THE MAXIMAL SUBGROUPS AND THE COMPLEXITY OF THE FLOW SEMIGROUP OF
FINITE (DI)GRAPHS

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Abstract. The flow semigroup, introduced by John Rhodes, is an invariant for digraphs and a complete invariant for graphs. We refine and prove Rhodes’s conjecture on the structure of the maximal groups in the flow semigroup for finite, antisymmetric, strongly connected graphs.

Building on this result, we investigate and fully describe the structure and actions of the maximal subgroups of the flow semigroup acting on all but \(k\) points for all finite digraphs and graphs for all \(k \geq 1\). A linear algorithm is presented to determine these so-called ‘defect \(k\) groups’ for any finite (di)graph.

Finally, we prove that the complexity of the flow semigroup of a 2-vertex connected (and strongly connected digraph with \(n\) vertices is \(n - 2\), completely confirming Rhodes’s conjecture for such (di)graphs.

1. Introduction

John Rhodes in [9] introduced the flow semigroup, an invariant for graphs and digraphs. In the case of graphs, this is a complete invariant determining the graph up to isomorphism. The flow semigroup is the semigroup of transformations of the vertices generated by elementary collapsings corresponding to the edges of the (di)graph. (See Section 2 for precise definitions.)

A maximal subgroup of this semigroup for a finite (di)graph \(D = (V_D, E_D)\) acts by permutations on all but \(k\) of its vertices \((1 \leq k \leq |V_D| - 1)\) and is called a “defect \(k\) group”. The set of defect \(k\) groups

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of a (di)graph is also an invariant. For each fixed \( k \), they are all iso-
"morphic to each other in the case of (strongly) connected (di)graphs. 
Rhodes formulated a conjecture on the structure of these groups for 
strongly connected digraphs whose edge relation is anti-symmetric in 
[9, Conjecture 6.51i (2)–(4)]. We show that his conjecture was correct, 
and we prove it here in sharper form (Theorems 1 and 27). Moreover, 
extending this result further, we fully determine the defect \( k \) groups 
for all finite graphs and digraphs (Theorem 2).

The structure of the argument is as follows. First, a maximal group 
in the flow semigroup of a digraph \( D \) is the direct product of maximal 
groups of the flow semigroups of its strongly connected components. 
Thus one needs only to consider strongly connected digraphs. The de-
fect \( k \) group of \( D \) consists of elements of the flow semigroup permut-
ing all but \( k \) vertices. It turns out, that if \( D \) is a strongly connected di-
graph, then the defect \( k \) group (up to isomorphism) does not depend on 
the choice of the vertices it acts on. Further, for a strongly connected 
digraph, its flow semigroup is the same as the flow semigroup of the 
simple graph obtained by “forgetting” the direction of the edges. This 
is detailed in Section 2 and is based on [9, p. 159–169]. Thus, one only 
needs to consider the defect \( k \) groups of the flow semigroup for simple 
connected graphs.

In Section 3 we list some useful lemmas and determine the defect 
\( k \) group of a cycle. Further, in Section 4 we lay some group theory 
groundwork by determining the permutation group \( T_{k,l,m} \) generated by 
two cycles \((a_1, \ldots, a_k, b_1, \ldots, b_l)\) and \((a_1, \ldots, a_k, c_1, \ldots, c_m)\). Then in 
Section 5 we first determine the defect 1 group of 2-vertex connected 
graphs, then of arbitrary simple connected graphs by proving

**Theorem 1** (Defect 1 group for simple connected graphs). Let \( \Gamma \) be a 
**simple connected graph of \( n \) vertices, and let \( \Gamma_1, \ldots, \Gamma_m \) be its 2-vertex 
connected components. Then the defect 1 group of \( \Gamma \) is the direct pro-
duct of the defect 1 groups of \( \Gamma_i \) (1 \( \leq i \leq m \)). If \( \Gamma \) is 2-vertex connected, 
then its defect 1 group is isomorphic (as a permutation group) to

\[ \text{(1) the cyclic group } Z_{n-1} \text{ if } \Gamma \text{ is a cycle;} \]
\[ \text{(2) } T_{2,2,2} \text{ (that is } S_5 \text{ acting sharply 3-transitively on 6 points), if } \Gamma \text{ is the exceptional graph (see Figure 1);} \]
\[ \text{(3) } S_{n-1} \text{ or } A_{n-1} \text{, otherwise. Further, the defect 1 group is } A_{n-1} \text{ if and only if } \Gamma \text{ is bipartite.} \]

In particular, Rhodes's conjecture (as phrased in [9, Conjecture 6.51i (2)]
for strongly connected, antisymmetric digraphs) about the defect 1 
group holds, and more generally: the defect 1 group of the flow semi-
group of a simple connected graph is indeed the product of cyclic, 
alternating and symmetric groups of various orders. Applying Theo-
rem 1, a straightforward linear algorithm can be given to determine
the direct components of the defect 1 group of an arbitrary connected graph (see Section 7).

Then in Section 6 we determine the defect $k$ groups of arbitrary graphs by considering the so-called $k$-components (maximal subgraphs for which the defect $k$ group is the full symmetric group) and prove

**Theorem 2.** Let $k \geq 2$, $\Gamma$ be a simple connected graph of $n$ vertices, $n > k$. If $\Gamma$ is a cycle, then its defect $k$ group is the cyclic group $\mathbb{Z}_{n-k}$. Otherwise, let $\Gamma_1, \ldots, \Gamma_m$ be the $k$-components of $\Gamma$, and let $\Gamma_i$ have $n_i$ vertices. Then the defect $k$ group of $\Gamma$ is the direct product of the defect $k$ groups of $\Gamma_i$ ($1 \leq i \leq m$), thus it is isomorphic (as a permutation group) to

$$S_{n_1-k} \times \cdots \times S_{n_m-k}.$$ 

In Section 7 we provide a linear algorithm (in the number of edges of $\Gamma$) to determine the $k$-components of an arbitrary connected graph.

Rhodes further conjectured [9, Conjecture 6.51i (1)] that the complexity of the flow semigroup of a strongly connected, antisymmetric digraph on $n$ vertices is $n - 2$. We confirm this conjecture in Section 8 when the digraph is 2-vertex connected:

**Theorem 3.** Let $\Gamma = (V, E)$ be a 2-vertex connected graph with $n \geq 2$ vertices. Then $\#G(S_\Gamma) = n - 2$.

Finally, we prove some bounds on the complexity of flow semigroups in the remaining cases, and state some open problems on the complexity of flow semigroups at the end of Section 8.

2. Flow semigroup of digraphs

For notions in graph theory we refer to [4, 6], in group theory to [12] in permutation groups to [1, 5], in semigroup theory to [2, 3]. For a digraph $D = (V_D, E_D)$, the flow semigroup $S = S_D$ is defined by

$$S = S_D = \langle e_{uv} \mid uv \in E_D \rangle,$$
where $e_{uv}$ is the elementary collapsing corresponding to the directed edge $uv \in E_D$, that is, for every $x \in V_D$ we have

$$x \cdot e_{uv} = xe_{uv} = \begin{cases} v, & \text{if } x = u, \\ x, & \text{otherwise}. \end{cases}$$

Note that in this paper functions act on the right, therefore permutations are multiplied from left to right. Further, for a set $X \subseteq V_D$ and a semigroup element $s \in S_D$ we define

$$Xs = \{ xs \mid x \in X \}.$$

A maximal subgroup of $S_D$ is a subgroup such that it is not contained properly in any other subgroup of $S_D$. In order to determine the maximal subgroups of $S_D$, one can make several reductions by [9, Proposition 6.51f]. First, one only needs to consider the maximal subgroups of $S_{D_i}$ for the strongly connected components $D_i$ of $D$. Strongly connected components are maximal induced subgraphs such that any vertex can be reached from any other vertex by a directed path.

**Lemma 4** ([9, Proposition 6.51f (1)]). Let $D$ be a digraph, then every maximal subgroup of $S_D$ is (isomorphic to) the direct product of maximal subgroups of $S_{D_i}$, where the $D_i$ are the strongly connected components of $D$.

An element $s \in S$ is of defect $k$ if $|V_D s| = |V_D| - k$. Let $V_k = \{ v_1, v_2, \ldots, v_k \} \subseteq V_D$. The defect $k$ group $G_k = G_{k,V_k}$ associated to $V_k$ (called the defect set) is generated by all elements of $S$ restricted to $V \setminus V_k$ which permute the elements of $V \setminus V_k$ and move elements of $V_k$ to elements of $V \setminus V_k$:

$$G_k = G_{k,V_k} = \langle s \mid_{V \setminus V_k} : s \in S, (V \setminus V_k)s = V \setminus V_k, V_ks \subseteq V \setminus V_k \rangle.$$

Now, $G_k$ is a permutation group acting on $V \setminus V_k$. For this reason $V \setminus V_k$ is called the permutation set of $G_k$. In general, the defect $k$ group $G_k$ can depend on the choice of $V_k$. However, by [9, Proposition 6.51f (2)] it turns out that if the graph is strongly connected then the defect $k$ group $G_k$ is unique up to isomorphism.

**Lemma 5** ([9, Proposition 6.51f (2)]). Let $D$ be a strongly connected digraph. Let $V_k, V'_k \subseteq D$ be subsets of nodes such that $|V_k| = |V'_k| = k$. Then $G_{k,V_k} \simeq G_{k,V'_k}$ as permutation groups.

Further, the case of strongly connected graphs can be reduced to the case of simple graphs. Let $\Gamma = (V, E)$ be a simple (undirected) graph, we define $S_\Gamma$ by considering $\Gamma$ as a directed graph where every edge is directed both ways. Namely, let $D_\Gamma = (V, E_D)$ be the directed graph on vertices $V$ such that both $uv \in E_D$ and $vu \in E_D$ if and only if the undirected edge $uv \in E$. Then let $S_\Gamma = S_{D_\Gamma}$.

Further, for every digraph $D = (V_D, E_D)$, one can associate an undirected graph $\Gamma$ by "forgetting" the direction of edges in $D$. Precisely,
Let $\Gamma_D = (V_D, E_D)$ be the undirected graph such that $uv \in E$ if and only if $uv \in E_D$ or $vu \in E_D$. The following lemma shows that if a digraph $D$ is strongly connected then the semigroup $S_D$ corresponding to $D$ and the semigroup $S_{\Gamma_D}$ corresponding to the simple graph $\Gamma_D$ are the same.

Lemma 6 ([9, Lemma 6.51b]). Let $D$ be an arbitrary digraph. Then

$$e_{ab} \in S_D \iff \begin{cases} a \to b \text{ is an edge in } D, \\
 b \to a \text{ is an edge in a directed cycle in } D. \end{cases}$$

In particular, if $D$ is strongly connected then $S_D = S_{\Gamma_D}$.

Proof. Let $b \to a \to u_1 \to \cdots \to u_{n-1} \to b$ be a directed cycle in $D$. Then an easy calculation shows that

$$e_{ab} = (e_{ba}e_{u_{n-1}}b e_{u_{n-2}}u_{n-1}e_{u_1}u_2 e_{au_1})^n.$$ 

For the other direction, assume $e_{ab} = e_{uv}s$ for some $s \in S_D$. Then $e_{uv}s$ moves $u$ and $v$ to the same vertex, while $e_{ab}$ moves only $a$ and $b$ to the same vertex. Thus $\{a, b\} = \{u, v\}$.

Therefore, in the following we only consider simple, connected, undirected graphs $\Gamma = (V, E)$, that is no self-loops or multiple edges are allowed. Further, $\Gamma$ is 2-edge connected if removing any edge does not disconnect $\Gamma$. Rhodes's conjecture [9, Conjecture 6.51i (2)-(4)] is about strongly connected, antisymmetric digraphs. Note that by [11] a strongly connected antisymmetric digraph becomes a 2-edge connected graph after forgetting the directions.

