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The hypersimplex canonical forms and the momentum amplituhedron-like logarithmic forms

Tomasz Łukowski* and **Jonah Stalknecht**

Department of Physics, Astronomy and Mathematics, University of Hertfordshire, Hatfield, Hertfordshire, AL10 9AB, United Kingdom

E-mail: t.lukowski@herts.ac.uk and j.stalknecht@herts.ac.uk

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Abstract

In this paper we provide a formula for the canonical differential form of the hypersimplex $\Delta_{k,n}$ for all n and k. We also study the generalization of the momentum amplituhedron $\mathcal{M}_{n,k}$ to $m = 2$, which has been conjectured to share many properties with the hypersimplex, and we provide counterexamples for these conjectures. Nevertheless, we find interesting momentum amplituhedron-like logarithmic differential forms in the $m = 2$ version of the spinor helicity space, that have the same singularity structure as the hypersimplex canonical forms.

Keywords: positive geometries, momentum amplituhedron, hypersimplex, scattering amplitudes

(Some figures may appear in colour only in the online journal)

1. Introduction

Geometry has always played an essential role in physics, and it continues to be crucial in many recently developed branches of theoretical and high-energy physics. In recent years, this statement has been supported by the introduction of positive geometries [1] that encode a variety of observables in quantum field theories [2–5], and beyond [6–8], see [9] for a comprehensive review. These recent advances have also renewed the interest in well-established and very well-studied geometric objects, allowing us to look at them in a completely new way. One essential new ingredient introduced by positive geometries is that to every convex polytope,

*Author to whom any correspondence should be addressed.

one can associate a meromorphic differential form with the property that it is singular on all boundaries of the polytope, and the divergence is logarithmic. Moreover, when each boundary is approached, an appropriately defined residue operation allows one to find the differential form of the boundary with the same properties. This process can be repeated and eventually one arrives at a zero-dimensional boundary with a trivial 0-form equal ± 1. Such canonical forms can be found for every convex polytope and for more complicated 'convex' shapes in Grassmannian spaces, which has been conjectured for the amplituhedron [2] and the momentum amplituhedron [4]. Many well-known convex polytopes made their recent appearance in physics in the context of positive geometries, the primary example given by the associahedron featured in the bi-adjoint ϕ^3 scalar field theory [3] or, more generally, generalized permutahedra discussed in [10]. More recently, another well-known polytope, the hypersimplex $\Delta_{k,n}$, also has become relevant in the positive geometry story. It was conjectured in [11] and consequently proven in [12] that a particular class of hypersimplex subdivisions are in one-to-one correspondence with the tilings of the amplituhedron $\mathcal{A}^{(2)}_{n,k}$, which is a prototypical example of a positive geometry. Moreover, it was conjectured that its spinor helicity cousin, $\mathcal{M}^{(2)}_{n,k}$, which is a generalization of the momentum amplituhedron $\mathcal{M}_{n,k}$ [4], shares many properties with the hypersimplex. This paper focuses on the latter statement and tries to verify whether it is correct. To this extent, we start by treating the hypersimplex as a positive geometry and finding its canonical differential form. In particular, the hypersimplex $\Delta_{k,n}$ can be defined as the image of the positive Grassmannian through the (algebraic) moment map [13] (see also [11]). Using this fact, we find a simple expression for the hypersimplex canonical form, which can be obtained by summing push-forwards of canonical forms of particular cells in the positive Grassmannian $G_+(k,n)$. The momentum amplituhedron $\mathcal{M}^{(2)}_{n,k}$ has also been defined as the image of the same positive Grassmannian using a linear map $\Phi_{(\Lambda,\widetilde{\Lambda})}$ [11], which we will define in the main text. After taking the same collection of positroid cells in the positive Grassmannian, and summing their push-forwards through the $\Phi_{(\Lambda,\widetilde{\Lambda})}$ map, we find a simple logarithmic differential form in spinor helicity space, that has the same singularity structure as the hypersimplex canonical form. However, it is not the canonical form of $\mathcal{M}^{(2)}_{n,k}$. Moreover, we show that $\mathcal{M}^{(2)}_{n,k}$ does not possess the desired properties conjectured in [11].

The paper is organized as follows: in section 2 we recall the definition of hypersimplex, describe its boundary structure and define positroid tilings. We also provide a previously unknown formula for its canonical differential form. In section 3 we recall the definition of the momentum amplituhedron $\mathcal{M}^{(2)}_{n,k}$ introduced in [11] and find a logarithmic differential form defined on the $m = 2$ version of the spinor helicity space, that has the same singularity structure as the hypersimplex canonical form. We also comment on the validity of the conjectures in section 11 of [11]. We end the paper with a summary and outlook, and appendices containing the definitions of positive geometries and push-forwards, and proofs of some of our statements from the main text.

2. Hypersimplex

The hypersimplices $\Delta_{k,n}$ form a two-parameter family of convex polytopes that appears in various algebraic and geometric contexts. In particular, they have been used to classify points in the Grassmannian $G(k,n)$ by studying their images through the moment map [13]. This naturally leads to a notion of matroid polytopes and matroid subdivisions [14–16], which are in turn related to the tropical Grassmmanian [14, 17, 18]. When the Grassmannian $G(k,n)$ is replaced by its positive part $G_+(k,n)$, the moment map image of $G_+(k,n)$ is still the hypersimplex $\Delta_{k,n}$, and one can use it to study positroid polytopes [19], positroid subdivisions [11, 20, 21] and their relation to the positive tropical Grassmannian [22]. In this paper we look at the hypersimplex $\Delta_{k,n}$ from the point of view of positive geometries[1]. As the main result of this section, we provide an explicit expression for the canonical differential form for $\Delta_{k,n}$ for all n and k.

2.1. Definitions

We denote by e_i the standard basis vectors in $\mathbb{R}^n$. The hypersimplex $\Delta_{k,n}$ is then defined[2] as the convex hull of the indicator vectors $e_I = \sum_{i\in I} e_i$ where I is a k-element subset of $[n] \equiv \{1, 2, \ldots, n\}$. Since for all $x = (x_1, \ldots, x_n) \in \Delta_{k,n}$ we have $x_1 + \cdots + x_n = k$, the hypersimplex $\Delta_{k,n}$ lives in an $(n-1)$-dimensional affine subspace inside $\mathbb{R}^n$. Moreover, the hypersimplex $\Delta_{k,n}$ is identical to the hypersimplex $\Delta_{n-k,n}$ after the replacement $I \leftrightarrow [n]\backslash I$. We refer to this symmetry as a *parity symmetry*.

Equivalently, the hypersimplex $\Delta_{k,n}$ can be defined as the image of the positive Grassmannian $G_+(k, n)$ through the moment map [13]. For a given n and $0 \leqslant k \leqslant n$, the *Grassmannian* $G(k, n)$ is the space of all k-dimensional subspaces of $\mathbb{R}^n$. Each element of $G(k, n)$ can be viewed as a maximal rank $k \times n$ matrix modulo $GL(k)$ transformations, whose rows span the k-dimensional space. We denote by $\binom{[n]}{k}$ the set of all k-element subsets of $[n]$. Then for $I \in \binom{[n]}{k}$, we define $p_I(C)$ to be the $k \times k$ minor formed of columns of C labelled by elements of I. We call these variables the *Plücker variables*, and they are defined up to an overall rescaling by a non-zero constant. The *positive Grassmannian* $G_+(k, n)$ is the set of all elements $C \in G(k, n)$ for which $p_I(C) \geqslant 0$ for all $I \in \binom{[n]}{k}$. Finally, we define the *moment map*

$$\mu : G(k, n) \to \mathbb{R}^n, \tag{2.1}$$

as

$$\mu(C) = \frac{\sum_I |p_I(C)|^2 e_I}{\sum_I |p_I(C)|^2}. \tag{2.2}$$

Then, the hypersimplex is the image of the (positive) Grassmannian

$$\Delta_{k,n} = \mu(G(k, n)) = \mu(G_+(k, n)). \tag{2.3}$$

If we restrict our attention to the positive Grassmannian $G_+(k, n)$, we can instead use the algebraic moment map [23]

$$\tilde{\mu}(C) = \frac{\sum_I p_I(C) e_I}{\sum_I p_I(C)}, \tag{2.4}$$

which will significantly simplify our calculations in the following. Most importantly, we have

$$\Delta_{k,n} = \tilde{\mu}(G_+(k, n)), \tag{2.5}$$

see [11] for more details.

An important fact we will use later is that the positive Grassmannian $G_+(k, n)$ has a natural decomposition into cells of all dimensions [24]. For a subset $M \subset \binom{[n]}{k}$, we denote by S_M the subset of all elements in the positive Grassmannian $G_+(k, n)$ such that their Plücker variables

[1] For an introduction on positive geometries, we refer the reader to [1], we also collect some basic information in appendix A.

