A rapidly rotating perfectly conducting sphere and the electrodynamics of a neutron star

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Summary –. We discuss the electrodynamics of a rapidly rotating perfectly conducting magnetized sphere rotating with constant angular velocity about an axis which is inclined to the magnetic dipole field. An exact special relativistic solution is obtained for the electromagnetic field in the interior. The exterior solution is also determined and matched to the interior by using the appropriate boundary conditions. A formula for the relativistic energy emission from the sphere is derived, and compared with the known non-relativistic formula. It is shown that in the slow rotation limit, all the relativistic solutions and formulae obtained reduce to the known non-relativistic expressions in the literature. Some of the main objectives of this work are to obtain the expressions for the electromagnetic field vectors in the interior and exterior and to see how large the special relativistic corrections are in the energy emission formula. The latter, but also other expressions and formulae, may be of relevance to rapidly rotating neutron stars, because it will be found that the classical non-relativistic formula for the energy emission, is inaccurate for such stars.

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1. Introduction

The electromagnetic field of a slowly rotating magnetized star, was discussed by Deutsch [1] and this classic work was used in a number of astrophysical applications. Belinsky et al. [2], obtained special relativistic solutions for a rotating magnetic dipole and their paper is an elaborated version of a relativistic magnetized star model, or of a millisecond pulsar, given earlier by Belinsky and Ruffini [3]. More recently, we had discussed the special relativistic electrodynamics of a rapidly rotating perfectly conducting sphere in relation to pulsar electrodynamics [4]. A treatment of a *slowly* rotating magnetized neutron star in the context of *general relativity,* was given by Rezzolla et al. [5]. In addition to the references in [2] and [3], millisecond pulsars were also discussed in [6] and [7]. We shall denote the magnetic and electric intensities by $\mathbf{H}$ and $\mathbf{E}$ respectively, with $\mathbf{B}$ and $\mathbf{D}$ the corresponding inductions. The non-relativistic expressions for the components of $\mathbf{B}$ and $\mathbf{E}$ in eqs. (48)-(53) of [5], were assumed to be independent of z. This may be clearly seen, if we transform these expressions to the coordinates $(ct,\ x,\ y,\ z)$. This assumption was also made in [4] where the expressions (48)-(53) of [5] were used as a first approximation for the components of $\mathbf{B}$ and $\mathbf{E}$.

We shall now give a brief description of the aims of the work of [4] and of the present work and note some of their differences. The scope of the work of [4] was limited because its objectives were to obtain exact interior and exterior solutions for a conducting rotating sphere under the above restrictive assumption. As a consequence of this assumption, both, the interior and exterior exact special relativistic solutions obtained in [4] contain the Bessel function of the first kind $J_1(kr\sin\theta)$ in its terms. Here, $(r,\ \theta,\ \phi)$ are spherical polar coordinates with associated unit vectors $\left(\hat{\mathbf{r}},\ \hat{\boldsymbol{\theta}},\ \hat{\boldsymbol{\varphi}}\right)$, the sphere is assumed to rotate about the *z*-axis with angular velocity of constant magnitude Ω and $k \equiv \Omega / c$. The connection between the work of [4] and rotating stars, is through the slow rotation limit of the exact interior relativistic solution given by eqs. (18)-(19) of [4]. In this limit the latter eqs. reduce to the approximations (48)-(53) of [5]. These eqs. will also be shown to be the slow rotation limits of the exact relativistic expressions we shall derive in Section 2. It is easy to see that eqs. (48)-(53) of [5] do not satisfy all of Maxwell's eqs. in the interior. The role of these approximations in the derivation of the Deutsch solutions, will become clear in Section 6. By *'Deutsch solutions'* we shall mainly mean the expressions for the components of $\mathbf{H}$ and $\mathbf{E}$ in the Appendix of [1]. The exact exterior solutions of [4], do not reduce to the approximate solutions (54)-(59) of [5] or to the Deutsch solutions of [1]. It is one of the objectives of this work to obtain special relativistic interior and exterior solutions for $\mathbf{B}$ and $\mathbf{E}$ that are not based on the above restrictive assumption and this is a major difference between the work of [4] and the present work. In the slow rotation limit, both, the exact interior and exterior solutions of the present work, will then reduce to the known approximate non-relativistic solutions of [1] and [5].

Instead of the Bessel function of the first kind J_1 with argument $kr\sin\theta$, we shall have spherical Bessel functions of various kinds and orders but with the same argument kr. As is the case in the works of [1, 4, 5], it will be seen that both, the interior and exterior solutions we shall obtain, are the sums of two different independent solutions. One of them has '$\cos\chi$' as a factor in its terms, and the other has '$\sin\chi$' as a factor, where χ is the angle between the dipole magnetic field and the rotation axis. We shall refer to them as the '$\cos\chi$' and '$\sin\chi$' solutions respectively. By removing the assumption which is implicit in eqs. (48)-(53) of [5], we shall obtain in this work new and different interior and exterior '$\sin\chi$' solutions, but the '$\cos\chi$' solutions will be the same as in [4]. Equally importantly, we shall derive a formula for the relativistic energy emission, which we shall compare with the corresponding well-known non-relativistic formula. This, we did not do in [4]. The new '$\sin\chi$' solutions in the interior and exterior will lead to significant new results, the most important of which is perhaps the relativistic energy emission formula. We have described some of the aims of the work in [4] and of the present work and also some of their important differences.

As in Deutsch [1], we shall assume that the material of the sphere has infinite conductivity. The relativistic and non-relativistic electrodynamics of such a sphere is given in [8] and [9] respectively, for the case where the rotation axis is aligned with the dipole magnetic field and so $\chi = 0$. We shall take the magnetic permeability and electric permittivity of the sphere to be those of the vacuum [1] and use units in which $\mathbf{E} = \mathbf{D}$ and $\mathbf{H} = \mathbf{B}$. These are the same units as in [2, 3]. Our choice of units corresponds to setting $\varepsilon_0 = \mu_0 = c^{-1}$ in the Deutsch paper [1], as it is also done in [2]. The radius of the star is denoted by R in [5] and by a in the present work and in [1, 4, 8, 9]. The symbol B_0 in the present work and in [4, 9], but also H_0 in [8], denotes the Deutsch parameter $R_1(a)$ in [1], so that B_0 and H_0 correspond to $R_1(a)$. The magnitude of the angular velocity is denoted by Ω here and in [4, 5, 8], but in [1, 2, 3, 9], it is denoted by ω. The symbol μ is used for the total dipole moment in [2, 3, 5] and in terms of our symbols we have $\mu = a^3 B_0 / 2$.

