Sample size determination for interval estimation of the prevalence of a sensitive attribute under non-randomized response models

Abstract

A sufficient number of participants should be included to adequately address the research interest in the surveys with sensitive questions. In this article, sample size formulas/iterative algorithms are developed from the perspective of controlling the confidence interval width of the prevalence of a sensitive attribute under four non-randomized response models, i.e., Crosswise Model, Parallel Model, Poisson Item Count Technique and Negative Binomial Item Count Technique. In contrast to the conventional approach for sample size determination, our sample size formulas/algorithms explicitly incorporate an assurance probability of controlling the width of a confidence interval within the pre-specified range. The performance of the proposed methods is evaluated with respect to the empirical coverage probability, empirical assurance probability and confidence width. Simulation results show that all formulas/algorithms are effective and hence are recommended for practical applications. A real example is used to illustrate the proposed methods.

Keywords— Assurance probability, confidence interval, non-randomized response models, sample size determination, sensitive attribute.

1 Introduction

When interviewers engage in survey research involving sensitive attributes, the direct questioning approach may introduce non-response bias and response bias, as respondents may refuse to answer or provide inaccurate responses. To address these biases and enhance the reliability of survey data, Warner[1] proposed the implementation of a randomized response technique (RRT), designed to obfuscate individual responses through the utilization of a randomizing device. In Warner's conceptualization of the RRT, the randomized response model presents the respondent with a binary choice between two questions that are complementary in nature. The pivotal element is that the interviewer remains unaware of which specific question the respondent is answering, owing to the unpredictable nature introduced by the randomizing device. This safeguard ensures the privacy of the interviewee. It is noteworthy that both questions posed within Warner's RRT framework pertain to the sensitive attribute. Subsequently, various adaptations and refinements to Warner's model have emerged over the years to enhance its applicability and effectiveness. For instance, Mangat *et al.*[2] proposed a two-stage randomized response model that necessitates the use of two randomizing devices. Abul-Ela *et al.*[3] and Bourke[4] expanded Warner's model to the Randomized Response Technique (RRT) with three mutually exclusive answers. Horvitz *et al.*[5] and Greenberg *et al.*[6] devised a randomized response model incorporating non-sensitive questions, while Christofides[7] introduced a randomized response technique for estimating the proportion of respondents possessing two sensitive characteristics simultaneously. It is noteworthy that randomized response models are often perceived as lacking reproducibility, incurring high costs, fostering low trust, and presenting challenges in comprehension (Tian *et al.*[8]). To address these limitations, Swensson[9] introduced a combination-question technique mandating two independent random samples to supplant the use of any randomization device. Takahasi *et al.*[10] substituted randomization devices with neutral auxiliary questions, achieving non-randomization and enhanced privacy protection for respondents. Yu *et al.*[11] proposed the crosswise and triangle models. The former is regarded as the non-randomized version of the Warner model, introducing a non-sensitive binary variable and a simpler survey format. The latter represents a variant of the crossover design model designed to mitigate its relative inefficiency. Both models necessitate one category to be non-sensitive (represented by Y=0) and are unsuitable for situations involving two or more sensitive categories (represented by Y=1), such as income, number of sexual partners, and loyalty or disloyalty to a boss. To overcome these limitations, Tian[12] introduced the parallel model, considered the non-randomized version of the uncorrelated randomized response model proposed by Horvitz *et al.*[5]. Theoretical comparisons between the parallel model and the crossover design and triangle models demonstrate that, across most parameter ranges, the parallel model is more efficient, affords superior privacy protection, and possesses a broader range of applications.

Miller[13] introduced the item count technique (ICT) as a non-randomized alternative to the randomized response model. In this approach, respondents are randomly assigned to either an experimental or control group. The experimental group is presented with K non-sensitive questions and one sensitive question, while the control group is exclusively asked the K non-sensitive questions. However, the ICT model exhibits a design flaw, where in the true status of the respondent is inevitably exposed when K+1 "yes" responses are obtained from the experimental group. This compromises privacy and may elicit dishonest responses.To address this critical issue, Tian *et al.*[14] proposed two novel models, namely the Poisson ICT Model and the Negative Binomial ICT Model. These models replace several non-sensitive questions following a binomial distribution with a single non-sensitive question eliciting responses in the form of non-negative integers. The development of these models is grounded in the assumption of counting data distribution. Experimental results demonstrate that the proposed methods offer accurate parameter estimation and confidence intervals, effectively mitigating the limitations inherent in the ICT model.

Determining the optimal sample size is an essential step in conducting survey research. Chow *et al.*[15] proposed two methods, namely the precision analysis and power analysis methods, to determine the sample size. The precision analysis method predefines the maximum acceptable level of the Type I error rate by specifying a confidence interval, while the power analysis method controls the Type II error rate β to determine the sample size. Tian *et al.*[8] extended the application of both methods to the Crosswise Model, Triangular Model, and Parallel Model introduced by Yu *et al.*[11] and Tian[12], providing sample size calculation formulas under the condition of expected power of $1-\beta$. Additionally, Tian *et al.*[14] offered approximations for determining the sample size of two novel ICT models, namely the Poisson ICT and Negative Binomial ICT, based on the precision and power analysis methods proposed by Chow *et al.*[15]. However, several scholars have suggested that confidence intervals are more informative than simple hypothesis testing in assessing the accuracy and precision of statistical data. Beal[16], Bristol[17], Goodman *et al.*[18] and Rumke[19] proposed using the expected width of the confidence interval to determine the sample size. Bland[20] also recommended using the width of the confidence interval instead of power to determine the sample size in a medical study. Zou[21] suggested that the determination of the required number of participants for estimating intra-class correlation coefficient in studies of confidence level should be based on the expected width of the confidence interval. Ulrich *et al.*[22] derived the statistical powers for the Wald test under Warner's model, unrelated question model, item count model and cheater detection model, their corresponding sample size requirements that can achieve a desired power for the Wald test with a predetermined type I error rate can be readily obtained. While determining the sample size is a crucial step in survey research and to some extent, the success of a survey depends on it, most studies have focused on determining the sample size from the perspective of testing power for surveys on sensitivity issues under non-random response models. Research on sample size determination from the perspective of interval estimation is still limited. Expanding on this idea, Qiu *et al.*[23] computed the width of the confidence interval ~~which~~ ~~regulates~~ the proportion of sensitive features at a particular confidence level for the non-randomized triangular model, and derived an approximate formula for the sample size with a given level of assurance. Qiu *et al.*[24] further obtained sample size formulas for four random response models, namely the Warner model, the unrelated question model, the item count technique model, and the cheating detection model. In this paper, we introduce a novel approach to address the challenge of controlling the width of the confidence interval for the prevalence of sensitive attributes at a specific confidence level within the context of four non-randomized response models, namely the Crosswise Model, Parallel Model, Poisson Item Count Technique, and Negative Binomial Item Count Technique. Specifically, we derive closed-form sample size formulas to achieve this objective for the former two models. For the latter two models, which do not allow for a closed-form formula, we propose an iterative algorithm to determine the required sample size. Importantly, our methodology incorporates a probability-based framework to ensure predetermined precision. These contributions bear significant implications for statistical inference in the estimation of sensitive attribute prevalence.