[2] Alternatively, the hypersimplex $\Delta_{k,n}$ can be defined as the intersection of the hyperplane $x_1 + \cdots + x_n = k$ with the unit cube in $\mathbb{R}^n$.

are positive, $p_I > 0$, for $I \in M$, and they vanish, $p_I = 0$, for $I \notin M$. If $S_M \neq \emptyset$ then we call S_M a *positroid cell*. Positroid cells can be labelled by various combinatorial objects, most importantly by bounded affine permutations π on $[n]$ [25]. From now on we will use S_π instead of S_M to label positroid cells of the positive Grassmannian.

In the following, we will adopt the notation from [11]. The closure of the image of the positroid cell S_π through the algebraic moment map $\tilde{\mu}$ is called a *positroid polytope* [19], and we denote it by $\Gamma_\pi = \overline{\tilde{\mu}(S_\pi)}$. We will be interested in a particular type of positroid polytopes: if the dimension of Γ_π is $n-1$ and $\tilde{\mu}$ is injective on S_π then we call Γ_π a *positroid tile*. We will use positroid tiles to define positroid tilings of the hypersimplex $\Delta_{k,n}$, which will allow us to find its canonical differential form $\omega_{k,n}$. One important property of this differential form is that it is logarithmically divergent on all facets of the hypersimplex $\Delta_{k,n}$. These facets are also positroid polytopes, of dimension $n-2$, and can be described using the underlying cell decomposition of the positive Grassmannian $G_+(k,n)$. In particular, for $1 < k < n-1$, there are exactly $2n$ boundaries of the hypersimplex $\Delta_{k,n}$, and they come in two types: $x_i = 0$ or $x_i = 1$, for $i = 1, \ldots, n$. In the former case, they are images of positroid cells S_π with $\dim S_\pi = (k-1)(n-k)$, and the positroid polytope Γ_π is isomorphic with the hypersimplex $\Delta_{k-1,n-1}$. In the latter case, we find positroid cells S_π with $\dim S_\pi = k(n-k-1)$, and Γ_π is identical with the hypersimplex $\Delta_{k,n-1}$. The exceptional cases are for $k = 1$ or $k = n-1$ when the hypersimplices $\Delta_{1,n}$ and $\Delta_{n-1,n}$ are just simplices, with only one type of boundaries: $x_i = 0$ for $k = 1$ and $x_i = 1$ for $k = n-1$. In all these cases, the permutations corresponding to boundary positroid polytopes can be found using the package `amplituhedronBoudaries` [26]. The package also provides an easy way to find the complete boundary stratification of the hypersimplex $\Delta_{k,n}$.

2.2. Hypersimplex canonical forms

We are now ready to explain how to find the canonical differential form $\omega_{k,n}$ for the hypersimplex $\Delta_{k,n}$. We will use the fact that all hypersimplices can be subdivided using a collection of positroid tiles whose interiors are non-overlapping and whose union equals $\Delta_{k,n}$. We will call such collection a *positroid tiling* of $\Delta_{k,n}$. Having found a positroid tiling of $\Delta_{k,n}$, the canonical differential form $\omega_{k,n}$ can be calculated as a sum of push-forwards through the algebraic moment map $\tilde{\mu}$ of the canonical forms of the corresponding positroid cells in the positive Grassmannian $G_+(k,n)$. More specifically, if $\mathcal{T} = \{\pi_1, \ldots, \pi_p\}$, with $S_{\pi_i} \subset G_+(k,n)$ a positroid cell for $i = 1, \ldots, p$, is a collection of bounded affine permutations for which $\{\Gamma_{\pi_1}, \ldots, \Gamma_{\pi_p}\}$ is a positroid tiling of $\Delta_{k,n}$, then

$$\omega_{k,n} = \sum_{\pi \in \mathcal{T}} \tilde{\mu}_* \, \omega_\pi, \tag{2.6}$$

where ω_π is the canonical form of the positroid cell S_π, and $\tilde{\mu}_*$ indicates the push-forward through $\tilde{\mu}$ defined in appendix A.

As already mentioned, the hypersimplex $\Delta_{k,n}$ reduces to a simplex for $k = 1$ or $k = n-1$. In these cases no tiling is required since the algebraic moment map is already injective, and we can take the push-forward of the top form on the positive Grassmannian $G_+(1,n)$ or $G_+(n-1,n)$. A simple calculation leads to the following canonical differential forms

$$\omega_{1,n} = \mathrm{d}\,\log\left(\frac{x_2}{x_1}\right) \wedge \ldots \wedge \mathrm{d}\,\log\left(\frac{x_n}{x_1}\right), \tag{2.7}$$

$$\omega_{n-1,n} = \mathrm{d}\,\log\left(\frac{1-x_2}{1-x_1}\right) \wedge \ldots \wedge \mathrm{d}\,\log\left(\frac{1-x_n}{1-x_1}\right). \tag{2.8}$$

These are just canonical differential forms on the projective space $\mathbb{P}^{n-1}$, with homogeneous coordinates $(x_1, \ldots, x_n)$ in the first case and $(y_1, \ldots, y_n) = (1 - x_1, \ldots, 1 - x_n)$ in the second case.

For $1 < k < n - 1$, the algebraic moment map $\tilde{\mu}$ is not injective anymore, and the image of the positive Grassmannian through $\tilde{\mu}$ covers the hypersimplex $\Delta_{k,n}$ infinitely many times. To find the canonical form $\omega_{k,n}$ we need to divide the hypersimplex into smaller non-overlapping pieces for which the algebraic moment map is injective, namely positroid tiles, such that their union equals $\Delta_{k,n}$. Such subdivisions have been extensively studied in [11], where they were related to subdivisions of the amplituhedron [2], and to the positive tropical Grassmannian [22]. For our purposes, we need to find a single positroid tiling for a given hypersimplex $\Delta_{k,n}$. There are various ways to find such tilings: for example using height vectors from the tropical positive Grassmannian [11], using the amplituhedron and T-duality [11], or using blade arrangements [21]. In the simplest non-trivial example, $\Delta_{2,4}$, one finds two positroid tilings:

- Positroid polytope $\Gamma_{\{3,5,4,6\}}$ with vertices $\{e_{\{1,2\}}, e_{\{1,3\}}, e_{\{1,4\}}, e_{\{2,3\}}, e_{\{2,4\}}\}$ and positroid polytope $\Gamma_{\{2,4,5,7\}}$ with vertices $\{e_{\{1,3\}}, e_{\{1,4\}}, e_{\{2,3\}}, e_{\{2,4\}}, e_{\{3,4\}}\}$, or
- Positroid polytope $\Gamma_{\{4,3,5,6\}}$ with vertices $\{e_{\{1,2\}}, e_{\{1,3\}}, e_{\{1,4\}}, e_{\{2,4\}}, e_{\{3,4\}}\}$ and positroid polytope $\Gamma_{\{3,4,6,5\}}$ with vertices $\{e_{\{1,2\}}, e_{\{1,3\}}, e_{\{2,3\}}, e_{\{2,4\}}, e_{\{3,4\}}\}$

Where we explicitly specified the affine permutations labelling cells in $G_+(2,4)$. Each of these polytopes is the image of a positroid cell S_π in the positive Grassmannian $G_+(2,4)$, and the algebraic moment map $\tilde{\mu}$ is injective on all of them. This allows us to invert $\tilde{\mu}$ on these cells, and to find the push-forward of the canonical forms for them. For each cell we find that the resulting differential form has singularities corresponding to spurious boundaries between polytopes in a tiling. For example, in the first positroid tilings above, we find a singularity at $x_1 + x_2 = 1$. However, this singularity disappears in the sum of terms, and we get a differential form in the so-called *local form*, with all singularities corresponding to the boundaries of the hypersimplex $\Delta_{2,4}$. We find the following explicit expression for $\omega_{2,4}$:

$$
\begin{aligned}
\omega_{2,4} = &\, \mathrm{d}\log\left(\frac{x_2}{x_1}\right) \wedge \mathrm{d}\log\left(\frac{x_3}{x_1}\right) \wedge \mathrm{d}\log\left(\frac{x_4}{x_1}\right) \\
&- \mathrm{d}\log\left(\frac{1-x_2}{x_1}\right) \wedge \mathrm{d}\log\left(\frac{x_3}{x_1}\right) \wedge \mathrm{d}\log\left(\frac{x_4}{x_1}\right) \\
&- \mathrm{d}\log\left(\frac{x_2}{x_1}\right) \wedge \mathrm{d}\log\left(\frac{1-x_3}{x_1}\right) \wedge \mathrm{d}\log\left(\frac{x_4}{x_1}\right) \\
&- \mathrm{d}\log\left(\frac{x_2}{x_1}\right) \wedge \mathrm{d}\log\left(\frac{x_3}{x_1}\right) \wedge \mathrm{d}\log\left(\frac{1-x_4}{x_1}\right) \\
&- \mathrm{d}\log\left(\frac{x_2}{1-x_1}\right) \wedge \mathrm{d}\log\left(\frac{x_3}{1-x_1}\right) \wedge \mathrm{d}\log\left(\frac{x_4}{1-x_1}\right).
\end{aligned}
\tag{2.9}
$$

