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# QUANTUM LOOP ALGEBRAS AND $\ell$ -ROOT OPERATORS

CHARLES YOUNG

ABSTRACT. Let  $\mathfrak{g}$  be a simple Lie algebra over  $\mathbb{C}$  and  $q \in \mathbb{C}^\times$  transcendental. We consider the category  $\mathcal{C}_{\mathcal{P}}$  of finite-dimensional representations of the quantum loop algebra  $U_q(\mathcal{L}\mathfrak{g})$  in which the poles of all  $\ell$ -weights belong to specified finite sets  $\mathcal{P}$ . Given the data  $(\mathfrak{g}, q, \mathcal{P})$ , we define an algebra  $\mathcal{A}$  whose raising/lowering operators are constructed to act with definite  $\ell$ -weight (unlike those of  $U_q(\mathcal{L}\mathfrak{g})$  itself). It is shown that there is a homomorphism  $U_q(\mathcal{L}\mathfrak{g}) \rightarrow \mathcal{A}$  such that every representation  $V$  in  $\mathcal{C}_{\mathcal{P}}$  is the pull-back of a representation of  $\mathcal{A}$ .

## 1. INTRODUCTION

Quantum loop algebras and their finite-dimensional representations have been a topic of interest for two decades at least: for a recent review see [CH10]. Besides their original setting in integrable quantum- and statistical-mechanical models, they appear in the contexts of algebraic geometry [GV93, Nak01, VV02, Nak04], combinatorics [JS10, LSS10], and cluster algebras [HL10, Nak11].

Given  $\mathfrak{g}$ , a simple Lie algebra over  $\mathbb{C}$ , and  $q \in \mathbb{C}^\times$  transcendental, let  $U_q(\mathcal{L}\mathfrak{g})$  be the corresponding quantum loop algebra and  $\mathcal{C}$  the category of its finite-dimensional representations. Let  $(U^+, U^0, U^-)$  be the triangular decomposition of  $U_q(\mathcal{L}\mathfrak{g})$  in Drinfeld's "new" realization [Dri88, Bec94]. The subalgebra  $U^0 \subset U_q(\mathcal{L}\mathfrak{g})$  is commutative, and any  $V \in \text{Ob}(\mathcal{C})$  can be decomposed into a direct sum of generalized eigenspaces of the generators of  $U^0$ . The eigenvalues are known as  $\ell$ -weights, and the  $q$ -character of  $V$  is by definition the formal sum of its  $\ell$ -weights [FR98, Kni95]. It is usually encoded as a Laurent polynomial  $\chi_q(V)$  in formal variables  $Y_{i,a}$ , where  $a \in \mathbb{C}^\times$  and where  $i$  runs over the set  $I$  of nodes of the Dynkin diagram of  $\mathfrak{g}$ . If one sends  $Y_{i,a} \mapsto y_i := e^{\omega_i}$  (where  $\omega_i$  are the fundamental weights) one recovers the usual formal character  $\chi(V)$  of  $V$  regarded as a  $U_q(\mathfrak{g})$ -module. In this sense  $q$ -characters refine the usual characters, and they have proven to be a powerful tool in understanding the structure of finite-dimensional representations [FM01, Her06, Her07a, MY12a, MY12b].

There is a fruitful analogy between the weight lattice  $P$  of  $\mathfrak{g}$ , and the so-called  $\ell$ -weight lattice of  $U_q(\mathcal{L}\mathfrak{g})$ , which is defined to be the free abelian group  $\mathcal{P}$  generated by the  $Y_{i,a}$ ,  $i \in I$ ,  $a \in \mathbb{C}^\times$ . Dominant  $\ell$ -weights are the monomials in the  $Y_{i,a}$ ,  $i \in I$ ,  $a \in \mathbb{C}^\times$ ; they form a free monoid  $\mathcal{P}^+ \subset \mathcal{P}$ . In the literature, dominant  $\ell$ -weights are usually called Drinfeld polynomials, and one of the first key results concerning  $\mathcal{C}$  was the classification of its irreducibles [CP94a, CP94b]: the isomorphism classes of irreducible modules in  $\mathcal{C}$  are in bijection with the dominant  $\ell$ -weights. We write  $L(\gamma)$  for the irreducible  $U_q(\mathcal{L}\mathfrak{g})$ -module with highest  $\ell$ -weight  $\gamma \in \mathcal{P}^+$ . Recall the corresponding classical result that the isomorphism classes of irreducible finite-dimensional  $\mathfrak{g}$ -modules, and  $U_q(\mathfrak{g})$ -modules, are in bijection with the dominant weights  $P^+ \subset P$ ; we write  $V(\omega)$  for the irreducible  $U_q(\mathfrak{g})$ -module

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with highest weight  $\omega \in P^+$ . The analogy between the weight theory of  $\mathfrak{g}$  and its quantum-loop counterpart goes further: there are also counterparts  $A_{i,a} \in \mathcal{P}$ ,  $i \in I$ ,  $a \in \mathbb{C}^\times$  of the simple roots  $\alpha_i \in P$ ,  $i \in I$ . We call them *simple  $\ell$ -roots*. Then, just as  $\chi(V(\omega)) \in e^\omega \mathbb{Z}[e^{-\alpha_i}]_{i \in I}$ , so also it is known that  $\chi_q(L(\gamma)) \in \gamma \mathbb{Z}[A_{i,a}^{-1}]_{i \in I, a \in \mathbb{C}^\times}$  [FM01].

However, at this stage the analogy with the usual weight theory breaks down, in the following important sense. Let  $x_i^\pm$  and  $k_i^{\pm 1}$ ,  $i \in I$ , be the usual Drinfeld-Jimbo generators of  $U_q(\mathfrak{g})$ . We are accustomed to thinking of  $x_i^\pm$  as step-operators between weight subspaces of a weight module  $V$  of  $U_q(\mathfrak{g})$ : if  $v \in V$  is a weight vector of weight  $\omega$  then  $x_i^\pm.v$  is again a weight vector, of weight  $\omega \pm \alpha_i$ . This follows, of course, from the defining relations

$$k_i x_j^\pm = q^{\pm B_{ij}} x_j^\pm k_i,$$

where  $B_{ij}$  is the symmetrized Cartan matrix. It is natural to ask for something similar in  $\mathcal{C}$ : namely, to each simple  $\ell$ -root  $A_{i,a}$  one would like to associate an operator “ $x_{A_{i,a}}^\pm$ ” in  $U^\pm$  such that for any  $V \in \text{Ob}(\mathcal{C})$  and any  $v \in V$  of  $\ell$ -weight  $\gamma$ , “ $x_{A_{i,a}}^\pm.v$ ” has  $\ell$ -weight  $\gamma A_{i,a}^{\pm 1}$ . But this is apparently too much to ask: the relevant commutation relation is

$$\phi_i^+(u) x_j^\pm(v) = \frac{q^{\pm B_{ij}} - uv}{1 - q^{\pm B_{ij}} uv} x_j^\pm(v) \phi_i^+(u),$$

and  $(A_{j,a}^{\pm 1})_i(u) := \frac{q^{\pm B_{ij}} - ua}{1 - q^{\pm B_{ij}} ua}$  (details are recalled in §2). Naively therefore, one can only obtain “ $x_{A_{i,a}}^\pm$ ” by “evaluating at  $z = a$ ” the formal series of generators  $x_i^\pm(z) := \sum_{r \in \mathbb{Z}} z^{-r} x_{i,r}^\pm$ . Such an infinite sum  $x_i^\pm(a) = \sum_{r \in \mathbb{Z}} a^{-r} x_{i,r}^\pm$  is ill-defined in a sense that is not merely technical: its would-be matrix representatives have singular entries (c.f. equation (1.1) below).

The lack of such operators “ $x_{A_{i,a}}^\pm$ ” in  $U_q(\mathcal{L}\mathfrak{g})$  is a problem. On the one hand it makes it hard to use algebraic techniques to prove statements about  $q$ -characters: for example, the combinatorial Frenkel-Mukhin algorithm [FM01] is believed to yield the correct  $q$ -character for a much larger class of representations than the class for which one can currently prove that it does so. At the same time it means that the combinatorial structure the  $q$ -character – which is in some sense elegant and sparse, certainly when compared to that of the usual formal character – can in practice be hard to lift to the level of representation theory. Ideally, one would like to have a general procedure to go from the  $q$ -character of a representation to an explicit  $\ell$ -weight basis, and for that one needs raising/lowering operators adapted to the  $\ell$ -weight decomposition.

In the present work, it is argued that this obstruction can be circumvented by working not with  $U_q(\mathcal{L}\mathfrak{g})$  directly, but rather with a new algebra  $\mathcal{A}$  whose raising/lowering operators act with definite  $\ell$ -weight by construction.  $\mathcal{A}$  will be defined in such a way that there is a homomorphism of algebras  $U_q(\mathcal{L}\mathfrak{g}) \rightarrow \mathcal{A}$ , allowing  $\mathcal{A}$ -modules to be pulled back to recover  $U_q(\mathcal{L}\mathfrak{g})$ -modules.

The key observation (Propositions 3.1 and 3.7) that makes such a construction possible is the following. For any finite-dimensional representation, say  $V \in \text{Ob}(\mathcal{C}_\mathcal{P})$ , the matrix representatives  $\rho(x_i^\pm(z)) \in \text{End}(V)[[z, z^{-1}]]$  take a very specific form. Namely,

$$\rho(x_i^\pm(z)) = \sum_{a \in \mathcal{P}_i} \sum_{m=0}^M E_{i,a,m}^\pm \frac{a^m}{m!} \left( \frac{\partial}{\partial a} \right)^m \delta \left( \frac{a}{z} \right), \quad (1.1)$$

for certain maps  $E_{i,a,m}^\pm \in \text{End}(V)$ . Here the  $\mathcal{P}_i$  are finite collections of points in  $\mathbb{C}^\times$ , and the upper limit  $M$  is related to the maximal dimension of the  $\ell$ -weight spaces. Both depend on the choice of representation  $V$ . The  $\delta$ -function  $\delta(a/z)$  is the formal distribution  $\sum_{r \in \mathbb{Z}} (a/z)^r$ , and the important point is that the dependence on the mode number  $r$  of  $x_{i,r}^\pm$  is solely through these formal  $\delta$ -functions and their derivatives (which, intuitively speaking, have “support” at the points  $a \in \mathcal{P}_i$ ). It is then natural to ask what algebraic relations are obeyed by the maps  $E_{i,a,m}^\pm$ . The interesting relations are the ones that are independent of  $V$ , and, by abstracting these, we arrive at the definition (§4.1) of  $\mathcal{A}$ .

We shall work in subcategories  $\mathcal{C}_{\mathcal{P}}$  of  $\mathcal{C}$  (see Definition 3.2) whose objects have  $q$ -characters lying in  $\mathbb{Z}[Y_{i,a}, Y_{i,aq_i}^{-1}]_{i \in I, a \in \mathcal{P}_i}$ , where  $(\mathcal{P}_i)_{i \in I}$  are arbitrary finite subsets of  $\mathbb{C}^\times$ . Given any finite collection of finite-dimensional representations there always exists a choice of  $\mathcal{P}$  such that they all belong to  $\mathcal{C}_{\mathcal{P}}$ ; to this extent, to study finite-dimensional representations it suffices to study such subcategories. Our definition of  $\mathcal{A}$  depends on  $\mathcal{P}$ , as well as on  $\mathfrak{g}$  and  $q$ . (Particular categories of this form played a prominent role recently in [HL10].) The main result of the paper, Theorem 4.4, is that a homomorphism  $U_q(\mathcal{L}\mathfrak{g}) \rightarrow \mathcal{A}$  exists and that every representation in  $\mathcal{C}_{\mathcal{P}}$  arises as a pull-back of a representation of  $\mathcal{A}$ .

The  $E_{i,a,m}^\pm$ , now to be thought of as abstract generators of  $\mathcal{A}$ , are the desired operators of definite  $\ell$ -weight  $A_{i,a}^{\pm 1}$ . Meanwhile the “Cartan” generators  $(\phi_{i,\pm r}^\pm)_{i \in I, r \in \mathbb{Z}}$  of  $U^0$  are mapped, in  $\mathcal{A}$ , to generators  $H_{i,a,m}$ , whose eigenvalues encode partial fraction decompositions of rational  $\ell$ -weights. The appearance of the extra label  $m \in \mathbb{Z}_{\geq 0}$  is closely related to a second sense in which the analogy between  $\ell$ -weights and the usual weight theory breaks down: whereas the  $(k_i)_{i \in I}$  can be simultaneously diagonalized, the  $(\phi_{i,\pm r}^\pm)_{i \in I, r \in \mathbb{Z}}$  cannot; the  $\ell$ -weight encodes only their generalized eigenvalues with multiplicities. In highest weight representations, further information about their Jordan chains can now be read off from the commutation relations between  $H_{i,a,m}$ ’s and  $E_{i,a,m}^-$ ’s, cf. the examples in §6.2.

Various natural questions about  $\mathcal{A}$  immediately arise and would be interesting to investigate. Here we shall merely note some of them.

- One would like to construct a basis for  $\mathcal{A}$ . Let  $\mathcal{A}^\pm$  and  $\mathcal{A}^0$  be the subalgebras of  $\mathcal{A}$  generated by the  $E_{i,a,m}^\pm$  and  $H_{i,a,m}$  respectively. In the present paper we establish that  $\mathcal{A} = \mathcal{A}^- \cdot \mathcal{A}^0 \cdot \mathcal{A}^+$  (Proposition 6.1). Because, for simplicity, we work with finite sets  $\mathcal{P}_i$ , the stronger statement  $\mathcal{A} \cong_{\mathbb{C}} \mathcal{A}^- \otimes \mathcal{A}^0 \otimes \mathcal{A}^+$  is actually false (Proposition 6.2), but were one to work instead with sets such that  $q^{B_{ij}}\mathcal{P}_i = \mathcal{P}_j$  for all  $i, j \in I$ , then it is plausible that one would have such a triangular decomposition. Cf. Remark 3.6. When  $\mathfrak{g} \neq \mathfrak{sl}_2$ , the form of the Serre relations, cf. Remark 4.1, makes even the construction of bases of  $\mathcal{A}^\pm$  an interesting problem. (A basis of  $\mathcal{A}^\pm$  when  $\mathfrak{g} = \mathfrak{sl}_2$  is given in Proposition 6.4.)
- It is well known that the untwisted quantum affine algebra  $U_q(\widehat{\mathfrak{g}})$ , and hence also  $U_q(\mathcal{L}\mathfrak{g})$ , has, in addition to the standard Drinfeld-Jimbo coalgebra structure, also a “Drinfeld current coproduct”. The latter is not a coproduct in a strict sense because it contains (in e.g.  $\Delta x_i^+(z) = 1 \otimes x_i^+(z) + x_i^+(z) \otimes \phi_i^-(1/z)$ ) formal sums that are ill-defined. See e.g. [Her07b]. In view of the role of formal  $\delta$ -functions in the homomorphism  $U_q(\mathcal{L}\mathfrak{g}) \rightarrow \mathcal{A}$ , outlined above, it is reasonable to hope that in  $\mathcal{A}$  an analog of this coproduct can be made well-defined.

- For simplicity we assume  $\mathfrak{g}$  is simple but it is known [Jin98, Nak01, Her05] that the quantum affinization of  $U_q(\mathfrak{g})$  can be defined whenever  $\mathfrak{g}$  has symmetrizable Cartan matrix (so in particular when  $\mathfrak{g}$  is an affine Lie algebra). It would be interesting to establish whether the approach of the present paper goes through in that case too.
- In simply-laced cases, there is a homomorphism from  $U_q(\mathcal{L}\mathfrak{g})$  to the Grothendieck ring of the category of equivariant coherent sheaves on a certain Steinberg-type variety (endowed with the convolution product) [Nak01]. It would be interesting to understand whether this homomorphism factors through (some generalization of)  $\mathcal{A}$ . More tentatively, one can hope that  $\mathcal{A}$  allows for an algebraic approach to results that to date rely on geometrical input: notably the algorithm of [Nak04] which, in the spirit of the Kazhdan-Lusztig conjecture, gives in principle the  $q$ -character of every irreducible of  $\mathcal{C}$ .

This paper is structured as follows. Background results about  $U_q(\mathcal{L}\mathfrak{g})$  and its finite-dimensional representations are recalled in §2. Then the first key result, namely the motivating observation – (1.1), above – concerning the action of the raising/lowering operators on  $\ell$ -weight modules is proved in §3, which also gives the definition of the categories  $\mathcal{C}_{\mathcal{P}}$ . The algebra  $\mathcal{A}$  itself and main theorem, Theorem 4.4, are given in section §4. The proof of Theorem 4.4 is in §5. Finally in §6 we note some first examples and properties of the algebra  $\mathcal{A}$ ; in particular that it appears to have a natural “rational limit”, in the spirit of Yangians.

