_i(k) = gx_i(l)$. In this case, we let $V_{i,j}$ be any subset of $U_{i,j}$ such that $k \in V_{i,j}$ and $l \notin V_{i,j}$.

If x_i is injective on $U_{i,j}$ and there exists $C \in \mathcal{A}_j$ such that $x_i(U_{i,j}) \cap C$ is infinite, then we define $V_{i,j}$ to be any infinite coinfinite subset of $U_{i,j} \cap x_i^{-1}(C)$. In this case, if $g \in F_{\mathcal{A}_j}$, then gx_i restricted to $U_{i,j} \cap x_i^{-1}(C)$ is almost equal to the constant function with value 0 or 1. Hence $f \notin F_{\mathcal{A}_j} x_i$, as required.

If x_i is injective on $U_{i,j}$ and $x_i(U_{i,j}) \cap C$ is finite for all $C \in \mathcal{A}_j$, then we define $V_{i,j}$ to be any infinite coinfinite subset of $U_{i,j}$. In this case, as above, if $g \in F_{\mathcal{A}_j}$, then gx_i restricted to $U_{i,j}$ is almost equal to the constant function with value 0 or 1, and so $f \notin F_{\mathcal{A}_j} x_i$.

We complete the definition of f by setting $f(n) = 0$ for all $n \in \mathbb{N} \setminus A$. From our construction, $f^{-1}(1) \subseteq A$ and $f \notin F_{\mathcal{A}_j} x_i$ for all $i, j \in \mathbb{N}$, as required. □

Proof of Theorem 6.1. Let $\mathcal{A}_0$ be any almost disjoint family of cardinality $2^{\aleph_0}$ containing all the finite subsets of $\mathbb{N}$. Then for all countable $X \subseteq \mathbb{N}^{\mathbb{N}}$, by Lemma 6.2, there exists $f \in \mathcal{C}$ such that $f \notin \langle F_{\mathcal{A}_0}, X\rangle$. In particular, $F_{\mathcal{A}_0} \prec \mathcal{C} \approx \mathfrak{F}_2$.

We define by transfinite recursion a chain $(F_{\mathcal{A}_\alpha})_{\alpha<\aleph_1}$ such that $\mathcal{A}_\alpha$ is a countable union of almost disjoint families and $F_{\mathcal{A}_\alpha} \prec F_{\mathcal{A}_\beta} \prec \mathcal{C}$ for all ordinals $\alpha < \beta < \aleph_1$.

Assume that $\alpha < \aleph_1$ and that we have defined countable unions $\mathcal{A}_\beta$ of almost disjoint families for all $\beta < \alpha$. Let $\mathcal{B}_\alpha = \bigcup_{\beta<\alpha} \mathcal{A}_\beta$, let $\mathcal{A} = (A_\lambda)_{\lambda<2^{\aleph_0}}$ be an almost disjoint family of subsets of $\mathbb{N}$, and let $(X_\lambda)_{\lambda<2^{\aleph_0}}$ be the countable subsets of $\mathbb{N}^{\mathbb{N}}$. Since every $\mathcal{A}_\beta$, $\beta < \alpha$, is a countable union of almost disjoint families and α is a countable ordinal, it follows that $\mathcal{B}_\alpha$ is a countable union of almost disjoint families. By Lemma 6.2, for all $\lambda < 2^{\aleph_0}$ there exists $C_\lambda \subseteq A_\lambda$ such that $f_{C_\lambda} \notin \langle F_{\mathcal{B}_\alpha}, X_\lambda\rangle$. Let $\mathcal{A}_\alpha = \mathcal{B}_\alpha \cup \{C_\lambda : \lambda < 2^{\aleph_0}\}$. Then $\{C_\lambda : \lambda < 2^{\aleph_0}\}$ is an almost disjoint family, since if $\lambda \neq \lambda'$, then $C_\lambda \cap C_{\lambda'} \subseteq A_\lambda \cap A_{\lambda'}$ and the latter is finite since $\mathcal{A}$ is an almost disjoint family. Hence $\mathcal{A}_\alpha$ is a countable union of almost disjoint families. In particular, by Lemma 6.2, $F_{\mathcal{A}_\alpha} \prec \mathcal{C}$. By construction, $F_{\mathcal{B}_\alpha} \leq F_{\mathcal{A}_\alpha} \not\leq F_{\mathcal{B}_\alpha}$ and so $F_{\mathcal{B}_\alpha} \prec F_{\mathcal{A}_\alpha}$. It follows that $F_{\mathcal{A}_\beta} \leq F_{\mathcal{B}_\alpha} \prec F_{\mathcal{A}_\alpha}$ for all $\beta < \alpha$. □

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