THE MAXIMAL SUBGROUPS AND THE COMPLEXITY OF THE FLOW SEMIGROUP OF FINITE (DI)GRAPHS

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Dedicated to John Rhodes on the occasion of his 80th birthday.

Abstract. The flow semigroup, introduced by John Rhodes, is an invariant for digraphs and a complete invariant for graphs. After collecting together previous partial results, we refine and prove Rhodes's conjecture on the structure of the maximal groups in the flow semigroup for finite, antisymmetric, strongly connected digraphs.

Building on this result, we investigate and fully describe the structure and actions of the maximal subgroups of the flow semigroup acting on all but k points for all finite digraphs and graphs for all $k \geq 1$. A linear algorithm (in the number of edges) is presented to determine these so-called 'defect k groups' for any finite (di)graph.

Finally, we prove that the complexity of the flow semigroup of a 2-vertex connected (and strongly connected di)graph with n vertices is $n-2$, completely confirming Rhodes's conjecture for such (di)graphs.

1. Introduction

John Rhodes in [9] introduced the *flow semigroup*, an invariant for graphs and digraphs (that is, isomorphic flow semigroups correspond to isomorphic digraphs). In the case of graphs, this is a complete invariant determining the graph up to isomorphism. The flow semigroup is the semigroup of transformations of the vertices generated by elementary

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collapsings corresponding to the edges of the (di)graph. An elementary collapsing corresponding to the directed edge uv is a map on the vertices moving u to v and acting as the identity on all other vertices. (See Section 2 for all the precise definitions.)

A maximal subgroup of this semigroup for a finite (di)graph $D = (V_D, E_D)$ acts by permutations on all but k of its vertices ($1 \leq k \leq |V_D| - 1$) and is called a "defect k group". The set of defect k groups of a (di)graph is also an invariant. For each fixed k, they are all isomorphic to each other in the case of (strongly) connected (di)graphs. Rhodes formulated a conjecture on the structure of these groups for strongly connected digraphs whose edge relation is anti-symmetric in [9, Conjecture 6.51i (2)–(4)]. We show that his conjecture was correct, and we prove it here in sharper form. Moreover, extending this result, we fully determine the defect k groups for all finite graphs and digraphs.

Rhodes further conjectured [9, Conjecture 6.51i (1)] that the Krohn–Rhodes complexity of the flow semigroup of a strongly connected, anti-symmetric digraph D on n vertices is $n-2$. We confirm this conjecture when the digraph is 2-vertex connected, and bound the complexity in the remaining cases.

The structure of the argument is as follows. First, a maximal group in the flow semigroup of a digraph D is the direct product of maximal groups of the flow semigroups of its strongly connected components. Thus one needs only to consider strongly connected digraphs. It turns out, that if D is a strongly connected digraph, then the defect k group (up to isomorphism) does not depend on the choice of the vertices it acts on. Furthermore, for a strongly connected digraph, its flow semigroup is the same as the flow semigroup of the simple graph obtained by "forgetting" the direction of the edges. This is detailed in Section 2 and is based on [9, p. 159–169]. Thus, one only needs to consider the defect k groups of the flow semigroup for simple connected graphs.

In Section 3 we list some useful lemmas and determine the defect k group of a cycle. In Section 4 we prove that the defect 1 group of arbitrary simple connected graph is the direct product of the defect 1 groups of its 2-vertex connected components. The defect 1 group of an arbitrary 2-vertex connected graph Γ has been determined by Wilson [15]. He proved that the defect 1 group is either A_{n-1} or S_{n-1}, unless Γ is a cycle or the exceptional graph displayed in Figure 1.

In particular, Rhodes's conjecture (as phrased for strongly connected, antisymmetric digraphs in [9, Conjecture 6.51i (2)]) about the defect 1 group holds, and more generally: the defect 1 group of the flow semigroup of a simple connected graph is indeed the product of cyclic, alternating and symmetric groups of various orders. A straightforward linear algorithm is given to determine the direct components of the defect 1 group of an arbitrary connected graph (see Section 6).

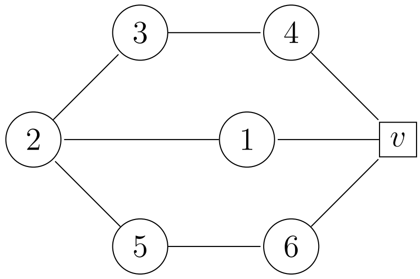


FIGURE 1. Exceptional graph

In Section 5 we determine the defect k groups ($k \geq 2$) of arbitrary graphs by considering the so-called maximal k-subgraphs (maximal subgraphs for which the defect k group is the full symmetric group) and prove that the defect k group of a graph is the direct product of the defect k groups of the maximal k-subgraphs (i.e. of full symmetric groups). In Section 6 we provide a linear algorithm (in the number of edges of Γ) to determine the maximal k-subgraphs of an arbitrary connected graph. Finally, in Section 7 we confirm [9, Conjecture 6.51i (1)] about the Krohn–Rhodes complexity of digraphs when the digraph is 2-vertex connected, and we prove some bounds on the complexity of the flow semigroup in the remaining cases. (See Section 7 for the definition of Krohn–Rhodes complexity.)

We have collected all these results into the following main theorem.

Theorem 1.

(1) *Let D be a digraph, then every maximal subgroup of S_D is (isomorphic to) the direct product of maximal subgroups of S_{D_i}, where the D_i are the strongly connected components of D.*

(2) *Let D be a strongly connected digraph. Let $V_k, V_k' \subseteq D$ be subsets of nodes such that $|V_k| = |V_k'| = k$. Let $G_{k,V_k}, G_{k,V_k'}$ be the defect k groups acting on $V \setminus V_k$ and $V \setminus V_k'$, respectively. Then $G_{k,V_k} \simeq G_{k,V_k'}$ as permutation groups.*

(2^{r}) *Let D be a strongly connected digraph, and Γ_D be the graph obtained from D by forgetting the direction of the edges in D. Then $S_D = S_{\Gamma_D}$.*

(3) *Let Γ be a simple connected graph of n vertices, and let $\Gamma_1, \ldots, \Gamma_m$ be its 2-vertex connected components. Then the defect 1 group of Γ is the direct product of the defect 1 groups of Γ_i ($1 \leq i \leq m$).*

(4) *Let Γ be a 2-vertex connected simple graph with $n \geq 2$ vertices. Then the defect 1 group of Γ is isomorphic (as a permutation group) to*

   (a) *the cyclic group Z_{n-1} if Γ is a cycle;*

   (b) *$S_5 \simeq PGL_2(5)$ acting sharply 3-transitively on 6 points, if Γ is the exceptional graph (see Figure 1);*

(c) *S_{n-1} or A_{n-1}, otherwise, where the defect 1 group is A_{n-1} if and only if Γ is bipartite.*

(4^{c}) *Let Γ be a 2-vertex connected simple graph with $n \geq 2$ vertices. Then the complexity of S_Γ is $\#_G(S_\Gamma) = n-2$.*

(4^{cc}) *Let Γ be a 2-edge connected simple graph with $n \geq 2$ vertices. Then for the complexity of S_Γ we have $n-3 \leq \#_G(S_\Gamma) \leq n-2$.*

(5) *Let $k \geq 2$, Γ be a simple connected graph of n vertices, $n > k$.*

(a) *If Γ is a cycle, then its defect k group is the cyclic group Z_{n-k}.*

(b) *Otherwise, let $\Gamma_1, \dots, \Gamma_m$ be the maximal k-subgraphs of Γ, and let Γ_i have n_i vertices. Then the defect k group of Γ is the direct product of the defect k groups of Γ_i ($1 \leq i \leq m$), thus it is isomorphic (as a permutation group) to*

$$S_{n_1-k} \times \cdots \times S_{n_m-k}.$$

Our main contribution to Theorem 1 are items (3), (4^{c}), (4^{cc}) and (5). Items (1), (2) and (2^{r}) (among some basic definitions and notations) are detailed in Section 2 and are based on [9, p. 159–169]. In Section 3 we list some useful lemmas and determine the defect k group of a cycle. Item (3) is proved in Section 4, while item (4) has already been proved by Wilson [15]. Then in Section 5 we prove item (5). In Section 6 we provide a linear algorithm (in the number of edges of Γ) to determine the maximal k-subgraphs of an arbitrary connected graph to help putting item (5) more into context. Finally, items (4^{c}) and (4^{cc}) are proved in Section 7.

East, Gadouleau and Mitchell [6] are currently looking into other properties of flow semigroups. In particular, they provide a linear algorithm (in the number of vertices of a digraph) for whether or not the flow semigroup contains a cycle of length m for a fixed positive integer m. Furthermore, they classify all those digraphs whose flow semigroups have any of the following properties: inverse, completely regular, commutative, simple, 0-simple, a semilattice, a rectangular band, congruence-free, is $\mathcal{K}$-trivial or $\mathcal{K}$-universal, where $\mathcal{K}$ is any of Green's $\mathcal{H}$-, $\mathcal{L}$-, $\mathcal{R}$-, or $\mathcal{J}$-relation, and when the flow-semigroup has a left, right, or two-sided zero.

Rhodes's original conjecture [9, Conjecture 6.51i] is about strongly connected, antisymmetric digraphs. By [11] a strongly connected antisymmetric digraph becomes a 2-edge connected graph after forgetting the directions. Therefore Theorem 1 almost completely settles Rhodes's conjecture [9, Conjecture 6.51i]. To completely settle the last remaining part of Rhodes's conjecture [9, Conjecture 6.51i (1)], one should find the complexity of the flow semigroups for the rest of the 2-edge connected graphs.

Problem 1. Determine the complexity of S_Γ for a 2-edge connected graph Γ which is not 2-vertex connected.

The smallest such graph is the "bowtie" graph:

Problem 2. Let Γ be the graph with vertex set $\{ u, v, w, x, y \}$ and edge set $\{ uv, vw, wu, wx, xy, yw \}$. Determine the complexity of S_Γ.

Ultimately, the goal is the determine the complexity for all flow semigroups.

Problem 3. Determine the complexity of S_Γ for an arbitrary finite graph (or digraph) Γ.

2. Flow semigroup of digraphs

For notions in graph theory we refer to [4, 7], in group theory to [12] in permutation groups to [1, 5], in semigroup theory to [2, 3].