2. The relativistic interior electrodynamics

We shall deal with a rapidly rotating perfectly conducting sphere without any direct reference to any star or neutron star. Aspects of the work which may be of relevance to rapidly rotating neutron stars, will be noted. The solutions for the exterior components of the magnetic induction **B** and electric intensity **E** obtained by Deutsch, are displayed in the Appendix of [1]. The interior solutions are not explicitly given, but they may be found in [5]. As already noted, these solutions are for a *slowly rotating* sphere and in this work we shall derive exact interior and exterior solutions for a *rapidly rotating* sphere using the same assumptions as Deutsch in [1]. Deformations of the sphere to an oblate spheroid due to centrifugal forces or to a prolate spheroid due to strong toroidal magnetic fields [6] will not be considered. Indeed, separate treatments are required for each one of these deformations involving the use of oblate and prolate spheroidal coordinates. In this work we shall ignore both of these distortions, but in Section 8 we shall briefly discuss ways to deal with them. There are effects in the electrodynamics that arise purely from *fast rotation* and these may be most efficiently investigated without the inclusion of these complications. Gravity was not considered in [1] and also there was no discussion of the detailed physics and structure of the interior. These will therefore be absent from our treatment as well. It is one of the objectives of this work, to see how different the special relativistic expressions are from the non-relativistic ones and how large the resulting corrections are, in the case of the energy emission.

Our treatment will be physically different from that of Belinsky et al. in [2], since both the interior and exterior electrodynamics will be given and the solutions will be matched on the boundary using the appropriate conditions. One of the most important quantities we shall obtain is the energy emission. It will be seen that this depends on the exterior field as well as on the interior one through the boundary conditions. Attention will be focussed on the energy emission in the relativistic and non-relativistic regimes. We shall further show that our eqs. and formulae reduce to the non-relativistic ones that are known in the literature.

The 3-vectors **B** and **E** are given by

$$(1) \quad \mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = \nabla A_0 - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t},$$

where $-A_0$ is the electric scalar potential and **A** the magnetic vector potential, with physical components A_r, A_θ, A_ϕ. We shall take A_0 to be time-independent, so that the Lorentz condition becomes

$$(2) \quad \nabla \cdot \mathbf{A} = 0.$$

The magnetic induction **B** is a solenoidal vector field, which we shall choose to be derived from the 3-vector potential **A** satisfying (2). The vectors **B** and **E** together with the electric charge and 3-current densities ρ_e and **j** satisfy Maxwell's eqs. in the form

(3) $\nabla\cdot\mathbf{B}=0,\quad \nabla\cdot\mathbf{E}=4\pi\rho_e,\quad \nabla\times\mathbf{B}=\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}+\frac{4\pi}{c}\mathbf{j},\quad \nabla\times\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}.$

Bearing in mind the time-independence of A_0 and eqs. (1) and (2), we may show that the second and third of eqs. (3), may be written in the forms

(4) $\nabla^2 A_0=4\pi\rho_e,$

(5) $\nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=-\frac{4\pi}{c}\mathbf{j}.$

It is important to note that in obtaining the interior solutions, we have to impose the requirements that their '$\cos\chi$' parts must be the same as in eqs. (18)-(19) of [4] and that their slow rotation limits, must give the non-relativistic solutions (48)-(53) of [5]. It was possible to show that these requirements are met, if we assume that (i) A_0 is time-independent and (ii) the electric charge density ρ_e and the non-zero components of the electric 3-current $\mathbf{j}$ are given by

(6) $2\pi\rho_e=-B_0k\Gamma^4\cos\chi,$

(7)
$$\frac{2\pi}{c}j_\theta=-\frac{5B_0}{kr^2}j_3\sin\chi\sin\lambda,$$
$$\frac{2\pi}{c}j_\phi=-\frac{5B_0}{kr^2}j_3\sin\chi\cos\theta\cos\lambda-B_0\Gamma^4k^2r\cos\chi\sin\theta.$$

Here, $\lambda\equiv\phi-\Omega t$ and j_n is the spherical Bessel function of the first kind of order n and argument kr. If the argument of j_n is kr, we shall simply write j_n. On the other hand, with $\alpha\equiv ka$ we shall always write $j_n(\alpha)$ for the value of j_n at $r=a$. The symbol Γ, denotes the Lorentz factor of the fluid at any point in $0\le r\le a$, $0\le\theta\le\pi$ and so we have $\Gamma=\left(1-k^2r^2\sin^2\theta\right)^{-1/2}$. It is easy to show that the 3-current $\mathbf{j}$ whose components are given by (7), is a solenoidal or transverse current [10] and so we have

(8) $\nabla\cdot\mathbf{j}=0.$

Since, according to (6), ρ_e is time-independent, eq. (8) is the eq. of continuity.

The components of the 4-potential A_ν may now be obtained by solving eqs. (4) and (5) with ρ_e as in eq. (6) and the components of $\mathbf{j}$ as in eqs. (7). Note that $\nu=0,\ 1,\ 2,\ 3$ gives the covariant components and $\nu=0,\ r,\ \theta,\ \phi$ gives the physical components of the 4-potential A_ν; the latter are the components to be used in this work. We shall find A_0 by solving the Poisson eq. (4), and $\mathbf{A}$ by solving the inhomogeneous vector wave eq. (5). The components of $\nabla^2\mathbf{A}$ on the left side of this eq. in spherical polar coordinates are too complicated to display here, but see Morse and Feshbach [11] or the Appendix here. It may be shown that if ρ_e and $\mathbf{j}$ are given by eqs. (6) and (7) respectively, then

$$
(9)\qquad
\begin{aligned}
A_0 &= \frac{B_0}{2k}\cos\chi \ln\left(1-k^2r^2\sin^2\theta\right),\\
A_r &= \frac{15B_0}{k^2r}\, j_2 \sin\chi \sin\theta\cos\theta\sin\lambda,\\
A_\theta &= -\frac{5B_0}{k}\left(j_1-\frac{2j_2}{kr}\right)\sin\chi\sin^2\theta\sin\lambda,\\
A_\phi &= -\frac{B_0}{2k^2}\frac{\cos\chi}{r\sin\theta}\ln\left(1-k^2r^2\sin^2\theta\right)
\end{aligned}
$$

are solutions of the Poisson eq. (4) and of the inhomogeneous vector wave eq. (5).

