

Cluster adjacency for $m = 2$ Yangian invariants

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ABSTRACT: We classify the rational Yangian invariants of the $m = 2$ toy model of $\mathcal{N} = 4$ Yang-Mills theory in terms of generalised triangles inside the amplituhedron $\mathcal{A}_{n,k}^{(2)}$. We enumerate and provide an explicit formula for all invariants for any number of particles n and any helicity degree k . Each invariant manifestly satisfies cluster adjacency with respect to the $\text{Gr}(2, n)$ cluster algebra.

KEYWORDS: Scattering Amplitudes, Supersymmetric Gauge Theory

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1 Introduction

Aspects of the $\text{Gr}(4, n)$ Grassmannian cluster algebras [1–3] have been found to play several still rather mysterious roles in the mathematical structure of scattering amplitudes in planar $\mathcal{N} = 4$ Yang-Mills theory [4, 5]. A simple toy model which serves as a nice playground for studying features of this cluster structure is the $m = 2$ version of the theory, where the momentum twistors describing the kinematic scattering data [6] are restricted to lie in a \mathbb{P}^1 subspace¹ of the usual \mathbb{P}^3 . The associated $\text{Gr}(2, n) \cong A_{n-3}$ cluster algebra² is completely understood [1]: its clusters are in one-to-one correspondence with the triangulations of an n -gon, and the coordinates in each cluster are in one-to-one correspondence with edges in the corresponding triangulation. The positive geometry [8] associated to these $m = 2$ amplitudes is the amplituhedron $\mathcal{A}_{n,k}^{(2)}$, which is an interesting object on its own, and its many interesting features were studied e.g. in [9–12].

In this paper we explore the conjecture [13, 14] that the poles of every rational Yangian invariant are given by cluster coordinates in a common cluster, a property referred to as *cluster adjacency* following [15]. The full $\mathcal{N} = 4$ Yang-Mills theory has a whole zoo of n -particle $N^k\text{MHV}$ Yangian invariants (see Chapter 12 of [16] for a discussion of their classification), and evidence supporting this conjecture is so far restricted to relatively small n and k . In contrast, in the $m = 2$ toy model, by using the amplituhedron formulation of scattering amplitudes [17], we are able to write down an explicit formula for all Yangian invariants for any n and k . Each $N^k\text{MHV}$ invariant is labelled by a collection of k non-intersecting triangles inside an n -gon, with denominator factors corresponding precisely to edges of the triangles. Consequently, the result manifestly satisfies cluster adjacency with respect to the $\text{Gr}(2, n)$ cluster algebra.

¹Note that this is quite different from restricting to two space-time dimensions.

²This algebra has also been found to govern the structure of $\mathcal{N} = 4$ Yang-Mills amplitudes in the multi-Regge limit [7].

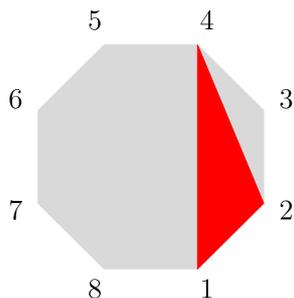


Figure 1. Triangle $T_{124}^{(8)}$.

2 Classification of Yangian Invariants for $m = 2$

Yangian invariants are basic building blocks for many amplitude-related quantities of interest (see for example [16, 18–25]). The classification of $\mathcal{N} = 4$ Yang-Mills invariants (i.e., $m = 4$) is discussed in section 12 of [16]. In this section we discuss the classification of the analogous set of Yangian invariants for $m = 2$. We will see that they can all be associated to the so-called generalised triangles, which are the building blocks for triangulations of the amplituhedron space $\mathcal{A}_{n,k}^{(2)}$. Then the Yangian invariants can be extracted from canonical differential forms with logarithmic singularities on all boundaries of generalised triangles. For $k = 1$ there is a unique type of Yangian invariants of the form (2.3) which trivially corresponds to a triangle in \mathbb{P}^2 [26]. For $k = 2$ there are two types of Yangian invariants, (2.8) and (2.9), which we will see correspond respectively to two non-intersecting triangles or to two triangles glued along an edge to form a quadrilateral. A general configuration of k non-intersecting triangles corresponds to the general N^k MHV Yangian invariant shown in (2.10).