This article is organized as follows. Sample size formulas/algorithms for the prevalence of a sensitive attribute for the aforementioned models (i.e., Crosswise Model, Parallel Model, Poisson Item Count Technique and Negative Binomial Item Count Technique) are derived in Section 2 and Section 3. The performance of the proposed methods is evaluated by simulation studies in Section 4. In Section 5, a real example for the investigation of premarital sexual practices in adolescents from Bogale *et al.* [25] is used to illustrate the accuracy of the estimated sample size formulas. A brief conclusion and discussion are given in Section 6.

2 Approximate Sample Sizes under Crosswise Model and Parallel Model

2.1 Crosswise Model and Parallel Model

Let Y represent a binary random variable indicating a sensitive attribute of interest, such as premarital sexual practices, where a value of 1 signifies "Have ever had premarital sexual intercourse," and 0 denotes "Have never had premarital sexual intercourse." The probability of Y being 1 is denoted as p = Pr(Y = 1). Additionally, let W be a non-sensitive binary attribute, independent of Y, such as "Is the first digit of your house number 1, 2, 3, 4, or 5?" The probability of W being 1 is known and is expressed as p = Pr(W = 1). According to "Benford's Law", Diekmann[26] reported that the probability of the first digit of a house number being 1, 2, 3, 4 or 5 is 0.778, specifically $p = \Pr(W = 1) = 0.778$.

The Crosswise Model proposed by Yu *et al.* [11] is reported in Table 1. In this design, each interviewee will be instructed to provide his/her response by placing a tick in the upper circle if he/she belongs to one of the two circles or putting a tick in the upper triangle if he/she belongs to one of the two triangles. ~~This design is mathematically equivalent Warner's model, so it is the non-randomized version of Warner's RRT.~~ Since W and Y are independent by design, the cell probabilities for the right side of Table 1 can be easily obtained by multiplying the marginal probabilities.

Table 1. Crosswise Model and the corresponding cell probabilities.

Category	W=0	W=1	Category	W=0	W=1	Marginal
Y=0	○	△	Y=0	$(1-\pi)(1-p)$	$(1-\pi)p$	$1-\pi$
Y=1	△	○	Y=1	$\pi(1-p)$	πp	π
			Marginal	$1-p$	p	1

Note: Interviewees are instructed to put a tick in the upper circle if they have never had premarital sexual intercourse AND the first digit of their house number is not 1, 2, 3, 4 and 5. Alternatively, interviewees should put a tick in the upper circle if they have ever had premarital sexual intercourse AND the first digit of their house number is 1, 2, 3, 4 or 5. In all other cases, interviewees are directed to put a tick in the upper triangle.

Note that the Crosswise Model necessitates one category to be non-sensitive, rendering it unsuitable for situations where two categories are sensitive. To address this limitation, Tian[12] introduced the Parallel Model. In the parallel design, Y is the binary random variable representing a sensitive attribute of interest, while U and W are two non-sensitive dichotomous variables, such as (W): "Is the first digit of your house number 1, 2, 3, 4, or 5?" and (U): "Is the last digit of your cell phone number odd?". It is assumed that Y, U, and W are mutually independent, and the probability of the sensitive attribute is denoted as p = Pr(Y = 1), with known probabilities q = Pr(U = 1) and p = Pr(W = 1).

Under the parallel design, each interviewee is instructed to provide a response by connecting two circles with a straight line if they belong to one of the two circles, or connecting two triangles with a straight line if they belong to one of the two triangles. This design serves as a non-randomized version of the unrelated question model (Greenberg *et al.*[6]). Since Y, U, and W are mutually independent by design, the cell probabilities for the right side of Table 2 can be easily obtained by multiplying the marginal probabilities.

Table 2. Parallel Model and the corresponding cell probabilities.

Category	W=0	W=1	Category	W=0	W=1	Marginal
U=0	◯		U=0	$(1-q)(1-p)$		$1-q$
U=1	△		U=1	$q(1-p)$		q
			Marginal	$1-p$	p	1.0
Y=0		◯	Y=0		$(1-\pi)p$	$1-\pi$
Y=1		△	Y=1		πp	π
			Marginal	$1-p$	p	1.0

Note: Interviewees are instructed to connect the two triangles by a straight line if the first digit of their house number is not 1, 2, 3, 4, or 5 AND the last digit of their cell phone number is odd. Alternatively, interviewees should connect the two triangles by a straight line if the first digit of their house number is 1, 2, 3, 4, or 5 AND they have ever had premarital sexual intercourse. In all other cases, interviewees are directed to connect the two circles by a straight line.