Interestingly, this expression can also be understood in a different way: each three-form in (2.9) is a differential form of a three-dimensional simplex, where the boundaries of each simplex can be read off from the singularities of the form. Then (2.9) suggests that the hypersimplex $\Delta_{2,4}$ can be obtained from the simplex with boundaries at $x_i = 0$, $i = 1, \ldots, 4$, after removing from it four simplices $\mathcal{T}_l$ with boundaries $x_l = 1$, $x_j = 0$ for $j \neq l$, for $l = 1, \ldots, 4$. This is indeed a correct statement, as is illustrated in figure 1. Notice that (2.9) is not manifestly invariant under the parity symmetry $x_i \leftrightarrow (1 - x_i)$, which we would expect to be true for $\Delta_{2,4}$. In particular, a

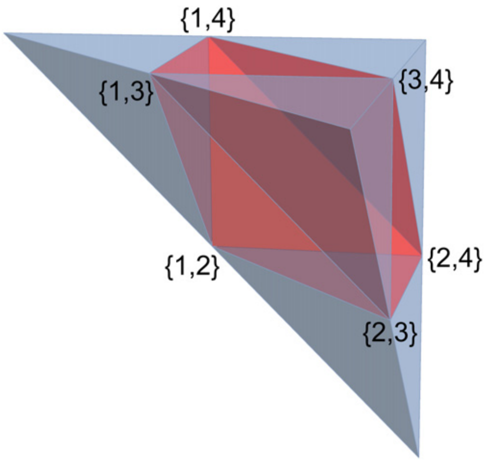

Figure 1. An interpretation of $\Delta_{2,4}$ using three-dimensional simplices. The big simplex has all facets on the hyperplanes $x_i = 0$, the four blue simplices have three facets on the hyperplanes $x_i = 0$, and one on a hyperplane $x_i = 1$. The projection from the 3D hypersurface in $\mathbb{R}^4$ to $\mathbb{R}^3$ is given by $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3)$.

parity conjugate version of (2.9) is

$$\begin{aligned}\omega_{2,4} &= \mathrm{d}\,\log\left(\frac{1-x_2}{1-x_1}\right)\wedge \mathrm{d}\,\log\left(\frac{1-x_3}{1-x_1}\right)\wedge \mathrm{d}\,\log\left(\frac{1-x_4}{1-x_1}\right)\\ &\quad - \mathrm{d}\,\log\left(\frac{1-x_2}{1-x_1}\right)\wedge \mathrm{d}\,\log\left(\frac{1-x_3}{1-x_1}\right)\wedge \mathrm{d}\,\log\left(\frac{x_4}{1-x_1}\right)\\ &\quad - \mathrm{d}\,\log\left(\frac{1-x_2}{1-x_1}\right)\wedge \mathrm{d}\,\log\left(\frac{x_3}{1-x_1}\right)\wedge \mathrm{d}\,\log\left(\frac{1-x_4}{1-x_1}\right)\\ &\quad - \mathrm{d}\,\log\left(\frac{x_2}{1-x_1}\right)\wedge \mathrm{d}\,\log\left(\frac{1-x_3}{1-x_1}\right)\wedge \mathrm{d}\,\log\left(\frac{1-x_4}{x_1}\right)\\ &\quad - \mathrm{d}\,\log\left(\frac{1-x_2}{x_1}\right)\wedge \mathrm{d}\,\log\left(\frac{1-x_3}{x_1}\right)\wedge \mathrm{d}\,\log\left(\frac{1-x_4}{x_1}\right).\end{aligned} \tag{2.10}$$

However, using the constraint $x_1 + x_2 + x_3 + x_4 = 2$, one can easily verify that (2.9) and (2.10) are the same. The formula (2.10) provides an alternative interpretation for $\Delta_{2,4}$ as a simplex with all boundaries at $x_i = 1$ with four smaller simplices removed.

Our study of the hypersimplex $\Delta_{2,4}$ can be easily generalized to $\Delta_{k,n}$ for any n and k. In all these cases we need to find a single positroid tiling of the hypersimplex $\Delta_{k,n}$, and to use the algebraic moment map $\tilde{\mu}$ to calculate the push-forward of differential forms on Grassmannian positroid cells, summing over the tiling. This allows us to find a general formula for the hypersimplex canonical form $\omega_{k,n}$. Our result has logarithmic singularities on all boundaries of the hypersimplex $\Delta_{k,n}$, which are of the form $\Delta_{k-1,n-1}$ or $\Delta_{k,n-1}$, and the residue when evaluated at these boundaries is $\omega_{k-1,n-1}$ and $\omega_{k,n-1}$, respectively.

Before writing down an explicit form for $\omega_{k,n}$, we need to introduce some notation which will allow us to write it in a concise way. Let us consider a $(d-1)$-dimensional bounded region with exactly d boundaries, where each boundary is of one of two types: boundaries at hyperplanes $a_i = 0, i = 1, \ldots, d$, and boundaries at hyperplanes $b_i = 0, i = 1, \ldots, d$. We know that a generic set of d hyperplanes in a $(d-1)$-dimensional space defines a simplex. Let us take $J \in \binom{[d]}{l}$ and denote by Σ_J the simplex bounded by hyperplanes $a_j = 0$ for $j \in [d]\backslash J$ and $b_{j'} = 0$ for $j' \in J$. The canonical differential form σ_J for the simplex Σ_J is then

$$\sigma_J = \bigwedge_{j=2}^{d} \mathrm{d}\,\log\left(\frac{\alpha_j}{\alpha_1}\right), \quad \text{where } \alpha_j = \begin{cases} a_j, & \text{for } j \notin J, \\ b_j, & \text{for } j \in J. \end{cases} \tag{2.11}$$

The choice of α_1 in the denominator is arbitrary, and any other α_j can be chosen at the cost of an overall factor $(-1)^{j+1}$. The simplex described above has $d-l$ facets of the form $a_i = 0$, and l facets of the form $b_i = 0$. It will prove useful to define a sum of the forms σ_J over all simplices with this distribution of facets:

$$\sigma_{l,d} := \sum_{J \in \binom{[d]}{l}} \sigma_J. \tag{2.12}$$

This sum over all simplices with a specific facet distribution enjoys useful properties. First of all, there is an inductive way to find $\sigma_{l,d}$ from $\sigma_{l,d-1}$ and $\sigma_{l-1,d-1}$:

$$\sigma_{l,d} = \sigma_{l,d-1} \wedge \mathrm{d}\,\log\left(\frac{a_d}{a_j}\right) + \sigma_{l-1,d-1} \wedge \mathrm{d}\,\log\left(\frac{b_d}{b_j}\right), \tag{2.13}$$

for any $j = 1, \ldots, d-1$. From this it immediately follows that:

$$\operatorname*{Res}_{a_d=0} \sigma_{l,d} = \sigma_{l,d-1}, \tag{2.14}$$

$$\operatorname*{Res}_{b_d=0} \sigma_{l,d} = \sigma_{l-1,d-1}, \tag{2.15}$$

where, by using (2.12), the right-hand side does no longer depend on a_d, b_d. More generally, we can take a residue for $a_j = 0$ or $b_j = 0$ for any $j = 1, \ldots, d$, and obtain similar formulae with the right-hand side relabelled. Also, let us notice that the parity symmetry that exchanges a_j with b_j leads to

$$\sigma_{l,d} \xleftrightarrow{a_j \leftrightarrow b_j} \sigma_{d-l,d}. \tag{2.16}$$

Finally, by expanding $\mathrm{d}\,\log\left(\alpha_j/\alpha_1\right) = \mathrm{d}\,\log\,\alpha_j - \mathrm{d}\,\log\,\alpha_1$ one can alternatively write (2.11) as:

$$\sigma_J = \sum_{j=1}^{d} (-1)^{j+1} \bigwedge_{i \in [d]\backslash\{j\}} \mathrm{d}\,\log(\alpha_i), \quad \alpha_i = \begin{cases} a_i, & \text{if } i \notin J, \\ b_i, & \text{if } i \in J. \end{cases} \tag{2.17}$$

Note that the d terms in the sum of (2.17) can be divided into two categories: there are l terms with l one-forms $\mathrm{d}\log b_i$'s and $d-l-1$ one-forms $\mathrm{d}\log a_i$'s, and there are $d-l$ terms with $l-1$ one-forms $\mathrm{d}\log b_i$'s and $d-l$ one-forms $\mathrm{d}\log a_i$'s. We introduce the notation

$$\tau_{l,d} := \sum_{j=1}^{d} (-1)^{j+1} \sum_{I \in \binom{[d]\backslash\{j\}}{l}} \bigwedge_{i \in [d]\backslash\{j\}} \mathrm{d}\,\log\,\alpha_i, \quad \alpha_i = \begin{cases} a_i, & \text{if } i \notin I, \\ b_i, & \text{if } i \in I, \end{cases} \tag{2.18}$$

which is the sum over all terms with exactly l $\mathrm{d}\log b_i$'s and $d-l-1$ $\mathrm{d}\log a_i$'s with minus signs consistent with (2.17). It then follows that:

$$\sigma_{l,d} = \tau_{l,d} + \tau_{l-1,d}. \tag{2.19}$$

From this, it is easy to arrive at the following identity:

$$\sum_{l=0}^{d} (-1)^l \sigma_{l,d} = 0, \tag{2.20}$$

since the alternating sum makes the terms in (2.19) telescope, and we use the fact that $\tau_{-1,d} = \tau_{d,d} = 0$.

