University of Hertfordshire UH 25

Research Archive

Citation for published version:

Maxime Oliva and Ole Steuernagel, 'Structures far below the sub-Planck scale in quantum phase space through superoscillations', *Physical Review A*, A95, 052112, published 15 May 2017.

DOI:

Document Version:

This is the Accepted Manuscript version.
The version in the University of Hertfordshire Research Archive may differ from the final published version. **Users should always cite the published version of record**.

Copyright and Reuse:

Enquiries

If you believe this document infringes copyright, please contact the Research & Scholarly Communications Team at rsc@herts.ac.uk

Structures far below sub-Planck scale in quantum phase-space through superoscillations

Maxime Oliva, Ole Steuernagel

School of Physics, Astronomy and Mathematics, University of Hertfordshire, Hatfield, AL10 9AB, UK
(Dated: April 27, 2017)

In 2001, Zurek derived the generic minimum scale a_Z for the area of structures of Wigner's quantum phase distribution. Here we show by construction, using superoscillatory functions, that the Wigner distribution can locally show regular spotty structures on scales much below Zurek's scale a_Z. The price to pay for the presence of such structures is their exponential smallness. For the case we construct there is no increased interferometric sensitivity from the presence of patches with superoscillatory structure in phase-space.

I. INTRODUCTION

Based on the concept of interferences in phase-space [1], Zurek established that the minimum scale a_Z for the area of structures of quantum phase distributions can, for one-dimensional quantum systems, be as small as $a_Z \approx \hbar^2/A$, where A is the action representing the area of support of a system's Wigner distribution [2]. This was surprising [3], since Heisenberg's uncertainty principle was interpreted to limit the area of spots in phase-space to approximately $\hbar/2$.

When a bandwidth-limited signal contains segments which oscillate faster than what its spectrum suggests, it "superoscillates" [4].

Superoscillations have first been noticed in physics in the framework of weak measurement by Aharonov et al. [5], and then studied in detail by Aharonov et al. [6], Berry et al. [4, 7], and others [8]. They have since been used experimentally, for example in super-resolution microscopy [9].

Translated to quantum phase-space: Wigner distributions can show regular spotty structures much below Zurek's scale a_Z; but such states cannot obviously be exploited for higher resolution in measurements.

II. ZUREK'S FUNDAMENTAL PHASE-SPACE TILES

The Wigner distribution of a "Schrödinger's cat" state of squeezed states $G(x,p) = (\pi\hbar)^{-1}e^{-x^2/\xi^2 - \frac{p^2\xi^2}{\hbar^2}}$, with squeezing parameter ξ, is

$$W(x,p) = \frac{G(x-\Delta x, p) + G(x+\Delta x, p)}{2} \quad (1a)$$

$$+ G(x,p)\cos\left(\frac{2p}{\hbar}\Delta x\right). \quad (1b)$$

Zurek's compass state is a coherent sum of two Schödinger's cat states rotated by $\pi/2$ with respect to each other. Starting from such compass states, Fig. 1 **(a)**, Zurek showed that *"Wigner functions can, and generally will, develop phase-space structures on scales as small as, but not generally smaller than"* [3]
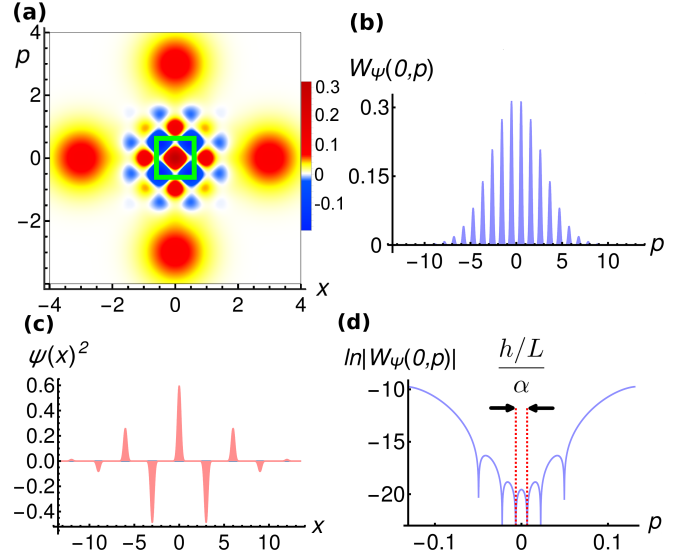

FIG. 1. (Color online) **(a)** shows a compass state [3] for $L = P = 6$ and $\xi = 1$ (atomic units [a.u.], $\hbar = 1$, are used in all figures), the green frame borders its Zurek tile with area a_Z. **(b)**, **(c)** and **(d)** are obtained using $N = 8$, $\alpha = 10$, $\xi = \frac{1}{4}$, $\Delta x = 3$ and $\hbar = 1$. **(c)** shows the squared wave function $\Psi(x)^2$ and its sign changes due to the complex coefficients in $\Psi(x)$ [Eq. (5)]. **(b)** shows $W_\Psi(0,p)$, and **(d)** the logarithm of $|W_\Psi(0,p)|$ near the origin, in a panel of width $h/L \approx 0.26$, in which the two red dashed lines bracket superoscillatory structure of length $\frac{h/L}{\alpha} = \frac{h/L}{10}$ [see Eq. (8)].

$$a_Z = \frac{h}{P} \times \frac{h}{L} = \frac{h^2}{A}. \quad (2)$$

Here, P and L are the phase-space distances between the squeezed states along the momentum and position axes respectively.