2.1 Review

We recall that the (tree-level) amplituhedron $\mathcal{A}_{n,k}^{(m)}$ is defined [17] as the image of the positive Grassmannian $G_+(k, n)$ under the linear map

$$C \in G_+(k, n) \quad \rightarrow \quad Y = C \cdot Z^T \in G(k, k + m), \quad (2.1)$$

for generic positive matrix $Z \in M_+(k + m, n)$. If \mathcal{C} is a positroid cell in $G_+(k, n)$, $Z(\mathcal{C})$ is its image under the amplituhedron map, and $\Omega_{\mathcal{C}}$ is the unique canonical differential form [8] on $G(k, k + m)$ with logarithmic singularities (only) on the boundaries of $Z(\mathcal{C})$. Then $\Omega_{\mathcal{C}}$ provides the Yangian invariant associated to \mathcal{C} defined directly in the amplituhedron space, as explained in [25]. Alternatively, Yangian invariants can be represented as certain residues, or contour integrals, of the top form on $\text{Gr}_+(k, n)$ [18, 19]. The connection between these two ways of representing Yangian invariants is laid out in section 7 of [17]. For our purposes, the significant advantage of using the amplituhedron construction is that it enables us to write down the completely general formula for $\Omega_{\mathcal{C}}$, given below in (2.10).

In the remainder of this paper we specialise to $m = 2$. The Yangian invariants considered in this paper correspond to $2k$ -dimensional positroid cells in $G_+(k, n)$ whose images under the amplituhedron map are also $2k$ -dimensional. We refer to the images of such

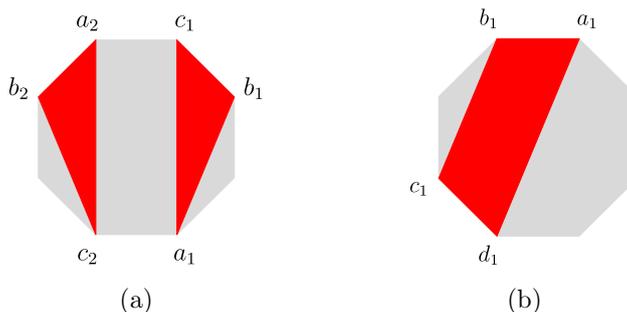


Figure 2. Labels for the $k = 2$ generalised triangles: (a) $\mathcal{T}_{a_1, b_1, c_1; a_2, b_2, c_2}^{(n)}$, (b) $\mathcal{T}_{a_1, b_1, c_1, d_1}^{(n)}$.

In boundary cases some indices could be repeated (for example c_1 could equal a_2 in the first matrix). Although each type of cell is four-dimensional in $G_+(2, n)$, only the first has a four-dimensional image in $G(2, 4)$; it is easy to check that cells of the second or third type have images of dimension three or two, respectively. Therefore, we are interested only in cells parametrised by matrices with three non-zero entries in columns $\{a_1, b_1, c_1\}$ in the first row and three non-zero entries in columns $\{a_2, b_2, c_2\}$ in the second row. We naturally label such a cell by a pair of triangles $T_{a_1 b_1 c_1}^{(n)}$ and $T_{a_2 b_2 c_2}^{(n)}$ inside an n -gon. The triangles must be non-intersecting, since otherwise the matrix $C_{a_1, b_1, c_1; a_2, b_2, c_2}$ would not be positive.

There are two types of generalised triangles parametrised by $C_{a_1, b_1, c_1; a_2, b_2, c_2}$, depending on the choice of indices. If the index sets $\{a_1, b_1, c_1\}$ and $\{a_2, b_2, c_2\}$ have at most one element in common, then the two triangles $T_{a_1 b_1 c_1}^{(n)}$ and $T_{a_2 b_2 c_2}^{(n)}$ intersect at most at a single point. This type of configuration is shown in the left panel of figure 2. Note that the image of this cell in $G(2, 4)$, which we denote by $\mathcal{T}_{a_1, b_1, c_1; a_2, b_2, c_2}^{(n)}$, has six codimension-one boundaries, regardless of whether or not the triangles share a vertex.