Armed with this formalism we can now set $a_i = x_i$ and $b_i = 1 - x_i$, and write the canonical differential form $\omega_{k,n}$ for the hypersimplex $\Delta_{k,n}$ for general n and k as:

$$\omega_{k,n} = \sum_{l=0}^{k-1} (-1)^l \sigma_{l,n} = \sum_{l=0}^{n-k+1} (-1)^{n-l} \sigma_{n-l,n}. \tag{2.21}$$

The equality between these two expressions comes from (2.20) and the fact that on the support of the hypersimplex constraint $x_1 + \cdots + x_n = k$ we have:

$$\sigma_{k,n} = 0. \tag{2.22}$$

As mentioned before, the alternating minus signs have the effect that terms telescope when expanded using (2.19). This allows us to write the hypersimplex form as a single term:

$$\omega_{k,n} = \tau_{k-1,n}. \tag{2.23}$$

We will sketch a proof of formulae (2.21) and (2.23) in appendix B.1.

Using the properties of the forms σ and τ, we can immediately read off the following properties for the hypersimplex canonical forms:

$$\omega_{k,n} \xleftrightarrow{x_i \leftrightarrow 1-x_i} \omega_{n-k,n}, \tag{2.24}$$

$$\operatorname*{Res}_{x_n=0} \omega_{k,n} = \omega_{k,n-1}, \tag{2.25}$$

$$\operatorname*{Res}_{x_n=1} \omega_{k,n} = \omega_{k-1,n-1}. \tag{2.26}$$

This reflects the proper structure of hypersimplex boundaries, and the fact that $\Delta_{k,n}$ is parity dual to $\Delta_{n-k,n}$.

We summarize this section by rewriting the results we obtained above for $k = 1, k = n-1$, and $n = 4, k = 2$ using this generalized notation. For the cases when the hypersimplex is a simplex, namely $k = 1$ and $k = n-1$, we can write

$$\omega_{1,n} = \sigma_{0,n} = \sigma_{\emptyset}, \tag{2.27}$$

$$\omega_{n-1,n} = \sigma_{n,n} = \sigma_{[n]}. \tag{2.28}$$

For $n = 4$, $k = 2$ we simply find

$$\omega_{2,4} = \sigma_{0,4} - \sigma_{1,4} = \sigma_{\emptyset} - \sigma_{\{1\}} - \sigma_{\{2\}} - \sigma_{\{3\}} - \sigma_{\{4\}}, \tag{2.29}$$

where the second expression supports the discussion after formula (2.9). We can also see a similar behaviour for higher n, for example for $\Delta_{2,n}$ we find

$$\omega_{2,n} = \sigma_{0,n} - \sigma_{1,n} = \sigma_{\emptyset} - \sum_{i=1}^{n} \sigma_{\{i\}}, \tag{2.30}$$

where each $\sigma_{\{i\}}$ corresponds to an $(n-1)$-dimensional simplex with one facet at $x_i = 1$ and all other facets at $x_j = 0$ for $j \neq i$. This discussion motivates us to rewrite the general formula (2.21) in a slightly different way:

$$\omega_{k,n} = \sigma_{\emptyset} - \left(\sum_{I_1 \in \binom{[n]}{1}} \sigma_{I_1} - \left(\sum_{I_2 \in \binom{[n]}{2}} \sigma_{I_2} - \left(\ldots - \sum_{I_{k-1} \in \binom{[n]}{k-1}} \sigma_{I_{k-1}} \right) \right) \right), \tag{2.31}$$

where each term is the canonical form of a simplex. This is equivalent to the following set-theoretic statement for $\Delta_{k,n}$:

$$\Delta_{k,n} = \overline{\Sigma_{\emptyset} \setminus \left(\bigcup_{I_1 \in \binom{[n]}{1}} \Sigma_{I_1} \setminus \left(\bigcup_{I_2 \in \binom{[n]}{2}} \Sigma_{I_2} \setminus \left(\ldots \setminus \bigcup_{I_{k-1} \in \binom{[n]}{k-1}} \Sigma_{I_{k-1}} \right) \right) \right)}, \tag{2.32}$$

which to our knowledge has not been previously known. A sketch of a proof that this indeed holds for all hypersimplices is provided in appendix B.2.

3. Momentum amplituhedron

The momentum amplituhedron $\mathcal{M}_{n,k}$ is a positive geometry introduced in [4] to describe tree-level scattering amplitudes in $\mathcal{N} = 4$ sYM in spinor helicity space. Its counterpart in momentum twistor space is the amplituhedron $\mathcal{A}_{n,k}$ [2], which has a natural generalization $\mathcal{A}_{n,k}^{(m)}$ beyond the case relevant to physics, labelled by an integer m, with $m = 4$ corresponding to the physical case. It was observed in [11] that a natural generalization also exists for the momentum amplituhedron for even m, and the authors of [11], including one of the authors of this paper, suggested a possible definition for $\mathcal{M}_{n,k}^{(m)}$ for even m. In particular, they conjectured in section 11 of their paper that, for $m = 2$, the momentum amplituhedron $\mathcal{M}_{n,k}^{(2)}$ shares many properties with the hypersimplex $\Delta_{k,n}$. Their main conjecture stated that the positroid tilings of the hypersimplex $\Delta_{k,n}$ are in one-to-one correspondence with positroid tilings of the momentum amplituhedron $\mathcal{M}_{n,k}^{(2)}$. Based on this, it was found in [26] that the boundary stratification of the momentum amplituhedron $\mathcal{M}_{n,k}^{(2)}$ is analogous to the boundary stratification of the hypersimplex $\Delta_{k,n}$. In this section we show that both statements are not correct and find their counterexamples.

Despite the fact that the definition of $\mathcal{M}_{n,k}^{(2)}$ in [11] does not provide an object closely related to the hypersimplex $\Delta_{k,n}$, we find interesting differential forms that can be naturally defined in the space introduced there. These differential forms have properties analogous to the hypersimplex canonical forms $\omega_{k,n}$ we studied in section 2. They are not, however, canonical forms of the momentum amplituhedron $\mathcal{M}_{n,k}^{(2)}$ defined in [11].

3.1. Definition of $m = 2$ momentum amplituhedron

We follow the notation in [11] and provide the definition of the momentum amplituhedron $\mathcal{M}^{(2)}_{n,k}$. It relies on two matrices Λ and $\widetilde{\Lambda}$, encoding the 'external data':

$$\Lambda = (\Lambda_1\Lambda_2\ldots\Lambda_n) \in M(n-k+1,n), \qquad \widetilde{\Lambda} = (\widetilde{\Lambda}_1\widetilde{\Lambda}_2\ldots\widetilde{\Lambda}_n) \in M(k+1,n). \tag{3.1}$$

One assumes that Λ is a positive matrix, i.e. all its maximal minors are positive, and $\widetilde{\Lambda}$ is a twisted positive matrix, i.e. the matrix describing its orthogonal complement is a positive matrix. Then, the $m = 2$ momentum amplituhedron $\mathcal{M}^{(2)}_{n,k}$ is defined as the image of the positive Grassmannian $G_+(k,n)$ through the map specified by these matrices:

$$\Phi_{(\Lambda,\widetilde{\Lambda})} : G_+(k,n) \to G\,(n-k, n-k+1) \times G\,(k,k+1)\,, \qquad C \mapsto (Y,\widetilde{Y}), \tag{3.2}$$

where

$$Y^A_\alpha = c^\perp_{\alpha i}\Lambda^A_i\,, \qquad \widetilde{Y}^{\dot{A}}_{\dot{\alpha}} = c_{\dot{\alpha}i}\widetilde{\Lambda}^{\dot{A}}_i. \tag{3.3}$$

We use $C = \{c_{\dot{\alpha}i}\} \in G_+(k,n)$, and $C^\perp = \{c^\perp_{\alpha i}\}$ is the orthogonal complement of C. The image of the positive Grassmannian naturally lives in an $(n-1)$-dimensional subspace of the $(n-k+k=n)$-dimensional space $G\,(n-k,n-k+1) \times G\,(k,k+1)$ specified by the 'momentum conservation'-like identity:

$$\sum_{i=1}^{n} \left(Y^\perp \cdot \Lambda\right)_i \left(\widetilde{Y}^\perp \cdot \widetilde{\Lambda}\right)_i = 0, \tag{3.4}$$

where $Y^\perp \in G(1,n-k-1)$ and $\widetilde{Y}^\perp \in G(1,k+1)$ are the orthogonal complements of Y and $\widetilde{Y}$, respectively. Similar to the $m = 4$ momentum amplituhedron $\mathcal{M}^{(4)}_{n,k}$ in [4], we define the 'spinor helicity' variables $\lambda, \tilde{\lambda}$ as:

$$\lambda_i := \langle Yi\rangle = \epsilon_{A_1A_2\ldots A_{n-k}A_{n-k+1}} Y^{A_1}_1 Y^{A_2}_2 \ldots Y^{A_{n-k}}_{n-k} \Lambda^{A_{n-k+1}}_i, \tag{3.5}$$

$$\tilde{\lambda}_i := [\widetilde{Y}i] = \epsilon_{\dot{A}_1\dot{A}_2\ldots\dot{A}_k\dot{A}_{k+1}} \widetilde{Y}^{\dot{A}_1}_1 \widetilde{Y}^{\dot{A}_2}_2 \ldots \widetilde{Y}^{\dot{A}_k}_k \widetilde{\Lambda}^{\dot{A}_{k+1}}_i. \tag{3.6}$$