The magnetic induction $\mathbf{B}$ and electric intensity $\mathbf{E}$ are now calculated by using eqs. (1) and (9). Their non-zero components may be shown to be

$$
(10)\qquad
\begin{aligned}
B_r &= B_0\Gamma^2\cos\chi\cos\theta + \frac{5B_0}{kr}\left(j_1-\frac{2j_2}{kr}\right)\sin\chi\sin\theta\cos\lambda,\\
B_\theta &= -B_0\Gamma^2\cos\chi\sin\theta + \frac{15B_0 j_2}{k^2r^2}\sin\chi\cos\theta\cos\lambda,\\
B_\phi &= 5B_0\left(j_2\sin^2\theta-\frac{3j_2}{k^2r^2}\right)\sin\chi\sin\lambda.
\end{aligned}
$$

$$
(11)\qquad
\begin{aligned}
E_r &= -B_0\Gamma^2 kr\cos\chi\sin^2\theta + \frac{15B_0 j_2}{kr}\sin\chi\sin\theta\cos\theta\cos\lambda,\\
E_\theta &= -B_0\Gamma^2 kr\cos\chi\sin\theta\cos\theta - 5B_0\left(j_1-\frac{2j_2}{kr}\right)\sin\chi\sin^2\theta\cos\lambda.
\end{aligned}
$$

An arbitrary constant factor in the components A_r and A_θ on which subsequent expressions depend, was taken as $5B_0/k$ so that in the slow rotation limit, the components of $\mathbf{B}$ and $\mathbf{E}$ give the corresponding non-relativistic expressions in eqs. (48)-(53) of [5]. We note that the forms of $\mathbf{B}$ and $\mathbf{E}$ in eqs. (10)-(11) are consistent with the assumption that the material of the sphere is a perfect conductor, because we have

$$
(12)\qquad E_r = kr\sin\theta B_\theta, \qquad E_\theta = -kr\sin\theta B_r
$$

and these, are the component eqs. of

$$
(13)\qquad \mathbf{E} = -\frac{1}{c}\mathbf{u}\times\mathbf{B},
$$

where $u_r = 0,\ u_\theta = 0,\ u_\phi = \Omega r\sin\theta$ are the physical components of the 3–velocity $\mathbf{u}$ at any point x^ν inside the sphere or on its boundary. The form of $\mathbf{E}$ in eq. (13) implies infinite conductivity. It may be shown that ρ_e, $\mathbf{j}$, $\mathbf{B}$ and $\mathbf{E}$ whose components are given in eqs. (6)-(7) and (10)-(11) respectively, satisfy Maxwell's eqs. (3) and in addition $\mathbf{A}$ and $\mathbf{j}$ satisfy eqs. (2) and (8) and so they are solenoidal.

Regarding the geometry of the interior electromagnetic field, we first note that $\mathbf{B}$ and $\mathbf{E}$, are perpendicular, because it may be shown that

$$
(14)\qquad \mathbf{E}\cdot\mathbf{B} = 0.
$$

Also, we find the following limits as $\Omega \to 0$:

$$\lambda \to \phi, \quad \Gamma \to 1, \quad j_1 \to 0, \quad j_2 \to 0, \quad j_1/kr \to 1/3, \quad j_2/kr \to 0, \quad j_2/k^2r^2 \to 1/15.$$

Using these expressions, eqs. (10) and (11) for the components of $\mathbf{B}$ and $\mathbf{E}$ give

$$B_r = B_0(\cos\chi\cos\theta + \sin\chi\sin\theta\cos\phi), \quad B_\theta = B_0(-\cos\chi\sin\theta + \sin\chi\cos\theta\cos\phi),$$
$$B_\phi = -B_0\sin\chi\sin\phi; \quad E_r = 0, \quad E_\theta = 0, \quad E_\phi = 0.$$

The magnitudes of $\mathbf{B}$ and $\mathbf{E}$ therefore, become B_0 and 0 respectively and we deduce that if it were not for the rotation, we would have a uniformly magnetized sphere at rest and no electric field. The part $-(c/2\pi)B_0\Gamma^4k^2r\cos\chi\sin\theta$ of the current j_ϕ in eqs. (7), is the convection current associated with the charge density ρ_e in eq. (6), because it may easily be seen that this current is $\rho_e\Omega r\sin\theta$, $\Omega r\sin\theta$ being the magnitude of the linear velocity at any point in $0 \le r \le a$, $0 \le \theta \le \pi$.

3. The relativistic exterior electrodynamics

We shall now deal with the exact exterior solutions for the components of $\mathbf{B}$ and $\mathbf{E}$. We shall take the '$\cos\chi$' terms of these components to be the same as in eqs. (25)-(26) of [4]. The '$\sin\chi$' parts of $\mathbf{B}$ and $\mathbf{E}$ are obtained from the expressions for the field vectors of the scattered wave in [12]. Due to the form of our interior solution in eqs. (10)-(11), only two of the constants involved in the linear combinations of the exterior solutions will be non-zero and the solutions reduce to just one or, at most, two terms. Consequently, the solution will be of the same form as the '$\sin\chi$' solution in the Appendix of [1]. Note however that, as we shall see, the constants L and M will be different. The complete solution is the sum of the '$\cos\chi$'solution from [4] and the '$\sin\chi$' solution. With j_n, y_n and $h_n^{(1)} \equiv j_n + iy_n$ the spherical Bessel functions of the first second and third kinds respectively of order n and argument kr and a prime denoting differentiation with respect to kr, the complete solution may therefore be expressed in terms of the equations

$$B_r = \cos\chi\sum_{n=1}^{\infty} D_{2n-1}2n(2n-1)\frac{a^{2n}}{r^{2n+1}}P_{2n-1}(\cos\theta) + B_0\sin\chi\sin\theta\,\mathrm{Re}\left[L\frac{h_1^{(1)}}{kr}e^{i\lambda}\right], \tag{15}$$

$$B_\theta = \frac{\cos\chi}{\sin\theta}\sum_{n=1}^{\infty} D_{2n-1}\frac{2n(2n-1)^2}{4n-1}\frac{a^{2n}}{r^{2n+1}}\left[P_{2n-2}(\cos\theta) - P_{2n}(\cos\theta)\right] + \frac{1}{2}B_0\sin\chi\cos\theta\,\mathrm{Re}\left[\left\{Mh_2^{(1)} + L\frac{\left(krh_1^{(1)}\right)'}{kr}\right\}e^{i\lambda}\right], \tag{16}$$