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## 2. BACKGROUND: QUANTUM LOOP ALGEBRAS AND $\ell$ -WEIGHTS

**2.1. Cartan data.** Let  $\mathfrak{g}$  be a simple Lie algebra over  $\mathbb{C}$  and  $\mathfrak{h}$  a Cartan subalgebra of  $\mathfrak{g}$ . We identify  $\mathfrak{h}$  and  $\mathfrak{h}^*$  by means of the invariant inner product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g}$  normalized such that the square length of the maximal root equals 2. With  $I$  a set of labels of the nodes of the Dynkin diagram of  $\mathfrak{g}$ , let  $\{\alpha_i\}_{i \in I}$  be a set of simple roots, and  $\{\omega_i\}_{i \in I}$ , the corresponding set of fundamental weights. Let  $C = (C_{ij})_{i,j \in I}$  be the Cartan matrix. We have

$$2 \langle \alpha_i, \alpha_j \rangle = C_{ij} \langle \alpha_i, \alpha_i \rangle, \quad 2 \langle \alpha_i, \omega_j \rangle = \delta_{ij} \langle \alpha_i, \alpha_i \rangle.$$

Let  $r^\vee$  be the maximal number of edges connecting two vertices of the Dynkin diagram of  $\mathfrak{g}$ . Thus  $r^\vee = 1$  if  $\mathfrak{g}$  is of types A, D or E,  $r^\vee = 2$  for types B, C and F and  $r^\vee = 3$  for  $G_2$ . Let  $r_i = \frac{1}{2} r^\vee \langle \alpha_i, \alpha_i \rangle$ . The numbers  $(r_i)_{i \in I}$  are relatively prime integers. We set

$$D := \text{diag}(r_1, \dots, r_N), \quad B := DC;$$

the latter is the symmetrized Cartan matrix,  $B_{ij} = r^\vee \langle \alpha_i, \alpha_j \rangle$ . Let  $Q$  (resp.  $Q^+$ ) and  $P$  (resp.  $P^+$ ) denote the  $\mathbb{Z}$ -span (resp.  $\mathbb{Z}_{\geq 0}$ -span) of the simple roots and fundamental weights respectively.

Fix a transcendental  $q \in \mathbb{C}^\times$ . For each  $i \in I$  let

$$q_i := q^{r_i}.$$

Define the  $q$ -numbers,  $q$ -factorial and  $q$ -binomial:

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! := [n]_q [n-1]_q \cdots [1]_q, \quad \binom{n}{m}_q := \frac{[n]_q!}{[n-m]_q! [m]_q!}.$$

We use also  $\binom{n}{m} := \frac{n!}{(n-m)!m!}$  for the usual ( $q = 1$ ) binomial coefficients.

**2.2. Quantum loop algebras.** The *quantum loop algebra*  $U_q(\mathcal{L}\mathfrak{g})$  is the associative unital algebra over  $\mathbb{C}$  generated by

$$(x_{i,n}^\pm)_{i \in I, n \in \mathbb{Z}}, \quad (k_i^{\pm 1})_{i \in I}, \quad (h_{i,r})_{i \in I, r \in \mathbb{Z} \setminus \{0\}},$$

subject to the following relations. We arrange the generators into formal series

$$\begin{aligned} x_i^\pm(u) &:= \sum_{n \in \mathbb{Z}} x_{i,n}^\pm u^{-n} \in U_q(\mathcal{L}\mathfrak{g})[[u, u^{-1}]], \\ \phi_i^\pm(u) &= \sum_{n=0}^{\infty} \phi_{i,\pm n}^\pm u^{\pm n} := k_i^{\pm 1} \exp\left(\pm(q - q^{-1}) \sum_{m=1}^{\infty} h_{i,\pm m} u^{\pm m}\right) \in U_q(\mathcal{L}\mathfrak{g})[[u^{\pm 1}]], \end{aligned} \quad (2.1)$$

and set

$$\delta(u) := \sum_{n \in \mathbb{Z}} u^n \in \mathbb{C}[[u, u^{-1}]].$$

The defining relations of  $U_q(\mathcal{L}\mathfrak{g})$  are then [Dri88]  $k_i k_i^{-1} = 1$  and

$$[\phi_i^\pm(u), \phi_j^\pm(v)] = 0, \quad [\phi_i^\pm(u), \phi_j^\mp(v)] = 0, \quad (2.2)$$

$$[x_i^+(u), x_j^-(v)] = \delta_{ij} \delta\left(\frac{u}{v}\right) \frac{\phi_i^+(1/u) - \phi_i^-(1/u)}{q_i - q_i^{-1}}, \quad (2.3)$$

$$(1 - q^{\pm B_{ij}} uv) \phi_i^+(u) x_j^\pm(v) = (q^{\pm B_{ij}} - uv) x_j^\pm(v) \phi_i^+(u), \quad (2.4)$$

$$(1/(uv) - q^{\pm B_{ij}}) \phi_i^-(u) x_j^\pm(v) = (q^{\pm B_{ij}}/(uv) - 1) x_j^\pm(v) \phi_i^-(u), \quad (2.5)$$

$$(u - q^{\pm B_{ij}} v) x_i^\pm(u) x_j^\pm(v) = (q^{\pm B_{ij}} u - v) x_j^\pm(v) x_i^\pm(u), \quad (2.6)$$

together with Serre relations

$$\sum_{\pi \in \Sigma_s} \sum_{r=0}^s (-1)^r \binom{s}{r}_{q_i} x_i^\pm(w_{\pi(1)}) \cdots x_i^\pm(w_{\pi(r)}) x_j^\pm(z) x_i^\pm(w_{\pi(r+1)}) \cdots x_i^\pm(w_{\pi(s)}) = 0, \quad (2.7)$$

for all  $i \neq j$ , where  $s = 1 - C_{ij}$  and  $\Sigma_s$  is the symmetric group on  $s$  letters.

Relations (2.4–2.5) are often written  $[h_{i,n}, x_{j,m}^\pm] = \pm \frac{1}{n} [n B_{ij}]_q x_{j,n+m}^\pm$ ; see e.g. [Her05] for a proof that they are equivalent.

In the present work it is useful have a slightly different presentation of  $U_q(\mathcal{L}\mathfrak{g})$ . Let us define

$$\Phi_i(u) = \sum_{k \in \mathbb{Z}} \Phi_{i,k} u^k := \frac{\phi_i^+(u) - \phi_i^-(u)}{q_i - q_i^{-1}} \in U_q(\mathcal{L}\mathfrak{g})[[u, u^{-1}]]. \quad (2.8)$$

Here, as already in relation (2.3), the right-hand side is to be interpreted by extending  $\phi_i^\pm(u)$  from a series in  $U_q(\mathcal{L}\mathfrak{g})[[u^{\pm 1}]]$  to one in  $U_q(\mathcal{L}\mathfrak{g})[[u, u^{-1}]]$  by setting  $\phi_{i,\mp k}^\pm = 0$  for all  $k \in \mathbb{Z}_{\geq 1}$ . So (2.8) is

equivalent to

$$\Phi_{i,0} := \frac{\phi_{i,0}^+ - \phi_{i,0}^-}{q_i - q_i^{-1}} = \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}, \quad \Phi_{i,\pm k} := \pm \frac{\phi_{i,\pm k}^\pm}{q_i - q_i^{-1}}, \quad k \in \mathbb{Z}_{\geq 1}. \quad (2.9)$$

**Proposition 2.1.**  $U_q(\mathcal{L}\mathfrak{g})$  is the associative unital algebra over  $\mathbb{C}$  generated by

$$(x_{i,n}^\pm)_{i \in I, n \in \mathbb{Z}}, \quad (k_i^{\pm 1})_{i \in I}, \quad (\Phi_{i,r})_{i \in I, r \in \mathbb{Z}},$$

subject to the following relations:  $k_i k_i^{-1} = 1$ ,

$$\Phi_{i,0} = \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}, \quad (2.10)$$

$$[\Phi_i(u), \Phi_j(v)] = 0, \quad [\Phi_i(u), k_j] = 0, \quad k_i k_j = k_j k_i, \quad (2.11)$$

$$\left[ x_i^+(u), x_j^-(v) \right] = \delta_{ij} \delta \left( \frac{u}{v} \right) \Phi_i \left( \frac{1}{u} \right), \quad (2.12)$$

$$k_i x_j^\pm(v) = x_j^\pm(v) k_i q^{\pm B_{ij}}, \quad (2.13)$$

$$(u - q^{\pm B_{ij}} v) \Phi_i(1/u) x_j^\pm(v) = (q^{\pm B_{ij}} u - v) x_j^\pm(v) \Phi_i(1/u), \quad (2.14)$$

$$(u - q^{\pm B_{ij}} v) x_i^\pm(u) x_j^\pm(v) = (q^{\pm B_{ij}} u - v) x_j^\pm(v) x_i^\pm(u) \quad (2.15)$$

together with the Serre relations as above.

*Proof.* The defining relations (2.2–2.7) involve the  $h_{i,r}$  only through the  $\phi_{i,\pm r}^\pm$ . For every  $i \in I$ , each  $h_{i,\pm r}$  can be expressed in terms of  $(\phi_{i,\pm s}^\pm)_{s \leq r}$  using (2.1) and an induction on  $r > 0$ . So we are free to regard the  $(\phi_{i,\pm r}^\pm)_{i \in I, r \in \mathbb{Z}_{\geq 1}}$  as generators in place of the  $(h_{i,r})_{i \in I, r \in \mathbb{Z}_{\neq 0}}$ , and then in turn to replace the  $(\phi_{i,\pm r}^\pm)_{i \in I, r \in \mathbb{Z}_{\geq 1}}$  by  $(\Phi_{i,r})_{i \in I, r \in \mathbb{Z}_{\neq 0}}$  according to (2.9). For later convenience, we also include  $(\Phi_{i,0})_{i \in I}$  as generators; consequently we must impose the linear relation (2.10) as a defining relation.

It is clear that (2.11) and (2.12) are equivalent to, respectively, (2.2) and (2.3). The relation (2.15) is copied verbatim. It remains to check that (2.13–2.14) is equivalent to (2.4–2.5). The  $u^0$  terms of (2.4) and (2.5) are, respectively, the relations involving  $k_i^{+1}$  and  $k_i^{-1}$  in (2.13). As written, (2.4) and (2.5) are equations of formal series in  $U_q(\mathcal{L}\mathfrak{g})[[u, v, v^{-1}]]$  and  $U_q(\mathcal{L}\mathfrak{g})[[u^{-1}, v, v^{-1}]]$  respectively.<sup>1</sup> But we may regard them both as equations of formal series in  $U_q(\mathcal{L}\mathfrak{g})[[u, u^{-1}, v, v^{-1}]]$ , c.f. the definition (2.8–2.9) of  $\Phi_i(u)$  above,

$$(1 - q^{\pm B_{ij}} uv) \phi_i^+(u) x_j^\pm(v) = (q^{\pm B_{ij}} - uv) x_j^\pm(v) \phi_i^+(u),$$

$$(1 - q^{\pm B_{ij}} uv) \phi_i^-(u) x_j^\pm(v) = (q^{\pm B_{ij}} - uv) x_j^\pm(v) \phi_i^-(u)$$

and subtract one from the other to find (2.14). Finally, we check that all remaining relations in (2.4–2.5) can be recovered from (2.13) and (2.14). The latter unpacks to give

$$\Phi_{i,r} x_{j,s}^\pm - q^{\pm B_{ij}} \Phi_{i,r-1} x_{j,s-1}^\pm = q^{\pm B_{ij}} x_{j,s}^\pm \Phi_{i,r} - x_{j,s-1}^\pm \Phi_{i,r-1}.$$

<sup>1</sup>Consequently (2.4), for example, is equivalent to  $\phi_i^+(u) x_j^\pm(v) = \frac{q^{\pm B_{ij}} - uv}{1 - q^{\pm B_{ij}} uv} x_j^\pm(v) \phi_i^+(u)$ , with the understanding that one must expand  $\frac{q^{\pm B_{ij}} - uv}{1 - q^{\pm B_{ij}} uv}$  for small  $u$  and equate powers of  $u$ .

For all  $r \in \mathbb{Z}_{\geq 2}$  this is

$$\phi_{i,r}^+ x_{j,s}^\pm - q^{\pm B_{ij}} \phi_{i,r-1}^+ x_{j,s-1}^\pm = q^{\pm B_{ij}} x_{j,s}^\pm \phi_{i,r}^+ - x_{j,s-1}^\pm \phi_{i,r-1}^+,$$

while for  $r = 1$  it is

$$\phi_{i,1}^+ x_{j,s}^\pm - q^{\pm B_{ij}} (k_i - k_i^{-1}) x_{j,s-1}^\pm = q^{\pm B_{ij}} x_{j,s}^\pm \phi_{i,1}^+ - x_{j,s-1}^\pm (k_i - k_i^{-1}),$$

and given that  $q^{\pm B_{ij}} k_i^{-1} x_{j,s-1}^\pm = x_{j,s-1}^\pm k_i^{-1}$  by (2.13) and  $k_i k_i^{-1} = 1$ , this is

$$\phi_{i,1}^+ x_{j,s}^\pm - q^{\pm B_{ij}} k_i x_{j,s-1}^\pm = q^{\pm B_{ij}} x_{j,s}^\pm \phi_{i,1}^+ - x_{j,s-1}^\pm k_i.$$

Thus we recover all the relations in (2.4). Similarly, by considering  $r \in \mathbb{Z}_{\leq 0}$  we recover all the relations in (2.5).  $\square$

**2.3. Rational  $\ell$ -weights and  $q$ -characters.** Let  $\mathcal{C}$  denote the category whose objects are finite-dimensional representations of  $U_q(\mathcal{L}\mathfrak{g})$  and whose morphisms are  $U_q(\mathcal{L}\mathfrak{g})$ -module maps.

Every representation  $V \in \text{Ob}(\mathcal{C})$  is a direct sum of its generalized eigenspaces for the mutually commuting  $\phi_{i,r}^\pm$ :

$$V = \bigoplus_{\gamma} V_{\gamma}, \quad \gamma = (\gamma_{i,\pm r}^\pm)_{i \in I, r \in \mathbb{Z}_{\geq 0}}, \quad \gamma_{i,\pm r}^\pm \in \mathbb{C}$$

where

$$V_{\gamma} = \{v \in V : \exists k \in \mathbb{N}, \forall i \in I, m \geq 0 \quad (\phi_{i,\pm m}^\pm - \gamma_{i,\pm m}^\pm)^k \cdot v = 0\}.$$

If  $\dim(V_{\gamma}) > 0$ ,  $\gamma$  is called an  $\ell$ -weight of  $V$ . Given  $v \in V$  we write  $v_{\gamma}$  for the component of  $v$  in  $V_{\gamma}$ . Note that  $\gamma_{i,0}^+ \gamma_{i,0}^- = 1$  necessarily. Let us write the  $\ell$ -weight  $\gamma$  as a formal series

$$\gamma_i^\pm(u) := \sum_{r=0}^{\infty} u^{\pm r} \gamma_{i,\pm r}^\pm.$$

Whenever  $\gamma$  is an  $\ell$ -weight of a finite-dimensional representation (and more generally whenever  $\gamma$  is an  $\ell$ -weight of a representation in the larger category  $\hat{\mathcal{O}}$  of [MY12c]) the series  $\gamma_i^+(u)$  and  $\gamma_i^-(u)$  are the Laurent expansions, about 0 and  $\infty$  respectively, of a complex-valued rational function  $\gamma_i(u) \in \mathbb{C}(u)$  with the property that  $\gamma_i(0)\gamma_i(\infty) = 1$ . Following [MY12c], we call such  $\ell$ -weights *rational*, and henceforth we shall not distinguish between a rational  $\ell$ -weight and its corresponding tuple of rational functions.

Rational  $\ell$ -weights form an abelian group, the group operation being (component-wise) multiplication of (tuples of) rational functions. Let  $\mathcal{R}$  denote this group and  $\mathbb{Z}\mathcal{R}$  its integral group ring. The  $q$ -character of a  $U_q(\mathcal{L}\mathfrak{g})$ -module  $V$  is by definition the formal sum

$$\chi_q(V) := \sum_{\gamma \in \mathcal{R}} \dim(V_{\gamma}) \gamma \in \mathbb{Z}\mathcal{R}.$$

The  $\ell$ -weights of finite-dimensional representations actually [FR98] belong to the subgroup of  $\mathcal{R}$  generated by rational  $\ell$ -weights  $Y_{i,a}$ ,  $i \in I$ ,  $a \in \mathbb{C}^\times$ , defined by

$$(Y_{i,a})_i(u) := q_i \frac{1 - q_i^{-2} u a}{1 - u a}, \quad (Y_{i,a})_j(u) := 1 \text{ for all } j \neq i.$$



The  $q$ -character of a finite-dimensional representation thus belongs to the ring  $\mathbb{Z}[Y_{i,a}, Y_{i,a}^{-1}]_{i \in I, a \in \mathbb{C}^\times}$  of formal Laurent polynomials. Moreover, when  $U_q(\mathcal{L}\mathfrak{g})$  is endowed with the standard Hopf algebra structure, the  $q$ -character map defines an injective homomorphism of rings [FR98]

$$\chi_q : \text{Rep}(\mathcal{C}) \rightarrow \mathbb{Z}[Y_{i,a}, Y_{i,a}^{-1}]_{i \in I, a \in \mathbb{C}^\times} \quad (2.16)$$

from the Grothendieck ring  $\text{Rep}(\mathcal{C})$ .