Let us set some notations. By cycle we will mean a simple cycle, that is a closed walk with no repetition of edges or vertices except for the starting and ending vertex. A path is a walk with no repetition of edges or vertices. By subgraph $\Gamma' = (V', E')$ we mean a graph for which $V' \subseteq V$, $E' \subseteq E$. If $\Gamma'$ is an induced subgraph, that is $E'$ consists of all edges from $E$ with both endpoints in $V'$, then we explicitly indicate it. The letters $k, l, m$ and $n$ will denote nonnegative integers. The number of vertices of $\Gamma$ is usually denoted by $n$, while $k$ will denote the size of the defect set. Usually we denote the defect $k$ group of a graph $\Gamma$ by $G_k$ or $G_{\Gamma}$, depending on the context. We try to heed to the convention of using $u$, $v$, $w$, $x$, $y$ as vertices of graphs, $V$ as the set of vertices, $E$ as the set of edges. Further, the flow semigroup is mostly denoted by $S$, its elements are denoted by $s$, $t$, $g$, $h$, $p$, $q$. In Section 4 we use $x$ and $y$ for denoting permutations. The cyclic group of $m$ elements is denoted by $Z_m$.

3. Preliminaries

Let $\Gamma = (V, E)$ be a simple, connected (undirected) graph, and for every $1 \leq k \leq |V| - 1$, let $G_k$ denote its defect $k$ group for some $V_k \subseteq V$, ...
Lemma 7 ([9, Fact 6.51c]). Let \( s \in S \) be of defect \( k \). If \( s e_{uv} \) is of defect \( k \), as well, then \( u \notin V s \) or \( v \notin V s \).

Further, it is not too hard to see that every defect 1 permutation arises from the permutations generated by cycles (in the graph) containing the defect point.

Lemma 8 ([9, Proposition 6.51e]). Let \( \Gamma \) be a connected graph, and let \( G_1 \) denote its defect 1 group, such that the defect is \( v \in V \). Then

\[
G_1 = \langle (u_1, \ldots, u_k) \mid (u_1, \ldots, u_k, v) \text{ is a cycle in } \Gamma \rangle.
\]

These yield that the defect \( k \) group of the \( n \)-cycle graph is cyclic:

Lemma 9. The defect \( k \) group of the \( n \)-cycle is isomorphic to \( \mathbb{Z}_{n-k} \).

Proof. Let \( x_1, x_2, \ldots, x_n \) be the (clockwise) consecutive elements of the cycle \( \Gamma = (V, E) \). If \( s \in S \) is an element of defect \( k \) then by Lemma 7 we have that \( s e_{x_i x_{i+1}} \) is of defect \( k \) if and only if \( x_i \notin V s \) or \( x_{i+1} \notin V s \). This means that if \( u_1, u_2, \ldots, u_{n-k} \) are the (clockwise) consecutive elements of \( V s \) in the cycle and \( s e_{x_i x_{i+1}} \) is of defect \( k \), as well, then

\[
u_1 e_{x_i x_{i+1}}, u_2 e_{x_i x_{i+1}}, \ldots, u_{n-k} e_{x_i x_{i+1}} \]

are the (clockwise) consecutive elements of \( V s e_{x_i x_{i+1}} \). Thus the cyclic ordering of these elements cannot be changed. Hence \( G_k \) is isomorphic to a subgroup of \( \mathbb{Z}_{n-k} \).

Now, assume that \( v_1, v_2, \ldots, v_k, u_1, u_2, \ldots, u_{n-k} \) are the consecutive elements of \( \Gamma \), and the defect set is \( V_k = \{ v_1, \ldots, v_k \} \). Let

\[
\begin{align*}
s_1 &= e_{v_1 v_2} \cdots e_{v_j v_{j+1}} \cdots e_{v_k v_1}, \\
s_2 &= e_{u_{n-k} v_k} e_{u_{n-k-1} u_{n-k}} \cdots e_{u_j u_{j+1}} \cdots e_{u_1 u_2} e_{v_k u_1}, \\
s &= s_1 s_2.
\end{align*}
\]

It easy to check that

\[
v_1 s = u_1, \quad u_1 s = u_2, \ldots, u_j s = u_{j+1}, \ldots, u_{n-k} s = u_1.
\]

Therefore \( s, s^2, \ldots, s^{n-k} \) are distinct elements of \( G_k \), hence \( G_k \simeq \mathbb{Z}_{n-k} \). \( \square \)

4. Finite permutation groups generated by two cycles

In this section we investigate the group \( T_{k,l,m} \) which is generated by two overlapping cycles. Let \( k, l, m \) be non-negative integers, \( n = k + l + m \geq 1 \), let

\[
x = (a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_l), \\
y = (a_1, a_2, \ldots, a_k, c_1, c_2, \ldots, c_m),
\]
and we define
\[ T_{k,l,m} = \langle x, y \rangle \]
as the subgroup of the symmetric group \( S_n \) generated by the cycles \( x \) and \( y \). The elements \( a_1, \ldots, a_k, b_1, \ldots, b_l, c_1, \ldots, c_m \) are pairwise distinct. We prove that the group \( T_{k,l,m} \) is either the symmetric or alternating group of degree \( n \), apart from a few exceptions.

**Theorem 10.** Let \( k, l, m \) be non-negative integers, \( n = k + l + m \geq 1 \), and let \( T_{k,l,m} \) be the group defined in (1). Then one of the following holds.

1. If \( k = 0 \) or \( k + l = 1 \) or \( k + m = 1 \) then \( T_{k,l,m} \cong \mathbb{Z}_{k+l} \times \mathbb{Z}_{k+m} \);
2. if \( k \geq 1, k + l \) and \( k + m \) are both odd, then \( T_{k,l,m} = A_n \);
3. \( T_{3,2,1} \cong T_{2,2,2} \cong T_{3,1,2} \cong S_5 \), and this is a sharply 3-transitive action of \( S_5 \) on 6 elements;
4. \( T_{k,l,m} = S_n \), otherwise.

**Proof.** We follow the convention of permutations acting on the right, therefore we multiply permutations from left to right. Further, the conjugation of \( x \) by \( y \) is \( x^y = y^{-1}xy \).

If \( k + l = 1 \) then \( x = id \), if \( k + m = 1 \) then \( y = id \). If \( k = 0 \), then \( x \) and \( y \) are disjoint and thus \( T_{0,l,m} \cong \mathbb{Z}_l \times \mathbb{Z}_m \). From now on, we assume \( k \geq 1 \). The following technical lemma will help in handling the different cases.

**Lemma 11.** Let \( k, l, m \) be non-negative integers, and assume that \( k \geq 1 \). Then the following three groups are isomorphic as permutation groups:
\[ T_{k,l,m} \cong T_{l+1,k-1,m} \cong T_{m+1,l,k-1} \]

**Proof.** Now,
\[ x = (a_k, b_1, b_2, \ldots, b_l, a_1, \ldots, a_{k-1}), \]
and
\[ xy^{-1} = (a_k, b_1, b_2, \ldots, b_l, c_m, \ldots, c_1). \]
Therefore
\[ T_{l+1,k-1,m} \cong \langle x, xy^{-1} \rangle = \langle x, y \rangle = T_{k,l,m}, \]
and \( T_{m+1,l,k-1} \cong T_{k,l,m} \) follows by exchanging the roles of \( x \) and \( y \). □

We prove in Lemma 13 that the group \( T_{k,l,m} \) is 2-transitive, therefore primitive. For the proof of Theorem 10 we are going to use Jordan’s famous theorem on primitive permutation groups.

**Theorem 12** (Jordan, [5, Theorem 3.3E]). Let \( G \) be a primitive permutation group of degree \( n \). If \( G \) contains a 2-cycle, or a 3-cycle, or a \( p \)-cycle for some prime \( p \leq n-3 \), then \( G \) is either the whole symmetric groups \( S_n \) or the alternating group \( A_n \).
Finally, the cases $k = l = m = 2$ or $k = 3$, $\{l, m\} = \{2, 1\}$ will be handled separately in Lemma 14. First, we prove that $T_{k,l,m}$ is 2-transitive.

**Lemma 13.** If $k \geq 1$, then $T_{k,l,m}$ is a 2-transitive group on the elements

$$\Omega = \{ a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_l, c_1, c_2, \ldots, c_m \}.$$  

**Proof.** Let $\Omega_2$ be the set of the two-element subsets of $\Omega$. To prove 2-transitivity of $T_{k,l,m}$, we show that $T_{k,l,m}$ acts transitively on $\Omega_2$, and that there exists an element of $T_{k,l,m}$ that transposes two elements of $\Omega$.

Assume $l \geq m$, but for this lemma we do not assume $k \geq l$ or $k \geq m$.

First, assume $l \geq 1$, $m \geq 1$, and consider the orbit of the subset $\{b_1, c_1\}$. Then

$$\begin{align*}
\{ b_1, c_j \} &= \{ b_1, c_1 \} x^{i-1} y^{j-1}, & 1 \leq i \leq l, 1 \leq j \leq m, \\
\{ a_i, b_j \} &= \{ b_j, c_m \} y^i, & 1 \leq i \leq k, 1 \leq j \leq l, \\
\{ a_i, c_j \} &= \{ b_1, c_j \} x^i, & 1 \leq i \leq k, 1 \leq j \leq m, \\
\{ a_i, a_j \} &= \{ a_{k+i-j+1}, b_1 \} x^{-k+j-1}, & 1 \leq i < j \leq k, \\
\{ b_i, b_j \} &= \{ b_{j+i-j+1}, a_1 \} x^{-l+j-1}, & 1 \leq i < j \leq l, \\
\{ c_i, c_j \} &= \{ c_{m+i-j+1}, a_1 \} y^{-m+j-1}, & 1 \leq i < j \leq m.
\end{align*}$$

Further, the permutation $x^{i-1} y^{m-1} x y^k x$ transposes $b_1$ and $c_1$.

Finally, if $l \geq 1$ and $m = 0$, or if $m \geq 1$, $l = 0$, then by Lemma 11 we have $T_{k,l,0} \simeq T_{1,l,k-1}$, or $T_{k,0,m} \simeq T_{1,k-1,m}$. □

The roles of $l$ and $m$ are symmetric, and $T_{k,l,m} \simeq T_{k,m,l}$, thus we may assume $l \geq m$. Further, by Lemma 11 we may assume $k \geq l + 1$, otherwise we consider $T_{l+1,k-1,m}$ instead. For finding a 3-cycle, we need to consider several cases.