Let λ be the probability of marking the upper circle in the Crosswise Model or connecting the two triangles by a straight line in the Parallel Model. Then, $\lambda = \zeta\pi + \eta$, with $\zeta = 2p-1$, $\eta = 1-p$ for Crosswise Model and $\zeta = p$, $\eta = q(1-p)$ for Parallel Model, respectively. Therefore, $\pi = (\lambda-\eta)/\zeta$ for both models.

2.2 Confidence Intervals under Crosswise and Parallel Designs

Suppose that x out of n subjects mark the upper circle in Table 1 or connect the two triangles by a straight line in Table 2. The maximum likelihood estimate of λ is denoted as $\hat{\lambda} = x/n$ with the expectation $\mathrm{E}(\hat{\lambda}) = \lambda$ and variance $\mathrm{Var}(\hat{\lambda}) = \lambda(1-\lambda)/n = a_\lambda^2/n$, respectively. Therefore, the maximum likelihood estimate (MLE) of π is $\hat{\pi} = (\hat{\lambda}-\eta)/\zeta$ with a variance $\mathrm{Var}(\hat{\pi}) = a_\lambda^2/(n\zeta^2)$ under both models.

As shown in van den Hout, A.[27], the MLE $\hat{\pi}$ is equal to the estimate provided by the moment estimator as long as they are in the interior of the parameter space. While the estimate $\hat{\pi}$ may fall outside the range $[0,1]$, the probability of $\hat{\pi}$ exceeding this boundary diminishes with an increasing sample size (Ulrich *et al.*[22]). Additionally, the Expectation Maximization (EM) algorithm can be employed to obtain the Maximum Likelihood Estimator(MLE) of $\hat{\pi}$ when $\hat{\pi} < 0$ or $\hat{\pi} > 1$.

According to the Central Limits Theorem, the $(1-\alpha)100\%$ Wald confidence interval (CI) for π under the above two models can be obtained by

$$[\pi_{l,W}, \pi_{u,W}] = [\hat{\pi} - z_{\alpha/2}\cdot\frac{\hat{a}_\lambda}{\sqrt{n}|\zeta|}, \hat{\pi} + z_{\alpha/2}\cdot\frac{\hat{a}_\lambda}{\sqrt{n}|\zeta|}], \tag{1}$$

where $z_{\alpha/2}$ is the $1-\alpha/2$ quantile of the standard normal distribution, and $\hat{a}_\lambda = \sqrt{\hat{\lambda}(1-\hat{\lambda})}$.

As shown in Newcombe[28] and Agresti & Coull[29], the confidence interval derived from the Wilson method usually outperforms the Wald interval, especially for small sample sizes. Hence, we adopt the Wilson[30] method for constructing a $(1-\alpha)100\%$ confidence interval for the prevalence π, and the Wilson CI under the Crosswise and Parallel design is given by $[\pi_{l,Wi}, \pi_{u,Wi}]$, where

$$\pi_{l,Wi} = \frac{n\hat{\lambda} + z_{\alpha/2}^2/2 - \eta(n+z_{\alpha/2}^2) - z_{\alpha/2}\sqrt{n\hat{a}_\lambda^2 + z_{\alpha/2}^2/4}}{\zeta(n+z_{\alpha/2}^2)} \quad \text{and}$$

$$\pi_{u,Wi} = \frac{n\hat{\lambda} + z_{\alpha/2}^2/2 - \eta(n+z_{\alpha/2}^2) + z_{\alpha/2}\sqrt{n\hat{a}_\lambda^2 + z_{\alpha/2}^2/4}}{\zeta(n+z_{\alpha/2}^2)}$$

if $\zeta > 0$. If $\zeta < 0$, the confidence lower and upper limits are

$$\pi_{l,Wi} = \frac{n\hat{\lambda} + z_{\alpha/2}^2/2 - \eta(n+z_{\alpha/2}^2) + z_{\alpha/2}\sqrt{n\hat{a}_\lambda^2 + z_{\alpha/2}^2/4}}{\zeta(n+z_{\alpha/2}^2)} \text{ and}$$

$$\pi_{u,Wi} = \frac{n\hat{\lambda} + z_{\alpha/2}^2/2 - \eta(n+z_{\alpha/2}^2) - z_{\alpha/2}\sqrt{n\hat{a}_\lambda^2 + z_{\alpha/2}^2/4}}{\zeta(n+z_{\alpha/2}^2)},$$

respectively.

2.3 Sample Size Formulas under Crosswise and Parallel Designs

The half width of the $(1-\alpha)100\%$ Wald CI for π is given by

$$z_{\alpha/2} \cdot \frac{\hat{a}_\lambda}{\sqrt{n}|\zeta|}.$$

If our objective is to ensure that the half-width does not exceed ω with a probability of $1-\beta$, then the condition is given by:

$$\Pr(z_{\alpha/2} \cdot \frac{\hat{a}_\lambda}{\sqrt{n}|\zeta|} \leq \omega) \geq 1-\beta.$$

It can be shown that it is equivalent to

$$\Pr(\hat{a}_\lambda \leq \frac{\omega|\zeta|}{z_{\alpha/2}}\sqrt{n}) \geq 1-\beta.$$

According to the large sample theory and delta method, it can be shown that

$$\hat{a}_\lambda \sim N(a_\lambda, \frac{b_\lambda^2}{n}),$$

where $\mathrm{Var}(\hat{a}_\lambda) = b_\lambda^2/n$ with $b_\lambda = |1-2\lambda|/2$ (Please see Appendix for details). Thus, we have

$$\Pr(\frac{\hat{a}_\lambda - a_\lambda}{b_\lambda/\sqrt{n}} \leq \frac{\frac{\omega|\zeta|}{z_{\alpha/2}}\sqrt{n} - a_\lambda}{b_\lambda/\sqrt{n}}) \geq 1-\beta.$$

Therefore, the desired sample size n can be obtained by solving the following equation:

$$\frac{\omega|\zeta|}{z_{\alpha/2}}\sqrt{n} - a_\lambda = \frac{z_\beta b_\lambda}{\sqrt{n}},$$

where z_β is the $1-\beta$ quantile of the standard normal distribution.