An  $\ell$ -weight in  $\mathbb{Z}[Y_{i,a}, Y_{i,a}^{-1}]_{i \in I, a \in \mathbb{C}^\times}$  is called *dominant* if it is in  $\mathbb{Z}[Y_{i,a}]_{i \in I, a \in \mathbb{C}^\times}$ . A vector  $v$  of a  $U_q(\mathcal{L}\mathfrak{g})$ -module is an  $\ell$ -weight vector of  $\ell$ -weight  $\gamma$  if  $\phi_{i,\pm r}^\pm \cdot v = v \gamma_{i,\pm r}^\pm$ , for all  $i \in I$ ,  $r \in \mathbb{Z}_{\geq 0}$ , and it is a *highest*  $\ell$ -weight vector if in addition  $x_{i,r}^+ \cdot v = 0$  for all  $i \in I$ ,  $r \in \mathbb{Z}$ . We write  $L(\gamma)$  for the irreducible quotient of  $U_q(\mathcal{L}\mathfrak{g}) \cdot v$ . It is known [CP94b] that the map  $\gamma \mapsto L(\gamma)$  is a bijection from the set of dominant  $\ell$ -weights to the set of isomorphism classes of irreducible finite-dimensional  $U_q(\mathcal{L}\mathfrak{g})$ -modules, i.e. the simple objects of  $\mathcal{C}$ .

For each  $j \in I$  and  $a \in \mathbb{C}^\times$ , define  $A_{j,a} \in \mathcal{R}$  by

$$(A_{j,a})_i(u) = q^{B_{ji}} \frac{1 - q^{-B_{ji}} au}{1 - q^{B_{ji}} au}$$

for each  $i \in I$ . We call each  $A_{j,a}$  a *simple  $\ell$ -root*. The  $A_{j,a}$  are algebraically independent. (The reader should be warned that in [FR98, FM01]  $A_{j,a}$  was instead labelled  $A_{j,aa_j}$ .)

### 3. MOTIVATION, AND THE CATEGORIES $\mathcal{C}_{\mathcal{P}}$

It will be important for us that the raising/lowering operators  $x_{i,r}^\pm$  of  $U_q(\mathcal{L}\mathfrak{g})$  act in a rather specific way in all finite-dimensional representations (actually, in all  $\ell$ -weight modules). Proposition 3.1 establishes this property. Then, in §3.2, we introduce the subcategories  $\mathcal{C}_{\mathcal{P}}$  of  $\mathcal{C}$  that we shall work with throughout the rest of the paper.

#### 3.1. Action of the raising/lowering operators between $\ell$ -weight spaces.

**Proposition 3.1.** *Let  $V \in \text{Ob}(\mathcal{C})$ . Pick and fix any  $i \in I$ . Let  $(\mu, \nu)$  be a pair of  $\ell$ -weights of  $V$  such that  $x_{i,r}^\pm(V_\mu) \cap V_\nu \neq \{0\}$  for some  $r \in \mathbb{Z}$ . Then:*

- (i)  $\nu = \mu A_{i,a}^{\pm 1}$  for some  $a \in \mathbb{C}^\times$ , and moreover
- (ii) there exist bases  $(v_k)_{1 \leq k \leq \dim(V_\mu)}$  of  $V_\mu$  and  $(w_\ell)_{1 \leq \ell \leq \dim(V_\nu)}$  of  $V_\nu$ , and complex polynomials  $P_{k,\ell}^\pm(z)$ , with  $\deg(P_{k,\ell}^\pm) \leq k + \ell - 2$ , such that

$$(x_i^\pm(z) \cdot v_k)_\nu = \sum_{\ell=1}^{\dim(V_\nu)} w_\ell P_{k,\ell}^\pm \left( \frac{\partial}{\partial a} \right) \delta \left( \frac{a}{z} \right).$$

*Proof.* Let  $(v_k)_{1 \leq k \leq \dim V_\mu}$  be a basis of  $V_\mu$  in which the action of the  $\phi_{j,r}^+$  is upper-triangular, in the sense that for all  $j \in I$  and  $1 \leq k \leq \dim V_\mu$ ,

$$(\phi_j^+(u) - \mu_j^+(u)) \cdot v_k = \sum_{k' < k} v_{k'} \xi_j^{+,k,k'}(u),$$

for certain formal series  $\xi_j^{+,k,k'}(u) \in u\mathcal{C}[[u]]$ . (The leading order is  $u^1$ : recall that  $\phi_{j,0}^\pm = k_j^{\pm 1}$  act diagonally.) Let  $(w_k)_{1 \leq k \leq \dim V_\nu}$  be a basis of  $V_\nu$  in which the action of the  $\phi_{j,r}^+$  is lower-triangular,

in the sense that for all  $j \in I$  and  $1 \leq \ell \leq \dim V_\nu$ ,

$$(\phi_j^+(u) - \nu_j^+(u)) \cdot w_\ell = \sum_{\ell' > \ell} w_{\ell'} \zeta_j^{+, \ell, \ell'}(u),$$

for certain formal series  $\zeta_j^{+, \ell, \ell'}(u) \in u\mathbb{C}[[u]]$ .

Consider for definiteness the case of  $x_i^+(z)$  ( $x_i^-(z)$  is similar). For all  $1 \leq k \leq \dim(V_\mu)$ ,

$$(x_i^+(z) \cdot v_k)_\nu = \sum_{\ell=1}^{\dim(V_\nu)} \lambda_{k, \ell}(z) w_\ell$$

for some formal series  $\lambda_{k, \ell}(z) \in \mathbb{C}[[z, z^{-1}]]$  for each  $\ell$ ,  $1 \leq \ell \leq \dim(V_\nu)$ .

By (2.4), we have

$$(q^{B_{ij}} - uz)x_i^+(z) \left( \phi_j^+(u) - \mu_j^+(u) \right) \cdot v_k = \left( (1 - q^{B_{ij}}uz)\phi_j^+(u) - (q^{B_{ij}} - uz)\mu_j^+(u) \right) x_i^+(z) \cdot v_k$$

and, on resolving this equation in the basis above of  $V_\nu$  and taking the  $w_\ell$  component,

$$\begin{aligned} (q^{B_{ij}} - uz) \sum_{k'=1}^{k-1} \xi_j^{+, k, k'}(u) \lambda_{k', \ell}(z) &= \left( (1 - q^{B_{ij}}uz)\nu_j^+(u) - (q^{B_{ij}} - uz)\mu_j^+(u) \right) \lambda_{k, \ell}(z) \\ &+ (1 - q^{B_{ij}}uz) \sum_{\ell'=1}^{\ell-1} \lambda_{k, \ell'}(z) \zeta_j^{+, \ell', \ell}(u). \end{aligned} \quad (3.1)$$

Suppose  $(x_i^+(z) \cdot V_\mu)_\nu \neq 0$ . Then there is a smallest  $K$  such that  $(x_i^+(z) \cdot v_K)_\nu \neq 0$  and then a smallest  $L$  such that  $\lambda_{K, L}(z) \neq 0$ . So (3.1) gives, in particular,

$$0 = \left( (1 - q^{B_{ij}}uz)\nu_j^+(u) - (q^{B_{ij}} - uz)\mu_j^+(u) \right) \lambda_{K, L}(z). \quad (3.2)$$

This must hold for all  $j \in I$ . For each  $j \in I$ , (3.2) is an equation of the form  $0 = \lambda_{K, L}(v) \sum_{n=0}^{\infty} u^n (b_n^{(i)} + c_n^{(i)}v)$  for the formal series  $\lambda_{K, L}(v) \in \mathbb{C}[[v, v^{-1}]]$ , with  $b_n^{(i)}, c_n^{(i)} \in \mathbb{C}$  for all  $n \in \mathbb{Z}_{\geq 0}$ . There are non-zero solutions if and only if there is an  $a \in \mathbb{C}^\times$  such that  $b_n^{(i)}/c_n^{(i)} = -a$  for all  $n \in \mathbb{Z}_{\geq 0}$  and all  $j \in I$ . That is,

$$\nu_j^+(u) \left( \mu_j^+(u) \right)^{-1} = q^{B_{ij}} \frac{1 - q^{-B_{ij}}ua}{1 - q^{B_{ij}}ua} =: (A_{i, a})_j(u)$$

as an equality of power series in  $u$ .

This establishes (i), and equation (3.1) then rearranges to give

$$\begin{aligned} \frac{(1 - q^{2B_{ij}})\mu_j^+(u)u}{1 - q^{B_{ij}}ua} (z - a) \lambda_{k, \ell}(z) &= (q^{B_{ij}} - uz) \sum_{k'=1}^{k-1} \xi_j^{+, k, k'}(u) \lambda_{k', \ell}(z) \\ &- (1 - q^{B_{ij}}uz) \sum_{\ell'=1}^{\ell-1} \lambda_{k, \ell'}(z) \zeta_j^{+, \ell', \ell}(u). \end{aligned} \quad (3.3)$$

Part (ii) now follows from (3.3) by an induction on  $k + \ell$ : in the base case  $k + \ell = 2$ , the right hand side of equation (3.3) is zero, so the equation for  $\lambda_{1, 1}$  is  $(z - a)\lambda_{1, 1}(z) = 0$ , whose solutions are  $\lambda_{1, 1}(z) = P_{1, 1}\delta(a/z)$ , with  $P_{1, 1} \in \mathbb{C}$ . For the inductive step, pick  $(k, \ell)$  and suppose (ii) is true for all  $(k', \ell')$  such that  $k' + \ell' < k + \ell$ . Consider  $\lambda_{k, \ell}$ . It is enough to consider the equation obtained

by taking the leading power of  $u$  in (3.3), which is  $u^1$ . This equation is

$$(1 - q^{2B_{ij}})\mu_{j,0}^+(z-a)\lambda_{k,\ell}(z) = q^{B_{ij}} \sum_{k'=1}^{k-1} \lambda_{k',\ell}(z)\zeta_{j,1}^{+,k,k'} - \sum_{\ell'=1}^{\ell-1} \lambda_{k,\ell'}(z)\zeta_{j,1}^{+,\ell',\ell},$$

and every solution is of the form  $\lambda_{k,\ell}(z) = P_{k,\ell}(\frac{\partial}{\partial a})\delta(\frac{a}{z})$  with

$$\deg(P_{k,\ell}) \leq 1 + \max_{\substack{k' < k \\ \ell' < \ell}} (\deg(P_{k',\ell'})).$$

(To see this, note that  $(z-a)\delta(\frac{z}{a}) = 0$ ,  $(z-a)\frac{\partial}{\partial a}\delta(\frac{a}{z}) = \delta(\frac{z}{a})$ , and more generally for all  $m \in \mathbb{Z}_{\geq 1}$ ,  $(z-a)\frac{1}{m!}(\frac{\partial}{\partial a})^m\delta(\frac{z}{a}) = \frac{1}{(m-1)!}(\frac{\partial}{\partial a})^{m-1}\delta(\frac{z}{a})$ .) This completes the inductive step.  $\square$

*Remark.* Part (i) in the above was proved in [MY12c].

**3.2. Categories  $\mathcal{C}_{\mathcal{P}}$  of finite dimensional representations.** To get more control over the  $\ell$ -roots  $A_{i,a}$  that can appear in Proposition 3.1, it is useful to work not with the category  $\mathcal{C}$  of *all* finite dimensional representations of  $U_q(\mathcal{L}\mathfrak{g})$ , but rather with smaller categories in which only rational  $\ell$ -weights with poles at certain prescribed points are allowed, as follows.

**Definition 3.2.** Pick, for each  $i \in I$ , a set  $\mathcal{P}_i \subset \mathbb{C}^\times$  consisting of finitely many (pairwise distinct) points. Given such a tuple  $\mathcal{P} = (\mathcal{P}_i)_{i \in I}$ , let  $\mathcal{C}_{\mathcal{P}}$  denote the full subcategory of  $\mathcal{C}$  such that for any  $\ell$ -weight  $\gamma = (\gamma_i(u))_{i \in I}$  of any representation  $V \in \text{Ob}(\mathcal{C}_{\mathcal{P}})$ , and for each  $i \in I$ , the rational function  $\gamma_i(u)$  has no poles lying outside the set  $\{a^{-1} : a \in \mathcal{P}_i\}$ .

Equivalently,  $\mathcal{C}_{\mathcal{P}}$  is the full subcategory of  $\mathcal{C}$  whose objects have  $q$ -characters in  $\mathbb{Z}[Y_{i,a}, Y_{i,aq^2}^{-1}]_{i \in I, a \in \mathcal{P}_i}$ .

The following lemma shows that the definition could also be phrased in terms of the allowed  $\ell$ -roots  $A_{i,a}$ . (This is not trivial, since  $(A_{i,a}^{\pm 1})_j(u)$  has no pole at  $a^{-1}$ .)

**Lemma 3.3.** *Let  $V$  be a representation in  $\mathcal{C}_{\mathcal{P}}$  and  $A_{i,a}$  a simple  $\ell$ -root. If  $\mu$  and  $\mu A_{i,a}^{-1}$  are both  $\ell$ -weights of  $V$  then  $a \in \mathcal{P}_i$ .*

*Proof.* This follows from Theorem 5.1 of [FM01], which states that  $\chi_q(\mathcal{C})$  is contained in (in fact, is equal to) the intersection  $\bigcap_{i \in I} K_i$  of the sets  $K_i := \mathbb{Z}[Y_{j,a}^{\pm 1}]_{j \neq i; a \in \mathbb{C}^\times} \otimes \mathbb{Z}[Y_{i,b} + Y_{i,b}A_{i,b}^{-1}]_{b \in \mathbb{C}^\times}$ .  $\square$

By (2.16),  $\mathcal{C}_{\mathcal{P}}$  is closed under taking submodules, quotients, finite direct sums (and tensor products, when  $\mathcal{C}$  is made into a tensor category using the standard Hopf algebra structure on  $U_q(\mathcal{L}\mathfrak{g})$ ).

*Remark 3.4.* The categories  $\mathcal{C}_\ell$  of the paper [HL10] are of this form. (In [HL10], the definition of  $\mathcal{C}_\ell$  is given in terms of the allowed highest  $\ell$ -weights.)

*Remark 3.5.* Roughly speaking, the choices of  $\mathcal{P}$  for which the categories  $\mathcal{C}_{\mathcal{P}}$  are non-trivial are those in which the  $\mathcal{P}_i$  contain sufficiently long subsequences of geometric progressions in  $q$ . In much of §6 we specialize to choices of the form  $\mathcal{P}_i = \{aq^k : k \in \mathbb{Z}, 0 \leq k \leq K\}$  for all  $i \in I$ , with  $a \in \mathbb{C}^\times$  and  $K \in \mathbb{Z}_{\geq 0}$  fixed. The reader may find it helpful to keep such a choice of  $\mathcal{P}$  in mind throughout.

*Remark 3.6.* The restriction to finite sets of allowed poles is for technical simplicity in what follows. Suitably modified, the main arguments should go through for countable sets without accumulation points in  $\mathbb{C}^\times$ . Of particular interest are the blocks of the category  $\mathcal{C}$ . In simply-laced cases these

are of the form  $\mathcal{C}_{\mathcal{P}}$  with  $\mathcal{P}_i = aq^{2\mathbb{Z}+c(i)}$  for some  $a \in \mathbb{C}^\times$ , where  $c : I \rightarrow \{0, 1\}$  is a two-colouring of the Dynkin diagram [CM05]. In the non-simply-laced cases the blocks are still of the form  $\mathcal{C}_{\mathcal{P}}$  for certain countable sets  $\mathcal{P}_i$ ,  $i \in I$ .

The following is essentially a corollary of Proposition 3.1.

**Proposition 3.7.** *Let  $V \in \text{Ob}(\mathcal{C}_{\mathcal{P}})$ , with  $\rho : U_q(\mathcal{L}\mathfrak{g}) \rightarrow \text{End}(V)$  the representation homomorphism. Let  $M = 2 \max_{\mu} \dim(V_{\mu})$ . Then there exist linear maps  $E_{i,a,m}^{\pm}$  and  $H_{i,a,m}$  in  $\text{End}(V)$ , for each  $i \in I$ ,  $a \in \mathcal{P}_i$  and  $0 \leq m \leq M$ , such that*

$$\rho(x_i^{\pm}(z)) = \sum_{a \in \mathcal{P}_i} \sum_{m=0}^M E_{i,a,m}^{\pm} \frac{a^m}{m!} \left( \frac{\partial}{\partial a} \right)^m \delta \left( \frac{a}{z} \right), \quad (3.4)$$

$$\rho(\Phi_i(1/z)) = \sum_{a \in \mathcal{P}_i} \sum_{m=0}^M H_{i,a,m} \frac{a^m}{m!} \left( \frac{\partial}{\partial a} \right)^m \delta \left( \frac{a}{z} \right). \quad (3.5)$$

Moreover,  $E_{i,a,m}^{\pm}(V_{\mu}) \subseteq V_{\mu A_{i,a}^{\pm 1}}$  (and  $H_{i,a,m}(V_{\mu}) \subseteq V_{\mu}$ ) for each  $\ell$ -weight  $\mu$  of  $V$ .

*Proof.* Pick an  $i \in I$ .