On the other hand, the two triangles could share an edge, say $a_2 = b_1$, $b_2 = c_1$, and $d_2 = d_1$, such that they form a quadrilateral with vertices $\{a_1, b_1, c_1, d_1\}$, as shown in the right panel of figure 2. The image of this cell, which we denote by $\mathcal{T}_{a_1, b_1, c_1, d_1}^{(n)}$, only has four codimension-one boundaries in $G(2, 4)$. Note that there are two ways to form the same quadrilateral by joining triangles, namely $T_{a_1 b_1 c_1}^{(n)} \cup T_{a_1 c_1 d_1}^{(n)}$ or $T_{a_1 b_1 d_1}^{(n)} \cup T_{b_1 c_1 d_1}^{(n)}$. Employing a $GL(2)$ transformation, one can show that the two corresponding matrices $C_{a_1 b_1 c_1; a_1 c_1 d_1}$ and $C_{a_1 b_1 d_1; b_1 c_1 d_1}$ parametrise the same cell in $G_+(2, n)$. The fact that both the cell and its image are independent of the triangulation will become important in the following when we describe our labelling of generalised triangles.

As a concrete example, we provide the complete list of labels for generalised triangles for $k = 2$, $n = 5$ in figure 3.

The Yangian invariants associated to the six- and four-boundary types of generalised triangles are respectively

$$\Omega_{a_1 b_1 c_1; a_2 b_2 c_2} = \frac{\langle Y(a_1 b_1 c_1) \cap (a_2 b_2 c_2) \rangle^2}{\langle Y a_1 b_1 \rangle \langle Y b_1 c_1 \rangle \langle Y c_1 a_1 \rangle \langle Y a_2 b_2 \rangle \langle Y b_2 c_2 \rangle \langle Y c_2 a_2 \rangle}, \quad (2.8)$$

$$\Omega_{a_1 b_1 c_1 d_1} = \frac{\langle a_1 b_1 c_1 d_1 \rangle^2}{\langle Y a_1 b_1 \rangle \langle Y b_1 c_1 \rangle \langle Y c_1 d_1 \rangle \langle Y d_1 a_1 \rangle}, \quad (2.9)$$

where we define $\langle Y(a_1 b_1 c_1) \cap (a_2 b_2 c_2) \rangle = \langle Y_1 a_1 b_1 c_1 \rangle \langle Y_2 a_2 b_2 c_2 \rangle - \langle Y_2 a_1 b_1 c_1 \rangle \langle Y_1 a_2 b_2 c_2 \rangle$.

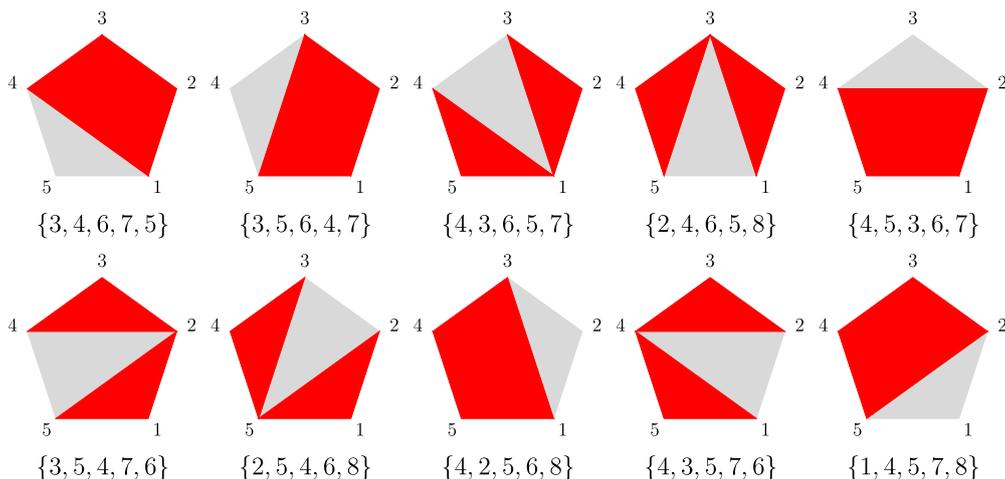


Figure 3. The 10 generalised triangles for $n = 5, k = 2$, together with the permutations that label the corresponding positroid cells in the notation of [16].