These λ and $\tilde{\lambda}$ variables satisfy a similar 'momentum conservation' identity:

$$\sum_{i=1}^{n} \lambda_i\tilde{\lambda}_i = 0. \tag{3.7}$$

3.2. Momentum amplituhedron-like logarithmic forms

Before discussing the geometry of $\mathcal{M}^{(2)}_{n,k}$, let us focus on differential forms that can be defined in the $(\lambda,\tilde{\lambda})$ space. Since the domains of the maps $\tilde{\mu}$ and $\Phi_{(\Lambda,\widetilde{\Lambda})}$ are the same, a natural question is what happens when we take a collection of positroid cells in the positive Grassmannian $G_+(k,n)$ that provides a positroid tiling of the hypersimplex $\Delta_{k,n}$, and evaluate their push-forwards using the momentum amplituhedron map $\Phi_{(\Lambda,\widetilde{\Lambda})}$ (3.2). An important observation is that this push-forward does not depend on the positivity conditions for the Λ and $\widetilde{\Lambda}$ matrices.

Taking any collection of $G_+(k,n)$ positroid cells labels $\mathcal{T}=\{\pi_1,\ldots,\pi_p\}$ that gives a positroid tiling of $\Delta_{k,n}$, we can define

$$\overline{\omega}_{n,k}=\sum_{\pi\in\mathcal{T}}(\Phi_{(\Lambda,\widetilde{\Lambda})})_*\,\omega_\pi, \tag{3.8}$$

where ω_π is the canonical form of the positroid cell S_π, and $(\Phi_{(\Lambda,\widetilde{\Lambda})})_*$ indicates the push-forward[3]. We have calculated $\overline{\omega}_{n,k}$ using positroid tilings of hypersimplices up to $n=7$, all k, and found that the answer is independent from the tiling. Moreover, it can be expressed using the notation we introduced in section 2.2. By taking $\sigma_{l,n}$ defined in (2.17), and substituting $a\to\lambda$, $b\to\tilde{\lambda}$, we can write the differential form $\overline{\omega}_{n,k}$ in (3.8) as:

$$\overline{\omega}_{n,k}=\sum_{l=0,2,4,\ldots}^{k-1}\sigma_{l,n},\quad \text{for } k \text{ odd}, \tag{3.9}$$

$$\overline{\omega}_{n,k}=\sum_{l=1,3,5,\ldots}^{k-1}\sigma_{l,n},\quad \text{for } k \text{ even}. \tag{3.10}$$

We believe that these formulae are true for any n and k. These can also be written in a more uniform way using the differential forms τ from (2.17) as

$$\overline{\omega}_{n,k}=\sum_{l=0}^{k-1}\tau_{l,n}. \tag{3.11}$$

Interestingly, these differential forms have properties similar to those we have found for the hypersimplex canonical forms $\omega_{k,n}$. In particular, they are parity symmetric when λ is exchanged with $\tilde{\lambda}$:

$$\overline{\omega}_{n,k}\xleftrightarrow{\lambda_i\leftrightarrow\tilde{\lambda}_i}\overline{\omega}_{n-k,k}. \tag{3.12}$$

This can be shown using a version of equation (2.20):

$$\sum_{l=0,2,4,\ldots}^{\leqslant n}\sigma_{l,n}=\sum_{l=1,3,5,\ldots}^{\leqslant n}\sigma_{l,n}, \tag{3.13}$$

and the fact that on the support of momentum conservation we have:

$$\sum_{l=0,2,4,\ldots}^{\leqslant n}\sigma_{l,n}=\sum_{l=1,3,5,\ldots}^{\leqslant n}\sigma_{l,n}=0,\quad \text{for } \sum_{i=1}^{n}\lambda_i\tilde{\lambda}_i=0. \tag{3.14}$$

Additionally, the differential form $\overline{\omega}_{n,k}$ has an identical singularity structure with the hypersimplex canonical forms $\omega_{k,n}$, namely:

$$\operatorname*{Res}_{\lambda_n=0}\overline{\omega}_{n,k}=\overline{\omega}_{n-1,k}, \tag{3.15}$$

$$\operatorname*{Res}_{\tilde{\lambda}_n=0}\overline{\omega}_{n,k}=\overline{\omega}_{n-1,k-1}. \tag{3.16}$$

[3] The signs of push-forwards are fixed such that the common singularities appearing in different terms, i.e. the spurious singularities, have a vanishing residue. We found that it is always possible to find such combinations of signs.

Analogous formulae are also true if we replace $\lambda_n, \tilde{\lambda}_n$ with any other $\lambda_i, \tilde{\lambda}_i$ for $i = 1, 2, \ldots, n$. These formulae indicate that the structure of singularities of the differential form $\overline{\omega}_{n,k}$ is exactly the same as the structure of singularities of $\omega_{k,n}$ in section 2.2, after we identify λ_i with x_i, and $\tilde{\lambda}_i$ with $1 - x_i$. In particular, there are exactly $2n$ singularities, n of which are of the form $\lambda_i = 0$, and n of which are of the form $\tilde{\lambda}_i = 0$. The residues at these singularities are given by differential forms $\overline{\omega}$ with lower labels as in (3.15) and (3.16), providing us with a recursive description akin to the one for the hypersimplex canonical forms $\omega_{k,n}$.

3.3. Geometry

Our calculations in the previous section pose a natural question whether there exists a geometric object for which $\overline{\omega}_{n,k}$ provides the canonical differential form. The first guess would be that this object must be the momentum amplituhedron defined in section 3.1. We have however checked that even in the first non-trivial example, for $n = 4, k = 2$, the momentum amplituhedron $\mathcal{M}^{(2)}_{4,2}$ defined above is not the correct geometry. Instead, one needs to modify the positivity conditions in the definition of $\mathcal{M}^{(2)}_{4,2}$ to get a geometry with $\overline{\omega}_{4,2}$ as the canonical form. Even after this modification, the final conjecture of section 11 in [11] is still not correct since, depending on the choice of external data, only one out of two positroid tilings of the hypersimplex $\Delta_{2,4}$ provides a tiling of such modified momentum amplituhedron. This can be attributed to the fact that, even with the modified positivity conditions, the region we define looks concave. We have found that for $k = 2$ and any n we can always find conditions for external data Λ and $\widetilde{\Lambda}$ such that the resulting geometry can be tiled using some, but not all, of the positroid tilings of the hypersimplex $\Delta_{2,n}$. Similar statement holds true for $k = n - 2$, as well as for $n = 6$ and $k = 3$. It is, however, not possible beyond these cases and therefore we conclude that $\Phi_{(\Lambda,\widetilde{\Lambda})}$ cannot be used to define a geometry for which $\overline{\omega}_{n,k}$ is the canonical differential.

Let us start by stating that for $k = 1$ and for $k = n - 1$ the momentum amplituhedron $\mathcal{M}^{(2)}_{n,k}$ is just a simplex. In these cases, the map $\Phi_{(\Lambda,\widetilde{\Lambda})}$ is injective and there is no need for any tiling. Then the canonical differential form for $\mathcal{M}^{(2)}_{n,1}$ is

$$\overline{\omega}_{n,1} = \mathrm{d}\,\log\left(\frac{\lambda_2}{\lambda_1}\right) \wedge \ldots \wedge \mathrm{d}\,\log\left(\frac{\lambda_n}{\lambda_1}\right), \tag{3.17}$$

and for $\mathcal{M}^{(2)}_{n,n-1}$ is

$$\overline{\omega}_{n,n-1} = \mathrm{d}\,\log\left(\frac{\tilde{\lambda}_2}{\tilde{\lambda}_1}\right) \wedge \ldots \wedge \mathrm{d}\,\log\left(\frac{\tilde{\lambda}_n}{\tilde{\lambda}_1}\right). \tag{3.18}$$

Trivially, the boundary stratifications of $\mathcal{M}^{(2)}_{n,1}$ and $\mathcal{M}^{(2)}_{n,n-1}$ are equivalent to the boundary stratifications of the hypersimplices $\Delta_{1,n}$ and $\Delta_{n-1,n}$, respectively.