We define the required maps  $E_{i,a,m}^{\pm}$  by giving their restrictions to  $\text{Hom}(V_{\mu}, V_{\nu})$  for each pair of  $\ell$ -weights  $\mu$  and  $\nu$  of  $V$ . If the restriction of  $\rho(x_i^{\pm}(z))$  to  $\text{Hom}(V_{\mu}, V_{\nu})[[z, z^{-1}]]$  is zero then we define the restriction of  $E_{i,a,m}^{\pm}$  to  $\text{Hom}(V_{\mu}, V_{\nu})$  to be zero too, for each  $a \in \mathcal{P}_i$ . Otherwise, we use Proposition 3.1. Part (i) asserts that there is an  $a \in \mathbb{C}^\times$  such that  $\nu = \mu A_{i,a}^{\pm 1}$ . By Lemma 3.3,  $a \in \mathcal{P}_i$ . Suppose, in part (ii) of Proposition 3.1, that  $P_{k,\ell}^{\pm}(x) = \sum_{m=0}^{k+\ell-2} P_{k,\ell,m}^{\pm} \frac{a^m}{m!} x^m$  and set  $P_{k,\ell,m}^{\pm} = 0$  for all  $m > k + \ell - 2$ . Then the restriction of  $E_{i,a,m}^{\pm}$  to  $\text{Hom}(V_{\mu}, V_{\nu})$  is given by

$$(E_{i,a,m}^{\pm} \cdot v_k)_{\nu} = \sum_{\ell=1}^{\dim(V_{\nu})} P_{k,\ell,m}^{\pm} w_{\ell},$$

while for all  $b \in \mathcal{P}_i \setminus \{a\}$  the restriction of  $E_{i,b,m}^{\pm}$  to  $\text{Hom}(V_{\mu}, V_{\nu})$  is zero.

Note that  $E_{i,a,m}^{\pm}(V_{\mu}) \subseteq V_{\mu A_{i,a}^{\pm 1}}$  by construction.

Next we show that the maps  $H_{i,a,m}$  exist too. From (3.4) we have in particular that  $\rho(x_{i,0}^-) = \sum_{a \in \mathcal{P}_i} E_{i,a,0}^-$ . It follows from relation (2.12) that  $[x_i^+(u), x_{i,0}^-] = \Phi_i(1/u)$ . Thus, given (3.4), we have

$$\rho(\Phi_i(1/u)) = \sum_{a,b \in \mathcal{P}_i} \sum_{m=0}^M \left( \frac{a^m}{m!} \left( \frac{\partial}{\partial a} \right)^m \delta \left( \frac{a}{u} \right) \right) [E_{i,a,m}^+, E_{i,b,0}^-].$$

Note that  $[E_{i,a,m}^+, E_{i,b,0}^-](V_{\mu}) \subseteq V_{\mu A_{i,a} A_{i,b}^{-1}}$  for all  $\ell$ -weights  $\mu$  of  $V$ , whereas by definition of  $\ell$ -weight,  $\rho(\Phi_i(1/u))(V_{\mu}) \subseteq V_{\mu}$ . So the terms with  $a \neq b$  on the right must sum to zero and can be dropped. Hence indeed

$$\rho(\Phi_i(1/u)) = \sum_{a \in \mathcal{P}_i} \sum_{m=0}^M \left( \frac{a^m}{m!} \left( \frac{\partial}{\partial a} \right)^m \delta \left( \frac{a}{u} \right) \right) H_{i,a,m}$$

as required, where we define  $H_{i,a,m} := [E_{i,a,m}^+, E_{i,a,0}^-] \in \text{End}(V)$ .  $\square$

4. THE ALGEBRA  $\mathcal{A}$  AND STATEMENT OF MAIN RESULT

We first define the algebra  $\mathcal{A}$  that will appear in the statement of Theorem 4.4 below. The defining relations are given in §4.1, and we then need to take a certain completion, as discussed in §4.2.

**4.1. Defining relations of  $\mathcal{A}$ .** Let  $\mathcal{P} = (\mathcal{P}_i)_{i \in I}$  be a tuple as in Definition 3.2. We define an algebra  $\mathcal{A}$ , depending on this choice of  $\mathcal{P}$ , and also on  $\mathfrak{g}$  and  $q$ , as follows. Let  $\mathcal{A}$  be the associative unital algebra over  $\mathbb{C}$  generated by

$$K_i^{\pm 1}, \quad E_{i,a,m}^{\pm}, \quad H_{i,a,m}, \quad i \in I, a \in \mathcal{P}_i, m \in \mathbb{Z}_{\geq 0},$$

subject to the following relations. For all  $i, j \in I$ ,  $a \in \mathcal{P}_i$ ,  $b \in \mathcal{P}_j$  and  $m, n \in \mathbb{Z}_{\geq 0}$ ,

$$K_i K_i^{-1} = 1, \quad K_i E_{j,b,m}^{\pm} = E_{j,b,m}^{\pm} q^{\pm B_{ij}} K_i, \quad K_i K_j = K_j K_i \quad (4.1)$$

$$K_i H_{j,b,m} = H_{j,b,m} K_i, \quad H_{i,a,m} H_{j,b,n} = H_{j,b,n} H_{i,a,m} \quad (4.2)$$

$$\left[ E_{i,a,m}^+, E_{j,b,n}^- \right] = \delta_{ij} \delta_{a,b} H_{i,a,m+n} \quad (4.3)$$

$$\begin{aligned} & (a - bq^{\pm B_{ij}}) E_{i,a,m}^{\pm} E_{j,b,n}^{\pm} + a E_{i,a,m+1}^{\pm} E_{j,b,n}^{\pm} - bq^{\pm B_{ij}} E_{i,a,m}^{\pm} E_{j,b,n+1}^{\pm} \\ = & (aq^{\pm B_{ij}} - b) E_{j,b,n}^{\pm} E_{i,a,m}^{\pm} + aq^{\pm B_{ij}} E_{j,b,n}^{\pm} E_{i,a,m+1}^{\pm} - b E_{j,b,n+1}^{\pm} E_{i,a,m}^{\pm} \end{aligned} \quad (4.4)$$

$$\begin{aligned} & (a - bq^{\pm B_{ij}}) H_{i,a,m} E_{j,b,n}^{\pm} + a H_{i,a,m+1} E_{j,b,n}^{\pm} - bq^{\pm B_{ij}} H_{i,a,m} E_{j,b,n+1}^{\pm} \\ = & (aq^{\pm B_{ij}} - b) E_{j,b,n}^{\pm} H_{i,a,m} + aq^{\pm B_{ij}} E_{j,b,n}^{\pm} H_{i,a,m+1} - b E_{j,b,n+1}^{\pm} H_{i,a,m} \end{aligned} \quad (4.5)$$

$$\sum_{a \in \mathcal{P}_i} H_{i,a,0} = \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}. \quad (4.6)$$

In addition, for each  $i \neq j$ , set  $s = 1 - C_{ij}$ ; then for all  $m_1, \dots, m_s, n \in \mathbb{Z}_{\geq 0}$  and all  $a \in \mathcal{P}_j$  such that  $\{aq^{\mp B_{ij} \mp (t-1)B_{ii}} : 1 \leq t \leq s\} \subset \mathcal{P}_i$ , we impose the relation

$$\sum_{r=0}^s (-1)^r \binom{s}{r}_{q_i} E_{i,aq^{\mp B_{ij} \mp r B_{ii}}, m_{r+1}}^{\pm} \cdots E_{i,aq^{\mp B_{ij} \mp (s-1)B_{ii}}, m_s}^{\pm} E_{j,a,n}^{\pm} E_{i,aq^{\mp B_{ij}}, m_1}^{\pm} \cdots E_{i,aq^{\mp B_{ij} \mp (r-1)B_{ii}}, m_r}^{\pm} = 0. \quad (4.7)$$

*Remark 4.1.* The relations (4.7) are analogs of the Serre relations. In contrast to (2.7), they do not involve a sum over the symmetric group  $\Sigma_s$ . It is interesting to note that they take the same form as the Serre relations for the finite-type quantum group, but “decorated” with shifts in “special

position", cf. §5.2 below. For clarity, let us list their explicit form for  $E^-$ , case-by-case:

$$\begin{array}{c} \circ \\ i \end{array} \quad \begin{array}{c} \circ \\ j \end{array} : 0 = E_{j,b,n}^- E_{i,b,m}^- - E_{i,b,m}^- E_{j,b,n}^-$$

$$\begin{array}{c} \circ \text{---} \circ \\ i \quad j \end{array} : 0 = E_{j,b,n}^- E_{i,bq^{-1},m_1}^- E_{i,bq,m_2}^- - [2]_q E_{i,bq,m_2}^- E_{j,b,n}^- E_{i,bq^{-1},m_1}^- + E_{i,bq^{-1},m_1}^- E_{i,bq,m_2}^- E_{j,b,n}^-$$

$$\begin{array}{c} \circ \text{---} \circ \\ i \quad j \end{array} : 0 = E_{j,b,n}^- E_{i,bq^{-2},m_1}^- E_{i,b,m_2}^- E_{i,bq^2,m_3}^- - [3]_q E_{i,bq^2,m_3}^- E_{j,b,n}^- E_{i,bq^{-2},m_1}^- E_{i,b,m_2}^- \\ + [3]_q E_{i,b,m_2}^- E_{i,bq^2,m_3}^- E_{j,b,n}^- E_{i,bq^{-2},m_1}^- - E_{i,bq^{-2},m_1}^- E_{i,b,m_2}^- E_{i,bq^2,m_3}^- E_{j,b,n}^- \\ 0 = E_{i,b,n}^- E_{j,bq^{-2},m_1}^- E_{j,bq^2,m_2}^- - [2]_q E_{j,bq^2,m_2}^- E_{i,b,n}^- E_{j,bq^{-2},m_1}^- + E_{i,bq^{-2},m_1}^- E_{i,bq^2,m_2}^- E_{j,b,n}^-$$

$$\begin{array}{c} \circ \text{---} \circ \\ i \quad j \end{array} : 0 = E_{j,b,n}^- E_{i,bq^{-3},m_1}^- E_{i,bq^{-1},m_2}^- E_{i,bq^1,m_3}^- E_{i,bq^3,m_4}^- - [4]_q E_{i,bq^3,m_4}^- E_{j,b,n}^- E_{i,bq^{-3},m_1}^- E_{i,bq^{-1},m_2}^- E_{i,bq^1,m_3}^- \\ + \binom{4}{2}_q E_{i,bq^1,m_3}^- E_{i,bq^3,m_4}^- E_{j,b,n}^- E_{i,bq^{-3},m_1}^- E_{i,bq^{-1},m_2}^- - [4]_q E_{i,bq^{-1},m_2}^- E_{i,bq^1,m_3}^- E_{i,bq^3,m_4}^- E_{j,b,n}^- E_{i,bq^{-3},m_1}^- \\ + E_{i,bq^{-3},m_1}^- E_{i,bq^{-1},m_2}^- E_{i,bq^1,m_3}^- E_{i,bq^3,m_4}^- E_{j,b,n}^- \\ 0 = E_{i,b,n}^- E_{j,bq^{-3},m_1}^- E_{j,bq^3,m_2}^- - [2]_q E_{j,bq^3,m_2}^- E_{i,b,n}^- E_{j,bq^{-3},m_1}^- + E_{i,bq^{-3},m_1}^- E_{i,bq^3,m_2}^- E_{j,b,n}^-$$

**4.2. Completion of  $\mathcal{A}$ .** Let  $\mathcal{I}_0 := \mathcal{A}$ . For each  $m \in \mathbb{Z}_{>0}$ , let  $\mathcal{I}_m \subset \mathcal{A}$  be the two-sided ideal generated by  $E_{i,a,n}^+$ ,  $E_{i,a,n}^-$  and  $H_{i,a,n}$ ,  $i \in I, a \in \mathcal{P}_i$  and  $n \geq m$ . These ideals form a strictly decreasing sequence

$$\mathcal{A} = \mathcal{I}_0 \supset \mathcal{I}_1 \supset \mathcal{I}_2 \supset \dots$$

**Example 4.2.** Suppose  $a \in \mathcal{P}_i, b \in \mathcal{P}_j$ . By relation (4.4),

$$E_{i,a,0}^- E_{j,aq^{-B_{ij}},0}^- = \frac{1}{1 - q^{-2B_{ij}}} \left( -E_{i,a,1}^- E_{j,aq^{-B_{ij}},0}^- + q^{-2B_{ij}} E_{i,a,0}^- E_{j,aq^{-B_{ij}},1}^- \right. \\ \left. + q^{-B_{ij}} E_{j,aq^{-B_{ij}},0}^- E_{i,a,1}^- - q^{-B_{ij}} E_{j,aq^{-B_{ij}},1}^- E_{i,a,0}^- \right).$$

Thus  $E_{i,a,0}^- E_{j,b,0}^- \in \mathcal{I}_1$  if  $aq^{-B_{ij}} = b$ . Note that  $E_{i,a,0}^- E_{j,aq^{-B_{ij}},0}^- \notin \mathcal{I}_2$ , because the third and final terms of the right-hand side cannot be further re-written in this way (although the first two terms can). We also have (when  $i = j$ ) that  $E_{i,a,0}^- E_{i,a,0}^- \in \mathcal{I}_1$ , because relation (4.4) yields

$$E_{i,a,0}^- E_{i,a,0}^- = \frac{1}{1 - q^{-B_{ii}}} \left( q^{-B_{ii}} E_{i,a,0}^- E_{i,a,1}^- - E_{i,a,1}^- E_{i,a,0}^- \right). \quad (4.8)$$

More generally, inspecting (4.4) and (4.5) one sees that for any finite linear combination  $S$  of monomials in the generators of  $\mathcal{A}$ , if  $S \neq 0$  then there is an  $n \in \mathbb{Z}_{\geq 1}$  such that  $S \notin \mathcal{I}_n$ . That is, we have

**Lemma 4.3.**  $\bigcap_{n=1}^{\infty} \mathcal{I}_n = \{0\}$ . □

By virtue of this we can, in what follows, work with infinite linear combinations of the form

$$S = \sum_{m=0}^{\infty} u_m, \quad u_m \in \mathcal{I}_m.$$

Such a sum is to be interpreted as the sequence of partial sums

$$S = (S_0, S_1, S_2, \dots), \quad S_M = \sum_{m=0}^M u_m,$$

and we declare that  $S = S'$  if and only if  $S_M \equiv S'_M \pmod{\mathcal{I}_{M+1}}$  for all  $M \in \mathbb{Z}_{\geq 0}$ .

More formally, the decreasing sequence of ideals  $(\mathcal{I}_m)_{m \geq 0}$  induces a topology on  $\mathcal{A}$ : namely the topology in which for each  $x \in \mathcal{A}$ , the sets  $\{x + \mathcal{I}_m : m \in \mathbb{Z}_{\geq 0}\}$  form a base of the open sets containing  $x$ . This topology is Hausdorff by the above lemma. And henceforth we work in the completion  $\varprojlim \mathcal{A}/\mathcal{I}_n$  of  $\mathcal{A}$  with respect to this topology. By a slight abuse, we continue to write  $\mathcal{A}$  for this completion.

**4.3. Main result.** We now state the main result of the paper.

**Theorem 4.4.** *There is a homomorphism of algebras  $\theta_{\mathcal{P}} : U_q(\mathcal{L}\mathfrak{g}) \rightarrow \mathcal{A}$  defined by  $k_i \mapsto K_i$  and*

$$\begin{aligned} x_i^{\pm}(z) &\mapsto \sum_{a \in \mathcal{P}_i} \sum_{m \in \mathbb{Z}_{\geq 0}} \mathbb{E}_{i,a,m}^{\pm} \frac{a^m}{m!} \left( \frac{\partial}{\partial a} \right)^m \delta \left( \frac{a}{z} \right), \\ \Phi_i(1/z) &\mapsto \sum_{a \in \mathcal{P}_i} \sum_{m \in \mathbb{Z}_{\geq 0}} \mathbb{H}_{i,a,m} \frac{a^m}{m!} \left( \frac{\partial}{\partial a} \right)^m \delta \left( \frac{a}{z} \right). \end{aligned}$$

*Moreover, every  $V \in \text{Ob}(\mathcal{C}_{\mathcal{P}})$  is the pull-back by  $\theta_{\mathcal{P}}$  of a finite-dimensional representation of  $\mathcal{A}$ .*

## 5. PROOF OF THEOREM 4.4

This section is devoted to the proof of Theorem 4.4. We first re-express the defining relations of  $\mathcal{A}$  in terms of formal generating series in §5.1. Then, much of the section is devoted to showing that the Serre relations (4.7) are equivalent to the, a priori stronger, relations (5.5) below. This is done in Propositions 5.3 and 5.4, and relies on the use of an identity due to Jing [Jin98]. With these preparations complete, Theorem 4.4 is proved in §5.3: the existence of the homomorphism in Proposition 5.6, and the “moreover” part in Proposition 5.8.

**5.1. Defining relations expressed through formal series.** For each  $G \in \{E^+, E^-, H\}$ ,  $i \in I$  and  $a \in \mathcal{P}_i$ , let us introduce a generating series

$$\mathbb{G}_{i,a}(z) := \sum_{m=0}^{\infty} \frac{z^m}{m!} \mathbb{G}_{i,a,m} \in \mathcal{A}[[z]]. \quad (5.1)$$

Given any formal series  $c \left( \frac{\partial}{\partial z} \right) = \sum_{k=0}^{\infty} c_k \left( \frac{\partial}{\partial z} \right)^k \in \mathbb{C}[[\frac{\partial}{\partial z}]]$ , the product  $c \left( \frac{\partial}{\partial z} \right) \mathbb{G}_{i,a}(z)$  is well-defined because for each  $m \in \mathbb{Z}$  the coefficient of  $z^m$  is a well-defined infinite sum, cf §4.2:

$$c \left( \frac{\partial}{\partial z} \right) \mathbb{G}_{i,a}(z) = \sum_{m=0}^{\infty} \frac{z^m}{m!} \sum_{n=0}^{\infty} c_n \mathbb{G}_{i,a,m+n}.$$

More generally, for any formal series  $c(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_K}) = \sum_{p,r \in \mathbb{Z}} c_{p_1, \dots, p_K} \prod_{k=1}^K (\frac{\partial}{\partial z_k})^{p_k} \in \mathbb{C}[[\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_K}]]$ , products of the form  $c(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_K}) \mathbf{G}_{i_1, a_1}^{(1)}(z_1) \dots \mathbf{G}_{i_K, a_K}^{(K)}(z_K)$  with each  $\mathbf{G}^{(k)} \in \{\mathbf{E}^+, \mathbf{E}^-, \mathbf{H}\}$  are well-defined.