The Zurek scale a_Z is, e.g., the phase-space area of one fundamental tile [3] ("Zurek tile") associated with a compass state, highlighted in Fig. 1 **(a)**.

III. THE SUPEROSCILLATING CROSS-STATE

Inspired by Zurek's compass state, we construct a "cross-state" featuring superoscillations in quantum phase-space. This state features small patches with regular structures on scales much smaller than a_Z, Fig. 2.

We use the superoscillating function [4, 6, 7]

$$f(x) = (\cos(x) + i\alpha \sin(x))^N, \ \alpha > 1, \ N \in \mathbb{N}. \quad (3)$$

For $\alpha = 1$, $f(x) = e^{iNx}$ is a regular plane wave. For $\alpha > 1$ and $N \gg 1$, $f(x)$ becomes superoscillatory [see Fig. 1 (d)]

$$f(x) = \sum_{j=0}^{N} C_j(N, \alpha) e^{i(N-2j)x}, \quad (4)$$

where $C_j(N, \alpha) = (-1)^j \binom{N}{j}(\alpha + 1)^{N-j}(\alpha - 1)^j/2^N$ are the Fourier coefficients [6].

To map $f(x)$ of Eq. (4) into phase-space, we use a superposition of suitably pairwise-displaced squeezed states $S(x) = (\pi\xi^2)^{-1/4} e^{-x^2/(2\xi^2)}$ [see Eq. (1)], to form

$$\Psi(x) = \Phi_0(x) + \frac{1}{\sqrt{2}} \sum_{\substack{j=-N/2 \\ j \neq 0}}^{N/2} (-i)^j \Phi_j(x), \quad (5)$$

where $\Phi_j(x) = K_j S(x - j\Delta x)$, $K_j = \sqrt{|D_j| / \sum_{l=0}^{N/2} D_l}$ and

$$D_j = \begin{cases} C_{N/2} & \text{if } j = 0, \\ C_{N/2+j} + C_{N/2-j} & \text{if } j \neq 0. \end{cases} \quad (6)$$

Here, N is even and Ψ contains $N + 1$ spikes, see Fig. 1 (c). The associated Wigner distribution W_Ψ contains a suitable combination of plane wave terms [Eq. (1b)] to emulate $f(x)$ of Eq. (4), see Fig. 1 (d).

An incoherent sum of two such Wigner distributions, rotated by $\pi/2$ with respect to each other, forms the desired cross-state $W_+(x, p) = [W_\Psi(x, p) + W_\Psi(-p, x)]/2$. This balanced mixed state features superoscillatory structures within Zurek tiles, on sub-Zurek scales [Fig. 2 (a)-inset, (b) and (c)].

One could use a coherent sum to form a cross, but this would lead to greater complexity in Fig. 2, which is unnecessary to illustrate our construction.

IV. SUBSTRUCTURES WITHIN ZUREK TILES

The area of sub-Planck structures of non-superoscillating states is limited by a_Z [3]. Now we show that regular structures on scales much smaller than a_Z can exist.
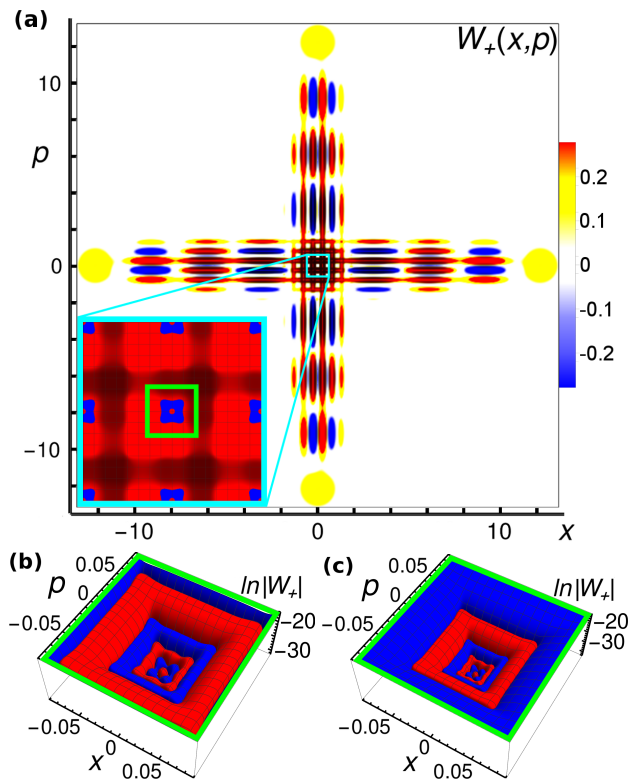