Beyond $k = 1$ and $k = n - 1$, the map $\Phi_{(\Lambda,\widetilde{\Lambda})}$ is not injective anymore, as was the case for the algebraic moment map $\tilde{\mu}$. There are, however, significant differences between the two geometries that we illustrate in detail in the simplest non-trivial case: $n = 4, k = 2$. Recall that the hypersimplex $\Delta_{2,4}$ is an octahedron depicted in figure 1, and it can be subdivided using pairs of positroid polytopes in two different ways. These positroid polytopes are images of some three-dimensional cells, labelled by π and π', in the positive Grassmannian $G_+(2, 4)$ through the algebraic moment map $\tilde{\mu}$. In particular, they have spurious boundaries along the hyperplanes $x_1 + x_2 = 1$ or $x_2 + x_3 = 1$. This can be easily seen by considering the shared co-dimension-one boundaries of cells S_π and $S_{\pi'}$ in the positive Grassmannian $G_+(2, 4)$, and studying images of such boundaries through the moment map. A similar analysis can be done using the $\Phi_{(\Lambda,\widetilde{\Lambda})}$ map and the images of the three-dimensional cells have spurious boundaries along $\lambda_1\tilde{\lambda}_1 + \lambda_2\tilde{\lambda}_2 = 0$ (two cells) or $\lambda_2\tilde{\lambda}_2 + \lambda_3\tilde{\lambda}_3 = 0$ (two cells). For a pair of cells to be a tiling of $\mathcal{M}^{(2)}_{4,2}$, their images must be disjoint. This means that close to their shared boundary, the images need to sit on the opposite sides of the surface $\lambda_1\tilde{\lambda}_1 + \lambda_2\tilde{\lambda}_2 = 0$ (resp. $\lambda_2\tilde{\lambda}_2 + \lambda_3\tilde{\lambda}_3 = 0$). For example, let us consider positroid cells in $G_+(2,4)$ labelled by the permutations $\pi_1 = \{3,4,6,5\}$ and $\pi_2 = \{4,3,5,6\}$ which are parametrised by matrices:

$$C_{\pi_1} = \begin{pmatrix} 1 & \alpha_3 & 0 & -\alpha_1 \\ 0 & 1 & \alpha_2 & 0 \end{pmatrix}, \qquad C_{\pi_2} = \begin{pmatrix} 1 & 0 & 0 & -\beta_1 \\ 0 & 1 & \beta_2 & \beta_3 \end{pmatrix} \tag{3.19}$$

with $\alpha_i > 0$ and $\beta_i > 0$. In the positive Grassmannian $G_+(2,4)$ these cells share a boundary $\overline{S_{\pi_1}} \cap \overline{S_{\pi_2}} = S_{4,3,6,5}$ that is the cell labelled by $\pi_{1\cap 2} = \{4,3,6,5\}$, parametrised by the matrix

$$C_{\pi_{1\cap 2}} = \begin{pmatrix} 1 & 0 & 0 & -\gamma_1 \\ 0 & 1 & \gamma_2 & 0 \end{pmatrix} \tag{3.20}$$

with $\gamma_i > 0$, whose image is the spurious boundary inside the surface $\lambda_2\tilde{\lambda}_2 + \lambda_3\tilde{\lambda}_3 = 0$. For the cell parametrized by the permutation π_1 we find

$$\lambda_2\tilde{\lambda}_2 + \lambda_3\tilde{\lambda}_3 = ([123] - \alpha_1[234])(-\langle 134\rangle + \alpha_2\langle 124\rangle)\alpha_2\alpha_3, \tag{3.21}$$

with $\alpha_i > 0$, where the points on the spurious boundary correspond to setting $\alpha_3 = 0$. For the cell parametrized by the permutation π_2 we find

$$\lambda_2\tilde{\lambda}_2 + \lambda_3\tilde{\lambda}_3 = -(\langle 234\rangle + \beta_1\langle 123\rangle)([124] + \beta_2[134])\beta_3, \tag{3.22}$$

with $\beta_i > 0$ and the spurious boundary corresponds to setting $\beta_3 = 0$. Then, assuming the positivity conditions for Λ and $\widetilde{\Lambda}$ from section 3.1, there exists an open set U in $S_{\pi_{1\cap 2}}$ such that $([123] - \gamma_1[234])(-\langle 134\rangle + \gamma_2\langle 124\rangle)\gamma_2 < 0$ and for any sufficiently small neighbourhood of U in S_{π_1}, the right-hand side of (3.21) is negative. This means that in the neighbourhood of the set U, the images of S_{π_1} and S_{π_2} are on the same side of the surface $\lambda_2\tilde{\lambda}_2 + \lambda_3\tilde{\lambda}_3 = 0$, and therefore they do intersect. Therefore they do not provide a positroid tiling of $\mathcal{M}^{(2)}_{4,2}$. This provides a counter-example to the statements in section 11 of [11].

By changing positivity conditions, it is possible to slightly modify the definition of the momentum amplituhedron to get a geometry for which $\overline{\omega}_{2,4}$ provides the correct canonical form. For example, if we assume

$$\begin{aligned} &[123] > 0, \qquad [124] > 0, \qquad [134] > 0, \qquad [234] < 0, \\ &\langle 123\rangle > 0, \qquad \langle 124\rangle > 0, \qquad \langle 134\rangle < 0, \qquad \langle 234\rangle > 0 \end{aligned} \tag{3.23}$$

then the $\Phi_{(\Lambda,\widetilde{\Lambda})}$ images of cells labelled by permutations π_1 and π_2 do not overlap, since $\lambda_2\tilde{\lambda}_2 + \lambda_3\tilde{\lambda}_3$ has opposite sign for all elements in these cells, and they subdivide the image of the positive Grassmannian $G_+(2,4)$. Therefore the logarithmic form $\overline{\omega}_{2,4}$ is the canonical form of this geometry. However, in this case, the images of the remaining two cells, $\pi_3 = \{2,4,5,7\}$ and $\pi_4 = \{3,5,4,6\}$, do overlap and they do not provide a subdivision of the image. This comes from the fact that the image of the positive Grassmannian $G_+(2,4)$ through $\Phi_{(\Lambda,\widetilde{\Lambda})}$ with the positivity conditions (3.23) looks concave and has the shape depicted in figure 2.

Using similar analysis, we have found that a similar behaviour is true for higher n. In particular, we found that the image of $G_+(k,n)$ through $\Phi_{(\Lambda,\widetilde{\Lambda})}$ with the positivity conditions described in section 3.1, cannot be tiled by the images of the same cells as for the hypersimplex. As in the

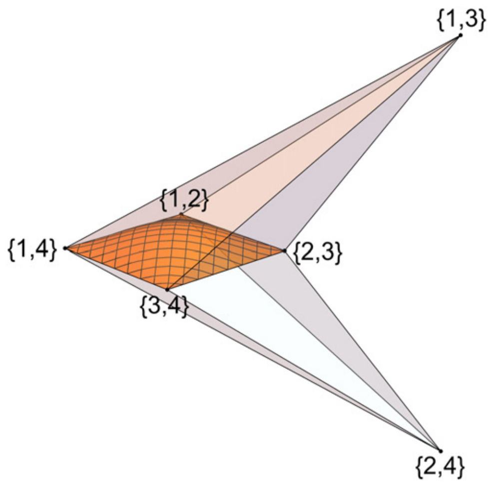

Figure 2. A schematic image of the positive Grassmannian $G_+(2,4)$ through the map $\Phi_{(\Lambda,\tilde{\Lambda})}$ with positivity conditions (3.23). The spurious boundary $\lambda_2\tilde{\lambda}_2 + \lambda_3\tilde{\lambda}_3 = 0$ is shown in orange. The labels at the vertices indicate the nonvanishing Plückers of the 0-dimensional positroid cells that map to them.

case described above, we checked that for $k = 2$ and the first few values of n, it is still possible to modify positivity conditions such that there exists a collection of cells in $G_+(2,n)$ that form a positroid tiling of $\Delta_{2,n}$ and their images through $\Phi_{(\Lambda,\tilde{\Lambda})}$ are disjoint. In all these cases, the differential form $\overline{\omega}_{n,2}$ from (3.9) is the canonical differential form of the corresponding image of $G_+(2,n)$ through $\Phi_{(\Lambda,\tilde{\Lambda})}$. Even this becomes impossible for higher k: we have computational evidence that for $n > 6$ and $2 < k < n-2$ there are no tilings of $\Delta_{k,n}$ for which the images through the map $\Phi_{(\Lambda,\tilde{\Lambda})}$ are disjoint. This shows that one cannot use the map $\Phi_{(\Lambda,\tilde{\Lambda})}$ to generate a region in the $(\lambda,\tilde{\lambda})$-space for which $\overline{\omega}_{n,k}$ is the canonical form.

4. Summary and outlook

In this paper, we have studied two geometries, the hypersimplex $\Delta_{k,n}$ and the generalization of the momentum amplituhedron $\mathcal{M}^{(2)}_{n,k}$ proposed in [11], from the point of view of positive geometries. We have provided two main results. One is the previously unknown formula (2.21) for the hypersimplex canonical form $\omega_{k,n}$. The formula has a natural interpretation as a set-theoretical decomposition of hypersimplex into simplices given in (2.32). Moreover, we provide a negative but important result stating that the generalization of the momentum amplituhedron suggested in [11] does not possess the desired properties. In particular, we have found counter-examples showing that the conjectures in section 11 of [11] regarding positroid tilings of $\mathcal{M}^{(2)}_{n,k}$ are not valid. It can be attributed to the fact that the momentum amplituhedron for $m = 2$ is 'concave'. This, in turn, is related to the fact that the momentum amplituhedron for $m = 2$ shares properties with the ordinary amplituhedron for $m = 1$. The latter is known to be concave and, in general, amplituhedra for odd m are less well-behaving than the ones for even m, see for example [27]. We predict that the momentum amplituhedron for $m = 2, 6, 10, \ldots$ will have similar behaviour, and the conjectures from section 11 of [11] will not hold in these cases. The question remains open on whether the conjectures are correct for m divisible by four, beyond $m = 4$.