The defining relations (4.3–4.5) are equivalent to<sup>2</sup>

$$\left[ \mathbf{E}_{i,a}^+(z), \mathbf{E}_{j,b}^-(w) \right] = \delta_{ij} \delta_{a,b} \mathbf{H}_{i,a}(z+w).$$

$$\begin{aligned} \left( a \left( 1 + \frac{\partial}{\partial z} \right) - bq^{\pm B_{ij}} \left( 1 + \frac{\partial}{\partial w} \right) \right) \mathbf{E}_{i,a}^{\pm}(z) \mathbf{E}_{j,b}^{\pm}(w) &= \left( aq^{\pm B_{ij}} \left( 1 + \frac{\partial}{\partial z} \right) - b \left( 1 + \frac{\partial}{\partial w} \right) \right) \mathbf{E}_{j,b}^{\pm}(w) \mathbf{E}_{i,a}^{\pm}(z). \\ \left( a \left( 1 + \frac{\partial}{\partial z} \right) - bq^{\pm B_{ij}} \left( 1 + \frac{\partial}{\partial w} \right) \right) \mathbf{H}_{i,a}(z) \mathbf{E}_{j,b}^{\pm}(w) &= \left( aq^{\pm B_{ij}} \left( 1 + \frac{\partial}{\partial z} \right) - b \left( 1 + \frac{\partial}{\partial w} \right) \right) \mathbf{E}_{j,b}^{\pm}(w) \mathbf{H}_{i,a}(z). \end{aligned}$$

Moreover we have the following.

**Proposition 5.1.** *Whenever  $a \neq bq^{\pm B_{ij}}$ ,*

$$\begin{aligned} \mathbf{E}_{i,a}^{\pm}(z) \mathbf{E}_{j,b}^{\pm}(w) &= \frac{aq^{\pm B_{ij}} \left( 1 + \frac{\partial}{\partial z} \right) - b \left( 1 + \frac{\partial}{\partial w} \right)}{a \left( 1 + \frac{\partial}{\partial z} \right) - bq^{\pm B_{ij}} \left( 1 + \frac{\partial}{\partial w} \right)} \mathbf{E}_{j,b}^{\pm}(w) \mathbf{E}_{i,a}^{\pm}(z) \\ \mathbf{H}_{i,a}(z) \mathbf{E}_{j,b}^{\pm}(w) &= \frac{aq^{\pm B_{ij}} \left( 1 + \frac{\partial}{\partial z} \right) - b \left( 1 + \frac{\partial}{\partial w} \right)}{a \left( 1 + \frac{\partial}{\partial z} \right) - bq^{\pm B_{ij}} \left( 1 + \frac{\partial}{\partial w} \right)} \mathbf{E}_{j,b}^{\pm}(w) \mathbf{H}_{i,a}(z) \end{aligned}$$

(where  $\frac{aq^{\pm B_{ij}} \left( 1 + \frac{\partial}{\partial z} \right) - b \left( 1 + \frac{\partial}{\partial w} \right)}{a \left( 1 + \frac{\partial}{\partial z} \right) - bq^{\pm B_{ij}} \left( 1 + \frac{\partial}{\partial w} \right)}$  is to be interpreted as a formal series in  $\frac{\partial}{\partial z}$  and  $\frac{\partial}{\partial w}$  by regarding them as small and expanding).

*Proof.* Let  $C(p, r) = (-1)^{r+1} \binom{p}{r} \frac{(bq^{\pm B_{ij}})^r a^{(p-r)}}{(bq^{\pm B_{ij}} - a)^p}$ . By induction, we find that for all  $P \in \mathbb{Z}_{\geq 0}$ ,

$$\begin{aligned} \mathbf{E}_{i,a,m}^{\pm} \mathbf{E}_{j,b,n}^{\pm} &= \mathbf{E}_{j,b,n}^{\pm} \mathbf{E}_{i,a,m}^{\pm} \frac{b - aq^{\pm B_{ij}}}{bq^{\pm B_{ij}} - a} \\ &+ \sum_{p=1}^P \sum_{r=0}^p \mathbf{E}_{j,b,n+r}^{\pm} \mathbf{E}_{i,a,m+p-r}^{\pm} \frac{q^{B_{ij}} - q^{-B_{ij}} (p-r) bq^{\pm B_{ij}} + ra}{bq^{\pm B_{ij}} - a} \frac{C(p, r)}{p} \\ &+ \sum_{r=0}^{P+1} \mathbf{E}_{j,b,n+r}^{\pm} \mathbf{E}_{i,a,m+P+1-r}^{\pm} \frac{(P+1-r)q^{B_{ij}} + rq^{-B_{ij}}}{P+1} C(P+1, r) \\ &- \sum_{r=0}^{P+1} \mathbf{E}_{i,a,m+P+1-r}^{\pm} \mathbf{E}_{j,b,n+r}^{\pm} C(P+1, r). \end{aligned}$$

The final two lines belong to  $\mathcal{L}_{P+1}$ . Therefore

$$\begin{aligned} \mathbf{E}_{i,a,m}^{\pm} \mathbf{E}_{j,b,n}^{\pm} &= \mathbf{E}_{j,b,n}^{\pm} \mathbf{E}_{i,a,m}^{\pm} \frac{b - aq^{\pm B_{ij}}}{bq^{\pm B_{ij}} - a} \\ &+ \sum_{p=1}^{\infty} \sum_{r=0}^p \mathbf{E}_{j,b,n+r}^{\pm} \mathbf{E}_{i,a,m+p-r}^{\pm} \frac{q^{B_{ij}} - q^{-B_{ij}} (p-r) bq^{\pm B_{ij}} + ra}{bq^{\pm B_{ij}} - a} \frac{C(p, r)}{p} \end{aligned}$$

<sup>2</sup>It is perhaps interesting to note that if instead of (5.1) one chose to work with geometric generating series  $\mathbf{G}_{i,a}(z) := \sum_{m=0}^{\infty} z^m \mathbf{G}_{i,a,m}$  then the first of these relations would read

$$\left[ \mathbf{E}_{i,a}^+(z), \mathbf{E}_{j,b}^-(w) \right] = \delta_{ij} \delta_{a,b} \frac{z \mathbf{H}_{i,a}(z) - w \mathbf{H}_{i,a}(w)}{z - w}.$$



and hence the result.  $\square$

**5.2. Sticking graphs and the Serre relations.** The “whenever” condition in Proposition 5.1 motivates the following definition. We say  $(i, a)$  *sticks to the left of*  $(j, b)$  if  $a = bq^{-B_{ij}}$ . Given a multiset  $V$  of elements of  $I \times \mathbb{C}^\times$ , define the *sticking graph* of  $V$  to be the directed graph  $(V, E)$  with vertex set  $V$  and directed edges

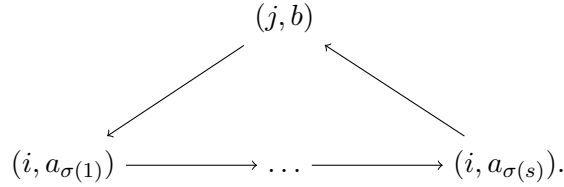
$$E = \{(i, a) \rightarrow (j, b) : (i, a) \text{ sticks to the left of } (j, b)\}. \quad (5.2)$$

Now, and for the remainder of this subsection, pick and fix  $i, j \in I$  such that  $i \neq j$ . We shall treat the Serre relations involving the nodes  $i$  and  $j$  of the Dynkin diagram. Let  $\Sigma_s$  be the symmetric group on  $s := 1 - C_{ij}$  letters, and let  $a_1, a_2, \dots, a_s \in \mathbb{C}^\times$  and  $b \in \mathbb{C}^\times$ .

**Lemma 5.2.** *If there is a  $\sigma \in \Sigma_s$  such that for each  $t \in \{1, \dots, s\}$ ,*

$$a_{\sigma(t)} = bq^{B_{ij} + (t-1)B_{ii}},$$

*then the sticking graph of  $\{(i, a_1), (i, a_2), \dots, (i, a_s), (j, b)\}$  is the directed cycle graph*



*Otherwise, it has no directed cycles.*

*Proof.* For the first part observe that if the given condition holds then  $a_{\sigma(s)} = q^{-B_{ij}}$ ; to see this note that  $B_{ii}C_{ij} = 2B_{ij}$ . So the sticking graph is indeed the cycle shown. The “otherwise” part can be seen by inspecting the Cartan matrices of  $A_1 \times A_1$ ,  $A_2$ ,  $B_2$ ,  $C_2$  and  $G_2$ .  $\square$

**Proposition 5.3.** *Suppose the sticking graph of  $\{(i, a_1), (i, a_2), \dots, (i, a_s), (j, b)\}$  is not a directed cycle graph. Then the relation (4.4) implies the relation*

$$\sum_{\pi \in \Sigma_s} \sum_{r=0}^s (-1)^r \binom{s}{r}_{q_i} \mathbf{E}_{i, a_{\pi(1)}}^\pm(z_{\pi(1)}) \dots \mathbf{E}_{i, a_{\pi(r)}}^\pm(z_{\pi(r)}) \mathbf{E}_{j, b}^\pm(w) \mathbf{E}_{i, a_{\pi(r+1)}}^\pm(z_{\pi(r+1)}) \dots \mathbf{E}_{i, a_{\pi(s)}}^\pm(z_{\pi(s)}) = 0. \quad (5.3)$$

*Proof.* Let us consider, for definiteness,  $\mathbf{E}^-$ ; the argument for  $\mathbf{E}^+$  is similar. By the preceding lemma, the sticking graph of  $\{(i, a_1), (i, a_2), \dots, (i, a_s), (j, b)\}$  has no directed cycles. Consequently, there must exist a permutation  $\tau \in \Sigma_s$  and a  $t \in \{1, \dots, s\}$  such that in the tuple

$$((i, a_{\tau(1)}), \dots, (i, a_{\tau(t)}), (j, b), (i, a_{\tau(t+1)}), \dots, (i, a_{\tau(s)}))$$

no element sticks to the left of any element preceding it; without loss of generality, suppose that  $\tau = \text{id}$ . It follows that we may use Proposition 5.1 to express every monomial appearing on the left of (5.3) in terms of the monomial

$$\mathcal{M} := \mathbf{E}_{i, a_1}^-(z_1) \dots \mathbf{E}_{i, a_t}^-(z_t) \mathbf{E}_{j, b}^-(w) \mathbf{E}_{i, a_{t+1}}^-(z_{t+1}) \dots \mathbf{E}_{i, a_s}^-(z_s).$$

Indeed, the left-hand side of (5.3) is

$$\begin{aligned}
 &= \sum_{\pi \in \Sigma_s} \sum_{r=0}^s (-1)^r \binom{s}{r}_{q_i} \prod_{\substack{n < m \\ \pi^{-1}(n) > \pi^{-1}(m)}} \frac{a_m q^{-B_{ii}} \left(1 + \frac{\partial}{\partial z_m}\right) - a_n \left(1 + \frac{\partial}{\partial z_n}\right)}{a_m \left(1 + \frac{\partial}{\partial z_m}\right) - a_n q^{-B_{ii}} \left(1 + \frac{\partial}{\partial z_n}\right)} \\
 &\quad \times \prod_{\substack{n > t \\ \pi^{-1}(n) \leq r}} \frac{a_n q^{-B_{ij}} \left(1 + \frac{\partial}{\partial z_n}\right) - b \left(1 + \frac{\partial}{\partial w}\right)}{a_n \left(1 + \frac{\partial}{\partial z_n}\right) - b q^{-B_{ij}} \left(1 + \frac{\partial}{\partial w}\right)} \prod_{\substack{n \leq t \\ \pi^{-1}(n) > r}} \frac{b q^{-B_{ij}} \left(1 + \frac{\partial}{\partial w}\right) - a_n \left(1 + \frac{\partial}{\partial z_n}\right)}{b \left(1 + \frac{\partial}{\partial w}\right) - a_n q^{-B_{ij}} \left(1 + \frac{\partial}{\partial z_n}\right)} \mathcal{M} \\
 &= \frac{1}{\Delta} \sum_{\pi \in \Sigma_s} \sum_{r=0}^s (-1)^r \binom{s}{r}_{q_i} \\
 &\quad \times \prod_{\substack{n < m \\ \pi^{-1}(n) < \pi^{-1}(m)}} \left(a_m \left(1 + \frac{\partial}{\partial z_m}\right) - a_n q^{-B_{ii}} \left(1 + \frac{\partial}{\partial z_n}\right)\right) \prod_{\substack{n < m \\ \pi^{-1}(n) > \pi^{-1}(m)}} \left(a_m q^{-B_{ii}} \left(1 + \frac{\partial}{\partial z_m}\right) - a_n \left(1 + \frac{\partial}{\partial z_n}\right)\right) \\
 &\quad \times \prod_{\substack{n > t \\ \pi^{-1}(n) > r}} \left(a_n \left(1 + \frac{\partial}{\partial z_n}\right) - b q^{-B_{ij}} \left(1 + \frac{\partial}{\partial w}\right)\right) \prod_{\substack{n > t \\ \pi^{-1}(n) \leq r}} \left(a_n q^{-B_{ij}} \left(1 + \frac{\partial}{\partial z_n}\right) - b \left(1 + \frac{\partial}{\partial w}\right)\right) \\
 &\quad \times \prod_{\substack{n \leq t \\ \pi^{-1}(n) \leq r}} \left(b \left(1 + \frac{\partial}{\partial w}\right) - a_n q^{-B_{ij}} \left(1 + \frac{\partial}{\partial z_n}\right)\right) \prod_{\substack{n \leq t \\ \pi^{-1}(n) > r}} \left(b q^{-B_{ij}} \left(1 + \frac{\partial}{\partial w}\right) - a_n \left(1 + \frac{\partial}{\partial z_n}\right)\right) \mathcal{M}, \quad (5.4)
 \end{aligned}$$

where

$$\begin{aligned}
 \Delta &:= \prod_{n < m} \left(a_m \left(1 + \frac{\partial}{\partial z_m}\right) - a_n q^{-B_{ii}} \left(1 + \frac{\partial}{\partial z_n}\right)\right) \\
 &\quad \times \prod_{n > t} \left(a_n \left(1 + \frac{\partial}{\partial z_n}\right) - b q^{-B_{ij}} \left(1 + \frac{\partial}{\partial w}\right)\right) \prod_{n \leq t} \left(b \left(1 + \frac{\partial}{\partial w}\right) - a_n q^{-B_{ij}} \left(1 + \frac{\partial}{\partial z_n}\right)\right).
 \end{aligned}$$

Now, given commuting indeterminates  $F_n$ ,  $1 \leq n \leq s$ , and  $G$ , define for each  $\pi \in \Sigma_s$  and each  $r \in \{1, \dots, s\}$ ,

$$\begin{aligned}
 A_{\pi, r}(F_1, \dots, F_s; G) &:= \prod_{\substack{n < m \\ \pi^{-1}(n) < \pi^{-1}(m)}} (F_m - q^{-B_{ii}} F_n) \prod_{\substack{n < m \\ \pi^{-1}(n) > \pi^{-1}(m)}} (q^{-B_{ii}} F_m - F_n) \\
 &\quad \times \prod_{\substack{n \\ \pi^{-1}(n) > r}} (F_n - q^{-B_{ij}} G) \prod_{\substack{n \\ \pi^{-1}(n) \leq r}} (q^{-B_{ij}} F_n - G).
 \end{aligned}$$

We then have the identity

$$\sum_{\pi \in \Sigma_s} \sum_{r=0}^s (-1)^r \binom{s}{r}_{q_i} A_{\pi, r}(F_1, \dots, F_s; G) = 0,$$

which can be verified by direct calculation with the aid of a computer algebra system case-by-case (i.e. for the Cartan matrices of  $A_1 \times A_1$ ,  $A_2$ ,  $B_2$ ,  $C_2$  and  $G_2$ ). It is actually a special instance of an identity due to Jing which holds for arbitrary symmetric generalized Cartan matrices, [Jin98], cf. also [DJ00] and [Her05]. One may check that (5.4) is equal to

$$\frac{1}{\Delta} \left( \sum_{\pi \in \Sigma_s} \sum_{r=0}^s (-1)^r \binom{s}{r}_{q_i} A_{\pi, r} \left( a_1 \left(1 + \frac{\partial}{\partial z_1}\right), \dots, a_s \left(1 + \frac{\partial}{\partial z_s}\right); b \left(1 + \frac{\partial}{\partial w}\right) \right) \right) \mathcal{M}.$$