A *semigroup* is a set with a binary associative multiplication. A *transformation* on a set X is a function $s\colon X \to X$. It *operates* (or *acts*) *on* X by mapping each $x \in X$ to some $x \cdot s \in X$. Here we write $x \cdot s$ or xs for transformation s applied to $x \in X$. A *transformation semigroup* S is a set of transformations $s \in S$ on some set X such that S is closed under (associative) function composition. Also, S itself is then said to operate or *to act on* the set X. Note that in this paper functions act on the right, therefore transformations are multiplied from left to right. Denoting by ss' the transformation of X obtained by first applying s and then s', we have $x \cdot ss' = (x \cdot s) \cdot s'$. If a semigroup element s acts on a set X, and for some $Y \supseteq X$ the action of s is not defined on $Y \setminus X$, then we may consider s acting on Y, as well, with the identity action on $Y \setminus X$.

A *permutation group* is a nonempty transformation semigroup G that contains only permutations and such that that if $g \in G$ then the inverse permutation g^{-1} is also in G. Furthermore, for a set $Y \subseteq X$ and a transformation s on X define

$$Ys = \{ ys \mid y \in Y \} .$$

A *subgroup* G of a transformation semigroup S is a subset of S whose transformations satisfy the (abstract) group axioms. It is not hard to show that if S is a transformation semigroup acting on X, then G contains a (unique) idempotent $e^2 = e$ (which does not generally act as the identity map on X), and furthermore distinct elements of G when restricted to Xe are distinct, permute Xe, and comprise a permutation group acting on Xe (see [9, p. 49]).

A *digraph* (V, E) is a set of *nodes* (or *vertices*) V, and a binary relation $E \subseteq V \times V$. An element $e = (u, v) \in E$ is called a *directed edge* from node u to node v, and also denoted uv. A *loop-edge* is an edge from a vertex to itself. A *graph* (V, E) is a set of nodes V and a symmetric binary relation $E \subseteq V \times V$. If $(u, v) \in E$, then uv is called an (undirected) edge. Such a graph is called *simple* if it has no loop-edges. *In this paper we consider only digraphs without loop-edges* *and simple graphs.* A *walk* is a sequence of vertices $(v_1, \ldots, v_n)$ such that $v_i v_{i+1}$ is a (directed) edge for all $1 \leq i \leq n-1$. By *cycle* we will mean a simple cycle, that is a closed walk with no repetition of vertices except for the starting and ending vertex. A *path* is a walk with no repetition of vertices. A (di)graph $\Gamma = (V, E)$ is (strongly) connected if there is a path from u to v for all distinct $u, v \in V$. By *subgraph* $\Gamma' = (V', E') \subseteq \Gamma$ we mean a graph for which $V' \subseteq V$, $E' \subseteq E$. If Γ' is an *induced subgraph*, that is E' consists of all edges from E with both endpoints in V', then we explicitly indicate it. A strongly connected component of a digraph Γ is a maximal strongly connected subgraph of Γ.

For a digraph $D = (V_D, E_D)$ without any loop-edges, the *flow semigroup* $S = S_D$ is the semigroup of transformations acting on V_D defined by

$$S = S_D = \langle e_{uv} \mid uv \in E_D \rangle ,$$

where e_{uv} is the *elementary collapsing* corresponding to the directed edge $uv \in E_D$, that is, for every $x \in V_D$ we have

$$x \cdot e_{uv} = xe_{uv} = \begin{cases} v, & \text{if } x = u, \\ x, & \text{otherwise.} \end{cases}$$

Thus, the flow semigroup of a (di)graph D is generated by idempotents (elementary collapsings) corresponding to the edges of D. The flow semigroup S_D is also called the *Rhodes semigroup of the (di)graph.*

A maximal subgroup of S_D is a subgroup that is not properly contained in any other subgroup of S_D. In order to determine the maximal subgroups of S_D, one can make several reductions by [9, Proposition 6.51f]. First, one only needs to consider the maximal subgroups of S_{D_i} for the strongly connected components D_i of D. Strongly connected components are maximal induced subgraphs such that any vertex can be reached from any other vertex by a directed path.

Lemma 2 ([9, Proposition 6.51f (1)])**.** *Let D be a digraph, then every maximal subgroup of S_D is (isomorphic to) the direct product of maximal subgroups of S_{D_i}, where the D_i are the strongly connected components of D.*

This is (1) of Theorem 1. An element $s \in S$ is *of defect k* if $|V_D s| = |V_D| - k$. Let $V_k = \{ v_1, v_2, \ldots, v_k \} \subseteq V_D$. The *defect k group G_{k,V_k}* associated to V_k (called the *defect set*) is generated by all elements of S restricted to $V_D \setminus V_k$ which permute the elements of $V_D \setminus V_k$ and move elements of V_k to elements of $V_D \setminus V_k$:

$$G_{k,V_k} = \left\langle s \restriction_{V_D \setminus V_k} : \, s \in S, (V_D \setminus V_k)s = V_D \setminus V_k, V_k s \subseteq V_D \setminus V_k \right\rangle ,$$

where $s \restriction_{V_D \setminus V_k}$ denotes the restriction of the transformation s onto the set $V_D \setminus V_k$. Now, G_{k,V_k} is a permutation group acting on $V_D \setminus V_k$.

For this reason $V_D \setminus V_k$ is called the *permutation set* of G_{k,V_k}, and the elements of G_{k,V_k} are sometimes called *defect k permutations*. Furthermore, if the defect set contains only one vertex v, then by abuse of notation we write *defect v* or *defect point v* instead of defect $\{ v \}$. In general, the defect k group G_{k,V_k} can depend on the choice of V_k. However, by [9, Proposition 6.51f (2)] it turns out that if the graph is strongly connected then the defect k group G_k is unique up to isomorphism.

Lemma 3 ([9, Proposition 6.51f (2)]). *Let D be a strongly connected digraph. Let $V_k, V'_k \subseteq V_D$ be subsets of nodes such that $|V_k| = |V'_k| = k$. Then the action of G_{k,V_k} on $V_D \setminus V_k$ is equivalent to that of G_{k,V'_k} on $V_D \setminus V'_k$. That is, $G_{k,V_k} \simeq G_{k,V'_k}$ as permutation groups.*

This is (2) of Theorem 1. By Lemma 3, we may write G_k instead of G_{k,V_k} without any loss of generality. Furthermore, the case of strongly connected graphs can be reduced to the case of simple graphs. Let $\Gamma = (V, E)$ be a simple (undirected) graph, we define S_Γ by considering Γ as a directed graph where every edge is directed both ways. Namely, let $D_\Gamma = (V, E_D)$ be the directed graph on vertices V such that both $uv \in E_D$ and $vu \in E_D$ if and only if the undirected edge $uv \in E$. Then let $S_\Gamma = S_{D_\Gamma}$.

Furthermore, for every digraph $D = (V_D, E_D)$, one can associate an undirected graph Γ by "forgetting" the direction of edges in D. Precisely, let $\Gamma_D = (V_D, E)$ be the undirected graph such that $uv \in E$ if and only if $uv \in E_D$ or $vu \in E_D$. The following lemma due to Nehaniv and Rhodes shows that if a digraph D is strongly connected then the semigroup S_D corresponding to D and the semigroup S_{Γ_D} corresponding to the simple graph Γ_D are the same. Moreover, Lemma 4 immediately implies that the transformation semigroup S_D is an invariant for digraphs and a complete invariant for (simple) graphs: That is, isomorphic digraphs have the isomorphic flow semigroups, and graphs are isomorphic if and only if their flow semigroups are isomorphic as transfromation semigroups.

Lemma 4 ([9, Lemma 6.51b]). *Let D be an arbitrary digraph. Then*

$$e_{ab} \in S_D \iff \begin{cases} a \to b \text{ is an edge in } D, \text{ or} \\ b \to a \text{ is an edge in a directed cycle in } D. \end{cases}$$

In particular, if D is strongly connected then $S_D = S_{\Gamma_D}$.

Proof. Let $b \to a \to u_1 \to \cdots \to u_{n-1} \to b$ be a directed cycle in D. Then an easy calculation shows that

$$e_{ab} = \left(e_{ba} e_{u_{n-1}b} e_{u_{n-2}u_{n-1}} \ldots e_{u_1u_2} e_{au_1}\right)^n .$$

For the other direction, assume $e_{ab} = e_{uv}s$ for some $s \in S_D$. Then $e_{uv}s$ moves u and v to the same vertex, while e_{ab} moves only a and b to the same vertex. Thus $\{ a, b \} = \{ u, v \}$. □

This is (2^{r}) of Theorem 1. Therefore, in the following we only consider simple, connected, undirected graphs $\Gamma = (V, E)$, that is no self-loops or multiple edges are allowed. Furthermore, Γ is 2-edge connected if removing any edge does not disconnect Γ. Rhodes's conjecture [9, Conjecture 6.51i (2)–(4)] is about strongly connected, antisymmetric digraphs. Note that by [11] a strongly connected antisymmetric digraph becomes a 2-edge connected graph after forgetting the directions.

Let us fix some notation. The letters k, l, m and n will denote nonnegative integers. The number of vertices of Γ is usually denoted by n, while k will denote the size of the defect set. Usually we denote the defect k group of a graph Γ by G_k or G_Γ, depending on the context. We try to heed the convention of using u, v, w, x, y as vertices of graphs, V as the set of vertices, E as the set of edges. Furthermore, the flow semigroup is mostly denoted by S, its elements are denoted by s, t, g, h, p, q. The cyclic group of m elements is denoted by Z_m.

We will need the notion of an open ear, and open ear decomposition.

Definition 5. Let Γ be an arbitrary graph, and let Γ' be a proper subgraph of Γ. A path $(u, c_1, \ldots, c_m, v)$ is called a Γ'-*ear* (or *open ear*) with respect to Γ, if $u, v \in \Gamma'$, $u \neq v$, and either $m = 0$ and the edge $uv \notin \Gamma'$, or $c_1, \ldots, c_m \in \Gamma \setminus \Gamma'$. An *open ear decomposition* of a graph is a partition of its set of edges into a sequence of subsets, such that the first element of the sequence is a cycle, and all other elements of the sequence are open ears of the union of the previous subsets in the sequence.

A connected graph Γ with at least k vertices is *k-vertex connected* if removing any $k-1$ vertices does not disconnect Γ. By [14] a graph is 2-vertex connected if and only if it is a single edge or it has an open ear decomposition.

3. Preliminaries

Let $\Gamma = (V, E)$ be a simple, connected (undirected) graph, and for every $1 \leq k \leq |V|-1$, let G_k denote its defect k group for some $V_k \subseteq V$, $|V_k| = k$. Let $S = S_\Gamma$ be the flow semigroup of Γ. The following is immediate.

Lemma 6 ([9, Fact 6.51c]). *Let $s \in S$ be of defect k. If se_{uv} is of defect k, as well, then $u \notin Vs$ or $v \notin Vs$.*

Furthermore, it is not too hard to see that every defect 1 permutation arises from the permutations generated by cycles (in the graph) containing the defect point.