Solving the above equation yields

$$\boldsymbol{n_W} = [\frac{a_\lambda + [a_\lambda^2 + 4\omega|\zeta|z_\beta/z_{\alpha/2} \cdot b_\lambda]^{1/2}}{2\omega|\zeta|/z_{\alpha/2}}]^2. \tag{2}$$

In particular, when $\beta = 0.5$, it is the conventional sample size, which is given by

$$\boldsymbol{n_{W,0.5}} = \frac{a_\lambda^2}{[\omega\zeta/z_{\alpha/2}]^2}. \tag{3}$$

On the other hand, given the values of n, ζ, η and π, the assurance probability can be obtained by

$$\Phi(\frac{\frac{\omega|\zeta|}{z_{\alpha/2}\sqrt{n}} - a_\lambda}{b_\lambda/\sqrt{n}}),$$

where $\Phi(\cdot)$ is the distribution function of the standard normal distribution.

Similarly, in order to control the half width of the Wilson CI for π no larger than ω with probability $1-\beta$, the half width should satisfy

$$\Pr(\frac{z_{\alpha/2}\sqrt{n\hat{a}_\lambda^2+z_{\alpha/2}^2/4}}{|\zeta|(n+z_{\alpha/2}^2)}\leq\omega)\geq 1-\beta,$$

i.e.,

$$\Pr(\hat{a}_\lambda^2\leq\frac{4\zeta^2(n+z_{\alpha/2}^2)^2\omega^2-z_{\alpha/2}^4}{4nz_{\alpha/2}^2})\geq 1-\beta.$$

By using the delta method, the variance of $\hat{a}_\lambda^2$ can be given by $\mathrm{Var}(\hat{a}_\lambda^2)=4a_\lambda^2b_\lambda^2/n$, then the asymptotical distribution of $\hat{a}_\lambda^2$ is $\hat{a}_\lambda^2\sim\mathrm{N}(a_\lambda^2,4a_\lambda^2b_\lambda^2/n)$ (Please see Appendix for details). Therefore, we have the following equation:

$$\frac{4\zeta^2(n+z_{\alpha/2}^2)^2\omega^2-z_{\alpha/2}^4}{4nz_{\alpha/2}^2}-a_\lambda^2=2z_\beta\sqrt{a_\lambda^2b_\lambda^2/n}.$$

The above equation can be simplified as the following quartic equation with respect to $n+z_{\alpha/2}^2$:

$$a(n+z_{\alpha/2}^2)^4+b(n+z_{\alpha/2}^2)^3+c(n+z_{\alpha/2}^2)^2+d(n+z_{\alpha/2}^2)+e=0, \tag{4}$$

where

$$\begin{aligned}
&a=16\omega^4\zeta^4,\\
&b=-32z_{\alpha/2}^2\omega^2\zeta^2a_\lambda^2,\\
&c=8z_{\alpha/2}^4[2a_\lambda^4+4\omega^2\zeta^2a_\lambda^2-\omega^2\zeta^2],\\
&d=8z_{\alpha/2}^4a_\lambda^2[z_{\alpha/2}^2-4z_{\alpha/2}^2a_\lambda^2-8z_\beta^2b_\lambda^2],\ \text{and}\\
&e=z_{\alpha/2}^6[16z_{\alpha/2}^2a_\lambda^4+64z_\beta^2a_\lambda^2b_\lambda^2-8z_{\alpha/2}^2a_\lambda^2+z_{\alpha/2}^2].
\end{aligned}$$

The eigenvalue method can be used to find the roots of the above quartic equation. Let $n_{\max}$ be the maximum real root of Equation (4) with respect to $n+z_{\alpha/2}^2$, and the approximate sample size is denoted as $\boldsymbol{n_{Wi}}$. Then $\boldsymbol{n_{Wi}}$ is the minimum integer that is not smaller than $n_{\max}-z_{\alpha/2}^2$. Especially, when $\beta=0.5$, the approximate sample size n is given by

$$\boldsymbol{n_{Wi,0.5}}=\frac{z_{\alpha/2}^2[a_\lambda^2+\sqrt{a_\lambda^4+\omega^2\zeta^2(1-4a_\lambda^2)}]}{2\omega^2\zeta^2}-z_{\alpha/2}^2. \tag{5}$$

On the other hand, given the values of n, ζ, η and π, the assurance probability can be obtained by

$$\Phi(\frac{(\frac{4\zeta^2(n+z_{\alpha/2}^2)^2\omega^2-z_{\alpha/2}^4}{4nz_{\alpha/2}^2}-a_\lambda^2)}{2\sqrt{a_\lambda^2b_\lambda^2}/n}),$$

where $\Phi(\cdot)$ is the distribution function of the standard normal distribution.

Hence, if there exists a general relationship between the binomial probability λ and the probability of a sensitive attribute π for any (non)randomized response design, expressed as $\lambda=\zeta\pi+\eta$ with ζ and η being known, the sample size formula explicitly incorporating an assurance probability to control the width of a Wald Confidence Interval (CI) within the pre-specified range can be given by Equation (2) or (3). Furthermore, sample size estimation based on a Wilson CI can be obtained by solving Equation (4). In these formulas, it is sufficient to calculate the variances $\mathrm{Var}(\hat{\lambda})$ and $\mathrm{Var}(\hat{a}_\lambda)$ to obtain a_λ^2 and b_λ^2.