It therefore vanishes, as required.  $\square$

**Proposition 5.4.** *Given the relations (4.4), imposing the relation*

$$\sum_{\pi \in \Sigma_s} \sum_{r=0}^s (-1)^r \binom{s}{r}_{q_i} \mathbf{E}_{i, a_{\pi(1)}}^{\pm}(z_{\pi(1)}) \cdots \mathbf{E}_{i, a_{\pi(r)}}^{\pm}(z_{\pi(r)}) \mathbf{E}_{j, b}^{\pm}(w) \mathbf{E}_{i, a_{\pi(r+1)}}^{\pm}(z_{\pi(r+1)}) \cdots \mathbf{E}_{i, a_{\pi(s)}}^{\pm}(z_{\pi(s)}) = 0 \quad (5.5)$$

is equivalent to imposing the relation

$$\sum_{r=0}^s (-1)^r \binom{s}{r}_{q_i} \mathbf{E}_{i, bq^{\mp B_{ij} \mp r B_{ii}}}^{\pm}(z_{r+1}) \cdots \mathbf{E}_{i, bq^{\mp B_{ij} \mp (s-1) B_{ii}}}^{\pm}(z_s) \mathbf{E}_{j, b}^{\pm}(w) \mathbf{E}_{i, bq^{\mp B_{ij}}}^{\pm}(z_1) \cdots \mathbf{E}_{i, bq^{\mp B_{ij} \mp (r-1) B_{ii}}}^{\pm}(z_r) = 0. \quad (5.6)$$

*Proof.* By the preceding proposition, it is enough to consider the case in which the sticking graph of  $\{(i, a_1), (i, a_2), \dots, (i, a_s), (j, b)\}$  is a directed cycle graph. That is, cf Lemma 5.2, we can suppose, relabelling the  $a$ 's as necessary, that

$$a_t = bq^{\mp B_{ij} \mp (t-1) B_{ii}} = bq^{\pm B_{ij} \pm (s-t) B_{ii}}.$$

Obviously (5.5) is equivalent to

$$\sum_{r=0}^s (-1)^r \binom{s}{r}_{q_i} \sum_{\pi \in \Sigma_s} \mathbf{E}_{i, a_{\pi(r+1)}}^{\pm}(z_{\pi(r+1)}) \cdots \mathbf{E}_{i, a_{\pi(s)}}^{\pm}(z_{\pi(s)}) \mathbf{E}_{j, b}^{\pm}(w) \mathbf{E}_{i, a_{\pi(1)}}^{\pm}(z_{\pi(1)}) \cdots \mathbf{E}_{i, a_{\pi(r)}}^{\pm}(z_{\pi(r)}) = 0.$$

The left-hand side here is equal to (by Proposition 5.1)

$$\sum_{r=0}^s (-1)^r \binom{s}{r}_{q_i} \left( \sum_{\pi \in \Sigma_s} \mathcal{D}_{\pi} \right) \mathbf{E}_{i, a_{r+1}}^{\pm}(z_{r+1}) \cdots \mathbf{E}_{i, a_s}^{\pm}(z_s) \mathbf{E}_{j, b}^{\pm}(w) \mathbf{E}_{i, a_1}^{\pm}(z_1) \cdots \mathbf{E}_{i, a_r}^{\pm}(z_r)$$

where for each  $\pi \in \Sigma_s$ ,  $\mathcal{D}_{\pi} \in \mathbb{C}[[\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_s}, \frac{\partial}{\partial w}]]$ , and moreover  $\mathcal{D}_{\pi}$  has zero constant term for each  $\pi \neq \text{id}$ , while  $\mathcal{D}_{\text{id}} = 1$ . The result follows.  $\square$

### 5.3. Proof of Theorem 4.4.

**Lemma 5.5.** *Let  $u$  be an indeterminate. For all  $n \in \mathbb{Z}_{\geq 0}$ .*

$$u \left[ \frac{a^{n+1}}{(n+1)!} \left( \frac{\partial}{\partial a} \right)^{n+1} \delta \left( \frac{a}{u} \right) \right] = a \left[ \frac{a^{n+1}}{(n+1)!} \left( \frac{\partial}{\partial a} \right)^{n+1} \delta \left( \frac{a}{u} \right) + \frac{a^n}{n!} \left( \frac{\partial}{\partial a} \right)^n \delta \left( \frac{a}{u} \right) \right]$$

*Proof.* We have

$$\begin{aligned} u \left( \frac{\partial}{\partial a} \right)^{n+1} \delta \left( \frac{a}{u} \right) &= \left( \frac{\partial}{\partial a} \right)^{n+1} u \delta \left( \frac{a}{u} \right) = \left( \frac{\partial}{\partial a} \right)^{n+1} a \delta \left( \frac{a}{u} \right) \\ &= a \left( \frac{\partial}{\partial a} \right)^{n+1} \delta \left( \frac{a}{u} \right) + (n+1) \left( \frac{\partial}{\partial a} \right)^n \delta \left( \frac{a}{u} \right) \end{aligned}$$

and hence the result.  $\square$

**Proposition 5.6.** *The assignment  $k_i \mapsto \mathsf{K}_i$  and*

$$\begin{aligned} x_i^\pm(z) &\mapsto \sum_{a \in \mathcal{P}_i} \sum_{m \in \mathbb{Z}_{\geq 0}} \mathsf{E}_{i,a,m}^\pm \frac{a^m}{m!} \left( \frac{\partial}{\partial a} \right)^m \delta \left( \frac{a}{z} \right), \\ \Phi_i(1/z) &\mapsto \sum_{a \in \mathcal{P}_i} \sum_{m \in \mathbb{Z}_{\geq 0}} \mathsf{H}_{i,a,m} \frac{a^m}{m!} \left( \frac{\partial}{\partial a} \right)^m \delta \left( \frac{a}{z} \right) \end{aligned}$$

*extends to a homomorphism of algebras  $\theta_{\mathcal{P}} : U_q(\mathcal{L}\mathfrak{g}) \rightarrow \mathcal{A}$ .*

*Proof.* In fact the defining relations of  $\mathcal{A}$  are constructed by demanding that this be true. Consider the relation (2.6). On applying  $\theta_{\mathcal{P}}$  to the left-hand side we have

$$\sum_{\substack{a \in \mathcal{P}_i, \\ m \in \mathbb{Z}_{\geq 0}}} \sum_{\substack{b \in \mathcal{P}_j, \\ n \in \mathbb{Z}_{\geq 0}}} \left( \frac{a^m}{m!} \left( \frac{\partial}{\partial a} \right)^m \delta \left( \frac{a}{u} \right) \right) \left( \frac{b^n}{n!} \left( \frac{\partial}{\partial b} \right)^n \delta \left( \frac{b}{v} \right) \right) (u - q^{\pm B_{ij}} v) \mathsf{E}_{i,a,m}^\pm \mathsf{E}_{j,b,n}^\pm.$$

By Lemma 5.5, and the identity  $\delta(u/a)u = \delta(u/a)a$ , this is equal to

$$\begin{aligned} &\sum_{\substack{a \in \mathcal{P}_i, \\ m \in \mathbb{Z}_{\geq 0}}} \sum_{\substack{b \in \mathcal{P}_j, \\ n \in \mathbb{Z}_{\geq 0}}} \left( \frac{a^m}{m!} \left( \frac{\partial}{\partial a} \right)^m \delta \left( \frac{a}{u} \right) \right) \left( \frac{b^n}{n!} \left( \frac{\partial}{\partial b} \right)^n \delta \left( \frac{b}{v} \right) \right) \\ &\times \left[ (a - bq^{\pm B_{ij}}) \mathsf{E}_{i,a,m}^\pm \mathsf{E}_{j,b,n}^\pm + a \mathsf{E}_{i,a,m+1}^\pm \mathsf{E}_{j,b,n}^\pm - bq^{\pm B_{ij}} \mathsf{E}_{i,a,m}^\pm \mathsf{E}_{j,b,n+1}^\pm \right] \end{aligned} \quad (5.7)$$

and we see that for the images  $\theta_{\mathcal{P}}(x_{i,r}^\pm) \in \mathcal{A}$  to obey the relation (2.6) it suffices to impose (4.4). The relation (2.14) works in the same way.

On applying  $\theta_{\mathcal{P}}$  to (2.12) we find that the left-hand side is

$$\sum_{\substack{a \in \mathcal{P}_i, \\ m \in \mathbb{Z}_{\geq 0}}} \sum_{\substack{b \in \mathcal{P}_j, \\ n \in \mathbb{Z}_{\geq 0}}} \left( \frac{a^m}{m!} \left( \frac{\partial}{\partial a} \right)^m \delta \left( \frac{a}{u} \right) \right) \left( \frac{b^n}{n!} \left( \frac{\partial}{\partial b} \right)^n \delta \left( \frac{b}{v} \right) \right) \left[ \mathsf{E}_{i,a,m}^+, \mathsf{E}_{j,b,n}^- \right]$$

while the right-hand side is

$$\begin{aligned} &\delta_{ij} \delta \left( \frac{u}{v} \right) \sum_{\substack{a \in \mathcal{P}_i, \\ M \in \mathbb{Z}_{\geq 0}}} \left( \frac{a^M}{M!} \left( \frac{\partial}{\partial a} \right)^M \delta \left( \frac{a}{u} \right) \right) \mathsf{H}_{i,a,M} \\ &= \delta_{ij} \sum_{\substack{a \in \mathcal{P}_i, \\ M \in \mathbb{Z}_{\geq 0}}} \left( \frac{a^M}{M!} \left( \frac{\partial}{\partial a} \right)^M \delta \left( \frac{a}{u} \right) \delta \left( \frac{a}{v} \right) \right) \mathsf{H}_{i,a,M} \\ &= \delta_{ij} \sum_{\substack{a \in \mathcal{P}_i, \\ m \in \mathbb{Z}_{\geq 0}}} \sum_{\substack{b \in \mathcal{P}_j, \\ n \in \mathbb{Z}_{\geq 0}}} \delta_{a,b} \left( \frac{a^m}{m!} \left( \frac{\partial}{\partial a} \right)^m \delta \left( \frac{a}{u} \right) \right) \left( \frac{b^n}{n!} \left( \frac{\partial}{\partial b} \right)^n \delta \left( \frac{b}{v} \right) \right) \mathsf{H}_{i,a,m+n} \end{aligned}$$

using the Leibniz rule. Therefore (4.5) is a sufficient condition for the images  $\theta_{\mathcal{P}}(x_{i,r}^\pm)$  and  $\theta_{\mathcal{P}}(\Phi_{i,r})$  obey the relation (2.12).

To ensure that the relation (2.10) holds for the images  $\mathsf{K}_i = \theta_{\mathcal{P}}(k_i)$  and  $\theta_{\mathcal{P}}(\Phi_{i,r})$  it suffices to impose the linear relation (4.6).

Finally, consider the Serre relations, (2.7). The image under  $\theta_{\mathcal{P}}$  of the left side of (2.7) is

$$\begin{aligned} & \sum_{\pi \in \Sigma_s} \sum_{\substack{a_1, \dots, a_s \in \mathcal{P}_i, b \in \mathcal{P}_j \\ m_1, \dots, m_s, n \in \mathbb{Z}_{\geq 0}}} \left( \frac{b^n}{n!} \left( \frac{\partial}{\partial b} \right)^n \delta \left( \frac{b}{z} \right) \right) \prod_{t=1}^s \left( \frac{a_t^{m_t}}{m_t!} \left( \frac{\partial}{\partial a_t} \right)^{m_t} \delta \left( \frac{a_t}{w_{\pi(t)}} \right) \right) \\ & \times \sum_{r=0}^s (-1)^r \binom{s}{r}_{q_i} \mathbf{E}_{i, a_1, m_1}^{\pm} \cdots \mathbf{E}_{i, a_r, m_r}^{\pm} \mathbf{E}_{j, b, n}^{\pm} \mathbf{E}_{i, a_{r+1}, m_{r+1}}^{\pm} \cdots \mathbf{E}_{i, a_s, m_s}^{\pm}, \\ & = \sum_{\pi \in \Sigma_s} \sum_{\substack{a_1, \dots, a_s \in \mathcal{P}_i, b \in \mathcal{P}_j \\ m_1, \dots, m_s, n \in \mathbb{Z}_{\geq 0}}} \left( \frac{b^n}{n!} \left( \frac{\partial}{\partial b} \right)^n \delta \left( \frac{b}{z} \right) \right) \prod_{t=1}^s \left( \frac{a_{\pi(t)}^{m_{\pi(t)}}}{m_{\pi(t)}!} \left( \frac{\partial}{\partial a_{\pi(t)}} \right)^{m_{\pi(t)}} \delta \left( \frac{a_{\pi(t)}}{w_t} \right) \right) \\ & \times \sum_{r=0}^s (-1)^r \binom{s}{r}_{q_i} \mathbf{E}_{i, a_{\pi(1)}, m_{\pi(1)}}^{\pm} \cdots \mathbf{E}_{i, a_{\pi(r)}, m_{\pi(r)}}^{\pm} \mathbf{E}_{j, b, n}^{\pm} \mathbf{E}_{i, a_{\pi(r+1)}, m_{\pi(r+1)}}^{\pm} \cdots \mathbf{E}_{i, a_{\pi s}, m_{\pi s}}^{\pm}, \end{aligned}$$

and for this to vanish it is sufficient to impose the relations (5.5). But then, as Proposition 5.4 states, given (4.4), the relations (5.5) are equivalent to (4.7).  $\square$

It remains to prove the ‘‘moreover’’ part of Theorem 4.4. We shall need the following lemma.

**Lemma 5.7.** *Suppose  $V$  is a complex vector space. Let  $F_{a,m} \in \text{End}(V)$  be linear maps  $V \rightarrow V$ , where the label  $a$  is drawn from a finite set of distinct points  $\mathcal{P} \subset \mathbb{C}^{\times}$  and where  $m \in \{0, 1, \dots, M\}$  for some fixed  $M \in \mathbb{Z}_{\geq 0}$ . Let  $z$  be an indeterminate. Suppose  $\sum_{a \in \mathcal{P}} \sum_{m=0}^M \left( \frac{a^m}{m!} \left( \frac{\partial}{\partial a} \right)^m \delta \left( \frac{a}{z} \right) \right) F_{a,m} = 0$ . Then  $F_{a,m} = 0$  for all  $a \in \mathcal{P}$  and  $m \in \mathbb{Z}_{\geq 0}$ .*

*More generally, let  $F_{a_1, m_1; a_2, m_2; \dots; a_s, m_s} \in \text{End}(V)$  and let  $z_1, \dots, z_s$  be indeterminates. If*

$$\sum_{a_1, a_2, \dots, a_s \in \mathcal{P}} \sum_{m_1, m_2, \dots, m_s=0}^M F_{a_1, m_1; a_2, m_2; \dots; a_s, m_s} \prod_{t=1}^s \left( \frac{a_t^{m_t}}{m_t!} \left( \frac{\partial}{\partial a_t} \right)^{m_t} \delta \left( \frac{a_t}{z_t} \right) \right) = 0$$

*then the  $F_{a_1, m_1; a_2, m_2; \dots; a_s, m_s}$  are all zero.*

*Proof.* Let

$$X(a) := \left( a^n \binom{n+m-2}{m-1} \right)_{\substack{1 \leq n \leq M|\mathcal{P}| \\ 1 \leq m \leq M}}.$$

We shall establish the identity

$$\det \begin{pmatrix} X(a_1) & X(a_2) & \dots & X(a_{|\mathcal{P}|}) \end{pmatrix} = (a_1 a_2 \dots a_{|\mathcal{P}|})^{\frac{M(M+1)}{2}} \left( \prod_{i < j} (a_j - a_i) \right)^{M^2}. \quad (5.8)$$

The first part of the lemma follows from the fact that this determinant is not zero (consider equating coefficients of  $z^k$  for  $1 \leq k \leq M|\mathcal{P}|$  in the equation given, and letting  $\mathcal{P} = \{a_i^{-1}\}_{1 \leq i \leq |\mathcal{P}|}$ ). The identity (5.8) follows by symmetry arguments analogous to those used in proving the Vandermonde determinant formula. Namely, let  $D = \det \begin{pmatrix} X(a_1) & X(a_2) & \dots & X(a_{|\mathcal{P}|}) \end{pmatrix}$ . This must be equal to a polynomial in the  $a_i$  of order  $1 + 2 + \dots + M|\mathcal{P}| = \frac{M|\mathcal{P}|(M|\mathcal{P}|+1)}{2}$ . On symmetry grounds it must have a zero of order  $M^2$  at  $a_i = a_j$ , for every pair  $i \neq j$ , and a zero of order  $M + \binom{M}{2} = \frac{M(M+1)}{2}$  at  $a_i = 0$  for each  $i$ . Since  $\frac{M|\mathcal{P}|(M|\mathcal{P}|+1)}{2} = M^2 \frac{|\mathcal{P}|(|\mathcal{P}|-1)}{2} + |\mathcal{P}| \frac{M(M+1)}{2}$ , these are all the zeros of  $D$ , and (5.8) must hold up to a constant of proportionality. To see that this constant is unity, consider

the coefficient of  $a_1^{1+2+\dots+M} a_2^{(M+1)+\dots+2M} \dots a_{|\mathcal{P}|}^{((|\mathcal{P}|-1)M+1)+\dots+|\mathcal{P}|M}$ . This term comes only from the block on-diagonal part of the determinant sum. Hence the coefficient is 1 by virtue of the identity

$$\det \left( \binom{n+m+k-2}{m-1} \right)_{1 \leq n, m \leq M} = 1, \quad (5.9)$$

valid for all non-negative integers  $k$  (actually for all  $k \in \mathbb{C}$ ; see for example [CC05], Proposition 3.6, of which (5.9) is a special instance).