(17) $$B_\phi = \frac{1}{2}B_0 \sin\chi \,\mathrm{Re}\left[\left\{ M h_2^{(1)} \cos 2\theta + L\frac{\left(krh_1^{(1)}\right)'}{kr}\right\} ie^{i\lambda}\right],$$

(18) $$E_r = \cos\chi \sum_{n=1}^{\infty} C_{2n}(2n+1)\frac{a^{2n+1}}{r^{2n+2}}P_{2n}(\cos\theta) + \frac{3}{2}B_0 \sin\chi \sin 2\theta \,\mathrm{Re}\left[M\frac{h_2^{(1)}}{kr}e^{i\lambda}\right],$$

(19) $$E_\theta = \frac{\cos\chi}{\sin\theta}\sum_{n=1}^{\infty} C_{2n}\frac{2n(2n+1)}{4n+1}\frac{a^{2n+1}}{r^{2n+2}}\left[P_{2n-1}(\cos\theta) - P_{2n+1}(\cos\theta)\right] + \frac{1}{2}B_0 \sin\chi \,\mathrm{Re}\left[\left\{ M\frac{\left(krh_2^{(1)}\right)'}{kr}\cos 2\theta - Lh_1^{(1)}\right\} e^{i\lambda}\right],$$

(20) $$E_\phi = \frac{1}{2}B_0 \sin\chi \cos\theta \,\mathrm{Re}\left[\left\{ M\frac{\left(krh_2^{(1)}\right)'}{kr} - Lh_1^{(1)}\right\} ie^{i\lambda}\right].$$

Here, $P_n(\cos\theta)$ is the Legendre polynomial of order *n* and argument $\cos\theta$ and the constants $C_0,\ C_{2n},\ D_1,\ D_{2n+1},\ n=1,2,...$ were evaluated in eqs. (20), (23) and (22) of [8] (1), by using the continuity of the potentials $-A_0$ and A_ϕ of the '$\cos\chi$' solution at the boundary $r=a$. Note that the constant C_0 is required for the evaluation of D_1 in accordance with eq. (23) of [8]. Also in [8], the covariant components Φ_4 and Φ_3 were used in place of the physical components A_0 and A_ϕ which we use here. Furthermore, in [8], $\cos\chi = 1,\ \sin\chi = 0$ and so there is no '$\sin\chi$' solution, because the dipole magnetic field is aligned with the rotation axis and so $\chi = 0$.

The only non-zero constants *L* and *M* of the '$\sin\chi$' solution are evaluated from the boundary conditions, which require that $B_r,\ E_\theta$ and E_ϕ must be continuous on $r = a$ [1]. We find that their values are

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(1) Note the following trivial misprints in [8]: in the expression for Φ_4 in eq. (16), here denoted by A_0 , the power of Ω/c should be $2n-1$ and not $2(n-1)$; in the expression for $\sin^{2m}\theta$ below eq. (19), the summation should be from $n=0$ to $n=m$; in the expression for C_0 in eqs. (20), the factor 2^{m-1} must be replaced by 2^{2m-1}.

(21) $$L = \frac{5}{h_1^{(1)}(\alpha)}\left(j_1(\alpha) - \frac{2j_2(\alpha)}{\alpha}\right), \qquad M = \frac{5\alpha}{\left[\alpha h_2^{(1)}(\alpha)\right]'}\left(j_1(\alpha) - \frac{2j_2(\alpha)}{\alpha}\right)$$

where $\left[\alpha h_2^{(1)}(\alpha)\right]'$ is the derivative of $\left(krh_2^{(1)}\right)$ with respect to *kr* evaluated at $r = a$. The values of these constants are exact, because they are the result of matching the exact interior and exterior solutions in eqs. (10)-(11) and (15)-(20) respectively, on the boundary $r = a$. The surface charge density may be calculated from the discontinuity of E_r at $r = a$. Likewise, there will be surface current densities in the θ and ϕ directions which may be calculated from the discontinuities of B_ϕ and B_θ respectively at $r = a$. Finally, after some considerable algebra, it may be shown that Maxwell's equations in the exterior are satisfied exactly, in the form

(22) $$\nabla\cdot\mathbf{B} = 0, \qquad \nabla\cdot\mathbf{E} = 0, \qquad \nabla\times\mathbf{B} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}, \qquad \nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}.$$

The wave zone is the region *r*>*a* outside the sphere at distances that are large compared with a wavelength and it may therefore be defined by $r \gg c/\Omega$, or equivalently, $kr \gg 1$. Using the expressions for spherical Bessel functions of large arguments, it is sufficient for our purposes to use the asymptotic formulae in Born and Wolf [12], which are

(23) $$h_1^{(1)} = -\frac{e^{ikr}}{kr}, \quad h_2^{(1)} = \frac{ie^{ikr}}{kr}, \quad \left(krh_1^{(1)}\right)' = -ie^{ikr}, \quad \left(krh_2^{(1)}\right)' = -e^{ikr}.$$

Excluding inverse powers of *r* higher than 2 and using the expressions in (23), the exterior solution in eqs. (15)-(20) is reduced to

$$B_r = -\frac{1}{k^2r^2}B_0 \sin\chi \sin\theta \,\mathrm{Re}\left[L\exp\{i(\lambda + kr)\}\right],$$