3 Approximate Sample Sizes under Poisson ICT and Negative Binomial ICT

3.1 Poisson ICT and Negative Binomial ICT

Obviously, for the item count technique proposed by Miller[13], the respondent's sensitive characteristic is inevitably exposed when $K+1$ "yes" responses are obtained from the experimental group if an item count design consists of K non-sensitive questions and one sensitive question. Therefore, Tian *et al.*[14] changed the K neutral questions to a single neutral question, and proposed the following design: n_c respondents are randomly assigned to the control group and receive a neutral question, for example, "How many times did you travel abroad last year?" or "How many online resumes do you need to submit to receive one interview invitation?". While the n_e respondents are randomly assigned to the experimental group and receive the same neutral question together with the sensitive question "Have you ever had sexual intercourse?", for example, "How many times did you travel abroad last year?" or "How many online resumes do you need to submit to receive one interview invitation?". Let Y be the answer to the sensitive question with $Y=1$ if the respondent possesses the sensitive characteristic and $Y=0$ otherwise, and X be the answer to the neutral question. Then, the respondents' answers under the experimental and control groups are $Z=Y+X$ and X, respectively. The parameter of interest is $\pi=\Pr(Y=1)$. Given that the variable X is a non-negative integer, we can assume that X follows either a Poisson distribution with parameter τ or a Negative Binomial distribution with parameters $r(>0)$ and p, denoted as $X\sim\text{Poisson}(\tau)$ with probability distribution:

$$\Pr(X=x)=\frac{\tau^x e^{-\tau}}{x!}, x=0,1,2,\cdots$$

or $X\sim\text{NBinomial}(r,p)$ with probability distribution:

$$\Pr(X=x)=\frac{\Gamma(x+r)}{x!\Gamma(r)}(1-p)^r p^x, x=0,1,2,\cdots.$$

The corresponding models are named as Poisson Item Count Technique (i.e., Poisson ICT) and Negative Binomial Item Count Technique (i.e., Negative Binomial ICT), respectively.

3.2 Confidence Intervals under Poisson and Negative Binomial ICTs

Let $\{x_i\}_{i=1}^{n_c}$ and $\{z_i\}_{j=1}^{n_e}$ be the observed data in the control and experimental groups, respectively. Thus, the moment estimate of π is given by

$$\hat{\pi}=\frac{1}{n_e}\sum_{j=1}^{n_e} z_j-\frac{1}{n_c}\sum_{i=1}^{n_c} x_i \tag{6}$$

and the variance of $\hat{\pi}$ is $\text{Var}(\hat{\pi})=\text{Var}(Z)/n_e+\text{Var}(X)/n_c$. Since X is independent of Y and $\text{Var}(Z)=\text{Var}(X)+\text{Var}(Y)=\sigma_x^2+\pi(1-\pi)$, we have

$$\text{Var}(\hat{\pi})=\frac{\sigma_x^2+\pi(1-\pi)}{n_e}+\frac{\sigma_x^2}{n_c}=\frac{\pi(1-\pi)}{n_e}+\sigma_x^2\left(\frac{1}{n_c}+\frac{1}{n_e}\right).$$

It is obvious that the above moment estimate $\hat{\pi}$ may fall outside the interval $[0,1]$. Therefore, we apply the EM algorithm to find the MLE of π by adding a latent data $Z_{\text{mis}}=\{y_1,\ldots,y_{n_e}\}$, which is the answer to the sensitive question in the experimental group. Therefore, the complete-data is $Z_{\text{com}}=\{Z_{\text{obs}},Z_{\text{mis}}\}$, where $Z_{\text{obs}}=\{x_1,\ldots,x_{n_c};z_1,\ldots,z_{n_e}\}$ denotes the observed data. Hence, the complete-data likelihood function is

$$L(\pi,\tau\mid Z_{\text{obs}},Z_{\text{mis}})=\pi^{\sum_{j=1}^{n_e}y_j}(1-\pi)^{n_e-\sum_{j=1}^{n_e}y_j}\times\left(\prod_{i=1}^{n_c}\frac{\tau^{x_i}\mathrm{e}^{-\tau}}{x_i!}\right)\left(\prod_{j=1}^{n_e}\frac{\tau^{z_j-y_j}\mathrm{e}^{-\tau}}{(z_j-y_j)!}\right).$$

for Poisson ICT and

$$L(\pi, p \mid Z_{\text{obs}}, Z_{\text{mis}}) \propto \pi^{\sum_{j=1}^{n_e} y_j}(1-\pi)^{n_e-\sum_{j=1}^{n_e} y_j} \times (1-p)^{(n_c+n_e)r} p^{\sum_{i=1}^{n_c} x_i + \sum_{j=1}^{n_e}(z_j - y_j)}$$

for Negative Binomial ICT, respectively. According to the methods proposed by Bliss & Fisher[31], the unknown parameter r can be estimated by $\hat{r} = \bar{x}^2/(s^2 - \bar{x})$, where $\bar{x} = \sum_{i=1}^{n_c} x_i/n_c$ is the sample mean, and $s^2 = \sum_{i=1}^{n_c}(x_i - \bar{x})^2/(n_c - 1)$ is the sample variance. The complete-data MLEs of π, τ and p are given by

$$\pi = \frac{1}{n_e}\sum_{j=1}^{n_e} y_j, \quad \tau = \frac{\sum_{i=1}^{n_c} x_i + \sum_{j=1}^{n_e}(z_j - y_j)}{n_c + n_e}, \quad p = \frac{\sum_{i=1}^{n_c} x_i + \sum_{j=1}^{n_e}(z_j - y_j)}{(n_c+n_e)\hat{r} + \sum_{i=1}^{n_c} x_i + \sum_{j=1}^{n_e}(z_j - y_j)}, \tag{7}$$

respectively. The E-step replaces $\{y_j\}_{j=1}^{n_e}$ by their conditional expectations

$$E(Y_j \mid Z_{\text{obs}}, \pi, \tau) = \frac{z_j \pi}{z_j \pi + \tau(1-\pi)}, \quad j = 1, \ldots, n_e \tag{8}$$