Lemma 7 ([9, Proposition 6.51e]). *Let Γ be a connected graph, and let G_1 denote its defect 1 group, such that the defect point is $v \in V$. Then*

$$G_1 = \langle (u_1, \ldots, u_k) \text{ as permutation} \mid (u_1, \ldots, u_k, v) \text{ is a cycle in } \Gamma \rangle .$$

These yield that the defect k group of the n-cycle graph is cyclic, proving items (4a) and (5a) of Theorem 1:

Lemma 8. *The defect k group of the n-cycle is isomorphic to Z_{n-k}.*

Proof. Let $x_1, x_2, \dots x_n$ be the consecutive elements of the cycle $\Gamma = (V, E)$. If $s \in S$ is an element of defect k then by Lemma 6 we have that $se_{x_i x_{i+1}}$ is of defect k if and only if $x_i \notin Vs$ or $x_{i+1} \notin Vs$. This means that if $u_1, u_2, \dots u_{n-k}$ are the consecutive elements of Vs in the cycle and $se_{x_i x_{i+1}}$ is of defect k, as well, then

$$u_1 e_{x_i x_{i+1}}, u_2 e_{x_i x_{i+1}}, \dots, u_{n-k} e_{x_i x_{i+1}}$$

are the consecutive elements of $Vse_{x_i x_{i+1}}$. Thus the cyclic ordering of these elements cannot be changed. Hence G_k is isomorphic to a subgroup of Z_{n-k}.

Now, assume that $v_1, v_2, \dots v_k, u_1, u_2, \dots u_{n-k}$ are the consecutive elements of Γ, and the defect set is $V_k = \{ v_1, \dots, v_k \}$. Let

$$\begin{aligned} s_1 &= e_{v_1 v_2} \dots e_{v_j v_{j+1}} \dots e_{v_{k-1} v_k}, \\ s_2 &= e_{u_{n-k} v_k} e_{u_{n-k-1} u_{n-k}} \dots e_{u_{j-1} u_j} \dots e_{u_1 u_2} e_{v_k u_1}, \\ s &= s_1 s_2. \end{aligned}$$

It easy to check that

$$v_i s = u_1, \quad u_1 s = u_2, \dots, u_j s = u_{j+1}, \dots, u_{n-k} s = u_1.$$

Therefore $s, s^2, \dots, s^{n-k}$ are distinct elements of G_k, hence $G_k \simeq Z_{n-k}$. □

4. Defect 1 groups

In this Section we prove item (3) of Theorem 1, which states that the defect 1 group of a simple connected graph is the direct product of the defect 1 groups of its 2-vertex connected components. This follows by induction on the number of 2-vertex connected components from Lemma 9. The case where Γ is 2-vertex connected (that is item (4) of Theorem 1) is covered by [15, Theorem 2].

Lemma 9. *Let Γ_1 and Γ_2 be connected induced subgraphs of Γ such that $\Gamma_1 \cap \Gamma_2 = \{ v \}$, where there are no edges in Γ between $\Gamma_1 \setminus \{ v \}$ and $\Gamma_2 \setminus \{ v \}$. Then the defect 1 group of $\Gamma_1 \cup \Gamma_2$ is the direct product of the defect 1 groups of Γ_1 and Γ_2.*

Proof. Let G_{Γ_i} denote the defect 1 group of Γ_i, where the defect point is v. By Lemma 7, G_Γ is generated by cyclic permutations corresponding to cycles through v in Γ. Now, $\Gamma_1 \cap \Gamma_2 = \{ v \}$, and every path between a node from Γ_1 and a node from Γ_2 must go through v, hence every cycle in Γ is either in Γ_1 or in Γ_2. Let $c_i^{(1)}, \dots, c_i^{(m_i)}$ be the permutations corresponding to the cycles in Γ_i $(i = 1, 2)$. Since these cycles do not involve v by Lemma 7, we have $c_1^{(j_1)}c_2^{(j_2)} = c_2^{(j_2)}c_1^{(j_1)}$ for all $1 \le j_i \le m_i$, $i = 1, 2$, thus

$$G_\Gamma = \left\langle c_1^{(1)}, \dots, c_1^{(m_1)}, c_2^{(1)}, \dots, c_2^{(m_2)} \right\rangle = \left\langle c_1^{(1)}, \dots, c_1^{(m_1)} \right\rangle \times \left\langle c_2^{(1)}, \dots, c_2^{(m_2)} \right\rangle = G_{\Gamma_1} \times G_{\Gamma_2}.$$

□

5. Defect k groups

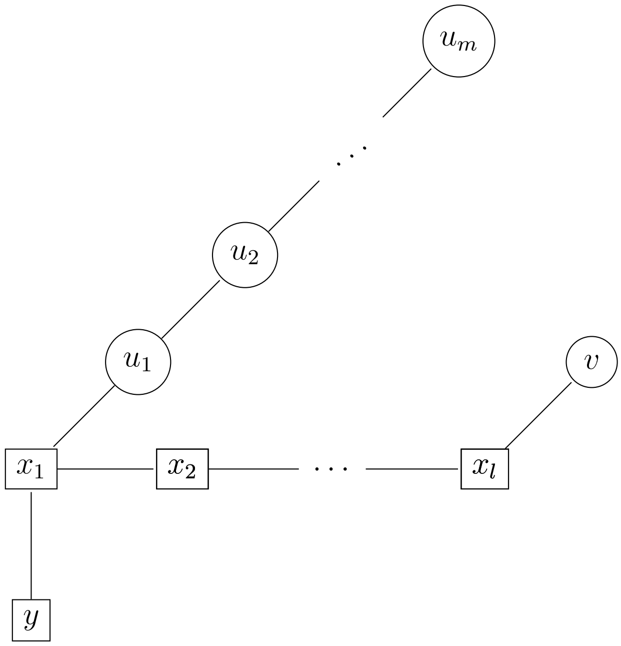
We prove item (5b) of Theorem 1 in this Section. In the following we assume $k \ge 2$, and every graph Γ is assumed to be simple connected. We start with some simple observations.

Lemma 10. *Let Γ be a connected graph, and let Γ' be a connected subgraph of Γ. If Γ' has at least $k+1$ vertices, then the defect k group of Γ contains a subgroup isomorphic (as a permutation group) to the defect k group of Γ'. Furthermore, if $\Gamma \setminus \Gamma'$ contains at least one vertex, and Γ' has at least k vertices, then the defect k group of Γ contains a subgroup isomorphic (as a permutation group) to the defect $k-1$ group of Γ'.*

Proof. Let $\Gamma = (V, E)$, $\Gamma' = (V', E')$. First, assume $|V'| \ge k+1$, and let $V_k = \{ v_1, \dots, v_k \} \subseteq V'$. Let G_{k,V_k} and G'_{k,V_k} be the defect k-groups of Γ and Γ'. Let $g \in G'_{k,V_k}$ be arbitrary. Then there exists $s \in S_{\Gamma'}$ with defect set V_k such that $s \restriction_{V' \setminus V_k} = g$. Now, $E' \subseteq E$, hence every elementary collapsing of Γ' is an elementary collapsing of Γ, as well, Thus $s \in S_\Gamma$, and s acts as the identity on $V \setminus V'$. Furthermore, if $s' \in S_{\Gamma'}$ is another element with defect set V_k such that $s' \restriction_{V' \setminus V_k} = g = s \restriction_{V' \setminus V_k}$, then $s' \in S_\Gamma$ with $s' \restriction_{V \setminus V_k} = s \restriction_{V \setminus V_k}$. Thus $\varphi \colon G'_{k,V_k} \to G_{k,V_k}$, $\varphi(g) = s \restriction_{V \setminus V_k}$ is a well defined injective homomorphism of permutation groups.

Second, assume $|V'| \ge k$, and let $V_{k-1} = \{ v_1, \dots, v_{k-1} \} \subseteq V'$. Let $v \in V \setminus V'$, and let $V_k = V_{k-1} \cup \{ v \}$. Let u be a neighbor of v and let $e = e_{vu}$. Let G_{k,V_k} be the defect k-group of Γ and let $G'_{k-1,V_{k-1}}$ be the defect $(k-1)$-group of Γ'. Let $g \in G'_{k-1,V_{k-1}}$ be arbitrary. Then there exists $s \in S_{\Gamma'}$ with defect set V_{k-1} such that $s \restriction_{V' \setminus V_{k-1}} = g$. Now, $es \in S_\Gamma$ has defect set V_k, and $es \restriction_{V \setminus V_k}$ acts as g on $V' \setminus V_{k-1}$, and acts as the identity on $V \setminus (V' \cup \{ v \})$. Furthermore, if $s' \in S_{\Gamma'}$ is another element with defect set V_{k-1} such that $s' \restriction_{V' \setminus V_{k-1}} = g = s \restriction_{V' \setminus V_{k-1}}$, then $es \restriction_{V \setminus V_k} = es' \restriction_{V \setminus V_k}$. As $g \in G'_{k-1,V_{k-1}}$ was arbitrary, we have that $\varphi \colon G'_{k-1,V_{k-1}} \to G_{k,V_k}$, $\varphi(g) = es \restriction_{V \setminus V_k}$ is a well defined injective homomorphism of permutation groups. □

Lemma 11. *Let $1 \le m \le l < k \le n-2$, and assume Γ contains the following subgraph:*



If V_k is a set of nodes of size k such that $y, x_1, \dots, x_l \in V_k$, and $v, u_i \notin V_k$ for some $1 \leq i \leq m$, then the defect k group G_{k,V_k} contains the transposition (u_i, v).

Proof. Let

$$r = \begin{cases} ss_1 e_{yx_1} e_{x_1u_1}, & \text{if } i = 1, \\ ss_1 \dots s_i p t t_{i-1} \dots t_1 q, & \text{if } i \geq 2, \end{cases}$$

where

$$\begin{aligned} s &= e_{vx_l} e_{x_lx_{l-1}} \dots e_{x_2x_1} e_{x_1y}, \\ s_1 &= e_{u_1x_1} e_{x_1x_2} \dots e_{x_{l-1}x_l} e_{x_lv}, \\ s_j &= e_{u_ju_{j-1}} \dots e_{u_2u_1} e_{u_1x_1} e_{x_1x_2} \dots e_{x_{l-j+1}x_{l-j+2}}, && (2 \leq j \leq m), \\ p &= e_{yx_1} e_{x_1u_1} e_{u_1u_2} \dots e_{u_{i-1}u_i}, \\ t &= e_{x_{l-i+2}x_{l-i+1}} \dots e_{x_2x_1} e_{x_1y}, \\ t_j &= e_{x_{l-j+2}x_{l-j+1}} \dots e_{x_2x_1} e_{x_1u_1} e_{u_1u_2} \dots e_{u_{j-1}u_j}, && (2 \leq j \leq m), \\ t_1 &= e_{vx_l} e_{x_lx_{l-1}} \dots e_{x_2x_1} e_{x_1u_1}, \\ q &= e_{yx_1} e_{x_1x_2} \dots e_{x_{l-1}x_l} e_{x_lv}. \end{aligned}$$

Then r transposes u_i and v and fixes all other vertices of Γ outside the defect set. □

Note that Lemma 11 is going to be useful whenever Γ contains a node with degree at least 3.