(24) $$B_\theta = \frac{1}{2kr}B_0 \sin\chi\cos\theta\,\mathrm{Re}\left[i(M - L)\exp\{i(\lambda + kr)\}\right],$$

$$B_\phi = -\frac{1}{2kr}B_0 \sin\chi\,\mathrm{Re}\left[(M\cos 2\theta - L)\exp\{i(\lambda + kr)\}\right].$$

$$E_r = \frac{3}{2k^2r^2}B_0 \sin\chi\sin 2\theta\,\mathrm{Re}\left[iM\exp\{i(\lambda + kr)\}\right],$$

(25) $$E_\theta = -\frac{1}{2kr}B_0 \sin\chi\,\mathrm{Re}\left[\left(M\cos 2\theta - L\right)\exp\{i(\lambda + kr)\}\right],$$

$$E_\phi = -\frac{1}{2kr}B_0 \sin\chi\cos\theta\,\mathrm{Re}\left[i\left(M - L\right)\exp\{i(\lambda + kr)\}\right].$$

It is noted that unlike the non-relativistic eqs. (13)-(14) in [1], both, $\sin(\lambda + kr)$ and $\cos(\lambda + kr)$ terms, are present in all the components of $\mathbf{B}$ and $\mathbf{E}$ in the wave zone. We shall demonstrate this for the case of the first of (24) for B_r as an example, by using the expression for L in eqs. (21); this gives

$$B_r = -\frac{5\left(j_1(\alpha) - \frac{2j_2(\alpha)}{\alpha}\right)}{\left(j_1^2(\alpha) + y_1^2(\alpha)\right)} \frac{1}{k^2 r^2} B_0 \sin\chi \sin\theta \left[j_1(\alpha)\cos(\lambda + kr) + y_1(\alpha)\sin(\lambda + kr) \right].$$

It will be seen later however, that in the slow rotation limit, eqs. (24) and (25) reduce to the non-relativistic eqs. (13) and (14) of [1].

4. The Poynting vector and the electromagnetic energy emission

The rate at which energy is radiated away from the sphere is

$$(26) \qquad -\frac{dW}{dt} = \frac{c}{4\pi} \int_S (\mathbf{E} \times \mathbf{B}) \cdot d\mathbf{S} + c \int_V \mathbf{j} \cdot \mathbf{E} dV$$

where S is a spherical surface of radius $r>a$ concentric with the sphere $r = a$ and V is the volume bounded by S. With $\mathbf{B} = \mathbf{H}$, the first integral on the right side of (26) is the flux of the Poynting vector $c\mathbf{E} \times \mathbf{B} / 4\pi$ over the surface S where the components of $\mathbf{B}$ and $\mathbf{E}$ in this integral are given in (15)-(20). The quantity $\mathbf{j} \cdot \mathbf{E}$ in the volume integral on the right side of (26), may be calculated by using the components of $\mathbf{j}$ and $\mathbf{E}$ in eqs. (7) and (11). This will not be necessary however, since we require the time average of (26) over the period of rotation $2\pi / \Omega$. It is easy to show, that the time-average of the volume integral will be zero.

The time-average of eq. (26) over a complete period, is therefore the time-average of the flux of the Poynting vector. Denoting this by $\bar{I}_{rel.}$ and with B_θ, B_ϕ and E_θ, E_ϕ as in eqs. (16), (17) and (19), (20) respectively, we have

$$(27) \qquad \bar{I}_{rel.} = \frac{\Omega c}{8\pi^2} \int_{t_1}^{t_2} dt \int_0^{2\pi} d\phi \int_0^{\pi} \left(E_\theta B_\phi - E_\phi B_\theta \right) r^2 \sin\theta d\theta,$$

with $t_1 = t$ and $t_2 = t + 2\pi / \Omega$. The calculations are quite involved, but this eq. eventually gives for the relativistic energy emission

$$(28) \qquad \bar{I}_{rel.} = \Lambda \left(\frac{2c}{3a^4} \mu^2 \sin^2\chi \right),$$

or since $\alpha = \Omega a / c$ this may also be written as

$$(29) \qquad \bar{I}_{rel.} = \frac{\Lambda}{\alpha^4} \left(\frac{2\Omega^4}{3c^3} \mu^2 \sin^2\chi \right)$$

where we have set

$$(30) \qquad \Lambda \equiv \frac{5}{2} \left(j_1(\alpha) - \frac{2j_2(\alpha)}{\alpha} \right)^2 \times \left[\frac{6}{[\alpha j_1(\alpha) - 2j_2(\alpha)]^2 + [\alpha y_1(\alpha) - 2y_2(\alpha)]^2} + \frac{10}{\alpha^2 [j_1^2(\alpha) + y_1^2(\alpha)]} \right].$$

In the derivation of the formulae (28)-(30) for the relativistic energy emission, we used the known results

$$y_1(\alpha)j_2(\alpha)-y_2(\alpha)j_1(\alpha)=\frac{1}{\alpha^2},\quad \alpha^2 h_1^{(1)}(\alpha)h_1^{(1)*}(\alpha)=\ \alpha^2[j_1^2(\alpha)+y_1^2(\alpha)],$$

$$[\alpha h_2^{(1)}(\alpha)]'[\alpha h_2^{(1)*}(\alpha)]'=[\alpha j_1(\alpha)-2j_2(\alpha)]^2+[\alpha y_1(\alpha)-2y_2(\alpha)]^2$$

where a star denotes complex conjugation. Also, we have used the relation $B_0=2\mu/a^3$ in the expressions for B_θ, B_ϕ and E_θ, E_ϕ in eqs. (16)-(17) and (19)-(20) respectively. The relativistic expressions for the energy emission are given by eqs. (28)-(30) and these formulae are of relevance to millisecond pulsars.

The ultra-relativistic limit of our energy emission formula is the limit of (29) as $\alpha\to 1$. This is finite, because it may be shown that the limit of Λ as $\alpha\to 1$ is finite. Using Maple we found that $\lim_{\alpha\to 1}\Lambda=0.405886$ correct to six decimal places; this leads to

$$\text{(31)}\qquad \bar{I}_{ultrarel.}=(0.405886)\left(\frac{2\Omega^4}{3c^3}\mu^2\sin^2\chi\right).$$

In the context of this work, the ultra-relativistic limit has no practical significance, since no surface velocity can reach the vacuum speed of light.