To summarise, the generalised triangles for $k = 2$ are in one-to-one correspondence with pairs of non-intersecting triangles $T_{a_1 b_1 c_1}^{(n)}$ and $T_{a_2 b_2 c_2}^{(n)}$ that share at most a vertex, together with quadrilaterals $\{a_1, b_1, c_1, d_1\}$ inside an n -gon. Furthermore, boundaries of generalised triangles, and hence singularities of respective Yangian invariants, are in correspondence with edges of these configurations. Therefore, the set of singularities of Yangian invariants for $k = 2$ correspond to a set of non-intersecting diagonals (and possible external edges) of an n -gon.

2.4 General k

For general k there exists a natural generalisation of the labelling we encountered for $k = 1, 2$. Generalised triangles are images of $2k$ -dimensional positroid cells inside $G_+(k, n)$ which can be parametrised by a matrix $C_{a_1 b_1 c_1; \dots; a_k b_k c_k}$ whose α^{th} row has non-zero entries only in columns $\{a_\alpha, b_\alpha, c_\alpha\}$, with $\alpha = 1, \dots, k$. To this set of indices we associate the union of the k non-intersecting triangles $T_{a_1 b_1 c_1}^{(n)}, \dots, T_{a_k b_k c_k}^{(n)}$ inside an n -gon. If no pair of triangles share a common edge then we denote the generalised triangle associated to this configuration by $\mathcal{T}_{a_1 b_1 c_1; \dots; a_k b_k c_k}^{(n)}$. If two triangles share a common edge then we combine them to form a quadrilateral. If there are more triangles sharing common edges then we remove all shared edges to further combine them into higher polygon inside the n -gon. Therefore, generalised triangles of the $m = 2$ amplituhedron are in one-to-one correspondence with sets of non-intersecting polygons inside an n -gon, with no pair of polygons sharing more than a single vertex. We have checked this statement up to high values of n and k using the `positroid` package [27] and we conjecture it is always true. Interestingly, we find that the intersection number (see [16, 27]) is $\Gamma = 1$ in each case. This contrasts with the situation for $m = 4$ Yangian invariants, where it is not uncommon to have $\Gamma > 1$.

Now it is possible to introduce a generalisation of the formulas (2.3), (2.8) and (2.9) for all Yangian invariants at general k . Let s be the total number of polygons, and for $j = 1, \dots, s$ let p_j be the number of edges of the j^{th} polygon. We denote the j^{th} polygon by

$P_{a_{j1}, \dots, a_{jp_j}}^{(n)}$, where $a_{j1}, a_{j2}, \dots, a_{jp_j}$ are its vertices. Since the total number of triangles has to be k and the j^{th} polygon contains $p_j - 2$ triangles, we must have $\sum_{j=1}^s (p_j - 2) = k$. Then the Yangian invariant associated to the collection of polygons $\{P_{a_{11}, \dots, a_{1p_1}}^{(n)}, \dots, P_{a_{s1}, \dots, a_{sp_s}}^{(n)}\}$ is

$$\Omega_{a_{11}, \dots, a_{1p_1}; \dots; a_{s1}, \dots, a_{sp_s}} = \frac{\langle Y(a_{11} \dots a_{1p_1}) \cap \dots \cap (a_{s1} \dots a_{sp_s}) \rangle^2}{(\langle Y a_{11} a_{12} \rangle \dots \langle Y a_{1p_1} a_{11} \rangle) \dots (\langle Y a_{s1} a_{s2} \rangle \dots \langle Y a_{sp_s} a_{s1} \rangle)}, \quad (2.10)$$