Lemma 12. *Let $k \geq 2$, $\Gamma' = (V', E')$ be such that $|V'| > k$ and its defect k group is transitive (e.g. if Γ' is a cycle with at least $k+1$ vertices). Let $\Gamma = (V' \cup \{ v \}, E' \cup \{ x_1v \})$ for a new vertex v and* *some $x_1 \in \Gamma'$, where the degree of x_1 in Γ' is at least 2. Then the defect k group of Γ is isomorphic to S_{n-k}.*

Proof. Let n be the number of vertices of Γ, then $n \geq k+2$. Let the vertices of Γ' be $y, x_1, x_2, \dots, x_{k-1}, u_1, u_2, \dots, u_{n-k-1}$ such that u_1 and y are neighbors of x_1 in Γ'. Let the defect set be $\{ y, x_1, \dots, x_{k-1} \}$. Applying Lemma 11 to the subgraph with vertices $\{ x_1, v, y, u_1 \}$ we obtain that the defect k group of Γ contains the transposition (u_1, v). Since the defect k group of Γ' is transitive and contained in the defect k group of Γ by Lemma 10, the defect k group of Γ contains the transposition (u_i, v) for all $1 \leq i \leq n-k-1$. Therefore, the defect k group of Γ is isomorphic to S_{n-k}. □

Motivated by Lemma 12, we define the *k-subgraphs* and the *maximal k-subgraphs* of a graph Γ.

Definition 13. Let Γ be a simple connected graph, $k \geq 2$. A connected subgraph $\Gamma' \subseteq \Gamma$ is called a *k-subgraph* if its defect k group is the symmetric group of degree $|\Gamma'| - k$. A k-subgraph is a *maximal k-subgraph* if it has no proper extension in Γ to a k-subgraph. Finally, we say that a k-subgraph Γ' is *nontrivial* if it contains a vertex having at least 3 distinct neighbors in Γ'.

Note that every maximal k-subgraph is an induced subgraph. A trivial k-subgraph is either a line on $k+1$ points or a cycle on $k+1$ or $k+2$ points. Furthermore, a trivial maximal k-subgraph cannot be a cycle by Lemma 12, unless the graph itself is a cycle. Finally, any connected subgraph of $k+1$ points is trivially a k-subgraph, thus every connected subgraph of $k+1$ points is contained in a maximal k-subgraph. Note that the intersection of two maximal k-subgraphs cannot contain more than k vertices:

Lemma 14. *Let Γ_1, Γ_2 be k-subgraphs such that $|\Gamma_1 \cap \Gamma_2| > k$. Then $\Gamma_1 \cup \Gamma_2$ is a k-subgraph, as well.*

Proof. Choose the defect set V_k such that $V_k \subsetneq \Gamma_1 \cap \Gamma_2$, and let $v \in (\Gamma_1 \cap \Gamma_2) \setminus V_k$. Then the symmetric groups acting on $\Gamma_1 \setminus V_k$ and $\Gamma_2 \setminus V_k$ are subgroups in the defect k group of $\Gamma_1 \cup \Gamma_2$. Thus, we can transpose every member of $\Gamma_i \setminus (V_k \cup \{ v \})$ with v. Therefore, the defect k group of $\Gamma_1 \cup \Gamma_2$ is the symmetric group on $(\Gamma_1 \cup \Gamma_2) \setminus V_k$. □

Lemma 15. *Let Γ be a simple connected graph, and let Γ' be a k-subgraph of Γ. Let $x_1 \in \Gamma'$, $v \notin \Gamma'$, and let $P = (x_1, x_2, \dots, x_l, v)$ be a shortest path between x_1 and v in Γ for some $l \leq k-1$. Assume that x_1 has at least 2 neighbors in Γ' apart from x_2. Then the subgraph $\Gamma' \cup P$ is a k-subgraph.*

Proof. First, consider the case $x_2, \dots, x_l \in \Gamma'$. Let u, y be two neighbors of x_1 in Γ' distinct from x_2, and choose the defect set V_k such that it contains $y, x_1, \ldots, x_l$ and does not contain u. By Lemma 11 the defect k group of $\Gamma' \cup \{ v \}$ contains the transposition (u, v). Furthermore, the defect k group of Γ' is the whole symmetric group on $\Gamma' \setminus V_k$. Thus, the defect k group of $\Gamma' \cup \{ v \}$ is the whole symmetric group on $(\Gamma' \setminus V_k) \cup \{ v \}$.

Now, if not all of $x_2, \ldots, x_l$ are in Γ', then, by the previous argument, one can add them (and then v) to Γ' one by one, and obtain an increasing chain of k-subgraphs. □

As a corollary, we obtain that every vertex of degree at least 3 together with at least two of its neighbors is contained in exactly one nontrivial maximal k-subgraph.

Corollary 16. *Let Γ be a simple connected graph with n vertices such that $n > k$, and let x_1 be a vertex having degree at least 3. Then there exists exactly one maximal k-subgraph Γ' containing x_1 such that x_1 has degree at least 2 in Γ'. Furthermore, Γ' is a nontrivial k-subgraph, and if Γ_{x_1} is the induced subgraph of the vertices in Γ that are of at most distance $k-1$ from x_1, then $\Gamma_{x_1} \subseteq \Gamma'$.*

Proof. Any connected subgraph of Γ with $k+1$ vertices containing x_1 and any two of its neighbors is a k-subgraph. Thus there exists at least one maximal k-subgraph containing x_1 and two of its neighbors.

Let Γ' be a maximal k-subgraph containing x_1 and at least two of its neighbors. Assume that $\Gamma_{x_1} \not\subseteq \Gamma'$. Let $v \in \Gamma_{x_1} \setminus \Gamma'$ be any vertex at a minimal distance from x_1, and let $P = (x_1, \ldots, x_l, v)$ be a shortest path between x_1 and v. If $l = 1$, then $P = (x_1, v)$. Now x_1 has at least two neighbors in Γ' apart from v, therefore $\Gamma' \cup P$ is a k-subgraph by Lemma 15, which contradicts the maximality of Γ'. Thus $l \geq 2$, in particular all neighbors of x_1 in Γ are in Γ', as well, and thus Γ' is a nontrivial k-subgraph. Hence x_1 has at least two neighbors in Γ' apart from x_2, therefore $\Gamma' \cup P$ is a k-subgraph by Lemma 15, which contradicts the maximality of Γ'. Thus $\Gamma_{x_1} \subseteq \Gamma'$.

Now, assume that Γ' and Γ'' are maximal k-subgraphs containing x_1 and at least two of its neighbors. Then $\Gamma_{x_1} \subseteq \Gamma'$ and $\Gamma_{x_1} \subseteq \Gamma''$. Note that either $\Gamma_{x_1} = \Gamma$ (and hence $|\Gamma_{x_1}| = n > k$), or there exists a vertex $v \in \Gamma$ which is of distance exactly k from x_1. Let $P = (x_1, \ldots, x_k, v)$ be a shortest path between x_1 and v, and let u and y be two neighbors of x_1 distinct from x_2. Then $\{ x_1, \ldots, x_k, y, u \} \subseteq \Gamma_{x_1}$, thus $|\Gamma_{x_1}| > k$. Therefore $|\Gamma' \cap \Gamma''| \geq |\Gamma_{x_1}| > k$, yielding $\Gamma' = \Gamma''$ by Lemma 14. □

Lemma 17. *Let Γ' be a nontrivial k-subgraph of Γ, and let P be a Γ'-ear. Then $\Gamma' \cup P$ is a (nontrivial) k-subgraph of Γ.*

Proof. Let Γ, Γ' and $P = (w_0, w_1, \ldots w_i, w_{i+1})$ be a counterexample, where i is minimal. There exists a shortest path $(w_0, y_1, \ldots, y_l, w_{i+1})$ in Γ' among those where the degree of some y_j or of w_0 or of w_{i+1} is at least 3 in Γ'. (At least one such path exists, because Γ' is connected, and is a nontrivial k-subgraph, hence contains a vertex of degree at least 3.) For easier notation, let $y_0 = w_0$, $y_{l+1} = w_{i+1}$. Let $y' \in \Gamma' \setminus \{ y_0, y_1, \dots, y_l, y_{l+1} \}$ be a neighbor of y_j; this exists, because the degree of y_j is at least 3, and otherwise a shorter path would exist between w_0 and w_{i+1}.

If $j + 1 \leq k - 1$ (that is $j \leq k - 2$), then by Lemma 15 the induced subgraph on $\Gamma' \cup \{ w_1 \}$ is a k-subgraph, thus $\Gamma' \cup \{ w_1 \}$ with the ear $(w_1, \dots, w_i, w_{i+1})$ is a counterexample with a shorter ear.

Similarly, if $l - j + 2 \leq k - 1$ (that is $l + 3 - k \leq j$), then by Lemma 15 the induced subgraph on $\Gamma' \cup \{ w_i \}$ is a k-subgraph, thus $\Gamma' \cup \{ w_i \}$ with the ear $(w_0, w_1, \dots, w_i)$ is a counterexample with a shorter ear.

Finally, if $k - 1 \leq j \leq l + 2 - k$, then $2k - 3 \leq l$. Let Γ'' be the cycle $P \cup (y_0, y_1, \dots, y_l, y_{l+1})$ together with y' and the edge $y_j y'$. Then Γ'' is a k-subgraph by Lemma 12, $|\Gamma' \cap \Gamma''| = l + 2 \geq 2k - 1 > k$, hence $\Gamma' \cup \Gamma'' = \Gamma' \cup P$ is a k-subgraph by Lemma 14. □

Corollary 18. *Let Γ be a simple connected graph with n vertices such that $n > k$, and assume that Γ is not a cycle. Suppose uv is an edge contained in a cycle of Γ. Then there exists exactly one maximal k-subgraph Γ' containing the edge uv. Furthermore, Γ' is a nontrivial k-subgraph, and if Γ_{uv} is the 2-edge connected component containing uv, then $\Gamma_{uv} \subseteq \Gamma'$.*

Proof. Any connected subgraph of Γ with $k + 1$ vertices containing the edge uv is a k-subgraph. Thus there exists at least one maximal k-subgraph Γ' containing the edge uv. We prove first that Γ' is a nontrivial k-subgraph, then prove $\Gamma_{uv} \subseteq \Gamma'$, and only after that do we prove that Γ' is unique.