The interpretation and significance of the formulae (28)-(31) as well as comparisons with the energy emission formulae of Belinsky et al. in [2] and with the classical (non-relativistic) results, will be made in Sections 5 and 6.

5. Rapidly rotating neutron stars

Most neutron stars, are observed to rotate slower than a few times per second [7]. There are neutron stars however, that can reach spin rates of hundreds of times a second. Millisecond pulsars are defined in [6] as those with spin periods less than $10\ \text{ms}$. We shall not give a comprehensive review of millisecond pulsars, because such reviews may be found in [2, 3, 7] and references cited therein. Some of these pulsars may have a large enough radius for linear speeds on the surface to reach appreciable fractions of the vacuum speed of light, resulting to significant relativistic effects near the stellar surface [3]. These effects have been described here by exact relativistic expressions. In particular, eqs. (28)-(30) are the relativistic energy emission formulae and eq. (31) gives their ultra-relativistic limit.

The electromagnetic field of a *rapidly rotating* magnetized sphere is an interesting problem in electrodynamics. Deutsch modelled a *slowly rotating* magnetized star as a sharply bounded perfectly conducting sphere in rigid rotation in vacuo [1]. Likewise, we shall model the electrodynamics of a *rapidly rotating* neutron star on the electrodynamics of such a sphere, which however, is *rapidly rotating*. All the relevant equations and formulae were obtained in the previous sections and they are appropriate for rapidly rotating neutron stars.

Another model of a millisecond pulsar is that of Belinsky et al. [2] and Belinsky and Ruffini [3]. They based their model on the electrodynamics of an infinitely thin magnetized rotating rod. They obtained special relativistic solutions for such a rapidly rotating magnetic dipole and their energy emission formula is

$$(32)\qquad \bar{I}^{Bel.}_{rel.}=\frac{1}{2}\left(\gamma^4+F\right)\left(\frac{2\Omega^4}{3c^3}\mu^2\sin^2\chi\right).$$

Here, we wrote Ω for the ω in [2], but we used $\gamma=\left(1-\beta^2\right)^{-1/2}$ for their Lorentz factor with $\beta=c^{-1}\Omega l\sin\chi,\ 2l$ being the length of their dipole. As for the function F, there is no expression for it in terms of elementary functions; it is given by eq. (24) of [2] and since it depends on β and χ, they have denoted it by $F=F(\beta,\chi)$.

Belinsky et al. in [2] calculated the ultra-relativistic limit of their energy emission formula in (32), which they define to be the limit of (32) as $\beta\to 1$ and they obtained the expression

$$(33)\qquad \bar{I}^{Bel.}_{ultrarel.}=\frac{1}{2}\gamma^4\left(\frac{2\Omega^4}{3c^3}\mu^2\sin^2\chi\right).$$

Our formula (29) and the formula (32), depend in the same way on $\sin\chi,$ through their common factor $\left(\Omega^4\mu^2\sin^2\chi\right)/3c^3$. Formula (32) however, has another dependence on χ through the factor $\left(\gamma^4+F\right),$ which also depends on β. In our formula (29), the factor $\left(\Lambda/\alpha^4\right)$ does not depend on $\chi,$ but only on our parameter α. Furthermore, the radius a of the star is absent from (32) and the other formulae in [2]. It follows that we can only make a meaningful comparison between our formula (29) and the formula (32) of Belinsky et al. in [2], in terms of their common factor, their ultra-relativistic limits and their slow rotation limits. Firstly, their common factor is $\left(\Omega^4\mu^2\sin^2\chi\right)/3c^3$. Secondly, it is clear that both, (32) and (33) diverge as $\beta\to 1,$ because $\lim_{\beta\to 1}\gamma^4=\infty.$ We must therefore conclude that the interpretation of the ultra-relativistic limit of the emission formula of Belinsky et al. in [2], must be that if β is sufficiently large so that F is negligible compared with $\gamma^4,$ then the energy emission is given by (33). Our ultra-relativistic limit on the other hand, is the limit of (29) as $\alpha\to 1$. This was calculated in eq. (31) and is finite. As for the slow rotation limits of the energy emission, it will be shown in Section 6, that the limit of our formula in (29) and of the formula (32) of Belinsky et al., are equal.

6. The slow rotation limits and the Deutsch solutions

To test the consistency of our approach and the validity of our equations, we shall derive the slow rotation limits of all the relativistic expressions we have obtained. This process should lead to the non-relativistic formulae that are known in the literature. In the case of slow rotation we have $\Omega r/c \ll 1$, or, equivalently, $kr \ll 1,$ for any r and θ in $0\le r\le a,\ 0\le\theta\le\pi$. We therefore take the Lorentz factor as $\Gamma=1,$ and use the expressions of the Bessel functions for small values of their arguments in [12]. This implies that for small values of the argument *x*, we may write

$$(34)\qquad \begin{aligned} &h_1^{(1)}=-\frac{i}{x^2}\,,\quad h_2^{(1)}=-\frac{3i}{x^3},\quad \left(xh_1^{(1)}\right)'=\frac{i}{x^2},\quad \left(xh_2^{(1)}\right)'=\frac{6i}{x^3}\,,\\ &j_1=\frac{x}{3},\quad j_2=\frac{x^2}{15},\quad j_3=\frac{x^3}{105},\quad y_1=-\frac{1}{x^2},\quad y_2=-\frac{3}{x^3}. \end{aligned}$$

Using these with $x = \alpha$, we calculate L and M in (21) and find the values

(35) $$L = i\alpha^3, \quad M = -i\frac{\alpha^5}{6}.$$

With $\Gamma = 1$, the appropriate expressions from (34) and with $x = kr$, we may calculate the slow rotation limits of the interior eqs. (10)-(11). We shall not display these limits here, because with $B_0 = 2\mu / a^3$, these are the same as the approximations (28)-(29) of [4] and with $B_0 = 2\mu / R^3$, they are the same as the non-relativistic eqs. (48)-(53) in [5]. Furthermore, with $\chi = 0$, they give the expressions (1a,b) of [9], but also the corresponding un-numbered eqs. of page 229 of [8].