where we have defined

$$\begin{aligned} &\langle Y(a_{11} \dots a_{1p_1}) \cap \dots \cap (a_{s1} \dots a_{sp_s}) \rangle \\ &= \epsilon_{\alpha_1^1 \dots \alpha_{p_s-2}^s} \langle Y_{\alpha_1^1 \dots \alpha_{p_1-2}^1} a_{11} \dots a_{1p_1} \rangle \dots \langle Y_{\alpha_1^s \dots \alpha_{p_s-2}^s} a_{s1} \dots a_{sp_s} \rangle, \end{aligned} \quad (2.11)$$

and $Y_{\alpha_1 \dots \alpha_r}^\perp$ parametrises the orthogonal complement of a collection of Y 's,

$$Y_{\alpha_1 \dots \alpha_r}^\perp = \epsilon_{\alpha_1 \dots \alpha_k} Y_{\alpha_{r+1}} \wedge \dots \wedge Y_{\alpha_k}. \quad (2.12)$$

We note that if $p_j = 3$ for all $j = 1, \dots, s$, i.e. all polygons are triangles, then (2.10) agrees with [8].⁴ Moreover, the formula (2.10) nicely encodes the geometry of the set of polygons of our labels. The building blocks of the numerator are s brackets, one for each polygon, with the j^{th} bracket containing all the vertices of the j^{th} polygon. Meanwhile the denominator has exactly s factors, and the j^{th} factor is a product of brackets of the type $\langle Yab \rangle$ over all edges $\{a, b\}$ of the j^{th} polygon. When the i^{th} and j^{th} polygon share an edge $\{a, b\}$, so that they combine to form a $(p_i + p_j - 2)$ -gon, then formula (2.10) nicely rearranges to give the Yangian invariant associated with the same set of polygons, but with the i^{th} and j^{th} polygon replaced by the merged $(p_i + p_j - 2)$ -gon. In particular, it can be shown that the numerator factorises and one gets an overall factor of $\langle Yab \rangle^2$, which cancels the singularities associated with the shared edge $\{a, b\}$ from the denominator, as expected.

To summarise, we see that in general, all singularities of a given Yangian invariant correspond to a subset of non-intersecting diagonals inside of an n -gon (and possibly external edges). In the appendix we provide an explicit enumeration of the number of n -particle N^k MHV Yangian invariants for $m = 2$.

3 A_{n-3} cluster adjacency

Tree-level *cluster adjacency* in $\mathcal{N} = 4$ Yang-Mills theory is the conjectured [13] property that every (rational) Yangian invariant has poles given by some collection of \mathcal{A} -coordinates of the $\text{Gr}(4, n)$ cluster algebra that can be found together in common cluster. So far, evidence supporting this conjecture is restricted to relatively small n and k . In [13] several examples were checked by explicitly identifying a suitable cluster for several relatively simple Yangian invariants. Later in [14] a computationally efficient method (first explained in [28]) for testing whether two cluster coordinates belong in a common cluster was used to provide further evidence for this conjecture for various somewhat higher n and k .

⁴Notice a typo in the numerator of formula (7.52) in [8]: the power should be 2 instead of k .

The natural generalisation of the cluster adjacency conjecture for general m would posit that the poles of every Yangian invariant are given by some subset of the \mathcal{A} -coordinates of some cluster of the $\text{Gr}(m, n)$ algebra. For $m = 2$ this is the same as the classic A_{n-3} algebra, whose structure is completely understood [1]. This algebra has $\frac{1}{n-1} \binom{2n-4}{n-2}$ clusters, each containing $2n-3$ \mathcal{A} -coordinates. These numbers are respectively the number of distinct triangulations of an n -gon, and the number of edges (including external edges) in any such triangulation. If we label the vertices of a regular n -gon by homogeneous coordinates z_1, \dots, z_n of n points in \mathbb{P}^1 and let $\langle ab \rangle = \epsilon_{AB} z_a^A z_b^B$, as usual, then the coordinates are enumerated as follows. Each cluster contains the n coordinates $\langle 12 \rangle, \langle 23 \rangle, \dots, \langle n1 \rangle$ corresponding to the external edges, together with precisely $n-3$ additional $\langle ab \rangle$'s that correspond to the internal edges of the triangulation.