First, assume $m = 0$. From

$$x = (a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_l),$$

$$y = (a_1, a_2, \ldots, a_k),$$

we have

$$
x^y = y^{-1} x y = (a_1 y, a_2 y, \ldots, a_k y, b_1 y, b_2 y, \ldots, b_l y) = (a_2, \ldots, a_k, a_1, b_1, b_2, \ldots, b_l),
$$

$$x^y x^{-1} = (a_1, a_k, b_l) \in T_{k,l,m}.$$
If $l = m = 1$, then
\[
x = (a_1, a_2, \ldots, a_k, b_1),
y = (a_1, a_2, \ldots, a_k, c_1),
\]
and we have
\[
xy^{-1} = (a_k, b_1, c_1) \in T_{k,l,m}.
\]
If neither $l = 1$, nor $m = 0$ hold, then we have $l \geq 2$, $m \geq 1$, $k \geq l+1$.
Note that if $k = 3$, then $m \geq 2$, otherwise we have $(k, l, m) = (3, 2, 1)$.
Assume $(k, l, m) \neq (3, 2, 1)$. Now, we have
\[
xy = y^{-1}xy = (a_1y, a_2y, \ldots, a_ky, b_1y, b_2y, \ldots, b_y)
= (a_2, \ldots, a_k, c_1, b_1, b_2, \ldots, b_t).
\]
Let
\[
s_1 = xyx^{-1} = (a_1, b_t) (a_k, c_1),
s_2 = s_1^{-1} = (b_1, b_{t-1}) (a_{k-1}, c_1),
s_3 = s_2^{-2} = \begin{cases} (b_t, b_{t-1}) (a_{k-3}, a_{k-1}), & \text{if } k \geq 4, \\
(b_t, b_{t-1}) (c_m, a_{k-1}), & \text{if } k = 3. \end{cases}
\]
If $k \geq 4$, then
\[
s_2 s_3 = (a_{k-3}, a_{k-1}, c_1) \in T_{k,l,m},
\]
otherwise $m \geq 2$, and
\[
s_2 s_3 = (a_{k-1}, c_1, c_m) \in T_{k,l,m}.
\]
Finally, we consider the remaining case $(k, l, m) = (3, 2, 1)$.

**Lemma 14.** $T_{3,2,1} \simeq T_{2,2,2} \simeq T_{3,1,2} \simeq S_5$, and this is a sharply 3-transitive action of $S_5$ on 6 elements.

**Proof.** By Lemma 11, we have $T_{3,2,1} \simeq T_{2,2,2} \simeq T_{3,1,2}$. We consider $T_{2,2,2}$ in the following. The symmetric group $S_5$ contains 6 Sylow 5-subgroups. Let them be
\[
P_1 = \langle (1, 2, 3, 4, 5) \rangle, \quad P_2 = \langle (1, 2, 4, 5, 3) \rangle, \quad P_3 = \langle (1, 2, 5, 3, 4) \rangle,
P_4 = \langle (1, 3, 2, 5, 4) \rangle, \quad P_5 = \langle (1, 2, 4, 5, 3) \rangle, \quad P_6 = \langle (1, 2, 5, 4, 3) \rangle.
\]
Let $g_1 = (1, 2, 3, 4)$, $g_2 = (5, 4, 3, 2)$. Now,
\[
\langle g_1, g_2 \rangle = \langle g_1, g_2^{-1} \rangle = \langle (2, 3, 4, 1), (2, 3, 4, 5) \rangle \simeq T_{3,1,1} \simeq S_5
\]
by the case $l = m = 1$. Let $\varphi : S_5 \rightarrow S_6$ be the conjugation action on these 6 Sylow subgroups. Then a straightforward calculation shows that
\[
\varphi (g_1) = (P_1, P_2, P_3, P_4), \quad \varphi (g_2) = (P_1, P_2, P_5, P_6).
\]
Thus, \( \langle \varphi(g_1), \varphi(g_2) \rangle \simeq T_{2,2,2} \), and is transitive on \( \{ P_1, \ldots, P_6 \} \). Further, \( \ker \varphi = \{ id \} \), otherwise it would contain \( A_5 \), and then the conjugation action could not have an element of order 4. Thus,

\[
S_5 \simeq \langle g_1, g_2 \rangle \simeq \langle \varphi(g_1), \varphi(g_2) \rangle \simeq T_{2,2,2}.
\]

Further, the stabilizer of \( P_6 \) in \( \varphi(G) \) has 20 elements, therefore contains an element of order 5, that is it contains a 5-cycle. Thus, \( \varphi(S_5) \) is sharply 3-transitive on the 6 Sylow 5-subgroups. □

This finishes the proof of Theorem 10. □

5. Defect 1 Groups

We now prove Theorem 1. Let us start with the exceptional case.

**Lemma 15.** Let \( \Gamma \) be the exceptional graph (Figure 1). Then for the defect 1 group of \( \Gamma \) we have \( G_\Gamma \simeq T_{2,2,2} \).

**Proof.** Let us denote the vertices of \( \Gamma \) as in Figure 1. Let \( v \) be the defect. Now, by Lemma 8 we have

\[
G_\Gamma = \langle (1,2,3,4), (1,2,5,6), (4,3,2,5,6) \rangle = \langle (1,2,3,4), (1,2,5,6) \rangle,
\]

because \( (4,3,2,5,6) = (1,2,5,6)(1,2,3,4)^{-1} \). Thus, \( G_\Gamma \simeq T_{2,2,2} \). □

We will need the notion of open ear, and open ear decomposition.

**Definition 16.** Let \( \Gamma \) be an arbitrary graph, and let \( \Gamma' \subset \Gamma \). A path \( (u,c_1,\ldots,c_m,v) \) is called a \( \Gamma' \)-ear (or open ear) with respect to \( \Gamma \), if \( u \neq v \in \Gamma' \), and either \( m = 0 \) and the edge \( uv \notin \Gamma' \), or \( c_1,\ldots,c_m \in \Gamma \setminus \Gamma' \). An open ear decomposition of a graph is a partition of its set of edges into a sequence of subsets, such that the first element of the sequence is a cycle, and all other elements of the sequence are open ears of the union of the previous subsets in the sequence.

First we consider the case, where \( \Gamma \) is 2-vertex connected. A connected graph \( \Gamma \) with at least \( k \) vertices is \( k \)-vertex connected if removing any \( k - 1 \) vertices does not disconnect \( \Gamma \). By [14] a graph is 2-vertex connected if and only if it is a single edge or it has an open ear decomposition. This result and Theorem 10 from Section 4 play a crucial role in proving Theorem 1.

**Proof of Theorem 1 if \( \Gamma \) is 2-vertex connected.** Let us consider an open ear decomposition of \( \Gamma \). We prove the statement by induction on the number of open ears. If \( \Gamma \) is a cycle, then its defect 1 group is isomorphic to \( Z_{n-1} \) by Lemma 8. Further, if \( \Gamma \) is the exceptional graph, then its defect 1 group is \( T_{2,2,2} \) by Lemma 15.
Now, assume that $\Gamma$ is not the exceptional graph and is the union of a 2-vertex connected graph $\Gamma'$ and a $\Gamma'$-ear $(u, c_1, \ldots, c_m, v)$, where $u, v \in \Gamma'$, $u \neq v$, $c_1, \ldots, c_m \notin \Gamma'$. Let the defect 1 group of $\Gamma'$ be denoted by $G_{\Gamma'}$, and the defect 1 group of $\Gamma$ be denoted by $G_{\Gamma}$, where the defect is $v$ (the defect 1 groups for different vertices are isomorphic by Lemma 5). We prove that $G_{\Gamma} \geq A_{n-1}$. Let $v, a_1, \ldots, a_k$ be a shortest path in $\Gamma'$ from $v$ to $u = a_k$, and let $y$ denote the permutation $y = (a_1, \ldots, a_k, c_1, \ldots, c_m)$.

If $\Gamma'$ is a cycle, then let us denote the vertices of $\Gamma$ according to this cycle by $v, a_1, \ldots, a_k, b_1, \ldots, b_l$. Let $x = (a_1, \ldots, a_k, b_1, \ldots, b_l)$. Then

$$G_{\Gamma} \geq \langle x, y \rangle = T_{k,l,m} \geq A_{n-1},$$

by Theorem 10. Note that $G_{\Gamma}$ is 2-transitive on $\Gamma \setminus \{v\}$.

Assume now, that $\Gamma'$ is not a cycle. Similarly as in Lemma 13, we prove that $G_{\Gamma}$ is 2-transitive on $\Gamma \setminus \{v\}$. By induction, $G_{\Gamma'}$ is 2-transitive on $\Gamma' \setminus \{v\}$. It is enough to prove that $G_{\Gamma}$ acts transitively on the two-element subsets of $\Gamma \setminus \{v\}$, because we can transpose two elements of $\Gamma'$ by the 2-transitivity of $G_{\Gamma'}$. Let $\Gamma \setminus \{a_1, \ldots, a_k, c_1, \ldots, c_m, v\} = \{b_1, \ldots, b_l\}$. Note that $k \geq 1$, $l \geq 1$. If $m = 0$, then $G_{\Gamma'}$ is already two-transitive on $\Gamma \setminus \{v\}$. Otherwise, we determine the orbit of $\{c_1, b_1\}$. Now, we have $\{c_1, b_1\} y^{-1} = \{a_k, b_1\}$, and thus (by the 2-transitivity of $G_{\Gamma'}$) all $\{w_1, w_2\}$ ($w_1, w_2 \in \Gamma'$) are in the orbit of $\{c_1, b_1\}$. Further, $\{c_1, b_1\} y^{i-1} = \{c_i, b_1\}$ for every $1 \leq i \leq m$, and thus all $\{c_i, w\}$ ($w \in \Gamma'$) are in the orbit of $\{c_1, b_1\}$ by the transitivity of $G_{\Gamma'}$. Finally, for $1 \leq i < j \leq m$, we have $\{c_i, c_j\} = \{a_k, c_{j-1}\} y^j$.

Now, $G_{\Gamma}$ is 2-transitive, and $G_{\Gamma'} \subseteq G_{\Gamma}$ contains a 3-cycle by induction and Theorem 10, unless $\Gamma'$ is the exceptional graph. Therefore, $A_{n-1} \leq G_{\Gamma}$ by Jordan's theorem (Theorem 12). If $\Gamma'$ is the exceptional graph (see Figure 1), then note that $k \leq 3$. In particular, if $m = 0$, then $y$ is either a 2-cycle or a 3-cycle, thus $A_{n-1} \leq G_{\Gamma}$ by Jordan's theorem (Theorem 12). Further, if $m \geq 2$, then $n \geq 8$, and $G_{\Gamma'}$ already has a 5-cycle, thus $A_{n-1} \leq G_{\Gamma}$ by Jordan's theorem (Theorem 12). Finally, if $m = 1$, then $n - 1 = 7$, and $G_{\Gamma} \leq S_7$ acts transitively on 7 points, hence 7 divides the order of $G_{\Gamma}$. Further, $|G_{\Gamma}| = 120$ divides $|G_{\Gamma}|$, and therefore $|S_7 : G_{\Gamma}| = \frac{7!}{120} = 6$. This yields $A_7 = A_{n-1} \leq G_{\Gamma}$.

Finally, note that $G_{\Gamma} \leq A_{n-1}$ if and only if every permutation corresponding to a cycle in $\Gamma$ is even, that is the length of every cycle in $\Gamma$ is even. This is equivalent to $\Gamma$ being bipartite [8].

Finally, Theorem 1 follows by induction on the number of 2-vertex connected components from Lemma 17.

**Lemma 17.** Let $\Gamma_1$ and $\Gamma_2$ be connected induced subgraphs of $\Gamma$ such that $\Gamma_1 \cap \Gamma_2 = \{v\}$, where there are no edges in $\Gamma$ between $\Gamma_1 \setminus \{v\}$ and $\Gamma_2 \setminus \{v\}$. Then the defect 1 group of $\Gamma_1 \cup \Gamma_2$ is the direct product of the defect 1 groups of $\Gamma_1$ and $\Gamma_2$. 