for Poisson ICT, and

$$E(Y_j \mid Z_{\text{obs}}, \pi, p) = \frac{z_j \pi}{z_j \pi + (1-\pi)(z_j + \hat{r} - 1)p}, \quad j = 1, \ldots n_e \tag{9}$$

for Negative Binomial ICT, respectively. The estimations of π, τ and p which are obtained by the above EM algorithm are denoted as $\hat{\pi}_{EM}$, $\hat{\tau}_{EM}$ and $\hat{p}_{EM}$, respectively. Thus, the variance of $\hat{\pi}$ can be estimated by

$$\widehat{Var}(\hat{\pi}) = \frac{\hat{\pi}_{EM}(1-\hat{\pi}_{EM})}{n_e} + \hat{\sigma}_x^2\left(\frac{1}{n_c} + \frac{1}{n_e}\right),$$

for Poisson ICT and Negative Binomial ICT, where $\hat{\sigma}_x^2$ is the estimation of σ_x^2, i.e., $\hat{\sigma}_x^2 = \hat{\tau}_{EM}$ for Poisson ICT and $\hat{\sigma}_x^2 = \hat{r}\hat{p}_{EM}/(1-\hat{p}_{EM})^2$ for Negative Binomial ICT.

Therefore, the $(1-\alpha)100\%$ Wald confidence interval for π is given by $[\pi_{l,W}, \pi_{u,W}]$, where

$$\pi_{l,W} = \hat{\pi}_{EM} - z_{\alpha/2}\left[\frac{\hat{\pi}_{EM}(1-\hat{\pi}_{EM})}{n_e} + \left(\frac{1}{n_c} + \frac{1}{n_e}\right)\hat{\sigma}_x^2\right]^{1/2}$$

and

$$\pi_{u,W} = \hat{\pi}_{EM} + z_{\alpha/2}\left[\frac{\hat{\pi}_{EM}(1-\hat{\pi}_{EM})}{n_e} + \left(\frac{1}{n_c} + \frac{1}{n_e}\right)\hat{\sigma}_x^2\right]^{1/2}.$$

The $(1-\alpha)100\%$ Wilson confidence interval for π is given by $[\pi_{l,Wi}, \pi_{u,Wi}]$, where

$$\pi_{l,Wi} = \frac{n_e\hat{\pi}_{EM} + z_{\alpha/2}^2/2 - z_{\alpha/2}[n_e\hat{\pi}_{EM}(1-\hat{\pi}_{EM}) + (1+\frac{n_e}{n_c})(n_e + z_{\alpha/2}^2)\hat{\sigma}_x^2 + z_{\alpha/2}^2/4]^{1/2}}{n_e + z_{\alpha/2}^2}$$

and

$$\pi_{u,Wi} = \frac{n_e\hat{\pi}_{EM} + z_{\alpha/2}^2/2 + z_{\alpha/2}[n_e\hat{\pi}_{EM}(1-\hat{\pi}_{EM}) + (1+\frac{n_e}{n_c})(n_e + z_{\alpha/2}^2)\hat{\sigma}_x^2 + z_{\alpha/2}^2/4]^{1/2}}{n_e + z_{\alpha/2}^2}.$$

3.3 Sample Size Determination Algorithm under Poisson and Negative Binomial ICTs

It is noted that the half widths of the $(1-\alpha)100\%$ Wald CI and Wilson CI are given by

$$z_{\alpha/2}\left[\frac{\hat{\pi}_{EM}(1-\hat{\pi}_{EM})}{n_e} + \left(\frac{1}{n_e} + \frac{1}{n_c}\right)\hat{\sigma}_x^2\right]^{1/2} \tag{10}$$

and

$$\frac{z_{\alpha/2}[n_e\hat{\pi}_{EM}(1-\hat{\pi}_{EM}) + (1+\frac{n_e}{n_c})(n_e + z_{\alpha/2}^2)\hat{\sigma}_x^2 + z_{\alpha/2}^2/4]^{1/2}}{n_e + z_{\alpha/2}^2}, \tag{11}$$

respectively.

To control the half widths of the Wald CI and Wilson CI no larger than ω with probability $1-\beta$, the desired sample sizes should satisfy

$$Pr(z_{\alpha/2}[\frac{\hat{\pi}_{EM}(1-\hat{\pi}_{EM})}{n_e}+(\frac{1}{n_e}+\frac{1}{n_c})\hat{\sigma}_x^2]^{1/2}\leq\omega)\geq 1-\beta \tag{12}$$

and

$$Pr(\frac{z_{\alpha/2}[n_e\hat{\pi}_{EM}(1-\hat{\pi}_{EM})+(1+\frac{n_e}{n_c})(n_e+z_{\alpha/2}^2)\hat{\sigma}_x^2+z_{\alpha/2}^2/4]^{1/2}}{n_e+z_{\alpha/2}^2}\leq\omega)\geq 1-\beta, \tag{13}$$

respectively.

To simplify the calculation, we just consider the balanced survey design (i.e. $\frac{n_e}{n}=0.5$ and $N=n_c+n_e$). The approximate sample size N that is required to achieve the desired probability of $1-\beta$ at level α can be obtained by solving Equation (12) and (13), respectively. However, no closed forms exist. Hence, the following algorithm is developed to find the solutions.

Algorithm 1 Algorithm of Sample Size Determination

Require: $N,\pi,\lambda,\omega,K,\beta$

Step 1: Generate K random samples $\boldsymbol{m}=\{x_1,\ldots,x_{n_c};z_1,\ldots,z_{n_e}\}$ for given N, π, ω, τ (for Poisson ICT) or (r,p) (for Negative Binomial ICT).