In this paper we have also provided interesting differential forms written directly in the $(\lambda, \tilde{\lambda})$ spinor helicity space, which have properties analogous to those of the hypersimplex canonical forms. This leads to the question of whether one can find a shape inside the $(\lambda, \tilde{\lambda})$ space with the canonical differential form given by $\overline{\omega}_{n,k}$. It is unclear from our explorations whether it will be possible, and it remains an interesting open problem.

Acknowledgments

We would like to thank Livia Ferro and Lauren Williams for useful discussions.

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

Appendix A. Definition of positive geometry and push-forward

Positive geometries [1] naturally live in complex projective spaces $\mathbb{P}^N$, and their real parts $\mathbb{P}^N(\mathbb{R})$. One defines X to be a complex projective algebraic variety of complex dimension D and $X(\mathbb{R})$ to be its real part, and one denotes by $X_{\geqslant 0} \subset X(\mathbb{R})$ an oriented set of real dimension D. A D-dimensional *positive geometry* is a pair $(X, X_{\geqslant 0})$ equipped with a unique non-zero differential D-form $\Omega(X, X_{\geqslant 0})$, called the *canonical form*, satisfying the following recursive axioms:

- For $D = 0$ we have that $X = X_{\geqslant 0}$ is a single real point and $\Omega(X, X_{\geqslant 0}) = \pm 1$ depending on the orientation of $X_{\geqslant 0}$.
- For $D > 0$ we have that every boundary component $(C, C_{\geqslant 0})$ of $(X, X_{\geqslant 0})$ is a positive geometry of dimension $D - 1$. Moreover, the form $\Omega(X, X_{\geqslant 0})$ is constrained by the residue relation

$$\mathrm{Res}_C\, \Omega(X, X_{\geqslant 0}) = \Omega(C, C_{\geqslant 0}), \tag{A.1}$$

along every boundary component C, and has no singularities elsewhere.

The *residue* operation Res_C for a meromorphic form ω on X is defined in the following way: suppose C is a subvariety of X and z is a holomorphic coordinate whose zero set $z = 0$ parametrizes C. Denote as u the remaining holomorphic coordinates. Then a simple pole of ω at C is a singularity of the form

$$\omega(u, z) = \omega'(u) \wedge \frac{\mathrm{d}z}{z} + \cdots, \tag{A.2}$$

where the ellipsis denotes terms smooth in the small z limit, and $\omega'(u)$ is a non-zero meromorphic form on the boundary component. One defines

$$\mathrm{Res}_C\, \omega := \omega'. \tag{A.3}$$

If there is no such simple pole then one defines the residue to be zero.

We also define what we mean by the push-forward of a differential form. We consider a surjective meromorphic map $\phi : A \to B$ of finite degree p, where A and B are complex manifolds of the same dimension. For a given point $b \in B$ we can find its pre-image, namely a collection of points a_i in A, $i = 1, \ldots, p$, satisfying $\phi(a_i) = b$. Taking a neighbourhood U_i of each point a_i and a neighbourhood V of b, we can define the inverse maps: $\psi_i = \phi|_{U_i}^{-1} : V \to U_i$. Then the push-forward of a meromorphic top form α on A through ϕ is a differential form β on B given by the sum over all solutions of the pull-backs through the inverse maps ψ_i:

$$\beta = \phi_* \alpha = \sum_{i=1}^{p} \psi_i^* \alpha, \tag{A.4}$$

where the pull-back of a differential form is a standard notion in differential geometry. In practice, one solves the equation $y = \phi(x)$ and for each solution $x = \psi_i(y)$ one substitutes the explicit expression for x into the differential form α, and then sums the resulting forms.

Appendix B. Proofs

B.1. Sketch of a proof for formulae (2.21) and (2.23)

In this appendix we will provide a sketch of a proof of formulae (2.21) and (2.23), which give explicit expressions for the canonical form of the hypersimplex $\Delta_{k,n}$. Since the two formulae easily follow from each other, it will be sufficient to only prove (2.23).

We want to show that the canonical form $\omega_{k,n}$ of the hypersimplex $\Delta_{k,n}$ is given by

$$\omega_{k,n} = \tau_{k-1,n}, \tag{B.1}$$

where

$$\tau_{l,n} := \sum_{j=1}^{n} (-1)^{j+1} \sum_{I \in \binom{[n]\setminus\{j\}}{l}} \bigwedge_{i \in [n]\setminus\{j\}} \mathrm{d}\,\log\,\alpha_i, \quad \alpha_i = \begin{cases} x_i, & \text{if } i \notin I, \\ 1 - x_i, & \text{if } i \in I. \end{cases} \tag{B.2}$$

Since we know the facet structure of the hypersimplex $\Delta_{k,n}$, and in particular the fact that they are hypersimplices $\Delta_{k-1,n-1}$ or $\Delta_{k,n-1}$, we have the following properties of $\omega_{k,n}$:

(a) $\omega_{k,n}, 1 < k < n-1$ has exactly $2n$ simple poles, n at $x_i = 0$ and n at $x_i = 1$ for $i = 1, \ldots, n$,

(b) $\operatorname{Res}_{x_i=0} \omega_{k,n} = \pm\omega_{k,[n]\setminus\{i\}}$,

(c) $\operatorname{Res}_{x_i=1} \omega_{k,n} = \pm\omega_{k-1,[n]\setminus\{i\}}$,

Where $\omega_{k,[n]\setminus\{i\}}$ is defined as $\omega_{k,n-1}$ with the variables x's relabelled in the following way: $x_j \to x_j$ if $j < i$, $x_j \to x_{j+1}$ if $j \geqslant i$. We will prove equation (B.1) by induction in n.

Assume that for some $n = m$, equation (B.1) holds for all $1 \leqslant k \leqslant m-1$. We want to find $\omega_{k,m+1}$, where for now we assume that $2 \leqslant k \leqslant m-1$. Using condition 2, we have that $\operatorname{Res}_{x_i=0}\omega_{k,m+1} = \pm\omega_{k,[m+1]\setminus\{i\}}$, and therefore $\omega_{k,m+1} = \pm\omega_{k,[m+1]\setminus\{i\}} \wedge \mathrm{d}\log x_i + \cdots$, where the ellipsis $\ldots$ indicates the terms that do not have a pole at $x_i = 0$. Note that $\omega_{k,[m+1]\setminus\{i\}}$ equals $\omega_{k,m}$ with x's relabelled, and therefore from the induction hypothesis we know that this is equal to $\tau_{k-1,[m+1]\setminus\{i\}}$ defined as $\tau_{k-1,m}$ with the x's relabelled as above. Then we have:

$$\omega_{k,m+1} = \pm\tau_{k-1,[m+1]\setminus\{i\}} \wedge \mathrm{d}\,\log\,x_i + \cdots \quad \text{for all } i \in [m+1]. \tag{B.3}$$

Note that $\operatorname*{Res}_{x_i=0} \tau_{k-1,m+1} = (-1)^{n-i}\tau_{k-1,[m+1]\setminus\{i\}}$, hence the terms in $\omega_{k,m+1}$ that have a residue at some $x_i = 0$ correspond to the terms that appear in $\tau_{k-1,m+1}$. Next, we use the fact that $\operatorname*{Res}_{x_i=0}\operatorname*{Res}_{x_{i+1}=0} \omega_{k,m+1} = -\operatorname*{Res}_{x_{i+1}=0}\operatorname*{Res}_{x_i=0} \omega_{k,m+1}$ to fix the relative signs of these terms. Importantly, we find that the relative signs do agree with those of $\tau_{k-1,m+1}$. Therefore, we conclude that $\omega_{k,m+1} = \pm\tau_{k-1,m+1} + \mathcal{P}$, where $\mathcal{P}$ is a term that does not have any poles at $x_i = 0$. Next, from point 3 and the explicit form of $\tau_{k-1,m+1}$, we see that $\mathcal{P}$ also cannot have any poles when $x_i = 1$. From point 1 we then conclude that $\mathcal{P}$ cannot have any poles at all, including at infinity and we find that $\mathcal{P} = 0$ and thus $\omega_{k,m+1} = \pm\tau_{k-1,m+1}$. Finally, without loss of generality, we can choose the orientation of $\Delta_{k,n}$ such that we have a positive sign. This argument holds for all $2 \leqslant k \leqslant m-1$. Since for $k = 1$ and $k = m$, hypersimplices $\Delta_{1,m+1}$ and $\Delta_{m,m+1}$ are just ordinary simplices, then we immediately see that (B.1) is also true in these cases. Therefore we have that $\omega_{k,m+1} = \tau_{k-1,m+1}$ for all $1 \leqslant k \leqslant m$. This concludes the inductive step. The base case can be checked explicitly for example for $n = 4, k = 2$ (see equation (2.9) and the surrounding discussion). Together, this proves the claim (B.1).