Let $G_\Gamma$ denote the defect 1 group of $\Gamma$, where the defect is $v$. By Lemma 8, $G_\Gamma$ is generated by cyclic permutations corresponding to cycles through $v$ in $\Gamma$. Now, $\Gamma_1 \cap \Gamma_2 = \{v\}$, and every path between a node from $\Gamma_1$ and a node from $\Gamma_2$ must go through $v$, hence every cycle in $\Gamma$ is either in $\Gamma_1$ or in $\Gamma_2$. Let $c_1^{(1)}, \ldots, c_1^{(m_1)}$ be the permutations corresponding to the cycles in $\Gamma_1$ ($i = 1, 2$). Then $c_1^{(j_1)}c_2^{(j_2)} = c_2^{(j_2)}c_1^{(j_1)}$ for all $1 \leq j_i \leq m_i$, $i = 1, 2$, thus $G_\Gamma = \langle c_1^{(1)}, \ldots, c_1^{(m_1)}, c_2^{(1)}, \ldots, c_2^{(m_2)} \rangle = G_{\Gamma_1} \times G_{\Gamma_2}$.

\[ \square \]

6. Defect $k$ groups

In the following we assume $k \geq 2$, and every graph $\Gamma$ is assumed to be simple connected. We start with some simple observations.

Lemma 18. Let $\Gamma' \subseteq \Gamma$ be connected graphs. Then the defect $k$ group of $\Gamma$ contains a subgroup isomorphic (as permutation group) to the defect $k$ group of $\Gamma'$. Further, if $\Gamma \setminus \Gamma'$ contains at least one vertex, then the defect $k$ group of $\Gamma$ contains a subgroup isomorphic (as permutation group) to the defect $k - 1$ group of $\Gamma'$.

Proof. Let $G_k$ be the defect $k$ group of $\Gamma$, $G'_k$ and $G'_{k-1}$ be the defect $k$ group and defect $k - 1$ group of $\Gamma'$, respectively. Now, every elementary collapsing of $\Gamma'$ is an elementary collapsing of $\Gamma$, as well. Thus $G'_k \leq G_k$ is clear.

Further, let $V_{k-1} = \{v_1, \ldots, v_{k-1}\} \subseteq \Gamma'$, $v$ be a vertex of $\Gamma \setminus \Gamma'$, and let $V_k = V_{k-1} \cup \{v\}$. Let $u$ be a neighbour of $v$ and let $e = vu$. Then for every permutation $g \in G'_{k-1}$ with defect set $V_{k-1}$ we have that $eg \in G_k$ with defect set $V_k$, where $eg$ is identical to $g$ on $\Gamma \setminus (\Gamma' \cup \{v\})$, and acts exactly the same on $\Gamma' \setminus V_{k-1}$ as $g$. In particular, $\varphi: G'_{k-1} \rightarrow G_k$, $\varphi(g) = eg$ is an injective homomorphism of permutation groups.

Lemma 19. Let $1 \leq m \leq l < k \leq n - 2$, and assume $\Gamma$ contains the following subgraph:
If the defect set contains $y, x_1, \ldots, x_l$, does not contain $v$, and does not contain $u_i$ for some $1 \leq i \leq m$, then the defect $k$ group contains the transposition $(u_i, v)$.

**Proof.** Let

$$r = \begin{cases} ss_1 e_{yx_1} e_{x_1 u_1}, & \text{if } i = 1, \\ ss_1 \ldots s_i p t t_{i-1} \ldots t_1 q, & \text{if } i \geq 2, \end{cases}$$

where

- $s = e_{vx_1} e_{x_1 x_{l-1}} \ldots e_{x_2 x_1} e_{x_1 y}$,
- $s_1 = e_{ux_1} e_{x_1 x_2} \ldots e_{x_{l-1} x_1} e_{x_1 v}$,
- $s_j = e_{u_j u_{j-1}} \ldots e_{u_2 u_1} e_{u_1 x_1} e_{x_1 x_2} \ldots e_{x_{l-j+2} x_{l-j+1}}$, \quad $(2 \leq j \leq m)$,
- $p = e_{yx_1} e_{x_1 u_1} e_{u_1 u_2} \ldots e_{u_{i-1} u_i}$,
- $t = e_{x_{l-1} x_{l-i+1}} \ldots e_{x_2 x_1} e_{x_1 y}$,
- $t_j = e_{x_{l-j+2} x_{l-j+1}} \ldots e_{x_{l-1} x_1} e_{x_1 u_1} e_{u_1 u_2} \ldots e_{u_{j-1} u_j}$, \quad $(2 \leq j \leq m)$,
- $t_1 = e_{ux_1} e_{x_1 x_{l-1}} \ldots e_{x_2 x_1} e_{x_1 u_1}$,
- $q = e_{yx_1} e_{x_1 x_2} \ldots e_{x_{l-1} x_1} e_{x_1 v}$.

Then $r$ transposes $u_i$ and $v$ and fixes all other vertices of $\Gamma$ outside the defect set. \hfill \Box

Note that Lemma 19 is going to be useful whenever $\Gamma$ contains a node with degree at least 3.

**Lemma 20.** Let $k \geq 2$, $\Gamma' = (V', E')$ be such that $|V'| > k$ and its defect $k$ group is transitive (e.g. $\Gamma'$ is a cycle with at least $k+1$ vertices). Let $\Gamma = (V' \cup \{v\}, E' \cup \{x_1 v\})$ for a new vertex $v$ and some $x_1 \in \Gamma'$.
where the degree of \( x_1 \) in \( \Gamma' \) is at least 2. Then the defect \( k \)-group of \( \Gamma \) is isomorphic to \( S_{n-k} \).

**Proof.** Let \( n \) be the number of vertices of \( \Gamma \), then \( n \geq k+2 \). Let the vertices of \( \Gamma' \) be \( y, x_1, x_2, \ldots, x_{k-1}, u_1, u_2, \ldots, u_{n-k-1} \) such that \( u_1 \) and \( y \) are neighbours of \( x_1 \) in \( \Gamma' \). Let the defect set be \( \{ y, x_1, \ldots, x_{k-1} \} \).

Applying Lemma 19 to the subgraph with vertices \( \{ x_1, v, y, u_1 \} \) we obtain that the defect \( k \)-group of \( \Gamma \) contains the transposition \( (u_1, v) \).

Since the defect \( k \)-group of \( \Gamma' \) is transitive, the defect \( k \)-group of \( \Gamma \) contains the transposition \( (u_i, v) \) for all \( 1 \leq i \leq n-k-1 \). Therefore, the defect \( k \)-group of \( \Gamma \) is isomorphic to \( S_{n-k} \). \( \square \)

Motivated by Lemma 20, we define the \( k \)-subgraphs and the \( k \)-components of a graph \( \Gamma \).

**Definition 21.** Let \( \Gamma \) be a simple connected graph, \( k \geq 2 \). A connected subgraph \( \Gamma' \subseteq \Gamma \) is called a \( k \)-subgraph if its defect \( k \)-group is the symmetric group of degree \( |\Gamma'| - k \). A \( k \)-subgraph is a \( k \)-component or a maximal \( k \)-subgraph if it has no proper extension in \( \Gamma \) to a \( k \)-subgraph.

Finally, we say that a \( k \)-subgraph or \( k \)-component \( \Gamma' \) is nontrivial if it contains a vertex having at least 3 neighbours in \( \Gamma' \).

Note that every \( k \)-component is an induced subgraph. A trivial \( k \)-subgraph is either a line on \( k+1 \) points or a cycle on \( k+1 \) or on \( k+2 \) points. Further, a trivial \( k \)-component cannot be a cycle by Lemma 20, unless the graph itself is a cycle. Finally, any connected subgraph of \( k+1 \) points is trivially a \( k \)-subgraph, thus every connected subgraph of \( k+1 \) points is contained in a \( k \)-component. Note that the intersection of two \( k \)-components cannot contain more than \( k \) vertices:

**Lemma 22.** Let \( \Gamma_1, \Gamma_2 \) be \( k \)-subgraphs such that \( |\Gamma_1 \cap \Gamma_2| > k \). Then \( \Gamma_1 \cup \Gamma_2 \) is a \( k \)-subgraph, as well.

**Proof.** Choose the defect set \( V_k \) such that \( V_k \subseteq \Gamma_1 \cap \Gamma_2 \), and let \( v \in (\Gamma_1 \cap \Gamma_2) \setminus V_k \). Then the symmetric groups acting on \( \Gamma_1 \setminus V_k \) and \( \Gamma_2 \setminus V_k \) are subgroups in the defect \( k \)-group of \( \Gamma_1 \cup \Gamma_2 \). Thus, we can transpose every member of \( \Gamma_1 \setminus (V_k \cup \{ v \}) \) with \( v \). Therefore, the defect \( k \)-group of \( \Gamma_1 \cup \Gamma_2 \) is the symmetric group on \( (\Gamma_1 \cup \Gamma_2) \setminus V_k \). \( \square \)

**Lemma 23.** Let \( \Gamma \) be a simple connected graph, \( \Gamma' \) be a \( k \)-subgraph of \( \Gamma \). Let \( x_1 \in \Gamma', v \notin \Gamma' \), and let \( P = (x_1, x_2, \ldots, x_l, v) \) be a shortest path between \( x_1 \) and \( v \) in \( \Gamma \) for some \( l \leq k-1 \). Assume that \( x_1 \) has at least 2 neighbours in \( \Gamma' \) apart from \( x_2 \). Then the subgraph \( \Gamma' \cup P \) is a \( k \)-subgraph.

**Proof.** First, consider the case \( x_2, \ldots, x_l \in \Gamma' \). Let \( u, y \) be two neighbours of \( x_1 \) in \( \Gamma' \) distinct from \( x_2 \), and choose the defect set \( V_k \) such that it contains \( y, x_1, \ldots, x_l \) and does not contain \( u \). By Lemma 19 the defect \( k \)-group of \( \Gamma' \cup \{ v \} \) contains the transposition \( (u, v) \). Further, the defect \( k \)-group of \( \Gamma' \) is the whole symmetric group on \( \Gamma' \setminus V_k \).
Thus, the defect $k$ group of $\Gamma' \cup \{v\}$ is the whole symmetric group on $(\Gamma' \setminus V_k) \cup \{v\}$.

Now, if not all of $x_2, \ldots, x_t$ are in $\Gamma'$, then, by the previous argument, one can add them (and then $v$) to $\Gamma'$ one by one, and obtain an increasing chain of $k$-subgraphs.

As a corollary, we obtain that every vertex of degree at least 3 with two neighbours is contained in exactly one nontrivial $k$-component.

**Corollary 24.** Let $\Gamma$ be a simple connected graph with $n$ vertices such that $n > k$, and let $x_1$ be a vertex having degree at least 3. Then there exists exactly one $k$-component $\Gamma'$ containing $x_1$ such that $x_1$ has degree at least $2$ in $\Gamma'$. Further, $\Gamma'$ is a nontrivial $k$-component, and if $\Gamma_x$ is the induced subgraph of the vertices in $\Gamma$ that are of at most distance $k-1$ from $x_1$, then $\Gamma_x \subseteq \Gamma'$.

**Proof.** Expanding $x_1$ and any two of its neighbours to an arbitrary connected subgraph of $\Gamma$ with $k+1$ points yields to a $k$-subgraph. Thus there exists at least one $k$-component containing $x_1$ and two of its neighbours.

Let $\Gamma'$ be a $k$-component containing $x_1$ and at least two of its neighbours. Assume that $\Gamma_{x_1} \not\subseteq \Gamma'$. Let $v \in \Gamma_1 \setminus \Gamma'$ be a point such that for a shortest path $P = (x_1, \ldots, x_t, v)$ between $x_1$ and $v$ we have that $l$ is minimal. If $l = 1$, then $P = (x_1, v)$. Now $x_1$ has at least two neighbours in $\Gamma'$ apart from $v$, therefore $\Gamma' \cup P$ is a $k$-subgraph by Lemma 23, which contradicts the maximality of $\Gamma'$. Thus $l \geq 2$, in particular all neighbours of $x_1$ in $\Gamma$ are in $\Gamma'$, as well, and thus $\Gamma'$ is a nontrivial $k$-component. Hence $x_1$ has at least two neighbours in $\Gamma'$ apart from $x_2$, therefore $\Gamma' \cup P$ is a $k$-subgraph by Lemma 23, which contradicts the maximality of $\Gamma'$. Thus $\Gamma_{x_1} \subseteq \Gamma'$.