Assume first that Γ' is a trivial k-subgraph. If Γ' were a cycle, then $\Gamma \setminus \Gamma'$ contains at least one vertex, because Γ' is an induced subgraph of Γ. Then Lemma 12 contradicts the maximality of Γ'. Thus Γ' is a line of $k + 1$ vertices. Let Γ_2 be a shortest cycle containing uv. Now, there must exist a vertex in $\Gamma \setminus \Gamma_2$, otherwise either $\Gamma = \Gamma_2$ would be a cycle, or there would exist an edge in $\Gamma \setminus \Gamma_2$ yielding a shorter cycle than Γ_2 containing the edge uv. Let $x_2 \in \Gamma \setminus \Gamma_2$ be a neighbor of a vertex in Γ_2. By Lemma 12 the induced subgraph on $\Gamma_2 \cup \{ x_2 \}$ is a k-subgraph. Thus $\Gamma' \not\subseteq \Gamma_2$, otherwise Γ' would not be a maximal k-subgraph. Let $x_1 \in \Gamma' \cap \Gamma_2$ be a vertex such that two of its neighbors are in Γ_2 and its third neighbor is some $x_2 \in \Gamma' \setminus \Gamma_2$. Note that every vertex in Γ' is of distance at most $k - 1$ from x_1, because $u, v \in \Gamma' \cap \Gamma_2$. Thus, if $|\Gamma_2| \geq k + 1$, then Γ_2 together with x_2 and the edge $x_1 x_2$ is a k-subgraph by Lemma 12, and hence $\Gamma_2 \cup \Gamma'$ is a k-subgraph by Lemma 15, contradicting the maximality of Γ'. Otherwise, if $|\Gamma_2| \leq k$, then every vertex in Γ_2 is of distance at most $k - 1$ from x_1, and hence $\Gamma_2 \cup \Gamma'$ is a k-subgraph by Lemma 15, contradicting the maximality of Γ'. Therefore Γ' is a nontrivial k-subgraph.

Now we show that the two-edge connected component $\Gamma_{uv} \subseteq \Gamma'$. Let Γ, Γ' be a counterexample to this such that the number of vertices of Γ_{uv} is minimal, and among these counterexamples choose one where the number of edges of Γ_{uv} is minimal. Using an ear-decomposition [11], Γ_{uv} is either a cycle, or there exists a 2-edge connected subgraph $\Gamma_1 \subseteq \Gamma_{uv}$ and there exists

(1) either a Γ_1-ear P such that $\Gamma_{uv} = \Gamma_1 \cup P$,
(2) or a cycle Γ_2 such that $|\Gamma_1 \cap \Gamma_2| = 1$ and $\Gamma_{uv} = \Gamma_1 \cup \Gamma_2$.

If Γ_{uv} is a cycle containing the edge uv, and $\Gamma_{uv} \not\subseteq \Gamma'$, then going along the edges of Γ_{uv}, one can find a Γ'-ear $P \subseteq \Gamma_{uv}$. Then $\Gamma' \cup P$ is a k-subgraph by Lemma 17, contradicting the maximality of Γ'. Thus Γ_{uv} is not a cycle. Let us choose Γ_1 from cases (1) and (2) so that it would have the least number of vertices.

Assume first that case (1) holds. By minimality of the counterexample, $\Gamma_1 \subseteq \Gamma'$. If $P \not\subseteq \Gamma'$, then going along the edges of P one can find a Γ'-ear $P' \subseteq P$. But then $\Gamma' \cup P'$ is a k-subgraph by Lemma 17, contradicting the maximality of Γ'.

Assume now that case (2) holds. Again, by induction, $\Gamma_1 \subseteq \Gamma'$. If $\Gamma_2 \not\subseteq \Gamma'$, then either $|\Gamma' \cap \Gamma_2| = 1$ or going along the edges of Γ_2 one can find a Γ'-ear $P' \subseteq \Gamma_2$. The latter case cannot happen, because then $\Gamma' \cup P'$ is a k-subgraph by Lemma 17, contradicting the maximality of Γ'. Thus $|\Gamma' \cap \Gamma_2| = 1$, and hence $\Gamma' \cap \Gamma_2 = \Gamma_1 \cap \Gamma_2$. Let $\Gamma_1 \cap \Gamma_2 = \{ x_1 \}$, and let v_1 be a neighbor of x_1 in $\Gamma_1 \setminus \Gamma_2$, and let v_2 be a neighbor of x_1 in $\Gamma_2 \setminus \Gamma_1$. If $|\Gamma_2| \leq k$, then Γ_2 can be extended to a connected subgraph of Γ having exactly $k+1$ vertices, which is a k-subgraph. If $|\Gamma_2| \geq k+1$, then $\Gamma_2 \cup \{ v_1 \}$ is a k-subgraph by Lemma 12. In any case, there exists a maximal k-subgraph $\Gamma'_2 \supseteq \Gamma_2$. For notational convenience, let Γ'_1 denote the maximal k-subgraph Γ' containing Γ_1. We prove that $\Gamma'_2 = \Gamma'_1 = \Gamma'$, thus Γ' contains Γ_2, contradicting that we chose a counterexample.

Now, both Γ_1 and Γ_2 contain at least two neighbors of x_1. Let $V_i \subseteq \Gamma_i$ be the set of vertices with distance at most $k-1$ from x_1 ($i \in \{ 1, 2 \}$). If $|\Gamma_i| \leq k$, then V_i contains all vertices of Γ_i, otherwise $|V_i| \geq k$ ($i \in \{ 1, 2 \}$). By Lemma 15, the induced subgraph on V_1 is contained in Γ'_2. Thus, if V_1 contains all vertices of Γ_1, then $\Gamma_1 \subseteq \Gamma'_2$, hence we have $\Gamma'_1 = \Gamma'_2$. Similarly, the induced subgraph on V_2 is contained in Γ'_1. Thus, if V_2 contains all vertices of Γ_2, then $\Gamma_2 \subseteq \Gamma'_1$, hence we have $\Gamma'_1 = \Gamma'_2$. Otherwise, $|\Gamma'_1 \cap \Gamma'_2| \geq |V_1| + |V_2| - |\{ x_1 \}| \geq 2k - 1 > k$, hence by Lemma 14 we have $\Gamma'_1 = \Gamma'_2$.

Finally, we prove uniqueness. Let Γ' and Γ'' be two maximal k-subgraphs containing the edge uv. Then both Γ' and Γ'' contain Γ_{uv}. If $\Gamma = \Gamma_{uv}$, then $\Gamma' = \Gamma_{uv} = \Gamma''$. Otherwise, there exists a vertex $x_2 \in \Gamma \setminus \Gamma_{uv}$ such that it has a neighbor $x_1 \in \Gamma_{uv}$. Note that x_1 has degree at least 3 in Γ. Let V_1 be the vertices of Γ of distance at most $k-1$ from x_1. Note that if V_1 does not contain all vertices of Γ, then $|V_1| > k$.

By 2-edge connectivity, $\Gamma_{uv} \subseteq \Gamma'$ contains at least two neighbors of x_1, thus $V_1 \subseteq \Gamma'$ by Lemma 15. Similarly, $\Gamma_{uv} \subseteq \Gamma''$ contains at least two neighbors of x_1, thus $V_1 \subseteq \Gamma''$ by Lemma 15. If V_1 contains all vertices of Γ, then $\Gamma' = \Gamma = \Gamma''$. Otherwise, $|\Gamma' \cap \Gamma''| \geq |V_1| > k$, and $\Gamma' = \Gamma''$ by Lemma 14. □

Recall that by [11] a strongly connected antisymmetric digraph becomes a 2-edge connected graph after forgetting the directions. Thus Rhodes's conjecture about strongly connected, antisymmetric digraphs [9, Conjecture 6.51i (3)–(4)] follows immediately from the following theorem on 2-edge connected graphs:

Theorem 19. *Let $n > k \geq 2$, Γ be a 2-edge connected simple graph having n vertices. If Γ is a cycle, then the defect k group is Z_{n-k}. If Γ is not a cycle, then the defect k group is S_{n-k}.*

Proof. If Γ is a cycle, then its defect k group is Z_{n-k} by Lemma 8. Since Γ is 2-edge connected with at least 3 vertices, every edge of Γ is contained in a cycle. Thus, if Γ is not a cycle, then the defect k group is S_{n-k} by Corollary 18. □

The final part of this section is devoted to prove item (5b) of Theorem 1. First, we define bridges in Γ:

Definition 20. A path $(x_1, \ldots, x_l)$ in a connected graph Γ for some $l \geq 2$ is called a *bridge* if the degree of x_i in Γ is 2 for all $2 \leq i \leq l-1$, and if $\Gamma \setminus \{ x_j x_{j+1} \}$ is disconnected for all $1 \leq j \leq l-1$. The *length* of the bridge $(x_1, \ldots, x_l)$ is l.

The intersection of maximal k-subgraphs turn out to be bridges:

Lemma 21. *Let Γ_1 and Γ_2 be distinct maximal k-subgraphs of the connected simple graph Γ. Assume that Γ is not a cycle. Then $\Gamma_1 \cap \Gamma_2$ is either empty, or is a bridge $(x_1, \ldots, x_l)$ such that*

1. *$l \leq k$, and*
2. *if $l \geq 2$ and $\Gamma_i \setminus \{ x_1, \ldots, x_l \}$ ($i \in \{ 1, 2 \}$) contains a neighbor of x_1 (resp. x_l), then Γ_i contains all neighbors of x_1 (resp. x_l),*

Proof. Note that Γ_1 and Γ_2 are induced subgraphs of Γ, thus so is $\Gamma_1 \cap \Gamma_2$.

We prove first that $\Gamma_1 \cap \Gamma_2$ is connected (or empty) if Γ_1 is a nontrivial maximal k-subgraph. Suppose that $u, v \in \Gamma_1 \cap \Gamma_2$ are in different components of $\Gamma_1 \cap \Gamma_2$ such that the distance between u and v is minimal in Γ_2. Due to the minimality, there exists a path $(u, x_1, \ldots, x_l, v)$ such that $x_1, \ldots, x_l \in \Gamma_2 \setminus \Gamma_1$. Then $P = (u, x_1, \ldots, x_l, v)$ is a Γ_1-ear, and $\Gamma_1 \cup P$ would be a k-subgraph by Lemma 17, contradicting the maximality of Γ_1. Thus $\Gamma_1 \cap \Gamma_2$ is connected. One can prove similarly that $\Gamma_1 \cap \Gamma_2$ is connected if Γ_2 is a nontrivial maximal k-subgraph.