B.2. Sketch of a proof of formula (2.32)

We now turn to equation (2.32). We want to prove that the following formula is correct:

$$\Delta_{k,n} = \overline{\Sigma_{\varnothing}\setminus\left(\bigcup_{I_1\in\binom{[n]}{1}}\Sigma_{I_1}\setminus\left(\bigcup_{I_2\in\binom{[n]}{2}}\Sigma_{I_2}\setminus\left(\ldots\setminus\bigcup_{I_{k-1}\in\binom{[n]}{k-1}}\Sigma_{I_{k-1}}\right)\right)\right)}, \quad \text{(B.4)}$$

where Σ_I, $I \in \binom{[n]}{I}$ is the simplex cut out by the hyperplanes $x_i = 0, i \notin I, x_i = 1, i \in I$ in the $n-1$ dimensional subspace of $\mathbb{R}^n$ defined by the relation $x_1 + x_2 + \cdots + x_n = k$.

To simplify our discussion, we introduce an alternative labelling for the simplices Σ_I: we denote by $\Sigma_{(b_1,b_2,\ldots,b_n)}$ with $b_i = 0, 1$, the simplex cut out by the hyperplanes $x_i = b_i$ (equivalent to $\Sigma_{\{i|b_i=1\}}$). Then, the inequalities that cut out these simplices are given by:

$$\Sigma_I = \Sigma_{(b_1,b_2,\ldots,b_n)} = \begin{cases} x_i \leqslant b_i, & \text{if } k < b \\ x_i \geqslant b_i, & \text{if } k > b \\ x_i = b_i, & \text{if } k = b, \end{cases} \quad \text{(B.5)}$$

where $I = \{i|b_i = 1\}$, and $b = b_1 + b_2 + \cdots + b_n = |I|$. This can be understood as follows: in $\mathbb{R}^n$ the hyperplanes $x_i = b_i$ intersect in the point $(b_1, b_2, \ldots, b_n)^{\mathrm{T}}$. Clearly, this point is above the hyperplane $x_1 + \cdots + x_n = k$ when $b > k$, hence we need $x_i \leqslant b_i$. When $b < k$, the hyperplanes $x_i = b_i$ in $\mathbb{R}^n$ intersect below the plane $x_1 + \cdots + x_n = k$, and we thus require $x_i \geqslant b_i$. When $b = k$ these hyperplanes $x_i = b_i$ in $\mathbb{R}^n$ intersect in the point $(b_1, b_2, \ldots, b_n)^{\mathrm{T}}$ which is on the plane $x_1 + \cdots + x_n = k$.

Now, it is easy to see that the intersection of two simplices $\Sigma_{(b_1,b_2,\ldots,b_n)}$ and $\Sigma_{(c_1,c_2,\ldots,c_n)}$ is just the simplex $\Sigma_{(d_1,d_2,\ldots,d_n)}$, where $d_i = b_i \wedge c_i$ if all three binary strings are 'above k': $b, c, d > k$, and $d_i = b_i \vee c_i$ if the three binary strings are 'below k': $b, c, d < k$ (where $b = \sum_i b_i, c = \sum_i c_i, d = \sum_i d_i$). Here, $\wedge$ and $\vee$ are binary 'and' and 'or' operations, respectively. If one of the three binary strings is on the opposite side of k from the other two, then the intersection is empty. In the language of $\Sigma_I, I \in \binom{[n]}{l}$, we can summarize this result as follows:

$$\Sigma_I \cap \Sigma_J = \begin{cases} \Sigma_{I\cup J}, & \text{If } |I|, |J|, |I \cup J| \leqslant k \\ \Sigma_{I\cap J}, & \text{If } |I|, |J|, |I \cap J| \geqslant k \\ \emptyset & \text{otherwise.} \end{cases} \tag{B.6}$$

An alternative way to define the hypersimplex $\Delta_{k,n}$ is as the intersection of the unit cube in $\mathbb{R}^n$ and the hyperplane $x_1 + x_2 + \cdots + x_n = k$. Explicitly, the hypersimplex is cut out by the inequalities:

$$0 \leqslant x_i \leqslant 1, \quad i = 1, \ldots, n, \tag{B.7}$$

constrained to the hypersurface $x_1 + x_2 + \cdots + x_n = k$. It is thus clear that the hypersimplex is completely contained inside the simplex $\Sigma_\emptyset = \Sigma_{(0,0,\ldots,0)} = \{x_i \geqslant 0, i = 1, \ldots, n\}$. We now look at the simplices with 1 and $(n-1)$ zeroes in their binary string, e.g. $\Sigma_{\{1\}} = \Sigma_{(1,0,\ldots,0)} = \{x_1 \geqslant 1, x_2 \geqslant 0, \ldots, x_n \geqslant 0\}$. It is clear that this simplex is a subset of $\Sigma_\emptyset$ and only intersects with the hypersimplex along the face $x_1 = 1$. The analogous result holds for all $\Sigma_{\{i\}}$. In fact, when we take away all simplices $\Sigma_{\{i\}}$ from $\Sigma_\emptyset$, the only region that is left will be the interior of the hypersimplex:

$$\Delta_{k,n} = \overline{\Sigma_\emptyset \setminus \left(\bigcup_{I\in\binom{[n]}{1}} \Sigma_I \right)}, \tag{B.8}$$

where it is necessary to take the closure on the rhs because we also subtract all boundaries of the hypersimplex.

In general, the simplices Σ_I, $I \in \binom{[n]}{1}$ overlap. Set-theoretically, this is not a problem and equation (B.8) works just fine. However, from the point of view of positive geometries there are certain regions that have been 'subtracted twice', hence the analogous equation does not hold for the canonical forms. Instead we would like to subtract the intersecting parts of $\bigcup_{I\in\binom{[n]}{1}} \Sigma_I$ before subtracting it from $\Sigma_\emptyset$:

$$\Delta_{k,n} = \overline{\Sigma_\emptyset \setminus \left(\bigcup_{I\in\binom{[n]}{1}} \Sigma_I \setminus \left(\bigcup_{J_1,J_2\in\binom{[n]}{1}} \Sigma_{J_1} \cap \Sigma_{J_2} \right)\right)}. \tag{B.9}$$

From the discussion above it is clear that, for example, $\Sigma_{\{1\}} \cap \Sigma_{\{2\}} = \Sigma_{\{1,2\}}$, and from this it is not difficult to see that $\bigcup_{J_1,J_2\in\binom{[n]}{1}} \Sigma_{J_1} \cap \Sigma_{J_2} = \bigcup_{I\in\binom{[n]}{2}} \Sigma_I$. However, we now run into the same problem that the simplices Σ_I, $I \in \binom{[n]}{2}$ can in general overlap. Continuing along the same line, the union of two set $I, J \in \binom{[n]}{2}$ can result in either an element of $\binom{[n]}{3}$ or $\binom{[n]}{4}$. But, since $\bigcup_{I\in\binom{[n]}{4}} \subset \bigcup_{I\in\binom{[n]}{3}}$, we find that $\bigcup_{I_1,I_2\in\binom{[n]}{2}} \Sigma_{I_1} \cap \Sigma_{I_2} = \bigcup_{J\in\binom{[n]}{3}} \Sigma_J$. In general, a similar reasoning implies that $\bigcup_{I_1,I_2\in\binom{[n]}{l}}\Sigma_{I_1}\cap\Sigma_{I_2}=\bigcup_{J\in\binom{[n]}{l+1}}\Sigma_J$. Thus, by continuously removing the intersection of simplices in the way explained above, we find

$$\Delta_{k,n}=\overline{\Sigma_{\emptyset}\backslash\left(\bigcup_{I_1\in\binom{[n]}{1}}\Sigma_{I_1}\backslash\left(\bigcup_{I_2\in\binom{[n]}{2}}\Sigma_{I_2}\backslash\cdots\backslash\left(\bigcup_{I_l\in\binom{[n]}{l}}\Sigma_{I_l}\right)\right)\right)}. \tag{B.10}$$

This formula holds 'set-theoretically' for all $l\leqslant k$, but we can refine the amount of overlap we subtract by increasing l. This process terminates naturally, since $\bigcup_{I_k\in\binom{[n]}{k}}\Sigma_{I_k}$ is not full-dimensional and is just the set of the vertices of $\Delta_{k,n}$. Then (B.4) is equivalent to (B.10) when we set $k=l$. While the hypersimplex $\Delta_{k,n}$ can be decomposed as (B.10) for any $l\leqslant k$, when $l=k$ in addition we have that the sets that appear at a given level in the formula (B.4) are disjoint when lower terms are subtracted. It allows us to translate this statement into an equality of canonical forms, namely (2.31).

ORCID iDs

Tomasz Łukowski https://orcid.org/0000-0002-4159-3573
Jonah Stalknecht https://orcid.org/0000-0002-0350-5114

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