Now, assume that $\Gamma'$ and $\Gamma''$ are $k$-components containing $x_1$ and at least two of its neighbours. Then $\Gamma_{x_1} \subseteq \Gamma'$ and $\Gamma_{x_1} \subseteq \Gamma''$. Note that either $\Gamma_{x_1} = \Gamma$ (and hence $|\Gamma_{x_1}| = n > k$), or there exists a vertex $v \in \Gamma$ which is of distance exactly $k$ from $x_1$. Let $P = (x_1, \ldots, x_t, v)$ be a shortest path between $x_1$ and $v$, and let $u$ and $y$ be two neighbours of $x_1$ distinct from $x_2$. Then $\{x_1, \ldots, x_t, y, u\} \subseteq \Gamma_{x_1}$, thus $|\Gamma_{x_1}| > k$. Therefore $|\Gamma' \cap \Gamma''| \geq |\Gamma_{x_1}| > k$, yielding $\Gamma' = \Gamma''$ by Lemma 22.

**Lemma 25.** Let $\Gamma'$ be a nontrivial $k$-subgraph of $\Gamma$, $P$ be a $\Gamma'$-ear. Then $\Gamma' \cup P$ is a (nontrivial) $k$-subgraph of $\Gamma$.

**Proof.** Let $\Gamma$, $\Gamma'$ and $P = (w_0, w_1, \ldots, w_i, w_{i+1})$ be a counterexample, where $i$ is minimal. There exists a shortest path $(w_0, y_1, \ldots, y_l, w_{i+1})$ in $\Gamma'$ among those where the degree of some $y_j$ or of $w_0$ or of $w_{i+1}$ is at least $3$ in $\Gamma'$. For easier notation, let $y_0 = w_0$, $y_{l+1} = w_{i+1}$. Let $y' \in \Gamma \setminus \{y_0, y_1, \ldots, y_l, y_{l+1}\}$ be a neighbour of $y_j$; this exists, otherwise a shorter path would exist between $w_0$ and $w_{i+1}$.
If \( j + 1 \leq k - 1 \) (that is \( j \leq k - 2 \)), then by Lemma 23 the induced subgraph on \( \Gamma' \cup \{ w_1 \} \) is a \( k \)-subgraph, thus \( \Gamma' \cup \{ w_1 \} \) with the ear \((w_1, \ldots, w_i, w_{i+1})\) is a counterexample with a shorter ear.

Similarly, if \( l - j + 2 \leq k - 1 \) (that is \( l + 3 - k \leq j \)), then by Lemma 23 the induced subgraph on \( \Gamma' \cup \{ w_i \} \) is a \( k \)-subgraph, thus \( \Gamma' \cup \{ w_i \} \) with the ear \((w_0, w_1, \ldots, w_i)\) is a counterexample with a shorter ear.

Finally, if \( k - 1 \leq j \leq l + 2 - k \), then \( 2k - 3 \leq l \). Let \( \Gamma'' \) be the cycle \( P \cup (y_0, y_1, \ldots, y_l, y_{l+1}) \) together with \( y' \) and the edge \( y_iy' \). Then \( \Gamma'' \) is a \( k \)-subgraph by Lemma 20, \(|\Gamma' \cap \Gamma''| = l + 2 \geq 2k - 1 > k\), hence \( \Gamma' \cup \Gamma'' = \Gamma' \cup \Gamma'' \) is a \( k \)-subgraph by Lemma 22.

**Corollary 26.** Let \( \Gamma \) be a simple connected graph with \( n \) vertices such that \( n > k \), and assume that \( \Gamma \) is not a cycle. Suppose \( uv \) is an edge contained in a cycle of \( \Gamma \). Then there exists exactly one \( k \)-component \( \Gamma' \) containing the edge \( uv \). Further, \( \Gamma' \) is a nontrivial \( k \)-component, and if \( \Gamma_{uv} \) is the 2-edge connected component containing \( uv \), then \( \Gamma_{uv} \subseteq \Gamma' \).

**Proof.** Expanding the edge \( uv \) to an arbitrary connected subgraph of \( \Gamma \) with \( k + 1 \) points yields to a \( k \)-subgraph. Thus there exists at least one \( k \)-component \( \Gamma' \) containing the edge \( uv \). We prove first that \( \Gamma' \) is a nontrivial \( k \)-component, then \( \Gamma_{uv} \subseteq \Gamma' \), and only after that do we prove that \( \Gamma' \) is unique.

Assume first that \( \Gamma' \) is a trivial \( k \)-component. If \( \Gamma' \) were a cycle, then \( \Gamma \setminus \Gamma' \) contains at least one vertex, because \( \Gamma' \) is an induced subgraph of \( \Gamma \). Then Lemma 20 contradicts the maximality of \( \Gamma' \). Thus \( \Gamma' \) is a line of \( k + 1 \) vertices. Let \( \Gamma_2 \) be a shortest cycle containing \( uv \). Now, there must exist a vertex in \( \Gamma \setminus \Gamma_2 \), otherwise either \( \Gamma = \Gamma_2 \) would be a cycle, or there would exist an edge in \( \Gamma \setminus \Gamma_2 \) yielding a shorter cycle than \( \Gamma_2 \) containing the edge \( uv \). Let \( x_2 \in \Gamma \setminus \Gamma_2 \) be a neighbour of a vertex in \( \Gamma_2 \). By Lemma 20 the induced subgraph on \( \Gamma_2 \cup \{ x_2 \} \) is a \( k \)-subgraph. Thus \( \Gamma' \subseteq \Gamma_2 \), otherwise \( \Gamma' \) would not be a maximal \( k \)-subgraph. Let \( x_1 \in \Gamma' \cap \Gamma_2 \) be a vertex such that two of its neighbours are in \( \Gamma_2 \) and its third neighbour is some \( x_2 \in \Gamma' \setminus \Gamma_2 \). Note that every vertex in \( \Gamma' \) is of distance at most \( k - 1 \) from \( x_1 \), because \( u, v \in \Gamma' \cap \Gamma_2 \). Thus, if \( |\Gamma_2| \geq k + 1 \), then \( \Gamma_2 \) together with \( x_2 \) and the edge \( x_1x_2 \) is a \( k \)-subgraph by Lemma 20, and hence \( \Gamma_2 \cup \Gamma' \) is a \( k \)-subgraph by Lemma 23, contradicting the maximality of \( \Gamma' \). Otherwise, if \( |\Gamma_2| \leq k \), then every vertex in \( \Gamma_2 \) is of distance at most \( k - 1 \) from \( x_1 \), and hence \( \Gamma_2 \cup \Gamma' \) is a \( k \)-subgraph by Lemma 23, contradicting the maximality of \( \Gamma' \). Therefore \( \Gamma' \) is a nontrivial \( k \)-component.

Now we show that the two-edge connected component \( \Gamma_{uv} \subseteq \Gamma' \). Let \( \Gamma, \Gamma' \) be a counterexample to this such that the number of vertices of \( \Gamma_{uv} \) is minimal, and among these counterexamples choose one where the number of edges of \( \Gamma_{uv} \) is minimal. Using an ear-decomposition [11], \( \Gamma_{uv} \) is either a cycle, or there exists a 2-edge connected subgraph \( \Gamma_1 \subseteq \Gamma_{uv} \) and there exists...
If $\Gamma_{uv}$ is a cycle containing the edge $uv$, and $\Gamma_{uv} \not\subseteq \Gamma'$, then going along the edges of $\Gamma_{uv}$, one can find a $\Gamma'$-ear $P \subseteq \Gamma_{uv}$. Then $\Gamma' \cup P$ is a $k$-subgraph by Lemma 25, contradicting the maximality of $\Gamma'$. Thus $\Gamma_{uv}$ is not a cycle. Let us choose $\Gamma_1$ from cases (1) and (2) so that it would have the least number of vertices.

Assume first that case (1) holds. By minimality of the counterexample, $\Gamma_1 \subseteq \Gamma'$. If $P \not\subseteq \Gamma'$, then going along the edges of $P$ one can find a $\Gamma'$-ear $P' \subseteq P$. But then $\Gamma' \cup P'$ is a $k$-subgraph by Lemma 25, contradicting the maximality of $\Gamma'$.

Assume now that case (2) holds. Again, by induction, $\Gamma_1 \subseteq \Gamma'$. If $\Gamma_2 \not\subseteq \Gamma'$, then either $|\Gamma' \cap \Gamma_2| = 1$ or going along the edges of $\Gamma_2$ one can find a $\Gamma'$-ear $P' \subseteq \Gamma_2$. The latter case cannot happen, because then $\Gamma' \cup P'$ is a $k$-subgraph by Lemma 25, contradicting the maximality of $\Gamma'$.

Thus, $\Gamma' \cap \Gamma_2 = \emptyset$, and hence by Lemma 22 we have

$$|\Gamma' \cap \Gamma_2| = 1.$$ 

Let $x_1$ be a neighbour of $\Gamma_1 \setminus \Gamma_2$, and let $v_2$ be a neighbour of $x_1$ in $\Gamma_2 \setminus \Gamma_1$. If $|\Gamma_2| \leq k$, then $\Gamma_2$ can be extended to a connected subgraph of $\Gamma$ having exactly $k + 1$ vertices, which is a $k$-subgraph. If $|\Gamma_2| \geq k + 1$, then $\Gamma_2 \cup \{v_1\}$ is a $k$-subgraph by Lemma 20. In any case, there exists a $k$-component $\Gamma'_2 \supseteq \Gamma_2$. For notational convenience, let $\Gamma'_1$ denote the $k$-component $\Gamma'$ containing $\Gamma_1$. We prove that $\Gamma'_1 = \Gamma'_2 = \Gamma'$, thus $\Gamma'$ contains $\Gamma_2$, contradicting that we chose a counterexample.

Now, both $\Gamma_1$ and $\Gamma_2$ contain at least two neighbours of $x_1$. Let $V_i \subseteq \Gamma_i$ be the set of vertices with distance at most $k - 1$ from $x_1$ ($i \in \{1, 2\}$). If $|\Gamma_i| \leq k$, then $V_i$ contains all vertices of $\Gamma_i$, otherwise $|V_i| \geq k$ ($i \in \{1, 2\}$). By Lemma 23, the induced subgraph on $V_i$ is contained in $\Gamma'_2$. Thus, if $V_1$ contains all vertices of $\Gamma_1$, then $\Gamma_1 \subseteq \Gamma'_2$, hence we have $\Gamma'_1 = \Gamma'_2$. Similarly, the induced subgraph on $V_2$ is contained in $\Gamma'_1$. Thus, if $V_2$ contains all vertices of $\Gamma_2$, then $\Gamma_2 \subseteq \Gamma'_1$, hence we have $\Gamma'_1 = \Gamma'_2$. Otherwise, $|\Gamma'_1 \cap \Gamma'_2| \geq |V_1| + |V_2| - |\{x_1\}| \geq 2k - 1 > k$, hence by Lemma 22 we have $\Gamma'_1 = \Gamma'_2$.