The ‘‘more generally’’ part follows by applying the first part  $s$  times.  $\square$

We can now complete the proof of Theorem 4.4.

**Proposition 5.8.** *Let  $V \in \text{Ob}(\mathcal{C}_{\mathcal{P}})$ . Then  $V$  is the pull-back via  $\theta_{\mathcal{P}}$  of a finite-dimensional representation of  $\mathcal{A}$ .*

*Proof.* Proposition 3.7 guarantees that the actions of the generators of  $x_i^{\pm}(z)$  and  $\Phi_i(z)$  on  $V$  are of the form (3.4–3.5) for some linear maps  $E_{i,a,m}^{\pm}$  and  $H_{i,a,m}$ . It is enough to show that these maps (together with the representatives of the  $k_i^{\pm 1}$ ) obey the defining relations of  $\mathcal{A}$ .

Consider the relation (4.4). Since the representatives of  $x_i^{\pm}(z)$  by assumption obey relation (2.6), we have

$$\begin{aligned} 0 &= \sum_{a,b \in \mathcal{P}_i} \sum_{m,n=0}^M \left( \frac{a^m}{m!} \left( \frac{\partial}{\partial a} \right)^m \delta \left( \frac{a}{u} \right) \right) \left( \frac{b^n}{n!} \left( \frac{\partial}{\partial b} \right)^n \delta \left( \frac{b}{v} \right) \right) \\ &\times \left[ (a - bq^{\pm B_{ij}}) E_{i,a,m}^{\pm} E_{j,b,n}^{\pm} + a E_{i,a,m+1}^{\pm} E_{j,b,n}^{\pm} - bq^{\pm B_{ij}} E_{i,a,m}^{\pm} E_{j,b,n+1}^{\pm}, \right. \\ &\left. -(aq^{\pm B_{ij}} - b) E_{i,a,m}^{\pm} E_{j,b,n}^{\pm} + aq^{\pm B_{ij}} E_{i,a,m+1}^{\pm} E_{j,b,n}^{\pm} - b E_{i,a,m}^{\pm} E_{j,b,n+1}^{\pm} \right] \end{aligned}$$

where now the sum is over finitely many values of  $m, n$ , in contrast to (5.7). Therefore Lemma 5.7 applies. Consequently the maps  $E_{i,a,m}^{\pm}$  must indeed satisfy (4.4). The remaining relations work in the same way: for brevity we omit the details, which amount to introducing upper limits  $M$  on the sums in the proof of Proposition 5.6 and replacing each ‘‘sufficient’’ statement with the corresponding ‘‘necessary’’ statement.  $\square$

*Remark 5.9.* In fact we have that, given any  $V \in \text{Ob}(\mathcal{C}_{\mathcal{P}})$ , there is an  $M$  (the  $M$  of Proposition 3.7) such that  $V$  is the pull-back not merely of a representation of  $\mathcal{A}$ , but of the ‘‘truncated’’ algebra  $\mathcal{A}/\mathcal{I}_{M+1}$ . And the proof above uses the fact that, for any given  $M$ , the homomorphism  $U_q(\mathcal{L}\mathfrak{g}) \rightarrow \mathcal{A}/\mathcal{I}_{M+1}$  to the truncated algebra has a right-inverse, as the identity (5.8) ensures. (This right-inverse is far from unique. For each  $i \in I$ , its codomain is spanned by the modes numbered  $-1, -2, \dots, -M|\mathcal{P}_i|$ . But this was a choice: any  $M|\mathcal{P}_i|$  distinct modes would have done.) However, the inverse of the matrix in (5.8) is not stable under changes in  $M$ , in the sense that the inverse of the restriction is not the restriction of the inverse, so we cannot take the inverse limit  $M \rightarrow \infty$  of these matrix inverses, c.f. §4.2. (Nor is it stable under the addition of more points to the set  $\mathcal{P}$ .)

6. PROPERTIES AND FIRST APPLICATIONS OF  $\mathcal{A}$ 

**6.1. Triangular decomposition: weak form.** Let  $\mathcal{A}^\pm$  be the subalgebra of  $\mathcal{A}$  generated by  $(\mathbf{E}_{i,a,m}^\pm)_{i \in I, a \in \mathcal{P}_i, m \in \mathbb{Z}_{\geq 0}}$ , and  $\mathcal{A}^0$  the subalgebra generated by  $(\mathbf{H}_{i,a,m})_{i \in I, a \in \mathcal{P}_i, m \in \mathbb{Z}_{\geq 0}}$  and  $(\mathbf{K}_i^{\pm 1})_{i \in I}$ .

**Proposition 6.1.**  $\mathcal{A} = \mathcal{A}^- \cdot \mathcal{A}^0 \cdot \mathcal{A}^+$ .

*Proof.* Given that  $\mathbf{E}_{i,a,m}^+ \mathbf{E}_{j,b,n}^- \in \mathcal{A}^- \cdot \mathcal{A}^+ + \mathcal{A}^0$  by relation (4.3), it suffices to check that

$$\mathbf{E}_{i,a,m}^+ \mathbf{H}_{j,b,n} \in \mathcal{A}^0 \cdot \mathcal{A}^+ \quad \text{and} \quad \mathbf{H}_{i,a,m} \mathbf{E}_{j,b,n}^- = \mathcal{A}^- \cdot \mathcal{A}^0.$$

Consider the second of these (the first is similar). Whenever  $a \neq bq^{-B_{ij}}$ , it follows from Proposition 5.1. It remains to consider  $\mathbf{H}_{i,a,m} \mathbf{E}_{j,aq^{B_{ij}},n}^-$ . Whenever  $m > 0$ , (4.5) can be used, in the form

$$\begin{aligned} \mathbf{H}_{i,a,m} \mathbf{E}_{j,aq^{B_{ij}},n}^- &= \mathbf{H}_{i,a,m-1} \mathbf{E}_{j,aq^{B_{ij}},n+1}^- \\ &+ (q^{-B_{ij}} - q^{B_{ij}}) \mathbf{E}_{j,aq^{B_{ij}},n}^- \mathbf{H}_{i,a,m-1} + q^{-B_{ij}} \mathbf{E}_{j,aq^{B_{ij}},n}^- \mathbf{H}_{i,a,m} - q^{B_{ij}} \mathbf{E}_{j,aq^{B_{ij}},n+1}^- \mathbf{H}_{i,a,m-1}. \end{aligned}$$

Note that here we are ‘‘solving downwards’’, in the sense that  $\mathbf{H}_{i,a,m} \mathbf{E}_{j,aq^{B_{ij}},n}^-$  is in  $\mathcal{I}_{\max(m,n)}$  and yet we are re-writing it in a form that is not manifestly so (if  $m \leq n$ ). Here it is useful to do so because by recursive application of this relation one has

$$\mathbf{H}_{i,a,m} \mathbf{E}_{j,aq^{B_{ij}},n}^- - \mathbf{H}_{i,a,0} \mathbf{E}_{j,aq^{B_{ij}},n+m}^- \in \mathcal{A}^- \cdot \mathcal{A}^0,$$

and it remains only to consider  $\mathbf{H}_{i,a,0} \mathbf{E}_{j,aq^{B_{ij}},n}^-$ ,  $n \in \mathbb{Z}_{\geq 0}$ . For such terms, (4.5) yields no relation. Instead, one substitutes for  $\mathbf{H}_{i,a,0}$  using relation (4.6):

$$\mathbf{H}_{i,a,0} \mathbf{E}_{j,aq^{B_{ij}},n}^- = \left( \frac{\mathbf{K}_i - \mathbf{K}_i^{-1}}{q_i - q_i^{-1}} - \sum_{b \in \mathcal{P}_i \setminus \{a\}} \mathbf{H}_{i,b,0} \right) \mathbf{E}_{j,aq^{B_{ij}},n}^-.$$

We have  $\sum_{b \in \mathcal{P}_i \setminus \{a\}} \mathbf{H}_{i,b,0} \mathbf{E}_{j,aq^{B_{ij}},n}^- \in \mathcal{A}^- \cdot \mathcal{A}^0$  by the arguments above; and  $\frac{\mathbf{K}_i - \mathbf{K}_i^{-1}}{q_i - q_i^{-1}} \mathbf{E}_{j,aq^{B_{ij}},n}^- \in \mathcal{A}^- \cdot \mathcal{A}^0$  by (4.2).  $\square$

However, since the tuple  $\mathcal{P} = (\mathcal{P}_i)_{i \in I}$  consists of finite sets, we also have the following.

**Proposition 6.2.**  $\mathcal{A} \not\cong_{\mathbb{C}} \mathcal{A}^- \otimes \mathcal{A}^0 \otimes \mathcal{A}^+$ .

*Proof.* Pick an  $i \in I$ . We have

$$\begin{aligned} \mathbf{K}_i \mathbf{E}_{i,a,m}^+ &= q_i^2 \mathbf{E}_{i,a,m}^+ \mathbf{K}_i = q_i^2 \mathbf{E}_{i,a,m}^+ \mathbf{K}_i^{-1} + q_i^2 (q_i - q_i^{-1}) \mathbf{E}_{i,a,m}^+ \sum_{b \in \mathcal{P}_i} \mathbf{H}_{i,b,0} \\ &= q_i^4 \mathbf{K}_i^{-1} \mathbf{E}_{i,a,m}^+ + q_i^2 (q_i - q_i^{-1}) \mathbf{E}_{i,a,m}^+ \sum_{b \in \mathcal{P}_i} \mathbf{H}_{i,b,0} \\ &= q_i^4 \mathbf{K}_i \mathbf{E}_{i,a,m}^+ - q_i^4 (q_i - q_i^{-1}) \sum_{b \in \mathcal{P}_i} \mathbf{H}_{i,b,0} \mathbf{E}_{i,a,m}^+ + q_i^2 (q_i - q_i^{-1}) \mathbf{E}_{i,a,m}^+ \sum_{b \in \mathcal{P}_i} \mathbf{H}_{i,b,0} \end{aligned}$$

Since  $\mathcal{P}_i$  is finite, there exists an  $a \in \mathcal{P}_i$  such that  $aq_i^{-2} \notin \mathcal{P}_i$ . Pick any such  $a$ . Then Proposition 5.1 allows  $\mathbf{E}_{i,a,m}^+ \mathbf{H}_{i,b,0}$  to be reexpressed as a sum of monomials  $\mathbf{H}_{i,b,n} \mathbf{E}_{i,a,m+k}^+$  with  $n, k \in \mathbb{Z}_{\geq 0}$ . On doing so, and re-arranging, one has the relation

$$0 = \frac{1}{q_i - q_i^{-1}} \mathbf{K}_i \mathbf{E}_{i,a,m}^+ + \sum_{b \in \mathcal{P}_i} \sum_{n,k=0}^{\infty} \mathbf{H}_{i,b,n} \mathbf{E}_{i,a,m+k}^+ \frac{a^k}{k!} \left( \frac{\partial}{\partial a} \right)^k \frac{b^n}{n!} \left( \frac{\partial}{\partial b} \right)^n \frac{b}{aq_i^{-2} - b}. \quad (6.1)$$

□

*Remark 6.3.* Another way to derive (6.1) is to consider the image of the relation

$$x_i^+(z)\phi_i^+(1/w) = q_i^{-2} \frac{1 - q_i^2 z/w}{1 - q_i^{-2} z/w} \phi_i^+(1/w)x_i^+(z) \quad (6.2)$$

under the homomorphism  $\theta_{\mathcal{P}}$ . We have

$$\theta_{\mathcal{P}}(\phi_i^+(1/w)) = \mathbf{K}_i + (q_i - q_i^{-1}) \sum_{b \in \mathcal{P}_i} \sum_{n=0}^{\infty} \mathbf{H}_{i,b,n} \frac{b^n}{n!} \left( \frac{\partial}{\partial b} \right)^n \frac{b/w}{1 - b/w}.$$

Comparing the formal partial fraction decompositions of the left- and right-hand sides of (6.2), one recovers (6.1) by considering the coefficients of the poles at  $w = aq_i^{-2}$  (which occur only on the right-hand side, provided  $aq_i^{-2} \notin \mathcal{P}_i$ ).

**6.2. Results and examples in rank 1.** For this subsection, set  $\mathfrak{g} = \mathfrak{sl}_2$ . The Dynkin diagram  $A_1$  has one node only and we omit its label  $i \in I = \{1\}$  throughout. We consider the case  $\mathcal{P} = \{a, aq^2, aq^4, \dots, aq^K\}$  and write, by abuse of notation,

$$\mathbf{E}_{k,m}^+ := \mathbf{E}_{aq^k,m}^+, \quad \mathbf{E}_{k,m}^- := \mathbf{E}_{aq^k,m}^-, \quad \mathbf{H}_{k,m} := \mathbf{H}_{aq^k,m}, \quad \mathbf{Y}_k := \mathbf{Y}_{aq^k}, \quad \mathbf{A}_k := \mathbf{A}_{aq^k}.$$

We have the principal gradation of  $\mathcal{A}$ ,

$$\mathcal{A}^{\pm} = \bigoplus_{r \in \mathbb{Z}_{\geq 1}} \mathcal{A}^{(\pm r)}, \quad \text{where } \mathcal{A}^{(\pm r)} := \{x \in \mathcal{A}^{\pm} : \mathbf{K}x\mathbf{K}^{-1} = q^{\pm 2r}x\}.$$

Let  $B_r$  be the set of all monomials of the form  $\mathbf{E}_{k_1,m_1}^- \mathbf{E}_{k_2,m_2}^- \dots \mathbf{E}_{k_r,m_r}^-$  such that, for each  $t \in \{1, 2, \dots, r-1\}$ ,  $k_t \leq k_{t-1}$  and if  $k_t = k_{t+1}$  then  $m_t < m_{t+1}$ .

**Proposition 6.4.**  $B_r$  is a  $\mathbb{C}$ -basis of  $\mathcal{A}^{(-r)}$ .

*Proof.*  $\mathcal{A}^{(-r)}$  is spanned by the monomials of the form  $\mathbf{E}_{k_1,m_1}^- \mathbf{E}_{k_2,m_2}^- \dots \mathbf{E}_{k_r,m_r}^-$ . By re-writing neighbouring factors according to the rules below, which follow from the defining relation (4.4), any such monomial can be expressed as a linear combination of elements of  $B_r$ .

Rule i):

$$\mathbf{E}_{k,m}^- \mathbf{E}_{k,m}^- \mapsto \frac{1}{1 - q^{-2}} (-\mathbf{E}_{k,m+1}^- \mathbf{E}_{k,m}^- + q^{-2} \mathbf{E}_{k,m}^- \mathbf{E}_{k,m+1}^-).$$

Rule ii): If  $k > l$ , or if  $k = l$  and  $m > n$ , then

$$\begin{aligned} \mathbf{E}_{k,m}^- \mathbf{E}_{l,n}^- \mapsto & \frac{1}{q^k - q^{\ell-2}} (-q^k \mathbf{E}_{k,m+1}^- \mathbf{E}_{l,n}^- + q^{\ell-2} \mathbf{E}_{k,m}^- \mathbf{E}_{l,n+1}^- \\ & + q^{k-2} \mathbf{E}_{l,n}^- \mathbf{E}_{k,m+1}^- - q^{\ell} \mathbf{E}_{l,n+1}^- \mathbf{E}_{k,m}^- + (q^{k-2} - q^{\ell}) \mathbf{E}_{l,n}^- \mathbf{E}_{k,m}^-). \end{aligned}$$

For elements of  $B_r$  no further re-writing is possible, and it is moreover clear that the defining relation (4.4) does not give any linear relations between elements of  $B_r$ . □

Suppose now that  $\gamma = (\gamma(u))$  is a dominant  $\ell$ -weight, cf. §2.3. The irreducible  $L(\gamma)$  is isomorphic to a tensor product of evaluation modules [CP91]. This provides an explicit basis for  $L(\gamma)$ , but one which is not generally compatible with its decomposition into  $\ell$ -weight spaces. In cases when all  $\ell$ -weight spaces have dimension one (the module  $L(\gamma)$  is then called thin/quasi-minuscule, and



in type  $A_1$  this happens precisely when all poles of  $\gamma(u)$  are simple) explicit  $\ell$ -weight bases can be found in [Tho04, YZ11]. However, when  $\gamma(u)$  has poles of higher order, it is a nontrivial task to find  $\ell$ -weight bases of  $L(\gamma)$  and thence, in particular, to determine the Jordan block structure of the generators  $\phi_{\pm r}^{\pm}$ .