Now we prove that $\Gamma_1 \cap \Gamma_2$ is connected, even if both Γ_1 and Γ_2 are trivial maximal k-subgraphs. As $\Gamma_1 \subsetneqq \Gamma$, Γ_1 cannot be a cycle hence must be a line $(x_1, \dots, x_{k+1})$. Note that the degree of x_i in Γ for $2 \le i \le k$ must be 2, otherwise a nontrivial maximal k-subgraph would contain x_i, and thus also Γ_1 by Corollary 16. In particular, if $\Gamma_1 \cap \Gamma_2$ is not connected, then $x_1, x_{k+1} \in \Gamma_1 \cap \Gamma_2$, $x_i \notin \Gamma_1 \cap \Gamma_2$ for some $2 \le i \le k$, and $\Gamma_1 \cup \Gamma_2$ would be a cycle. However, by Corollary 18, the edge x_1x_2 is contained in a unique nontrivial maximal k-subgraph, contradicting that it is also contained in the trivial maximal k-subgraph Γ_1.

Now, we prove (1). By Corollary 18, $\Gamma_1 \cap \Gamma_2$ cannot contain any edge uv which is contained in a cycle. As $\Gamma_1 \cap \Gamma_2$ is connected, it must be a tree. However, $\Gamma_1 \cap \Gamma_2$ cannot contain any vertex of degree at least 3 in $\Gamma_1 \cap \Gamma_2$, otherwise that vertex would be contained in a unique maximal k-subgraph by Corollary 16. Thus $\Gamma_1 \cap \Gamma_2$ is a path $(x_1, \dots, x_l)$. Now, $l \le k$ by Lemma 14, proving (1). Note that if any x_i $(2 \le i \le l-1)$ is of degree at least 3 in Γ, then $\{ x_{i-1}, x_i, x_{i+1} \}$ is contained in a unique maximal k-subgraph by Corollary 16, a contradiction. For (2) observe that at least two neighbors of x_1 (resp. x_l) are in Γ_i, and thus all its neighbors must be in Γ_i by Corollary 16. Finally, if $l \ge 2$ then $\Gamma \setminus \{ x_j x_{j+1} \}$ is disconnected for all $1 \le j \le l-1$ follows immediately from Corollary 18 and the fact that any edge that is not contained in any cycle disconnects the graph Γ. ☐

Edges of short maximal bridges (having length at most $k-1$) are contained in a unique maximal k-subgraph:

Lemma 22. *Let Γ be a simple connected graph with n vertices such that $n > k$, and let uv be an edge which is not contained in any cycle. Let $(x_1, \dots, x_l)$ be a longest bridge containing the edge uv. If $l \le k-1$, then uv is contained in a unique maximal k-subgraph Γ', and furthermore, Γ' is a nontrivial k-subgraph.*

Proof. As uv is not part of any cycle in Γ, uv is a bridge of length 2. Note that a longest bridge $(x_1, \dots, x_l)$ containing uv is unique, because as long as the degree of at least one of the path's end vertices is 2 in Γ, the path can be extended in that direction. The obtained path is the unique longest bridge containing uv.

Let Γ' be a maximal k-subgraph containing uv, and assume $l \le k-1$. Note that the distance of x_1 and x_l is $l-1 \le k-2$. As $|\Gamma| \ge k+1$, at least one of x_1 and x_l has degree at least 3 in Γ, say x_1. We distinguish two cases according to the degree of x_l.

Assume first that x_l is of degree 1. As Γ' is a connected subgraph having at least $k+1$ vertices, Γ' must contain x_1 and at least two of its neighbors. Then by Corollary 16 it contains all vertices of Γ of distance at most $k-1$ from x_1. In particular, Γ' must contain the bridge $(x_1, \dots, x_l)$. However, there is a unique (nontrivial) maximal k-subgraph Γ'_1 containing x_1 and two of its neighbors by Corollary 16, and thus $\Gamma' = \Gamma'_1$ is that unique maximal k-subgraph.

Assume now that x_l is of degree at least 3. As Γ' is a connected subgraph having at least $k+1$ vertices, Γ' must contain x_1 and at least two of its neighbors, or x_l and at least two of its neighbors. If Γ' contains x_1 and at least two of its neighbors, then by Corollary 16 it contains all vertices of Γ of distance at most $k-1$ from x_1. In particular, Γ' must contain the bridge $(x_1, \dots, x_l)$ and all of the neighbors of x_l. Similarly, one can prove that if Γ' contains x_l and two of its neighbors, then it also contains the bridge $(x_1, \dots, x_l)$ and all of the neighbors of x_1. However, there is a unique (nontrivial) maximal k-subgraph Γ'_1 containing x_1 and two of its neighbors by Corollary 16, and also a unique (nontrivial) maximal k-subgraph Γ'_l containing x_l and two of its neighbors by Corollary 16. Therefore Γ' must equal to both Γ'_1 and Γ'_l, and hence is unique. □

In particular, in non-cycle graphs trivial maximal k-subgraphs or intersections of two different maximal k-subgraphs consist of edges that are contained in long bridges (having length at least k). The key observation in proving item (5b) of Theorem 1 is that a defect k group cannot move a vertex across a bridge of length at least k:

Lemma 23. *Let $2 \le k \le l$, Γ_1 and Γ_2 be disjoint connected subgraphs of the connected graph Γ, and $(x_1, x_2, \dots, x_l)$ be a bridge in Γ such that $x_1 \dots, x_l \notin \Gamma_1 \cup \Gamma_2$, x_1 has only neighbors in Γ_1 (except for x_2), x_l has only neighbors in Γ_2 (except for x_{l-1}). Assume Γ has no more vertices than $\Gamma_1 \cup \Gamma_2 \cup (x_1, \dots, x_l)$. Let the defect set be $V_k = \{ x_1, \dots, x_k \}$. Then for any $u \in \Gamma_1$ and $v \in \Gamma_2$ there does not exist any permutation in G_{k,V_k} which moves u to v.*

Proof. Let $S = S_\Gamma$. Assume that there exists $u \in \Gamma_1$, $v \in \Gamma_2$, and a transformation $g \in S$ of defect V_k such that $g \restriction_{V \setminus V_k} \in G_{k,V_k}$ and $ug = v$. Let $s_0 \in G_{k,V_k}$ be the unique idempotent power of g, that is s_0 is a transformation of defect V_k that acts as the identity on $\Gamma \setminus V_k$. Then there exists a series of elementary collapsings $e_1, \dots, e_m$ such that $g = e_1 \dots e_m$. For every $1 \le d \le m$ let $s_d = s_0 e_1 \dots e_d$. Now, $s_m = s_0 e_1 \dots e_m = s_0 g = g s_0 = g$. In particular, both s_m and s_0 are of defect k, hence s_d is of defect k for all $1 \le d \le m$. Consequently, $|\Gamma_1 s_d| = |\Gamma_1|$, $|\Gamma_2 s_d| = |\Gamma_2|$ and $\Gamma_1 s_d \cap \Gamma_2 s_d = \emptyset$ for all $1 \le d \le m$.

For an arbitrary $s \in S$, let

$$i(s) = \begin{cases} 0, & \text{if } \Gamma_1 s \subseteq \Gamma_1, \\ l+1, & \text{if } \Gamma_1 s \not\subseteq \Gamma_1 \cup \{ x_1, \dots, x_l \}, \\ \min\limits_{1 \le i \le l} \{ \Gamma_1 s \subseteq \Gamma_1 \cup \{ x_1, \dots, x_i \} \}, & \text{otherwise.} \end{cases}$$

Similarly, let

$$j(s) = \begin{cases} l+1, & \text{if } \Gamma_2 s \subseteq \Gamma_2, \\ 0, & \text{if } \Gamma_2 s \not\subseteq \Gamma_2 \cup \{ x_1, \dots, x_l \}, \\ \max\limits_{1 \le j \le l} \{ \Gamma_2 s \subseteq \Gamma_2 \cup \{ x_j, \dots, x_l \} \}, & \text{otherwise.} \end{cases}$$

Note that for arbitrary $s \in S$ and elementary collapsing e, we have $|i(s) - i(se)| \le 1$, $|j(s) - j(se)| \le 1$. Furthermore, both $|i(s_d) - i(s_d e)| = 1$ and $|j(s_d) - j(s_d e)| = 1$ cannot happen at the same time for any $1 \le d \le m$, because that would contradict $\Gamma_1 s_d \cap \Gamma_2 s_d \ne \emptyset$.

For s_0 we have $i(s_0) = 0 < l+1 = j(s_0)$, for s_m we have $i(s_m) = l+1 \ge j(s_m)$. Let $1 \le d \le m$ be minimal such that $i(s_d) \ge j(s_d)$. Then $i(s_{d-1}) < j(s_{d-1})$. From s_{d-1} to s_d either i or j can change and by at most 1, thus $i(s_d) = j(s_d)$. If $i(s_d) = j(s_d) \in \{ 1, \dots, l \}$, then $x_{i(s_d)} \in \Gamma_1 s_d \cap \Gamma_2 s_d$, contradicting $\Gamma_1 s_d \cap \Gamma_2 s_d = \emptyset$. Thus $i(s_d) = j(s_d) \notin \{ 1, \dots, l \}$. Assume $i(s_d) = j(s_d) = l+1$, the case $i(s_d) = j(s_d) = 0$ can be handled similarly.

Now, $j(s_d) = l+1$ yields $\Gamma_2 s_d \subseteq \Gamma_2$. Furthermore, $|\Gamma_2 s_d| = |\Gamma_2|$, thus $\Gamma_2 s_d = \Gamma_2$. From $i(s_d) = l+1$ we have $\Gamma_1 s_d \cap \Gamma_2 \ne \emptyset$. Thus $\Gamma_1 s_d \cap \Gamma_2 s_d = \Gamma_1 s_d \cap \Gamma_2 \ne \emptyset$, a contradiction. □

Corollary 24. *Let Γ_1 and Γ_2 be connected subgraphs of Γ such that $\Gamma_1 \cap \Gamma_2$ is a length k bridge in Γ. Let $V_k = \Gamma_1 \cap \Gamma_2$ be the defect set. Let G_i be the defect k group of Γ_i, G be the defect k group of $\Gamma_1 \cup \Gamma_2$. Then*

$$G = G_1 \times G_2.$$

Proof. By Lemma 10 we have $G_1, G_2 \le G$. Since G_1 and G_2 act on disjoint vertices, their elements commute. Thus $G_1 \times G_2 \le G$. Now, V_k is a bridge of length k, thus by Lemma 23 (applied to the disjoint subgraphs $\Gamma_1 \setminus V_k$ and $\Gamma_2 \setminus V_k$) there exists no element of G moving a vertex from Γ_1 to Γ_2 or vice versa. Therefore $G \le G_1 \times G_2$. □

Finally, we are ready to prove item (5b) of Theorem 1.