We now find the slow rotation limit of the exact exterior solution in eqs. (15)-(20). It is first noted that because of the form of the approximate interior solution in (48)-(53) of [5], we must only use the first term $n = 1$ in the infinite series of the exterior solution in eqs. (15)-(20). This will exclude all the inverse powers of r higher than the fourth and will reduce the '$\cos\chi$' parts of the exact eqs. (15)-(20) precisely to the '$\cos\chi$' parts of Deutsch solution in the Appendix of [1]. If we now match this reduced solution to the approximate interior solution in (48)-(53) of [5], the boundary conditions regarding the continuity of the components B_r, E_θ and E_ϕ, give $L = \alpha / h_1^{(1)}(\alpha)$ and $M = \alpha^2 / \left(\alpha h_2^{(1)}(\alpha)\right)'$. We have shown how the Deutsch solutions in the Appendix of [1] were derived, where we have to note that $R_1(a) = B_0$ and $\mu_0 = \varepsilon_0 = c^{-1}$. This procedure was not described in [1], and the approximate interior solutions (48)-(53) of [5] were not given.

We stress the fact that the Deutsch solutions in the Appendix of [1], are valid for all values of r in $a \le r < \infty$ and that *'slow rotation limit'* refers to small values of kr, $kr \ll 1$, in the interior $0 \le r \le a$. In these cases, outside the star but near its surface, we may still have $kr \ll 1$ and so we may use the approximations (34) to show that the Deutsch solutions reduce to the eqs. (18)-(19) of [1], or to the eqs. (54)-(59) of [5]. If $\chi = 0$, these eqs. reduce to eqs. (2a,b) of [9] but also to the corresponding un-numbered eqs. on page 229 of [8].

The slow rotation limits of the relativistic eqs. (24)-(25) in the wave zone, are obtained by using eqs. (34) with $x = \alpha$ or the approximate values of L and M in (35). Powers of ka greater than 2, should be neglected in the final calculation and we have to set $\varepsilon_0 = \mu_0 = c^{-1}$ in all the eqs. of [1]. This procedure gives the solutions (13)-(14) of [1]. In this approximation $E_r = 0$, because from eqs. (25) it contains the factor M which by (35) is $M = -i\alpha^5 / 6$ and is therefore neglected. It follows that the vectors $\mathbf{B}$ and $\mathbf{E}$ are perpendicular, because with the approximation $E_r = 0$, eqs. (13)-(14) of [1] lead to $\mathbf{E} \cdot \mathbf{B} = 0$.

Finally, the slow rotation limit of the rate at which energy is radiated away from the star, is obtained from our exact fully relativistic formulae (28)-(30) by using the appropriate expressions from (34) with $x = \alpha$ and eqs. (35). In this procedure, we have to ignore the α^8 term as too small compared to the α^4 term. This leads to

(36) $$\bar{I}_{nonrel.} = \alpha^4 \left(\frac{2c}{3a^4} \mu^2 \sin^2 \chi \right).$$

Since $\alpha = \Omega a/c,$ this may also be written as

(37) $$\overline{I}_{nonrel.} = \frac{2\Omega^4}{3c^3}\mu^2 \sin^2 \chi.$$

The slow rotation limit of the formula (32) of the model of Belinsky et al. in [2], is the same as the limit (37) of our formula, because for small values of Ω, $\gamma^4 = F = 1.$ It follows from (28) and from (36), that if we multiply the factor $\left(2c\mu^2 \sin^2 \chi/3a^4\right)$ by Λ, we obtain the fully relativistic energy emission in eq. (28) and if we multiply the same factor by α^4, we obtain the non-relativistic energy emission in eq. (36). Furthermore, the fully relativistic expression in (29) is the same as the non-relativistic one in (37), times the factor $\left(\Lambda/\alpha^4\right)$.

7. Numerical results for the energy emission

The relativistic formula (28) for the energy emission with Λ given by (30), is very different from and more complicated than the non-relativistic formula (36). Despite this, for very small values of α, their numerical results are almost the same. The two sets of formulae begin to give different numerical results at higher values of α. For a star of a given radius, this means of course higher values of the magnitude Ω of the angular velocity. Thus, for values of α smaller than about $0.1,$ the magnitude of the percentage difference between the relativistic and non-relativistic energy emissions, is less than about 1.25, which may be regarded as negligible. Here, we have taken the relativistic value to be the *correct value* for the emission.

It was recently reported [7, 13], that a pulsar was discovered in the globular cluster of stars Terzan 5, located some 28,000 light years from Earth in the constellation of Sagittarius. The newly-discovered pulsar, named PSR J1748-2446ad, rotates with frequency 716 Hz and is the fastest spinning neutron star known. Taking its radius to be about 16 km as it was assumed in [13], we find that α is about 0.24. The fully relativistic formula (28), gives for the energy emission the value $(0.003095)(2c/3a^4)\mu^2 \sin^2 \chi.$ The non-relativistic formula (36), gives the value $(0.003318)(2c/3a^4)\mu^2 \sin^2 \chi$ for the energy emission and so the magnitude of the percentage difference between them is about 7.2. This difference cannot be regarded as negligible. In any case, it was noted in [7, 13] that there may be many more pulsars in Terzan 5 and other clusters, some of which may be spinning even faster than this new one. The numerical difference therefore, between the fully relativistic formula (28) and the non-relativistic formula in (36), may even be greater.

In [2, 3] the possibility is mentioned, for the existence of pulsars with periods as short as 0.5 milliseconds and 10 km radius. For such a pulsar we find that, $\alpha = 0.4192,$ which is an appreciable fraction of the vacuum speed of light. The fully relativistic formula (28) gives for the energy emission the value $(0.02537)(2c/3a^4)\mu^2 \sin^2 \chi.$ If we use the non-relativistic formula in (36) instead, we shall find the value $(0.03112)(2c/3a^4)\mu^2 \sin^2 \chi$. The magnitude of the percentage difference between these two values is about 22.66, which is considerable. In numerical calculations, the difference between the formulae (28) and (36) increases with increasing values of α. The limits of α^4 as $\alpha \to 0$ and as $\alpha \to 1,$ are 0 and 1 respectively. We also find that $\lim_{\alpha \to 0} \Lambda = 0$ and in Section 4 we found that correct to six decimal places, $\lim_{\alpha \to 1} \Lambda = 0.405886,$ which is the ultra-relativistic limit of Λ. From all these considerations and from the numerical calculations in this Section, it is obviously important to have a relativistic formula to calculate the energy emission from rapidly rotating neutron stars, because the known non-relativistic formulae (36) and (37) are inaccurate. The required formula was derived here, in eqs.(28) or (29) where Λ is given by (30).