Step 2: Approximate the half-widths and the assurance probabilities given in (10)-(11) and (12)-(13) based on the data generated in Step 1.

for $k=1$ to K **do**

Approximate the half-width as $\omega_k(N)$

end for

Approximate the probability as $p^*(N)=Pr\left(|\omega_k(N)-\omega|\leq 0.001\right)$

Step 3: Repeat Steps 1 and 2 via increase (or decrease) N by Bisection method if the approximate probability $p^*(N)$ is less (or greater) than $1-\beta$.

Step 4: Repeat Step 3 until the approximate probability $p^*(N)$ is close to $1-\beta$, i.e., $N=\min\{N:|p^*(N)-(1-\beta)|\leq 0.001\}$. The resulting N is the approximate sample size.

The approximate sample sizes based on Wald CI and Wilson CI obtained by the above algorithm are denoted as $\boldsymbol{n_W}$ and $\boldsymbol{n_{Wi}}$, respectively. When $\beta=0.5$, the corresponding sample sizes are denoted as $\boldsymbol{n_{W,0.5}}$ and $\boldsymbol{n_{Wi,0.5}}$, respectively.

4 Simulation Study

In this section, we evaluate the proposed methods for sample size determination via simulation studies. We consider different parameter settings at the confidence level $1-\alpha=0.95$ and assurance probability $1-\beta=0.95$ or 0.50 for the four non-randomized models. Ulrich *et al.*([22]) considered the parameter settings $p=0.3,0.6,0.8$ for the randomization probability p to assess the statistical power of randomized response models (i.e., Warner's model, unrelated question model). Similar to Ulrich *et al.*([22]), we also consider the same settings for the probability of non-sensitive binary attribute (i.e., p) for Crosswise and Parallel Models. And we consider the following parameter settings for different models:

(a) The Crosswise Model: (i) $p=0.3,0.6,0.8$; (ii) $\pi=0.04(0.04)0.16$; (iii) $\omega=25\%$ or 50% of π; i.e., a total of $3\times4\times2=24$ parameter combinations.

(b) The Parallel Model: $p = 0.75$ and (i) $q = 0.2(0.3)0.8$; (ii) $\pi = 0.04(0.04)0.16$; (iii) $\omega = 25\%$ or 50% of π; i.e., a total of $3 \times 4 \times 2 = 24$ parameter combinations.

(c) The Poisson ICT Model: (i) $\tau = 2, 3, 4$; (ii) $\pi = 0.04(0.04)0.16$; (iii) $\omega = 25\%$ or 50% of π; i.e., a total of $3 \times 4 \times 2 = 24$ parameter combinations.

(d) The Negative Binomial ICT Model: $r = 2$ and (i) $p = 0.6, 0.7, 0.8$; (ii) $\pi = 0.04(0.04)0.16$; (iii) $\omega = 25\%$ or 50% of π; i.e., a total of $3 \times 4 \times 2 = 24$ parameter combinations.

According to the approximate sample formulas or iterative algorithms given in Section 2, the estimated sample sizes can be obtained. Based on the estimated sample sizes, we can evaluate the performance of the estimated sample sizes by using the empirical coverage probability (ECP), empirical assurance probability (EAP), left non-coverage probability (LNCP) and right non-coverage probability (RNCP) of a $100(1-\alpha)\%$ CI for π. The confidence level is $1-\alpha = 0.95$ and the number of replications is set to $\mathrm{K} = 4000$. These indices providing an assessment of the precision of the sample size formulas are given by

(i) Empirical Assurance Probability (EAP)

$$\mathrm{EAP} = \frac{1}{K}\sum_{k=1}^{K} I\left(\pi_u^{(k)} - \pi_l^{(k)} \leq \omega\right),$$

where $\left(\pi_l^{(k)}, \pi_u^{(k)}\right)$ is the CI for π at the kth replication, and $I(\cdot)$ is the indicator function of the event that $\pi_u^{(k)} - \pi_l^{(k)} \leq \omega$.

(ii) Empirical Coverage Probability (ECP)

$$\mathrm{ECP} = \frac{1}{K}\sum_{k=1}^{K} I\left(\pi \in \left(\pi_l^{(k)}, \pi_u^{(k)}\right)\right).$$

(iii) Left and Right Non-coverage Probability (LNCP and RNCP)

$$\mathrm{LNCP} = \frac{1}{K}\sum_{k=1}^{K} I\left(\pi \leq \pi_l^{(k)}\right), \ \mathrm{RNCP} \quad = \frac{1}{K}\sum_{k=1}^{K} I\left(\pi \geq \pi_u^{(k)}\right)$$

Simulation results for the Crosswise Model, Parallel Model, PICT and NBICT are reported in Tables 3-6, respectively.

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Insert Table 3-6 about here

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According to Tables 3-6, all sample size determination methods for each model perform well in the sense that all CIs produce satisfactory empirical coverage probabilities and empirical assurance probabilities, and ~~have balance left and right non-coverage probabilities~~ based on the estimated sample sizes.

The simulation studies described above rely on the assumption that the expected prevalence is equal to the actual prevalence. In fact, we do not know the true prevalence before we conduct the trials. Therefore, we investigate the performance of the proposed methods when the expected prevalence is different from the true prevalence. We consider the following parameter settings: true prevalence $\pi = 0.165$, the expected prevalence $\pi_e = r\pi$ with $r = 0.40, 0.60, 0.80, 1.20$, and the half-widths of CI $\omega = 0.05, 0.10$. For other parameters of each models, we consider: (i) Crosswise Model: $p = 0.3$; (ii) Parallel Model: $p = 0.7$ and $q = 0.5$; (iii) Poisson ICT Model: $\tau = 3$; (iv)Negative Binomial ICT Model: $p = 0.7$. The confidence level $1-\alpha = 0.95$ and the assurance probability $1-\beta = 0.95$. Simulation results are reported in Table 7.