Proof of item (5b) of Theorem 1. If Γ is a cycle, then its defect k group is Z_{n-k} by Lemma 8. Otherwise, we prove the theorem by induction on the number of maximal k-subgraphs of Γ. If Γ is a maximal k-subgraph, then the theorem holds, and the defect k group of Γ is S_{n-k}. In the following we assume that Γ contains m-many maximal k-subgraphs for some $m \ge 2$, and that the theorem holds for all graphs with at most $(m-1)$-many maximal k-subgraphs.

We consider two cases. Assume first that there exists a degree 1 vertex $x_1 \in \Gamma$, such that there exists a path $(x_1, \dots, x_{k+1})$ which is a bridge. Let Γ_1 be the path $(x_1, \dots, x_{k+1})$, and let Γ_2 be $\Gamma \setminus \{ x_1 \}$. Now, Γ_1 is a trivial maximal k-subgraph, hence Γ_2 contains the same maximal k-subgraphs as Γ except Γ_1. Furthermore, Γ_2 is connected, and cannot be a cycle because the degree of x_2 in Γ_2 is 1. Let the sizes of the maximal k-subgraphs of Γ_2 be $n_2, \dots, n_m$, then by induction the defect k group of Γ_1 is $S_{n_2-k} \times \cdots \times S_{n_m-k}$. The size of Γ_1 is $n_1 = k+1$, its defect k-group is S_{n_1-k}. Furthermore, $\Gamma_1 \cap \Gamma_2$ is a bridge of length k. By Corollary 24 the defect k-group of Γ is $S_{n_1-k} \times S_{n_2-k} \times \cdots \times S_{n_m-k}$.

In the second case, no degree 1 vertex x_1 is in a path $(x_1, \dots, x_{k+1})$ which is a bridge. Then any maximal bridge $(x_1, \dots, x_l)$ with a degree 1 vertex x_1 has length $l \leq k$, and, as the bridge cannot be extended, x_l must have degree at least 3. Moreover, $(x_1, ..., x_l)$ lies in a maximal k-subgraph containing x_l and all its neighbors by Lemma 22 and Corollary 16. In particular every bridge in Γ of length at least $k+1$ occurs between nodes of degree at least 3. Hence every bridge of length at least $k+1$ occurs between two nontrivial maximal k-subgraphs by Corollary 16. For every vertex v having degree at least 3 in Γ, let Γ_v be the unique maximal k-subgraph containing v and all its neighbors (Corollary 16). By definition, these are all the nontrivial maximal k-subgraphs of Γ.

Let Γ^k be the graph whose vertices are the nontrivial maximal k-subgraphs, and $\Gamma_u\Gamma_v$ is an edge in Γ^k (for $\Gamma_u \neq \Gamma_v$) if and only if there exists a bridge in Γ between a vertex $u' \in \Gamma_u$ of degree at least 3 in Γ_u and a vertex $v' \in \Gamma_v$ of degree at least 3 in Γ_v. By Corollary 18, $\Gamma_u = \Gamma_v$ if u and v are in the same 2-edge connected component. As the 2-edge connected components of Γ form a tree, the graph Γ^k is a tree.

Now, Γ^k has m vertices. Let Γ_1 be a leaf in Γ^k, and let Γ_m be its unique neighbor in Γ^k. Let $x_1 \in \Gamma_1$ and $x_l \in \Gamma_m$ be the unique vertices of degree at least 3 in Γ_i $(i \in \{1, l\})$ such that there exists a bridge $P = (x_1, \dots, x_l)$ in Γ. Note that the length of P is at least k, otherwise $\Gamma_1 = \Gamma_m$ would follow by Lemma 22. Furthermore, any other bridge having an endpoint in Γ_1 must be of length at most k, because every degree 1 vertex is of distance at most $k-1$ from a vertex of degree at least 3. Thus every bridge other than P and having an endpoint in Γ_1 is a subset of Γ_1 by Corollary 16.

Let $\Gamma_2 = (\Gamma \setminus \Gamma_1) \cup P$. Now, Γ_1 is a maximal k-subgraph, Γ_2 has one less maximal k-subgraphs than Γ. Furthermore, Γ_2 is connected, because every bridge other than P and having an endpoint in Γ_1 is a subset of Γ_1. Finally, Γ_2 is not a cycle, because it contains the vertex x_1 which is of degree 1 in Γ_2. Let the sizes of the maximal k-subgraphs of Γ_2 be $n_2, \dots, n_m$, then by induction the defect k group of Γ_1 is $S_{n_2-k} \times \cdots \times S_{n_m-k}$. Let the size of Γ_1 be n_1, its defect k-group is S_{n_1-k}. Furthermore, $\Gamma_1 \cap \Gamma_2$ is a bridge of length k. By Corollary 24 the defect k-group of Γ is $S_{n_1-k} \times S_{n_2-k} \times \cdots \times S_{n_m-k}$. □

6. An algorithm to calculate the defect k group

Note that by items (3) and (4) of Theorem 1 the defect 1 group can be trivially computed in $O(|E|)$ time by first determining the 2-vertex connected components [8], and whether each is a cycle, the exceptional graph (Figure 1) or if not, whether or not it is bipartite.

For $k \geq 2$ one can check first if Γ is a cycle (and then the defect group is Z_{n-k}) or a path (and then the defect group is trivial). In the following, we give a linear algorithm (running in $O(|E|)$ time) to determine the maximal k-subgraphs ($k \geq 2$) of a connected graph Γ having n vertices, $|E|$ edges where at least one vertex is of degree at least 3.

During the algorithm we color the vertices. Let us call a maximal subgraph with vertices having the same color a *monochromatic component.* First, one finds all 2-edge connected components and the tree of two-edge connected components in $O(|E|)$ time using e.g. [13]. Color the vertices of the nontrivial (i.e. having size greater than 1) 2-edge connected components such that two distinct vertices have the same color if and only if they are in the same nontrivial 2-edge connected component. Furthermore, color the uncolored vertices having degree at least 3 by different colors from each other and from the colors of the 2-edge connected components. Then the monochromatic components are each contained in a unique nontrivial maximal k-subgraph by Corollaries 16 and 18 (a nontrivial maximal k-subgraph may contain more than one of these monochromatic components). Furthermore, the monochromatic components and the degree 1 vertices are connected by bridges. If any of the bridges connecting two monochromatic components is of length at most $k-1$, then recolor the two monochromatic components at the ends of the bridge and the vertices of the bridge by the same color, because these are contained in the same maximal k-subgraph by Corollary 16. Similarly, if any of the bridges connecting a monochromatic component and a degree 1 vertex is of length at most $k-1$, then recolor the monochromatic component and the vertices of the bridge by the same color, because these are contained in the same maximal k-subgraph by Lemma 22. Repeat recoloring along all bridges of length at most $k-1$ in $O(|E|)$ time. Then we obtain monochromatic components $\Gamma_1, \ldots, \Gamma_l$ connected by long bridges (i.e. bridges of length at least k), and possibly some long bridges to degree 1 vertices. Now, we have finished coloring.

For every $1 \leq i \leq l$, let Γ'_i be the induced subgraph having all vertices of distance at most $k-1$ from Γ_i, which can be obtained in $O(|E|)$ time by adding the appropriate $k-1$ vertices of the long bridges to the appropriate monochromatic component. Note that the obtained induced subgraphs are not necessarily disjoint. Then $\Gamma'_1, \ldots, \Gamma'_l$ are the nontrivial maximal k-subgraphs of Γ by Lemma 22. Again, by Lemma 22, the trivial maximal k-subgraphs of Γ are the paths containing exactly $k+1$ vertices in a long bridge. These can also be computed in $O(|E|)$ time by going through all long bridges. By item (5b) of Theorem 1, the defect k group of Γ as a permutation group is the direct product of the defect k groups of $\Gamma'_1, \dots \Gamma'_l$, and the defect k groups of the trivial maximal k-subgraphs.

7. Complexity of the flow semigroup of (di)graphs

In this section we apply our results and the complexity lower bounds of [10] to verify [9, Conjecture 6.51i (1)] for 2-vertex connected graphs. That is, we prove that the Krohn–Rhodes (or group-) complexity of the flow semigroup of a 2-vertex connected graph with n vertices is $n-2$ (item (4^{c}) of Theorem 1). Then we derive item (4^{cc}) of Theorem 1 as a further consequences of our results.

For standard definitions on wreath product of semigroups, we refer the reader to e.g. [9, Definition 2.2]. A finite semigroup S is called *combinatorial* if and only if every maximal subgroup of S has one element. Recall that the *Krohn–Rhodes (or group-) complexity of a finite semigroup* S (denoted by $\#_G(S)$) is the smallest non-negative integer n such that S is a homomorphic image of a subsemigroup of the iterated wreath product

$$C_n \wr G_n \wr \cdots \wr C_1 \wr G_1 \wr C_0,$$

where $G_1, \dots, G_n$ are finite groups, $C_0, \dots, C_n$ are finite combinatorial semigroups, and $\wr$ denotes the wreath product (for the precise definition, see e.g. [9, Definition 3.13]). The definition immediately implies that if a finite semigroup S is the homomorphic image of a subsemigroup of T, then $\#_G(S) \leq \#_G(T)$. More can be found on the complexity of semigroups in e.g. [9, Chapter 3]. We need the following results on the complexity of semigroups.

Lemma 25 ([9, Prop. 6.49(b)]). *The flow semigroup K_n of the complete graph on $n \geq 2$ vertices has $\#_G(K_n) = n-2$.*

Lemma 26 ([10, Sec. 3.7]). *The complexity of the full transformation semigroup F_n on n points is $\#_G(F_n) = n-1$.*

The well-known $\mathcal{L}$-*order* is a pre-order, i.e. a transitive and reflexive binary relation, on the elements of a semigroup S given by $s_1 \succeq_{\mathcal{L}} s_2$ if $s_1 = s_2$ or $ss_1 = s_2$ for some $s \in S$. The $\mathcal{L}$-*classes* of S are the equivalence classes of the $\mathcal{L}$-order. The $\mathcal{L}$-classes are thus partially ordered by $L_1 \succeq_{\mathcal{L}} L_2$ if and only if $SL_1 \cup L_1 \supseteq SL_2 \cup L_2$. One says that a finite semigroup S is a T_1-*semigroup* if it is generated by some $\succeq_{\mathcal{L}}$-chain of its $\mathcal{L}$-classes, i.e. if there exist $\mathcal{L}$-classes $L_1 \succeq_{\mathcal{L}} \cdots \succeq_{\mathcal{L}} L_m$ of S such that $S = \langle L_1 \cup \ldots \cup L_m \rangle$. Equivalently, S is a T_1-semigroup if there exist $U_i \subseteq L_i$ ($1 \leq i \leq m$) for such a chain of $\mathcal{L}$-classes of S such that $S = \langle U_1 \cup \ldots \cup U_m \rangle$.