Finally, we prove uniqueness. Let $\Gamma'$ and $\Gamma''$ be two $k$-components containing the edge $uv$. Then both $\Gamma'$ and $\Gamma''$ contain $\Gamma_{uv}$. If $\Gamma = \Gamma_{uv}$, then $\Gamma' = \Gamma_{uv} = \Gamma''$. Otherwise, there exists a vertex $x_2 \in \Gamma \setminus \Gamma_{uv}$ such that it has a neighbour $x_1 \in \Gamma_{uv}$. Note that $x_1$ has degree at least 3 in $\Gamma$. Let $V_1$ be the vertices of $\Gamma$ of distance at most $k - 1$ from $x_1$. Note that if $V_1$ does not contain all vertices of $\Gamma$, then $|V_1| > k$. By 2-edge connectivity, $\Gamma_{uv} \subseteq \Gamma'$ contains at least two neighbours of $x_1$, thus $V_1 \subseteq \Gamma'$ by Lemma 23. Similarly, $\Gamma_{uv} \subseteq \Gamma''$ contains at least two neighbours of $x_1$, thus $V_1 \subseteq \Gamma''$ by Lemma 23. If $V_1$ contains all vertices of $\Gamma$, then $\Gamma' = \Gamma = \Gamma''$. Otherwise, $|\Gamma' \cap \Gamma''| \geq |V_1| > k$, and $\Gamma' = \Gamma''$ by Lemma 22.  

□
Recall that by [11] a strongly connected antisymmetric digraph becomes a 2-edge connected graph after forgetting the directions. Thus Rhodes's conjecture about strongly connected, antisymmetric digraphs [9, Conjecture 6.5i (3)-(4)] follows immediately from the following theorem on 2-edge connected graphs:

**Theorem 27.** Let $n > k \geq 2$, $\Gamma$ be a 2-edge connected simple graph having $n$ vertices. If $\Gamma$ is a cycle, then the defect $k$ group is $Z_{n-k}$. If $\Gamma$ is not a cycle, then the defect $k$ group is $S_{n-k}$.

*Proof.* If $\Gamma$ is a cycle, then its defect $k$ group is $Z_{n-k}$ by Lemma 9. If $\Gamma$ is not a cycle, then the defect $k$ group is $S_{n-k}$ by Corollary 26. □

The final part of this section is devoted to prove Theorem 2.

**Lemma 28.** Let $\Gamma_1 \neq \Gamma_2$ be $k$-components of the connected simple graph $\Gamma$. Assume that $\Gamma$ is not a cycle. Then $\Gamma_1 \cap \Gamma_2$ is either empty, or is a path $(x_1, \ldots, x_l)$ such that

1. $l \leq k$,
2. the degree of $x_i$ is 2 in $\Gamma$ $(2 \leq i \leq l - 1)$,
3. if $l \geq 2$ and $\Gamma_1 \setminus \{x_1, \ldots, x_l\}$ contains a neighbour of $x_1$ (resp. $x_l$), then $\Gamma_1$ contains all neighbours of $x_1$ (resp. $x_l$),
4. if $l \geq 2$ then $\Gamma \setminus \{x_jx_{j+1}\}$ is disconnected for all $1 \leq j \leq l - 1$.

*Proof.* Note that $\Gamma_1$ and $\Gamma_2$ are induced subgraphs of $\Gamma$, thus so is $\Gamma_1 \cap \Gamma_2$.

We prove first that $\Gamma_1 \cap \Gamma_2$ is connected (or empty) if $\Gamma_1$ is a nontrivial $k$-component. Suppose that $u, v \in \Gamma_1 \cap \Gamma_2$ are in different components of $\Gamma_1 \cap \Gamma_2$ such that the distance between $u$ and $v$ is minimal in $\Gamma_2$. Due to the minimality, there exists a path $(u, x_1, \ldots, x_l, v)$ such that $x_1, \ldots, x_l \in \Gamma_2 \setminus \Gamma_1$. Then $P = (u, x_1, \ldots, x_l, v)$ is a $\Gamma$-ear, and $\Gamma_1 \cup P$ would be a $k$-subgraph by Lemma 25, contradicting the maximality of $\Gamma_1$. Thus $\Gamma_1 \cap \Gamma_2$ is connected. One can prove similarly that $\Gamma_1 \cap \Gamma_2$ is connected if $\Gamma_2$ is a nontrivial $k$-component.

Now we prove that $\Gamma_1 \cap \Gamma_2$ is connected, even if both $\Gamma_1$ and $\Gamma_2$ are trivial $k$-components. As $\Gamma_1 \subseteq \Gamma$, $\Gamma_1$ cannot be a cycle hence must be a line $(x_1, \ldots, x_{k+1})$. Note that the degree of $x_i$ in $\Gamma$ for $2 \leq i \leq k$ must be 2, otherwise a nontrivial $k$-component would contain $x_i$, and thus also $\Gamma_1$ by Corollary 24. In particular, if $\Gamma_1 \cap \Gamma_2$ is not connected, then $x_1, x_{k+1} \in \Gamma_1 \cap \Gamma_2$, $x_i \not\in \Gamma_1 \cap \Gamma_2$ for some $2 \leq i \leq k$, and $\Gamma_1 \cup \Gamma_2$ would be a cycle. However, by Corollary 26, the edge $x_1x_{k+1}$ is contained in a unique nontrivial $k$-component, contradicting that it is also contained in the trivial $k$-component $\Gamma_1$.

Now, we prove (1-4). By Corollary 26, $\Gamma_1 \cap \Gamma_2$ cannot contain any edge $uv$ which is contained in a cycle. As $\Gamma_1 \cap \Gamma_2$ is connected, it must be a tree. However, $\Gamma_1 \cap \Gamma_2$ cannot contain any vertex of degree at least 3 in $\Gamma_1 \cap \Gamma_2$, otherwise that vertex would be contained in a unique $k$-component by Corollary 24. Thus $\Gamma_1 \cap \Gamma_2$ is a path $(x_1, \ldots, x_l)$. 

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Now, \( l \leq k \) by Lemma 22, proving (1). For (2) note that if any \( x_i \) 
\((2 \leq i \leq l - 1)\) is of degree at least 3 in \( \Gamma \), then \( \{ x_{i-1}, x_i, x_{i+1} \} \) is 
contained in a unique \( k \)-component by Corollary 24, a contradiction. 
For (3) observe that at least two neighbours of \( x_1 \) (resp. \( x_l \)) are in \( \Gamma \), 
and thus all its neighbours must be in \( \Gamma \) by Corollary 24. Finally, (4) 
follows immediately from Corollary 26 and the fact that any edge that 
is not contained in any cycle disconnects the graph \( \Gamma \). \( \square \)

Motivated by the structure of intersections of \( k \)-components, we define bridges in \( \Gamma \):

**Definition 29.** A path \((x_1, \ldots, x_l)\) in a connected graph \( \Gamma \) for some 
\( l \geq 2 \) is called a bridge if the degree of \( x_i \) in \( \Gamma \) is 2 for all \( 2 \leq i \leq l - 1 \), 
and if \( \Gamma \setminus \{ x_j, x_{j+1} \} \) is disconnected for all \( 1 \leq j \leq l - 1 \). The length 
of the bridge \((x_1, \ldots, x_l)\) is \( l \).

Edges of short bridges (having length at most \( k - 1 \)) are contained 
in a unique \( k \)-component:

**Lemma 30.** Let \( \Gamma \) be a simple connected graph with \( n \) vertices such 
that \( n > k \), and let \( uv \) be an edge which is not contained in any cycle. 
Let \((x_1, \ldots, x_l)\) be a longest bridge containing the edge \( uv \). If \( l \leq k - 1 \), 
then \( uv \) is contained in a unique \( k \)-component \( \Gamma' \), and furthermore, \( \Gamma' \) 
is a nontrivial \( k \)-component.

**Proof.** As \( uv \) is not part of any cycle in \( \Gamma \), \( uv \) is a bridge of length 2. 
Note that a longest bridge \((x_1, \ldots, x_l)\) containing \( uv \) is unique, because 
as long as the degree of at least one of the path’s end vertices is 2 in \( \Gamma \), 
the path can be extended in that direction. The obtained path is the 
unique longest bridge containing \( uv \).

Let \( \Gamma' \) be a \( k \)-component containing \( uv \), and assume \( l \leq k - 1 \). Note 
that the distance of \( x_1 \) and \( x_l \) is \( l - 1 \leq k - 2 \). As \(|\Gamma| \geq k + 1 \), at least 
one of \( x_1 \) and \( x_l \) has degree at least 3 in \( \Gamma \), say \( x_1 \). We distinguish two 
cases according to the degree of \( x_1 \).

Assume first that \( x_1 \) is of degree 1. As \( \Gamma' \) is a connected subgraph 
having at least \( k + 1 \) vertices, \( \Gamma' \) must contain \( x_1 \) and at least two of 
its neighbours. Then by Corollary 24 it contains all vertices of \( \Gamma \) of 
distance at most \( k - 1 \) from \( x_1 \). In particular, \( \Gamma' \) must contain the bridge 
\((x_1, \ldots, x_1)\). However, there is a unique (nontrivial) \( k \)-component \( \Gamma' \), 
containing \( x_1 \) and two of its neighbours by Corollary 24, and thus 
\( \Gamma' = \Gamma' \) is that unique \( k \)-component.

Assume now that \( x_1 \) is of degree at least 3. As \( \Gamma' \) is a connected 
subgraph having at least \( k + 1 \) vertices, \( \Gamma' \) must contain \( x_1 \) and at least 
two of its neighbours, or \( x_1 \) and at least two of its neighbours. If \( \Gamma' \) 
contains \( x_1 \) and at least two of its neighbours, then by Corollary 24 it 
contains all vertices of \( \Gamma \) of distance at most \( k - 1 \) from \( x_1 \). In particular, \( \Gamma' \) must contain the bridge 
\((x_1, \ldots, x_1)\) and all of the neighbours of \( x_1 \). Similarly, one can prove that if \( \Gamma' \) contains \( x_1 \) and two of its
neighbours, then it also contains the bridge \((x_1, \ldots, x_l)\) and all of the neighbours of \(x_1\). However, there is a unique (nontrivial) \(k\)-component \(\Gamma'_1\) containing \(x_1\) and two of its neighbours by Corollary 24, and also a unique (nontrivial) \(k\)-component \(\Gamma'_2\) containing \(x_1\) and two of its neighbours by Corollary 24. Therefore \(\Gamma'\) must equal to both \(\Gamma'_1\) and \(\Gamma'_2\), and hence is unique. \(\square\)

In particular, in non-cycle graphs trivial \(k\)-components or intersections of two different \(k\)-components consist of edges that are contained in long bridges (having length at least \(k\)). The key observation in proving Theorem 2 is that a defect \(k\) group cannot move a vertex across a bridge of length at least \(k\):

**Lemma 31.** Let \(2 \leq k \leq l\), \(\Gamma_1\) and \(\Gamma_2\) be disjoint connected subgraphs of the connected graph \(\Gamma\), and \((x_1, x_2, \ldots, x_l)\) be a bridge in \(\Gamma\) such that \(x_1, \ldots, x_l \notin \Gamma_1 \cup \Gamma_2\), \(x_1\) has only neighbours in \(\Gamma_1\) (except for \(x_2\)), \(x_l\) has only neighbours in \(\Gamma_2\) (except for \(x_{l-1}\)). Assume \(\Gamma\) has no more vertices than \(\Gamma_1 \cup \Gamma_2 \cup \{x_1, \ldots, x_l\}\). Let the defect set be \(V_k = \{x_1, \ldots, x_k\}\), let the defect \(k\) group corresponding to \(V_k\) be \(G_k\). Then for any \(u \in \Gamma_1\) and \(v \in \Gamma_2\) there does not exist any permutation in \(G_k\) which moves \(u\) to \(v\).