Figure 1 shows three examples illustrating the use of the algebra  $\mathcal{A}$  in constructing such  $\ell$ -weight bases. In each case, the graph of the  $q$ -character is shown together with a basis of the representation consisting of vectors in  $B.v$ , where

$$B := \bigcup_r B_r$$

is a basis of  $\mathcal{A}^-$  by the above proposition, and where  $v$  is defined to be a simultaneous eigenvector of  $K$  and  $H_{k,m}$  such that  $E_{k,m}^+.v = 0$  for all  $k, m$ . The eigenvalues  $\lambda$  of  $K$  and  $\lambda_{k,m}$  of  $H_{k,m}$  are read off from the partial fraction decomposition of  $\gamma(u)$  according to – cf. §2.3 and (3.5) –

$$\gamma(u) = \lambda + \sum_{k,m} \lambda_{k,m} \frac{a^m}{m!} \left( \frac{\partial}{\partial a} \right)^m \frac{aq^k u}{1 - aq^k u}. \quad (6.3)$$

In each case, the action of the generators  $H_{k,m}$  and  $E_{k,m}^+$  in this basis can be computed using the defining relations and their known action on the highest weight vector  $v$ . For example, in the module  $L(Y_0^2 Y_2)$ , one has

$$E_{0,0}^+.(E_{0,1}^- E_{2,0}^- .v) = H_{0,1}^- E_{2,0}^- .v = -(q^2 - q^{-2}) E_{2,0}^- H_{0,0} .v = (q^2 - q^{-2}) E_{2,0}^- .v.$$

The action of the generators  $E_{k,m}^-$  is found by direct calculation, working in the induced  $\mathcal{A}$ -module  $\mathcal{A} \otimes_{\mathcal{A}^0 \otimes_{\mathcal{A}^+}} \mathbb{C}v$  whose irreducible quotient is  $L(\gamma)$ . In the first two examples in Figure 1, namely  $L(Y_0 Y_2^2)$  and  $L(Y_0^2 Y_2)$ , one finds that the set of all vectors in  $B.v$  that are not zero in  $L(\gamma)$  form a basis of  $L(\gamma)$ . It is natural to ask whether this property might be true in general (cf. Lusztig's canonical bases [Lus90] – see e.g. Theorem 14.2.5 in [CP94a] or Theorem 11.16 in [Jan96]). The third example,  $L(Y_0^3 Y_0^2)$ , shows, however, that this is not so. The  $\ell$ -weight space of  $\ell$ -weight  $Y_0^2 Y_4^{-1}$  has dimension 4, but one finds that 5 vectors of  $B.v \subset \mathcal{A} \otimes_{\mathcal{A}^0 \otimes_{\mathcal{A}^+}} \mathbb{C}v$  are not singular, namely

$$E_{0,0}^- E_{2,0}^- .v, \quad E_{0,0}^- E_{2,1}^- .v, \quad E_{0,1}^- E_{2,0}^- .v, \quad E_{0,1}^- E_{2,1}^- .v, \quad E_{0,2}^- E_{2,0}^- .v.$$

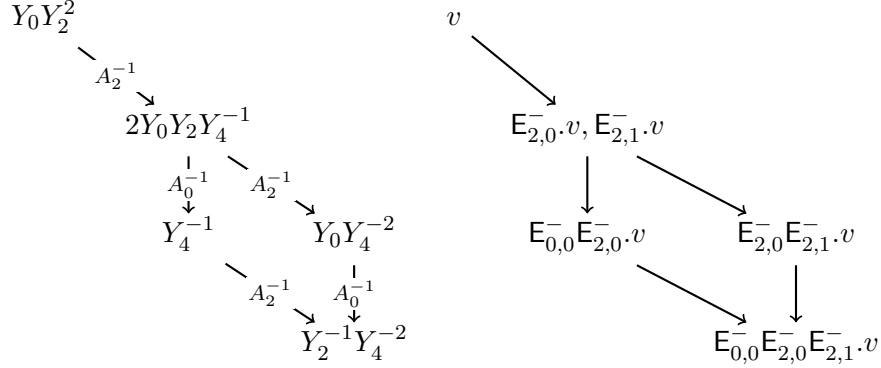
Some linear combination must therefore be singular. And indeed,  $E_{0,0}^+ E_{0,2}^- E_{2,0}^- .v = H_{0,2} E_{2,0}^- .v = H_{0,1} E_{2,1}^- .v$ , while also  $E_{0,0}^+ E_{0,1}^- E_{2,1}^- .v = H_{0,1} E_{2,1}^- .v$ . In this way one checks that  $(E_{0,2}^- E_{2,0}^- - E_{0,1}^- E_{2,1}^-) .v \notin B.v$  is a singular vector. Thus  $E_{0,2}^- .(E_{2,0}^- .v) = E_{0,1}^- E_{2,1}^- .v$  in  $L(Y_0^3 Y_2^2)$ .

**6.3. Comment on truncations and triangular decompositions.** Recall – see Remark 5.9 – that for every  $V \in \text{Ob}(\mathcal{C}_{\mathcal{P}})$  there is an  $M$  such that  $V$  is the pull-back of a representation of the truncated algebra  $\mathcal{A}/\mathcal{I}_{M+1}$ . Here we consider how highest weight  $U_q(\mathcal{L}\mathfrak{g})$ -modules can fail to be pull-backs of highest weight  $\mathcal{A}/\mathcal{I}_N$ -modules if  $N$  is too small.

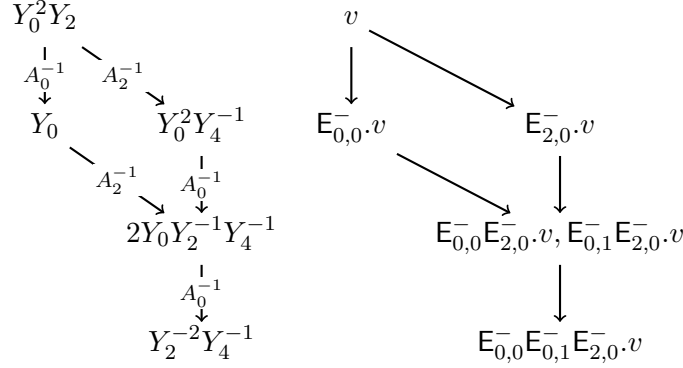
Let  $\gamma = (\gamma_i(u))_{i \in I}$  be the highest  $\ell$ -weight. If  $\gamma_i(u)$  has any pole of order  $> N$  then the problem is obvious, cf. (6.3). For instance, the first example in Figure 1 is not a pull-back of a representation of  $\mathcal{A}/\mathcal{I}_1$ . But there are more subtle possibilities, as the following pair of examples illustrate.

We set  $\mathcal{P} = \{aq^k : 0 \leq k \leq K\}$  for some  $a \in \mathbb{C}^\times$  and  $K \in \mathbb{Z}_{\geq 0}$ .

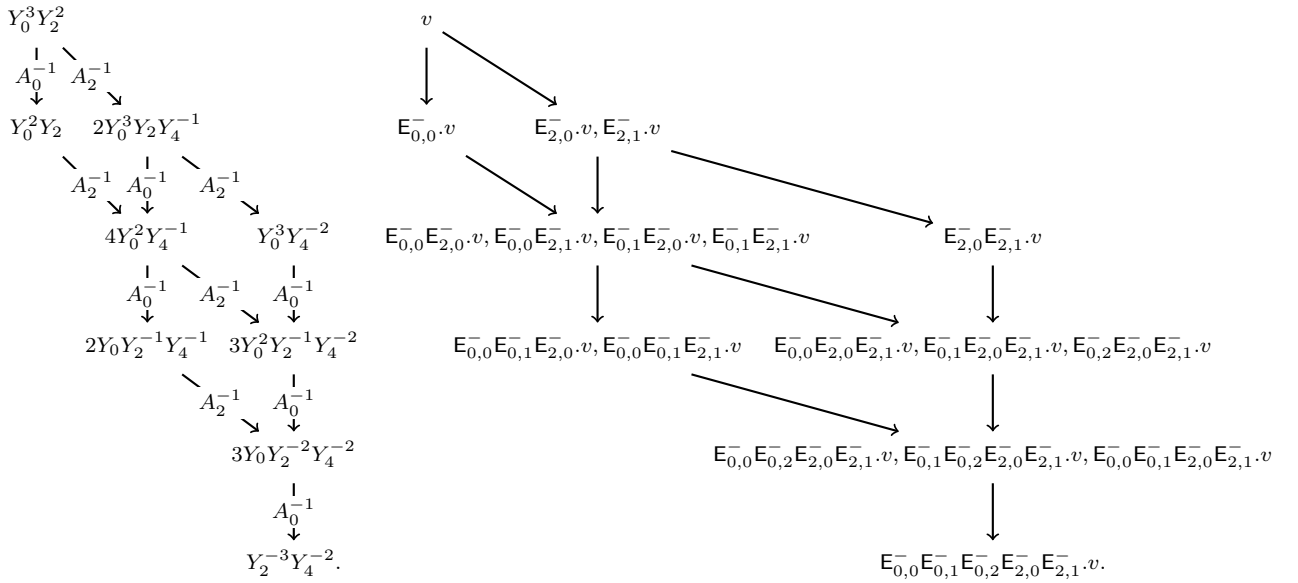
$$L(Y_0Y_2^2): K.v = q^3v, \quad H_{2,0}.v = (q^2 + 1 + q^{-2})v, \quad H_{2,1}.v = (q^2 - q^{-2})v.$$



$$L(Y_0^2Y_2): K.v = q^3v, \quad H_{0,0}.v = -v, \quad H_{2,0}.v = (q^2 + 2 + q^{-2})v.$$



$$L(Y_0^3Y_2^2): K.v = q^5v, \quad H_{0,0}.v = v, \quad H_{2,0}.v = (q^4 + q^2 + q^{-2} + q^{-4})v, \quad H_{2,1}.v = (q^4 + 2q^2 - 2q^{-2} - q^{-4})v.$$


 FIGURE 1. Examples of  $\ell$ -weight bases of simple  $U_q(\mathcal{Lsl}_2)$ -modules. See §6.2.

- Consider  $L(Y_{1,a}Y_{2,aq})$  in type  $A_2$ . All poles of the functions  $(Y_{1,a}Y_{2,aq})_i(u)$ ,  $i = 1, 2$ , are simple. Yet  $L(Y_{1,a}Y_{2,aq})$  is not thin (indeed,  $\chi_q(L(Y_{1,a}Y_{2,aq})) = Y_{1,a}Y_{2,aq} + Y_{2,aq}^2 Y_{1,aq}^{-1} + 2Y_{2,aq}Y_{2,aq}^{-1} + \dots$ ) so it should not be a pull-back of a representation of  $\mathcal{A}/\mathcal{I}_1$ . Suppose it were. Then this representation would have to be highest weight with highest weight vector  $v$  such that  $H_{1,a,0} \cdot v = v$ ,  $H_{2,aq,0} \cdot v = v$  and all others zero. Now, in  $\mathcal{A}/\mathcal{I}_1$ , one has the relation  $E_{1,a,0}^- H_{2,aq,0} = 0$ . Hence  $E_{1,a,0}^- H_{2,aq,0} \cdot v = 0$  and therefore  $E_{1,a,0}^- \cdot v = 0$ . But then  $H_{1,a,0} \cdot v = [E_{1,a,0}^+, E_{1,a,0}^-] \cdot v = 0$ , which is a contradiction unless  $v = 0$ .
- Similarly, consider  $L(Y_{1,a}Y_{3,aq^2})$  in type  $A_3$ . It is generated by a highest weight vector  $v$  such that  $H_{1,a,0} \cdot v = H_{3,aq^2,0} \cdot v = v$ . Hence  $E_{2,aq,0}^- E_{1,a,0}^- \cdot v = E_{2,aq,0}^- E_{1,a,0}^- H_{3,aq^2,0} \cdot v = E_{2,aq,0}^- H_{3,aq^2,0} E_{1,a,0}^- \cdot v = 0$  since  $E_{2,aq,0}^- H_{3,aq^2,0} = 0$  in  $\mathcal{A}/\mathcal{I}_1$ . But, once we compute  $H_{2,aq,0} E_{1,a,0}^- \cdot v = E_{1,a,0}^- \cdot v$ , we find that  $E_{1,a,0}^+ E_{2,aq,0}^+ E_{2,aq,0}^- E_{1,a,0}^- \cdot v = E_{1,a,0}^+ E_{1,a,0}^- \cdot v = H_{1,a,0} \cdot v = v$  which again is a contradiction unless  $v = 0$ .

What underlies these examples is the fact that in the truncated algebras  $\mathcal{A}/\mathcal{I}_N$  one has relations like  $E_{i,aq^{Bi},N-1}^- H_{i,a,N-1} = 0$ . This relation holds for all  $a$  such that  $\{a, aq^{Bi}\} \subset \mathcal{P}_i$ . (By contrast, in Proposition 6.1, the failure of triangularity had to do with “end” points, i.e. to points  $a \in \mathcal{P}_i$  such that for example  $aq_i^{-2} \notin \mathcal{P}_i$ .)

**6.4. A “rational limit” of  $\mathcal{A}$ .** Just as Yangians are related to quantum loop algebras, [GT10, GM10], one might expect there to be a “rational” version of  $\mathcal{A}$ , whose defining relations do not involve a deformation parameter. Here we merely note that a natural candidate can be defined.

For this subsection (only) let  $q = e^h$  with  $h$  an indeterminate. Pick finite sets  $\overline{\mathcal{P}}_i \subset \mathbb{C}$  for each  $i \in I$  and let  $\mathcal{P}_i = \{q^k : k \in \overline{\mathcal{P}}_i\}$ . We define new generators  $\overline{E}_{i,k,m}^\pm$ ,  $\overline{H}_{i,k,m}^\pm$  and  $\overline{H}_i$  by

$$\overline{E}_{i,k,m}^\pm := h^m \overline{E}_{i,q^k,m}^\pm, \quad \overline{H}_{i,k,m}^\pm := h^m \overline{H}_{i,q^k,m}^\pm, \quad \overline{K}_i := e^{hr_i \overline{H}_i}.$$

Note that  $\frac{\overline{K}_i - \overline{K}_i^{-1}}{q_i - q_i^{-1}} = \overline{H}_i + O(h)$ . If one then keeps the leading order in  $h$  of the defining relations in the definition, §4.1, of  $\mathcal{A}$ , one obtains the following definition. Let  $\overline{\mathcal{A}}$  be the associative unital algebra over  $\mathbb{C}$  generated by

$$\overline{E}_{i,k,m}^\pm, \quad \overline{H}_{i,k,m}, \quad \overline{H}_i, \quad i \in I, \quad k \in \overline{\mathcal{P}}_i, \quad m \in \mathbb{Z}_{\geq 0},$$

subject to the following relations for all  $i, j \in I$ ,  $k, \ell \in \overline{\mathcal{P}}_i$  and  $m, n \in \mathbb{Z}_{\geq 0}$ :

$$\begin{aligned} [\overline{H}_i, \overline{E}_{j,k,m}^\pm] &= \pm C_{ij} \overline{E}_{j,k,m}^\pm, & [\overline{H}_i, \overline{H}_j] &= 0 \\ [\overline{H}_i, \overline{H}_{j,k,m}] &= 0, & [\overline{H}_{i,k,m}, \overline{H}_{j,\ell,n}] &= 0 \\ [\overline{E}_{i,k,m}^+, \overline{E}_{j,\ell,n}^-] &= \delta_{ij} \delta_{k\ell} \overline{H}_{i,k,m+n} \\ (k - \ell \mp B_{ij}) \overline{E}_{i,k,m}^\pm \overline{E}_{j,\ell,n}^\pm &+ \overline{E}_{i,k,m+1}^\pm \overline{E}_{j,\ell,n}^\pm - \overline{E}_{i,k,m}^\pm \overline{E}_{j,\ell,n+1}^\pm \\ &= (k \pm B_{ij} - \ell) \overline{E}_{j,\ell,n}^\pm \overline{E}_{i,k,m}^\pm + \overline{E}_{j,\ell,n}^\pm \overline{E}_{i,k,m+1}^\pm - \overline{E}_{j,\ell,n+1}^\pm \overline{E}_{i,k,m}^\pm \\ (k - \ell \mp B_{ij}) \overline{H}_{i,k,m} \overline{E}_{j,\ell,n}^\pm &+ \overline{H}_{i,k,m+1} \overline{E}_{j,\ell,n}^\pm - \overline{H}_{i,k,m} \overline{E}_{j,\ell,n+1}^\pm \\ &= (k \pm B_{ij} - \ell) \overline{E}_{j,\ell,n}^\pm \overline{H}_{i,k,m} + \overline{E}_{j,\ell,n}^\pm \overline{H}_{i,k,m+1} - \overline{E}_{j,\ell,n+1}^\pm \overline{H}_{i,k,m} \end{aligned}$$

$$\sum_{k \in \mathbb{Z}} \bar{H}_{i,k,0} = \bar{H}_i,$$

together with Serre relations

$$\sum_{r=0}^s (-1)^r \binom{s}{r} \bar{E}_{i,k \mp B_{ij} \mp r B_{ii}, m_{r+1}}^{\pm} \cdots \bar{E}_{i,k \mp B_{ij} \mp (s-1) B_{ii}, m_s}^{\pm} \bar{E}_{j,k,n}^{\pm} \bar{E}_{i,k \mp B_{ij}, m_1}^{\pm} \cdots \bar{E}_{i,k \mp B_{ij} \mp (r-1) B_{ii}, m_r}^{\pm} = 0,$$

where  $s = 1 - C_{ij}$ , for each  $i \neq j$  such that  $B_{ij} \neq 0$ , for all  $m_1, \dots, m_s, n \in \mathbb{Z}_{\geq 0}$ , and for all  $k \in \bar{\mathcal{P}}_j$  such that  $\{k \mp B_{ij} \mp (t-1)B_{ii} : 1 \leq t \leq s\} \subset \bar{\mathcal{P}}_i$ .

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