Lemma 27 ([10, Lemma 3.5(b)]). *Let S be a noncombinatorial T_1-semigroup. Then*

$$\#_G(S) \geq 1 + \#_G(EG(S)),$$

where $EG(S)$ is the subsemigroup of S generated by all its idempotents.

Now we prove [9, Conjecture 6.51i (1)] for 2-vertex connected graphs.

Proof of item (4^c) of Theorem 1. Let Γ be a 2-vertex connected simple graph with $n \geq 2$ vertices. Let K_n denote the flow semigroup of the complete graph on vertices V, where $|V| = n$. Then $\#_G(S_\Gamma) \leq \#_G(K_n) = n - 2$ by Lemma 25. We proceed by induction on n. If $n \leq 3$, then Γ is a complete graph, and $\#_G(S_\Gamma) = n-2$ by Lemma 25. From now on we assume $n > 3$ and $\Gamma = (V, E)$.

Case 1. Assume first that Γ is not a cycle. Let (u, v) and (x, y) be two disjoint edges in Γ. Let G_1 be the defect 1 group with defect set $V \setminus \{u\}$ and idempotent e_{uv} as its identity element. Then $e_{uv} \succeq_{\mathcal{L}} e_{xy}e_{uv} = e_{uv}e_{xy}$. Let T be $\langle G_1 \cup \{e_{uv}e_{xy}\}\rangle$. Since $G_1 \succeq_{\mathcal{L}} \{e_{uv}e_{xy}\}$ is an $\mathcal{L}$-chain in T, T is a T_1-semigroup. Furthermore, T is noncombinatorial since G_1 is nontrivial. Thus, by Lemma 27

$$\#_G(T) \geq 1 + \#_G(EG(T)). \tag{1}$$

Let Γ' be the complete graph on $V \setminus \{u\}$. Let $a, b \in V \setminus \{u\}$ be arbitrary distinct vertices. By item (4) of Theorem 1, G_1 is 2-transitive. Let $\pi \in G_1$ be such that $\pi(x) = a$ and $\pi(y) = b$. There is a positive integer $\omega > 1$, with $\pi^\omega = e_{uv}$. In particular, e_{uv} commutes with π. Observe that

$$\pi^{\omega-1}e_{uv}e_{xy}\pi = e_{uv}\left(\pi^{\omega-1}e_{xy}\pi\right) = e_{uv}e_{ab}, \text{ and thus}$$
$$\left(\pi^{\omega-1}e_{xy}e_{uv}\pi\right)\restriction_{V\setminus\{u\}} = e_{ab}.$$

That is, we obtain the generators e_{ab} of $S_{\Gamma'}$ by restricting the idempotents $e_{uv}e_{ab} \in T$ to $V \setminus \{u\}$. Therefore, $S_{\Gamma'}$ is a homomorphic image of a subsemigroup of $EG(T)$, yielding

$$\#_G(EG(T)) \geq \#_G(S_{\Gamma'}).$$

By induction, $\#_G(S_{\Gamma'}) = n - 3$. Applying (1), we obtain $\#_G(T) \geq n-2$. Since T is a subsemigroup of S_Γ, we obtain $\#_G(S_\Gamma) \geq \#_G(T) \geq n - 2$.

Case 2. Assume now that Γ is the n-node cycle $(u, v_1, \ldots, v_{n-1})$. Then (u, v_1) and (v_2, v_3) are disjoint edges. Let $G_1 \simeq Z_{n-1}$ be the defect 1 group with defect set $V \setminus \{u\}$ and idempotent e_{uv_1} as its identity element. Let π be a generator of G_1 with cycle structure $(v_1, \ldots, v_{n-1})$. Then $e_{uv_1} \succeq_{\mathcal{L}} e_{v_2v_3}e_{uv_1} = e_{uv_1}e_{v_2v_3}$. Let T be $\langle G_1 \cup \{e_{uv_1}e_{v_2v_3}\}\rangle$. Since $G_1 \succeq_{\mathcal{L}} \{e_{uv_1}e_{v_2v_3}\}$ is an $\mathcal{L}$-chain in T, T is a T_1-semigroup. Furthermore, T is noncombinatorial since G_1 is nontrivial. Thus, by Lemma 27

$$\#_G(T) \geq 1 + \#_G(EG(T)). \tag{2}$$

Let Γ' be an $(n-1)$-node cycle with nodes $V \setminus \{ u \} = \{ v_1, \ldots, v_{n-1} \}$. Note that $e_{uv_1} = \pi^{n-1}$, and therefore e_{uv_1} commutes with π. Let $v_{i-1}, v_i, v_{i+1} \in V \setminus \{ u \}$ be three neighboring nodes in Γ', where the indices are in $\{ 1, \ldots, n-1 \}$ taken modulo $n-1$. Observe that

$$\pi^{n-2} e_{uv_1} e_{v_{i-1}v_i} \pi = e_{uv_1} \left(\pi^{n-2} e_{v_{i-1}v_i} \pi \right) = e_{uv_1} e_{v_i v_{i+1}}, \text{ and thus}$$
$$\left(\pi^{n-2} e_{uv_1} e_{v_{i-1}v_i} \pi \right) \restriction_{V \setminus \{ u \}} = e_{v_i v_{i+1}}.$$

That is, we obtain the generators $e_{v_i v_{i+1}}$ of $S_{\Gamma'}$ by restricting the idempotents $e_{uv_1} e_{v_i v_{i+1}} \in T$ to $V \setminus \{ u \}$. Therefore, $S_{\Gamma'}$ is a homomorphic image of a subsemigroup of $EG(T)$, yielding

$$\#_G \left(EG(T) \right) \geq \#_G \left(S_{\Gamma'} \right).$$

By induction, $\#_G \left(S_{\Gamma'} \right) = n-3$. Applying (2), we obtain $\#_G (T) \geq n-2$. Since T is a subsemigroup of S_Γ, we have $\#_G (S_\Gamma) \geq \#_G (T) \geq n-2$. □

Note that by Lemma 4 a strongly connected digraph has the same flow semigroup as the corresponding graph. Thus, item (4^{c}) of Theorem 1 proves Rhodes's conjecture [9, Conjecture 6.51i (1)] for 2-vertex connected strongly connected digraphs, as well. The following lemma bounds the complexity in the remaining cases.

Lemma 28. *Let k be the smallest positive integer such that for a graph Γ the flow semigroup S_Γ has defect k group S_{n-k}. Then $\#_G (S_\Gamma) \geq n-1-k$.*

Proof. Assume first $k = n-1$. Then the lemma holds trivially. From now on, assume $k \leq n-2$. Let uv be an edge in Γ. Let V_k be an arbitrary k-element subset of the vertex set V disjoint from $\{ u, v \}$. Let G_k be the defect k group with defect set V_k. Let S be the subsemigroup of S_Γ generated by G_k and e_{uv}. As $G_k \simeq S_{n-k}$, we have that S is the semigroup of all transformations on $V \setminus V_k$. Hence, $\#_G (S) = \#_G (F_{n-k}) = n-k-1$ by Lemma 26. Whence, $\#_G (S_\Gamma) \geq \#_G (S) = n-k-1$. □

By Theorem 19, it immediately follows that the complexity of the flow semigroup of a 2-edge connected graph Γ is at least $n-3$. Furthermore, $\#_G (S_\Gamma) \leq \#_G (K_n) = n-2$ by Lemma 25. This finishes the proof of item (4^{cc}) of Theorem 1.

References

[1] P. J. Cameron. *Permutation Groups*, volume 45 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1999.

[2] A. H. Clifford and G. B. Preston. *The Algebraic Theory of Semigroups. Vol. I.* Number 7 in Mathematical Surveys. American Mathematical Society, Providence, R.I., 1961.

[3] A. H. Clifford and G. B. Preston. *The Algebraic Theory of Semigroups. Vol. II.* Number 7 in Mathematical Surveys. American Mathematical Society, Providence, R.I., 1967.

[4] R. Diestel. *Graph Theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, Heidelberg, fourth edition, 2010.

[5] J. D. Dixon and B. Mortimer. *Permutation Groups*, volume 163 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1996.

[6] J. East, M. Gadouleau, and J. D. Mitchell. On transformation semigroups based on digraphs. 2017. submitted, `https://arxiv.org/abs/1704.00937`.

[7] R. L. Graham, M. Grötschel, and L. Lovász, editors. *Handbook of Combinatorics. Vol. 1, 2.* Elsevier Science B.V., Amsterdam; MIT Press, Cambridge, MA, 1995.

[8] J. Hopcroft and R. Tarjan. Algorithm 447: Efficient algorithms for graph manipulation. *Commun. ACM*, 16(6):372–378, June 1973.

[9] J. Rhodes. *Applications of Automata Theory and Algebra: Via the Mathematical Theory of Complexity to Biology, Physics, Psychology, Philosophy, and Games*. World Scientific Publishing Co. Pte. Ltd., Singapore, 2010. Edited by Chrystopher L. Nehaniv, with a foreword by Morris W. Hirsch. [Original Version: University of California at Berkeley Mathematics Library, 1971].

[10] J. Rhodes and B. R. Tilson. Lower bounds for complexity of finite semigroups. *J. Pure Appl. Algebra*, 1(1):79–95, 1971.

[11] H. E. Robbins. Questions, Discussions, and Notes: A Theorem on Graphs, with an Application to a Problem of Traffic Control. *Amer. Math. Monthly*, 46(5):281–283, 1939.

[12] D. J. S. Robinson. *A Course in the Theory of Groups*, volume 80 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1996.

[13] R. E. Tarjan. A note on finding the bridges of a graph. *Information Processing Lett.*, 2:160–161, 1973/74.

[14] H. Whitney. Non-separable and planar graphs. *Trans. Amer. Math. Soc.*, 34(2):339–362, 1932.

[15] R. M. Wilson. Graph puzzles, homotopy, and the alternating group. *J. Combinatorial Theory Ser. B*, 16:86–96, 1974.

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