**Proof.** Let \(S = S_\Gamma\). Assume that there exists \(u \in \Gamma_1\), \(v \in \Gamma_2\), and a transformation \(g \in S\) of defect \(V_k\) such that \(g|_{V_k} \in G_k\) and \(ug = v\). Let \(s_0 \in G_k\) be the unique idempotent power of \(g\), that is \(s_0\) is a transformation of defect \(V_k\) that acts as the identity on \(\Gamma \setminus V_k\). Then there exists a series of elementary collapsings \(e_1, \ldots, e_m\) such that \(g = e_1 \ldots e_m\). For every \(1 \leq d \leq m\) let \(s_d = s_0 e_1 \ldots e_d\). Now, \(s_m = s_0 e_1 \ldots e_m = s_0 g = g s_0 = g\). In particular, both \(s_m\) and \(s_0\) are of defect \(k\), hence \(s_d\) is of defect \(k\) for all \(1 \leq d \leq m\). Consequently, \(|\Gamma_1 s_d| = |\Gamma_1|\), \(|\Gamma_2 s_d| = |\Gamma_2|\) and \(\Gamma_1 s_d \cap \Gamma_2 s_d = \emptyset\) for all \(1 \leq d \leq m\).

For an arbitrary \(s \in S\), let

\[
i(s) = \begin{cases} 0, & \text{if } \Gamma_1 s \subseteq \Gamma_1, \\ l + 1, & \text{if } \Gamma_1 s \not\subseteq \Gamma_1 \cup \{x_1, \ldots, x_l\}, \\ \min_{1 \leq i \leq l} \{\Gamma_1 s \subseteq \Gamma_1 \cup \{x_1, \ldots, x_i\}\}, & \text{otherwise.} \end{cases}
\]

Similarly, let

\[
j(s) = \begin{cases} l + 1, & \text{if } \Gamma_2 s \subseteq \Gamma_2, \\ 0, & \text{if } \Gamma_2 s \not\subseteq \Gamma_2 \cup \{x_1, \ldots, x_l\}, \\ \max_{1 \leq j \leq l} \{\Gamma_2 s \subseteq \Gamma_2 \cup \{x_j, \ldots, x_l\}\}, & \text{otherwise.} \end{cases}
\]

Note that for arbitrary \(s \in S\) and elementary collapsing \(e\), we have \(|i(s) - i(se)| \leq 1\), \(|j(s) - j(se)| \leq 1\). Further, both \(|i(s_d) - i(s_d e)| = 1\) and \(|j(s_d) - j(s_d e)| = 1\) cannot happen at the same time for any \(1 \leq d \leq m\), because that would contradict \(\Gamma_1 s_d \cap \Gamma_2 s_d \neq \emptyset\).
For $s_0$ we have $i(s_0) = 0 < l + 1 = j(s_0)$, for $s_m$ we have $i(s_m) = l + 1 \geq j(s_m)$. Let $1 \leq d \leq m$ be minimal such that $i(s_d) \geq j(s_d)$. Then $i(s_{d-1}) < j(s_{d-1})$. From $s_{d-1}$ to $s_d$ either $i$ or $j$ can change and by at most 1, thus $i(s_d) = j(s_d)$. If $i(s_d) = j(s_d) \in \{1, \ldots, l\}$, then $x_i(s_d) \in \Gamma_1 s_d \cap \Gamma_2 s_d$, contradicting $\Gamma_1 s_d \cap \Gamma_2 s_d = \emptyset$. Thus $i(s_d) = j(s_d) \notin \{1, \ldots, l\}$. Assume $i(s_d) = j(s_d) = l + 1$, the case $i(s_d) = j(s_d) = 0$ can be handled similarly.

Now, $j(s_d) = l + 1$ yields $\Gamma_2 s_d \subseteq \Gamma_2$. Further, $|\Gamma_2 s_d| = |\Gamma_2|$, thus $\Gamma_2 s_d = \Gamma_2$. From $i(s_d) = l + 1$ we have $\Gamma_1 s_d \cap \Gamma_2 \neq \emptyset$. Thus $\Gamma_1 s_d \cap \Gamma_2 s_d = \Gamma_1 s_d \cap \Gamma_2 \neq \emptyset$, a contradiction. □

**Corollary 32.** Let $\Gamma_1$ and $\Gamma_2$ be connected subgraphs of $\Gamma$ such that $\Gamma_1 \cap \Gamma_2$ is a length $k$ bridge in $\Gamma$. Let $V_k = \Gamma_1 \cap \Gamma_2$ be the defect set. Let $G_1$ be the defect $k$ group of $\Gamma_1$, $G$ be the defect $k$ group of $\Gamma_1 \cup \Gamma_2$. Then

$$G = G_1 \times G_2.$$  

**Proof.** By Lemma 18 we have $G_1, G_2 \leq G$. Since $G_1$ and $G_2$ act on disjoint vertices, their elements commute. Thus $G_1 \times G_2 \leq G$. Now, $V_k$ is a bridge of length $k$, thus by Lemma 31 (applied to the disjoint subgraphs $\Gamma_1 \setminus V_k$ and $\Gamma_2 \setminus V_k$) there exists no element of $G$ moving a vertex from $\Gamma_1$ to $\Gamma_2$ or vice versa. Therefore $G \leq G_1 \times G_2$. □

Finally, we are ready to prove Theorem 2.

**Proof of Theorem 2.** If $\Gamma$ is a cycle, then its defect $k$ group is $Z_{n-k}$ by Lemma 9. Otherwise, we prove the theorem by induction on the number of $k$-components of $\Gamma$. If $\Gamma$ is a $k$-component, then the theorem holds, and the defect $k$ group of $\Gamma$ is $S_{n-k}$.

Otherwise, we consider two cases. Assume first that there exists a degree 1 vertex $x_1 \in \Gamma$, such that there exists a path $(x_1, \ldots, x_{k+1})$ which is a bridge. Let $\Gamma_1$ be the path $(x_1, \ldots, x_{k+1})$, and let $\Gamma_2 = \Gamma \setminus \{x_1\}$. Now, $\Gamma_1$ is a trivial $k$-component, hence $\Gamma_2$ contains one less $k$-component than $\Gamma$. Further, $\Gamma_2$ is connected, and cannot be a cycle because the degree of $x_2$ in $\Gamma_2$ is 1. Thus induction and Corollary 32 finishes the proof in this case.

In the second case, no degree 1 vertex $x_1$ is in a path $(x_1, \ldots, x_{k+1})$ which is a bridge. Then any maximal bridge $(x_1, \ldots, x_l)$ with a degree 1 vertex $x_1$ has length $l \leq k$, and, as the bridge cannot be extended, $x_1$ must have degree at least 3. Moreover, $(x_1, \ldots, x_l)$ lies in a $k$-component containing $x_1$ and all its neighbours by Lemma 30 and Corollary 24. In particular every bridge in $\Gamma$ of length at least $k + 1$ occurs between nodes of degree at least 3. Hence every bridge of length at least $k + 1$ occurs between two nontrivial $k$-components by Corollary 24. For every vertex $v$ having degree at least 3 in $\Gamma$, let $\Gamma_v$ be the unique $k$-component containing $v$ and all its neighbours (Corollary 24). By definition, these are all the nontrivial $k$-components of $\Gamma$. 

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Let $\Gamma^k$ be the graph whose vertices are the nontrivial $k$-components, and $\Gamma_u \Gamma_v$ is an edge in $\Gamma^k$ (for $\Gamma_u \neq \Gamma_v$) if and only if there exists a bridge in $\Gamma$ between a vertex $u' \in \Gamma_u$ of degree at least 3 in $\Gamma_u$ and a vertex $v' \in \Gamma_v$ of degree at least 3 in $\Gamma_v$. By Corollary 26, $\Gamma_u = \Gamma_v$ if $u$ and $v$ are in the same 2-edge connected component. As the 2-edge connected components of $\Gamma$ form a tree, the graph $\Gamma^k$ is a tree.

Assume $\Gamma^k$ has $m$ vertices. Let $\Gamma_1$ be a leaf in $\Gamma^k$, and let $\Gamma_m$ be its unique neighbour in $\Gamma^k$. Let $x_1 \in \Gamma_1$ and $x_l \in \Gamma_m$ be the unique vertices of degree at least 3 in $\Gamma_i$ ($i \in \{1, l\}$) such that there exists a bridge $P = (x_1, \ldots, x_l)$ in $\Gamma$. Note that the length of $P$ is at least $k$, otherwise $\Gamma_1 = \Gamma_m$ would follow by Lemma 30. Further, any other bridge having an endpoint in $\Gamma_1$ must be of length at most $k$, because every degree 1 vertex is of distance at most $k - 1$ from a vertex of degree at least 3. Thus every bridge other than $P$ and having an endpoint in $\Gamma_1$ is a subset of $\Gamma_1$ by Corollary 24.

Let $\Gamma_2 = (\Gamma \setminus \Gamma_1) \cup P$. Now, $\Gamma_1$ is a $k$-component, $\Gamma_2$ has one less $k$-component than $\Gamma$. Further, $\Gamma_2$ is connected, because every bridge other than $P$ and having an endpoint in $\Gamma_1$ is a subset of $\Gamma_1$. Finally, $\Gamma_2$ is not a cycle, because it contains the vertex $x_1$ which is of degree 1 in $\Gamma_2$. Thus induction and Corollary 32 finishes the proof in this case.

7. AN ALGORITHM TO CALCULATE THE DEFECT $k$ GROUP

Note that by Theorem 1 the defect 1 group can be trivially computed in $O(|E|)$ time by first determining the 2-vertex connected components [7], and whether each is a cycle, the exceptional graph (Figure 1) or if not, whether or not is bipartite.

For $k \geq 2$ one can check first if $\Gamma$ is a cycle (and then the defect group is $\mathbb{Z}_{n-k}$) or a path (and then the defect group is trivial). In the following, we give a linear algorithm (running in $O(|E|)$ time) to determine the $k$-components ($k \geq 2$) of a connected graph $\Gamma$ having $n$ vertices, $|E|$ edges where at least one vertex is of degree at least 3.

During the algorithm we color the vertices. Let us call a maximal subgraph with vertices having the same color a monochromatic component. First, one finds all 2-edge connected components and the tree of two-edge connected components in $O(|E|)$ time using e.g. [13]. Color the vertices of the 2-edge connected components such that two vertices have the same color if and only if they are in the same 2-edge connected component. Further, color the uncolored vertices having degree at least 3 by different colors from each other and from the colors of the 2-edge connected components. Then the monochromatic components are each contained in a unique nontrivial $k$-component by Corollaries 24 and 26 (a nontrivial $k$-component may contain more than one of these monochromatic components). Further, the monochromatic components and the degree 1 vertices are connected by bridges. If any of
the bridges connecting two monochromatic components is of length at most \( k - 1 \), then recolor the two monochromatic components at the ends of the bridge and the vertices of the bridge by the same color, because these are contained in the same \( k \)-component by Corollary 24. Similarly, if any of the bridges connecting a monochromatic component and a degree 1 vertex is of length at most \( k - 1 \), then recolor the monochromatic component and the vertices of the bridge by the same color, because these are contained in the same \( k \)-component by Lemma 30. Repeat recoloring along all bridges of length at most \( k - 1 \) in \( O(|E|) \) time. Then we obtain monochromatic components \( \Gamma_1, \ldots, \Gamma_l \) connected by long bridges (i.e. bridges of length at least \( k \)), and possibly some long bridges to degree 1 vertices. Now, we have finished coloring.

For every \( 1 \leq i \leq l \), let \( \Gamma_i' \) be the induced subgraph having all vertices of distance at most \( k - 1 \) from \( \Gamma_i \), which can be obtained in \( O(|E|) \) time by adding the appropriate \( k - 1 \) vertices of the long bridges to the appropriate monochromatic component. Note that the obtained induced subgraphs are not necessarily disjoint. Then \( \Gamma_1', \ldots, \Gamma_l' \) are the nontrivial \( k \)-components of \( \Gamma \) by Lemma 30. Again, by Lemma 30, the trivial \( k \)-components of \( \Gamma \) are the paths containing exactly \( k + 1 \) vertices in a long bridge. These can also be computed in \( O(|E|) \) time by going through all long bridges. By Theorem 2, the defect \( k \) group of \( \Gamma \) as a permutation group is the direct product of the defect \( k \) groups of \( \Gamma_1', \ldots, \Gamma_l' \), and the defect \( k \) groups of the trivial \( k \)-components.