8. Concluding remarks

The problem of the electrodynamics of a rapidly rotating perfectly conducting sphere apart from being of importance in its own right, is also of interest for millisecond pulsars. In this work, we have considered an idealized star as a sharply bounded perfectly conducting sphere in rigid rotation in vacuo [1]. We solved the Poisson eq. (4) for the electric scalar potential $-A_0$ and the inhomogeneous vector wave eq. (5) for the 3-vector potential $\mathbf{A}$ in the interior, for the case where the electric charge and current densities ρ_e and $\mathbf{j}$ are given by eqs. (6) and (7). The forms of these densities are not arbitrary, but they were chosen so that the resulting components of the 4-vector potential A_ν are as in eqs. (9). The interior magnetic induction $\mathbf{B}$ and electric intensity $\mathbf{E}$ calculated from the components of A_ν in (9) and the formulae in (1), are in eqs. (10)-(11). The slow rotation limits of $\mathbf{B}$ and $\mathbf{E}$, give the non-relativistic forms that are known in the literature. This was the ultimate aim of our choice of the forms of the charge and current densities ρ_e and $\mathbf{j}$ in eqs. (6) and (7). The components of $\mathbf{B}$ and $\mathbf{E}$ in the exterior were also calculated and are given in eqs. (15)-(20). We derived from these the relativistic eqs. (24)-(25) in the wave zone.

The rate at which energy is radiated away from the sphere in the relativistic regime was calculated and displayed in eqs. (28)-(30). The slow rotation limits of all these expressions and formulae were shown to be the same as the non-relativistic expressions and formulae that are known in the literature. In particular, the slow rotation limit (37) of the energy emission, is the same as the corresponding limit of the formula (32) in Belinsky et al. [2].

The existence of special relativistic effects is demonstrated in this work in terms of exact analytical results. In this respect, the formulae (28)-(30) for the energy radiated away from the star in the relativistic regime, are most important. If such emissions are calculated using the known non-relativistic formulae in eq. (36) or (37), the results of the calculations will be incorrect. It is therefore important to have an exact special relativistic formula to calculate the energy emission, since the greater the value of α , the greater the difference between the relativistic and non-relativistic calculations is going to be.

As it was seen in Section 7, if the non-relativistic formula (36) is used instead of the fully relativistic formula (28), the result for the 716 Hz pulsar in [7, 13], will be more than 7% wrong and for the pulsar described in [2, 3], it will be more than 22% wrong. We have thus shown that for a rapidly rotating star, the special relativistic corrections to the energy emission are large enough to be significant and this was demonstrated by exact analytical results in Section 4 and numerical calculations in Section 7.

We have considered here, an idealized model of a pulsar and consequently there are physical limitations in this work. The first one to note, is that it was assumed that the star has a sharply defined boundary separating it from the vacuum as was indeed also assumed in [1]. We had not therefore considered the plasma in the magnetosphere. Another limitation is that we ignored the deformation of the star by the centrifugal forces to an oblate spheroidal shape. The deformation to prolate spheroidal shape caused by strong toroidal magnetic fields was not considered either [6]. For each of these, separate treatments are required, using oblate and prolate spheroidal coordinates respectively [14]. For small deviations from sphericity, we may achieve higher accuracies by using the results already obtained and considering a boundary which is a nearly spherical surface. For example, we may assume a surface with eq. $r = a - \varepsilon P_2(\cos\theta),$ instead of the sphere $r = a$. Here ε is a small positive number which represents the deformation parameter and numerical values must be given for different pulsars. The value of ε, depends on the angular velocity, the gravitational acceleration due to the star and the elastic moduli [15]. If we set $\theta = \pi/2$ in $r = a - \varepsilon P_2(\cos\theta)$, we obtain $r = a + (\varepsilon/2)$ and $\theta = 0$ or $\theta = \pi$, give $r = a - \varepsilon$. The eq. $r = a - \varepsilon P_2(\cos\theta)$ therefore, represents a spheroid with equatorial radius $a + (\varepsilon/2)$, and polar radius $a - \varepsilon$ the axis of the spheroid being the $\theta = 0$ axis. Using the eq. of this spheroid, we may therefore calculate the perturbations in the various quantities. This is only meant as an illustration on how to deal with small departures from sphericity. There is no suggestion that the boundary of a pulsar and its equatorial and polar radii have to be given by the above equations. These are valid only if the pulsar is an oblate spheroid, in which case a numerical value for ε must be pre-assigned.

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Appendix

If Ψ and $\mathbf{A}$ are respectively scalar and vector functions and if $\mathbf{A} = A_r\hat{\mathbf{r}} + A_\theta\hat{\boldsymbol{\theta}} + A_\phi\hat{\boldsymbol{\varphi}}$, where A_r, A_θ, A_ϕ are the physical components of $\mathbf{A}$ in spherical polar coordinates r, θ, ϕ with associated unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\varphi}}$ and ∇^2 is the Laplacian operator, then we have [11]

(A1) $$\nabla^2\Psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2},$$

(A2) $$\begin{aligned}\nabla^2\mathbf{A} = \hat{\mathbf{r}}&\left[\nabla^2 A_r - \frac{2}{r^2}A_r - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(A_\theta\sin\theta\right) - \frac{2}{r^2\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right]\\ +\hat{\boldsymbol{\theta}}&\left[\nabla^2 A_\theta - \frac{A_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial A_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\phi}{\partial\phi}\right]\\ +\hat{\boldsymbol{\varphi}}&\left[\nabla^2 A_\phi - \frac{A_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\theta}{\partial\phi}\right].\end{aligned}$$

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