FIG. 2. (Color online) (a) Cross-state $W_+(x, p) = [W_\Psi(x, p) + W_\Psi(-p, x)]/2$ for $N = 4$, $\alpha = 6$, $\xi = 1$, and $\Delta x = 6$ ($\hbar = 1$, a.u.). In (a)'s inset and panels (b) and (c), red (positive) and blue (negative) regions depict superoscillatory structures contained within Zurek tiles (green frames). (b) and (c) show $\ln|W_+(x,p)|$ and demonstrate the scaling with a_{SO}, see Eq. (8). State parameters $N = 12$, $\xi = \frac{1}{4}$, $\Delta x = 3$, with $\alpha = 10$ in (b), and $\alpha = 16$ in (c), respectively.

The local expansion of Eq. (3) around the origin has the local superoscillatory plane wave form

$$f(x) = e^{N \ln[\cos(x) + i\alpha \sin(x)]} \approx e^{iN\alpha x} e^{N\alpha^2 x^2/2}. \quad (7)$$

Therefore, the superoscillating Wigner distribution W_Ψ contains interference terms, equivalent to expression (1b), proportional to $\cos(\frac{2p}{\hbar} \frac{N\Delta x}{2} \alpha) = \cos(p\frac{L}{\hbar}\alpha)$. Thus for W_+, analogously to Zurek's scale a_Z in the compass state Fig. 1 (a), superoscillatory structures with α-fold reduced length scales, Fig. 1 (d), yielding areas on the scale of

$$a_{SO} \approx \frac{h/P}{\alpha} \times \frac{h/L}{\alpha} \approx \frac{a_Z}{\alpha^2} \quad (8)$$

arise.

For these superoscillatory structures to show, the 'overspill' from the two adjacent squeezed states Φ_{-1} and Φ_{+1} has to be so small that their Wigner distributions obey

$$W_{\Phi_{-1}}(0, 0) + W_{\Phi_{+1}}(0, 0) \ll |W_\Psi(0, 0)|. \quad (9)$$

V.   CONCLUSION

We remind the reader of the fact that a quantum wave function cannot be *strictly* "bandwidth-limited", simultaneously in position and momentum. Our Wigner distributions are confined by a finite area A in phase-space, yet they feature regular sub-Zurek scale structures, in this sense they are superoscillating.

Zurek's compass states provide interferometric sensitivity at the Heisenberg limit. As Fig. 1 **(c)** illustrates, our superoscillating states change, under tiny displacements in x, p and t, in essentially the same way as regular compass states; they therefore do not perform better at the detection of small shifts or rotations.

At this stage, the formation of structures below the Zurek scale using superoscillations in phase-space is primarily a surprising curiosity. Superoscillating regions are known to be tiny in extent and amplitude [6–8]. This is why our superoscillating states cannot show sensitivity below the Heisenberg limit.

We have shown, to paraphrase Zurek's statement cited above, that phase-space structures are frequently as small as, but not generally smaller than a_Z. Yet, superoscillating states can generate localized small patches with regular structures on very much smaller scales.

An interesting open question raised by the existence of sub-Planck and sub-Zurek scale phase-space structures concerns their potential effects on simulations. Possibly, grids finer than commonly assumed for numerical calculations [10] have to be used.

[1] W. P. Schleich, *Quantum Optics in Phase Space* (Wiley-VCH, 2001).
[2] E. Wigner, Phys. Rev. **40**, 749 (1932).
[3] W. H. Zurek, Nature **412**, 712 (2001), 0201118.
[4] M. Berry, *Quantum Coherence and Reality; in celebration of the 60th Birthday of Yakir Aharonov*, edited by J. S. Anandan and J. L. Safko (World Scientific, Singapore, 1994) pp. 55–65.
[5] Y. Aharonov, J. Anandan, S. Popescu, and L. Vaidman, Phys. Rev. Lett. **64**, 2965 (1990).

[6] Y. Aharonov, F. Colombo, I. Sabadini, D. Struppa, and J. Tollaksen, J. Phys. A: Math. Theor. **44**, 365304 (2011).
[7] M. V. Berry and S. Popescu, J. Phys. A: Math. Theor. **39**, 6965 (2006).
[8] M. S. Calder and A. Kempf, J. Math. Phys. **46**, 012101 (2005).
[9] E. T. Rogers, J. Lindberg, T. Roy, S. Savo, J. E. Chad, M. R. Dennis, and N. I. Zheludev, Nature Mat. **11**, 432 (2012).
[10] R. Kosloff, J. Chem. Phys. **92**, 2087 (1988).