8. Complexity of the flow semigroup of (di)graphs

In this section we apply our results and the complexity lower bounds of [10] to verify [9, Conjecture 6.51i (1)] for 2-vertex connected graphs. That is, we prove that the Krohn–Rhodes (or group-) complexity of the flow semigroup of a 2-vertex connected graph with \( n \) vertices is \( n - 2 \). Then we derive further consequences of our results, and finish by stating some open problems.

For standard definitions on wreath product of semigroups, we refer the reader to e.g. [9, Definition 2.2]. A finite semigroup \( S \) is called combinatorial if and only if every maximal subgroup of \( S \) has one element. Recall that the Krohn–Rhodes (or group-) complexity of a finite semigroup \( S \) (denoted by \( \#_G(S) \)) is the smallest non-negative integer \( n \) such that \( S \) is a homomorphic image of a subsemigroup of the iterated wreath product

\[
C_n \wr G_n \wr \cdots \wr C_1 \wr G_1 \wr C_0,
\]

where \( G_1, \ldots, G_n \) are finite groups, \( C_0, \ldots, C_n \) are finite combinatorial semigroups, and \( \wr \) denotes the wreath product (for the precise definition, see e.g. [9, Definition 3.13]). The definition immediately implies
that if a finite semigroup $S$ is the homomorphic image of a subsemigroup of $T$, then $\#_G(S) \leq \#_G(T)$. More can be found on the complexity of semigroups in e.g. [9, Chapter 3]. We need the following results on the complexity of semigroups.

Lemma 33 ([9, Prop. 6.49(b)]). The flow semigroup $K_n$ of the complete graph on $n \geq 2$ vertices has $\#_G(K_n) = n - 2$.

Lemma 34 ([10, Sec. 3.7]). The complexity of the full transformation semigroup $F_n$ on $n$ points is $\#_G(F_n) = n - 1$.

The well-known $\mathcal{L}$-order is a pre-order, i.e. a transitive and reflexive binary relation, on the elements of a semigroup $S$ given by $s_1 \succeq \mathcal{L} s_2$ if $s_1 = s_2$ or $ss_1 = s_2$ for some $s \in S$. The $\mathcal{L}$-classes are the equivalence classes of the $\mathcal{L}$-order. We say that a finite semigroup $S$ is a $T_1$-semigroup if it is generated by some $\mathcal{L}$-chain of subsets of its $\mathcal{L}$-classes $L_1 \succeq \mathcal{L} \cdots \succeq \mathcal{L} L_m$, where $L_i \succeq \mathcal{L} L_{i+1}$ if and only if $SL_i \cup L_i \supseteq SL_{i+1} \cup L_{i+1}$ ($1 \leq i \leq m - 1$).

Lemma 35 ([10, Lemma 3.5(b)]). Let $S$ be a noncombinatorial $T_1$-semigroup. Then

$$\#_G(S) \geq 1 + \#_G(EG(S)),$$

where $EG(S)$ is the subsemigroup of $S$ generated by all its idempotents.

Now we prove [9, Conjecture 6.51i (1)] for 2-vertex connected graphs.

Proof of Theorem 3. Let $K_n$ denote the flow semigroup of the complete graph on vertices $V$, where $|V| = n$. Then $\#_G(S_T) \leq \#_G(K_n) = n - 2$ by Lemma 33. We proceed by induction on $n$. If $n \leq 3$, then $\Gamma$ is a complete graph, and $\#_G(S_T) = n - 2$ by Lemma 33. From now on we assume $n > 3$.

Case 1. Assume first that $\Gamma$ is not a cycle. Let $(u, v)$ and $(x, y)$ be two disjoint edges in $\Gamma$. Let $G_1$ be the defect 1 group with defect set $V \setminus \{u\}$ and idempotent $e_{uv}$ as its identity element. Then $e_{uv} \succeq \mathcal{L} e_{xy} e_{uv} = e_{uv} e_{xy}$. Let $T$ be $\langle G_1 \cup \{e_{uv} e_{xy}\} \rangle$. Since $G_1 \succeq \mathcal{L} \{e_{uv} e_{xy}\}$ is an $\mathcal{L}$-chain in $T$, $T$ is a $T_1$-semigroup. Further, $T$ is noncombinatorial since $G_1$ is nontrivial. Thus, by Lemma 35

$$\#_G(T) \geq 1 + \#_G(EG(T)).$$

Let $\Gamma’$ be the complete graph on $V \setminus \{u\}$. Let $a, b \in V \setminus \{u\}$ be arbitrary distinct vertices. By Theorem 1, $G_1$ is 2-transitive. Let $\pi \in G_1$ be such that $\pi(x) = a$ and $\pi(y) = b$. There is a positive integer $\omega > 1$, with $\pi^\omega = e_{uv}$. In particular, $e_{uv}$ commutes with $\pi$. Observe that

$$(\pi^{\omega-1} e_{uv} e_{xy} \pi e_{uv}) = e_{uv} (\pi^{\omega-1} e_{xy} \pi) = e_{uv} e_{ab},$$

and thus

$$e_{xy} (\pi^{\omega-1} e_{uv} \pi) \in V \setminus \{u\} = e_{ab}.$$
That is, we obtain the generators $e_{ab}$ of $S_T$ by restricting the idempotents $e_{uv}e_{ab} \in T$ to $V \setminus \{u\}$. Therefore, $S_T$ is a homomorphic image of a subsemigroup of $EG(T)$, yielding

$$\#_G(EG(T)) \geq \#_G(S_T).$$

By induction, $\#_G(S_T) = n - 3$. Applying (2), we obtain $\#_G(T) \geq n - 2$. Since $T$ is a subsemigroup of $S_T$, we obtain $\#_G(S_T) \geq \#_G(T) \geq n - 2$.

**Case 2.** Assume now that $\Gamma$ is the $n$-node cycle $(u, v_1, \ldots, v_{n-1})$. Then $(u, v_1)$ and $(v_2, v_3)$ are disjoint edges. Let $G_1 \simeq \mathbb{Z}_{n-1}$ be the defect 1 group with defect set $V \setminus \{u\}$ and idempotent $e_{uv_1}$ as its identity element. Let $\pi$ be a generator of $G_1$ with cycle structure $(v_1, \ldots, v_{n-1})$. Then $e_{uv_1} \geq e_{v_2v_3}e_{uv_1} = e_{uv_1}e_{v_2v_3}$. Let $T$ be $\langle G_1 \cup \{ e_{uv_1}e_{v_2v_3} \} \rangle$. Since $G_1 \geq \{ e_{uv_1}e_{v_2v_3} \}$ is an $\mathcal{L}$-chain in $T$, $T$ is a $T_1$-semigroup. Further, $T$ is noncombinatorial since $G_1$ is nontrivial. Thus, by Lemma 35

$$\#_G(T) \geq 1 + \#_G(EG(T)).$$

Let $\Gamma'$ be an $(n-1)$-node cycle with nodes $V \setminus \{u\} = \{v_1, \ldots, v_{n-1}\}$. Note that $e_{uv_1} = \pi^{n-1}$, and therefore $e_{uv_1}$ commutes with $\pi$. Let $v_{i-1}, v_i, v_{i+1} \in V \setminus \{u\}$ be three neighboring nodes in $\Gamma'$, where the indices are in $\{1, \ldots, n-1\}$ taken modulo $n-1$. Observe that

$$\pi^{n-2}e_{uv_1}e_{v_{i-1}v_i} = \pi^{n-2}e_{uv_1}\left(\pi^{n-2}e_{v_{i-1}v_i}\pi\right) = e_{uv_1}e_{v_{i-1}v_{i+1}},$$

and thus

$$\left(\pi^{n-2}e_{uv_1}e_{v_{i-1}v_i}\pi\right)_{V \setminus \{u\}} = e_{v_{i-1}v_{i+1}}.$$

That is, we obtain the generators $e_{v_{i-1}v_{i+1}}$ of $S_{T'}$ by restricting the idempotents $e_{uv_1}e_{v_{i-1}v_{i+1}} \in T$ to $V \setminus \{u\}$. Therefore, $S_{T'}$ is a homomorphic image of a subsemigroup of $EG(T)$, yielding

$$\#_G(EG(T)) \geq \#_G(S_{T'}).$$

By induction, $\#_G(S_{T'}) = n - 3$. Applying (3), we obtain $\#_G(T) \geq n - 2$. Since $T$ is a subsemigroup of $S_T$, we have $\#_G(S_T) \geq \#_G(T) \geq n - 2$.  

Note that by Lemma 6 a strongly connected digraph has the same flow semigroup as the corresponding graph. Thus, Theorem 3 proves Rhodes's conjecture [9, Conjecture 6.51i (1)] for 2-vertex connected strongly connected digraphs, as well. The following lemma bounds the complexity in the remaining cases.

**Lemma 36.** Let $k$ be the smallest positive integer such that for a graph $\Gamma$ the flow semigroup $S_\Gamma$ has defect $k$ group $S_{n-k}$. Then $\#_G(S_\Gamma) \geq n - 1 - k$.

**Proof.** Assume first $k = n - 1$. Then the lemma holds trivially. From now on, assume $k \leq n - 2$. Let $uv$ be an edge in $\Gamma$. Let $V_k$ be an arbitrary $k$-element subset of the vertex set $V$ disjoint from $\{u, v\}$. Let $G_k$ be the defect $k$ group with defect set $V_k$. Let $S$ be the subsemigroup
of $S_\Gamma$ generated by $G_k$ and $e_v$. As $G_k \simeq S_{n-k}$, we have that $S$ is the semigroup of all transformations on $V \setminus V_k$. Hence, $\#_G(S) = \#_G(F_{n-k}) = n - k - 1$ by Lemma 34. Whence, $\#_G(S_\Gamma) \geq \#_G(S) = n - k - 1$. 

Rhodes’s conjecture [9, Conjecture 6.51i (1)] is about strongly connected, antisymmetric digraphs. By [11] a strongly connected antisymmetric digraph becomes a 2-edge connected graph after forgetting the directions. By Theorem 27, it immediately follows that the complexity of the flow semigroup of a 2-edge connected graph is at least $n - 3$.

**Corollary 37.** Let $\Gamma$ be a 2-edge connected graph with $n \geq 3$ vertices. Then $n - 2 \geq \#_G(S_\Gamma) \geq n - 3$.

This leaves some questions open. To completely settle the last remaining part of Rhodes’s conjecture [9, Conjecture 6.51i (1)], one should find the complexity of the flow semigroups for the rest of the 2-edge connected graphs.

**Problem 1.** Determine the complexity of $S_\Gamma$ for a 2-edge connected graph $\Gamma$ which is not 2-vertex connected.

The smallest such graph is the “bowtie” graph:

**Problem 2.** Let $\Gamma$ be the graph with vertex set $\{u, v, w, x, y\}$ and edge set $\{uw, vw, wu, wx, xy, yw\}$. Determine the complexity of $S_\Gamma$.

Ultimately, the goal is to determine the complexity for all flow semigroups.

**Problem 3.** Determine the complexity of $S_\Gamma$ for an arbitrary finite graph (or digraph) $\Gamma$.

**References**


MAXIMAL SUBGROUPS AND COMPLEXITY OF THE FLOW SEMIGROUP


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