In light of this discussion it is now essentially obvious that every Yangian invariant in the $m = 2$ version of $\mathcal{N} = 4$ Yang-Mills theory, whose generic form is shown in (2.10), satisfies cluster adjacency. In order to make this statement manifest, one follows the prescription for extracting scattering amplitudes from amplituhedron differential forms discussed in [17]. It states that in order to project our results from the amplituhedron space parametrised by Z_a 's to the $m = 2$ “momentum twistor” space parametrised by z_a 's we need to set

$$Y = \begin{pmatrix} 0_{2 \times k} \\ 1_{k \times k} \end{pmatrix}. \tag{3.1}$$

Therefore, as far as the denominator of (2.10) is concerned, poles of amplituhedron Yangian invariants at $\langle Yab \rangle = 0$ translate directly into poles of momentum twistor Yangian invariants at $\langle ab \rangle = 0$, making the cluster adjacency manifest. The numerator of each Yangian invariant will in general be a rather non-trivial polynomial in the z_a 's and their Grassmann partners, but these are not relevant in the analysis of poles of a single Yangian invariant. The numerators only come into play when we combine different Yangian invariants into amplitudes, where they ensure the cancellation of spurious poles. In particular, the poles of a full tree-level amplitude are encoded by boundaries of the special form $\langle Yaa+1 \rangle = 0$, so in this sense full tree amplitudes could themselves be said to satisfy cluster adjacency (indeed an even stronger form of it, involving only external edges of the associated n -gon).

The fact that the connection between cluster adjacency and Yangian invariants is strikingly simple for $m = 2$ provides some circumstantial evidence in support of our hope that the same will be true for the apparently much more non-trivial case of $m = 4$ (or perhaps even for general m). It also gives support to the suggestion made in [14] that the connection between cluster adjacency and Yangian invariants might admit a mathematical explanation that is independent of the physics of scattering amplitudes. In this paper, we see evidence that this connection most likely originates from the geometry of the amplituhedron.

If there is to be, one day, an analytic proof of the tree-level cluster adjacency conjecture, it is natural to speculate that it may hinge on the fact that it is known [22] that every positive Yangian invariant can be written as Grassmannian integral [29] (more specifically, over the momentum twistor Grassmannian [18, 19]) over a contour associated to a positroid cell. Here, then, we have access to relatively simple situations in which both the *integrand*

$k \backslash n$	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1
1	1	4	10	20	35	56	84	120
2	0	1	10	48	161	434	1008	2100
3	0	0	1	20	161	824	3186	10152
4	0	0	0	1	35	434	3186	16840
5	0	0	0	0	1	56	1008	10152
6	0	0	0	0	0	1	84	2100
7	0	0	0	0	0	0	1	120
8	0	0	0	0	0	0	0	1
Total	2	6	22	90	394	1806	8558	41586

Table 1. The number of generalised triangles for $n < 11$.

(the natural top form on $\text{Gr}(k, n)$) and the *integral* (the resulting Yangian invariant) are both *rational functions*. It would be extremely exciting to learn what property of the former is responsible, after integration, for cluster adjacency of the latter. This could help point the way towards answering the long-standing, but much more complicated, question of how the cluster structure of integrands [16] in SYM theory is related, upon integration, to the cluster structure of the resulting polylogarithmic functions that appear in amplitudes.

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A Number of generalised triangles

We have tabulated the number of generalised triangles for $n < 11$ In table 1. Reading down the columns gives 1, 1, 1, 1, 4, 1, 1, 10, 10, 1, 1, 20, 48, 20, 1, 1, 35, 161, 161, 35, . . . which is sequence A175124 in [30]. It is generated by the coefficients of the inverse series of $\frac{x(1-pqx^2)}{(1+px)(1+qx)}$. The sequence of the total number of generalised triangles for given n : 2, 6, 22, 90, 394, . . ., is known as the sequence of large Schröder numbers. Interestingly, the definition of Schröder numbers as the number of all configurations of non-intersecting triangles in an n -gon, seems to be absent in the literature.

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