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Insert Table 7 about here

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According to Table 7, ~~we have the following observations:~~ (i) ECPs of all CIs are very close to the pre-specified confidence level for each model; (ii) ~~almost all CIs have satisfactory interval locations because the balanced left- and right- tailed errors~~; (iii) ECPs of the CIs may be influenced by the difference between the expected and true prevalence rates, i.e., the ECP of the CI decreases when the difference between the expected and true prevalence rates increases ; (iv) ECP of the CI is also influenced by the expected half-width of the interval (i.e., ω), i.e., the ECP of the CI increases with the increasing of the half-width.

5 Numerical Example

To illustrate the practicality and effectiveness of the proposed methods, we examine a study on premarital sexual practices among adolescents (refer to Bogale & Seme [25]). In this scenario, an AIDS researcher collects survey data to evaluate premarital sexual practices among in-school youths. The researcher estimates that approximately 19% of adolescents have had premarital sexual intercourse (i.e., $\pi = 0.19$). We then compute the required sample size for a new study, aiming for a 95% chance (i.e., $\beta = 0.05$) that the half-width of the 95% confidence interval (i.e., $\alpha = 0.05$) is no greater than 25% of the point estimate (i.e., $\omega = 0.25\pi$), considering various models discussed in this article.

5.1 Crosswise Model

Under the Crosswise Model, let us assume that two independent binary classification questions Y and W are considered, where the sensitive issue (Y) is *"Have you ever had sexual intercourse?"* and the non-sensitive issue (W) is "*Is the first digit of your house number 1, 2, 3, 4 or 5?*". The probability of the sensitive attribute is $\pi = 0.19$ and the probability of the non-sensitive issue is $p = 0.778$. Based on the formulas given in Section 2, the approximate sample size $\boldsymbol{n}_W = 1255$ based on Wald CI and $\boldsymbol{n}_{Wi} = 1251$ based on Wilson CI, respectively. The corresponding ECPs (EAPs) are 95.5% (96.6%) and 95.23% (95.84%) for Wald and Wilson methods, respectively. In contrast, for the conventional sample sizes (i.e., the assurance probability $1-\beta = 50\%$) required for a two-sided 95% confidence interval with expected width $\omega = 0.25\pi$ are $\boldsymbol{n}_{W,0.5} = 1213$ and $\boldsymbol{n}_{Wi,0.5} = 1210$, the corresponding ECPs (EAPs) are 95.42% (50.16%) and 94.97% (49.89%) for Wald and Wilson method, respectively.

5.2 Parallel Model

Under the Parallel Model, let us assume that three mutually independent binary classification questions Y, W and U are considered, where the sensitive issue (Y) is *"Have you ever had sexual intercourse?"*, and the non-sensitive issues (W) are *"Is the first digit of your house number 1, 2, 3, 4 or 5?"* and (U) *"Is the last digit cell phone number of your's is odd?"* (i.e., $p = 0.778, q = 0.5$). With $\pi = 0.19$, the approximate sample size $\boldsymbol{n}_W = 581$ based on Wald CI and $\boldsymbol{n}_{Wi} = 578$ based on Wilson CI, respectively. The corresponding ECPs (EAPs) are 94.38% (96.23%) and 95.21% (96.25%) for Wald and Wilson CIs, respectively. In contrast, for the conventional sample sizes (i.e., the assurance probability $1-\beta = 50\%$) required for a two-sided 95% confidence interval with expected width $\omega = 0.25\pi$ are $\boldsymbol{n}_{W,0.5} = 540$ and $\boldsymbol{n}_{Wi,0.5} = 537$, the corresponding ECPs (EAPs) are 94.79% (48.68%) and 94.68% (52.71%) for Wald and Wilson method, respectively.

5.3 Poisson ICT Model

Under Poisson ICT, let us assume that the researcher uses a neutral question that follows a Poisson distribution with parameter $\tau = 2$, for example, *"How many times did you travel abroad last year?"*. The number of respondents in control group is the same as that in experimental group, i.e., $n_c = n_e = \frac{1}{2}N$. With $\pi = 0.19$, the approximate sample size $\boldsymbol{n}_W = N = 14292$ based on Wald CI and $\boldsymbol{n}_{Wi} = N = 14275$ based on Wilson CI, respectively. The corresponding ECPs (EAPs) are 95.14% (94.38%) and 95.08% (94.18%) for Wald and Wilson methods, respectively. In contrast, the conventional sample sizes (i.e., the assurance probability $1-\beta = 50\%$) required for a two-sided 95% confidence interval only with expected width $\omega = 0.25\pi$ are $\boldsymbol{n}_{W,0.5} = N = 14148$ and $\boldsymbol{n}_{Wi,0.5} = N = 14129$, the corresponding ECPs (EAPs) are 94.83% (50.61%) and 95.09% (51.03%) for Wald and Wilson CIs, respectively.

5.4 Negative Binomial ICT

Under Negative Binomial ICT, let us assume that the researcher uses a neutral question that follows a negative binomial distribution with parameters $r = 2$ and $p = 0.7$, for example, *"How many online resumes do you need to submit to receive one interview invitation?"*. The number of respondents in the control group is the same as that in experimental group, i.e., $n_c = n_e = \frac{1}{2}N$. With $\pi = 0.19$, the approximate sample size $\boldsymbol{n}_W = N = 9325$ based on Wald CI and $\boldsymbol{n}_{Wi} = N = 9365$ based on Wilson CI, respectively. The corresponding ECPs (EAPs) are 95.22% (94.98%) and 94.92% (95.16%) for Wald and Wilson CIs, respectively. In contrast, the conventional sample sizes (i.e., the assurance probability $1-\beta = 50\%$) required for a two-sided 95% confidence interval only with expected width $\omega = 0.25\pi$ are $\boldsymbol{n}_{W,0.5} = N = 8875$ and $\boldsymbol{n}_{Wi,0.5} = N = 8836$; the corresponding ECPs (EAPs) are 95.14% (48.91%) and 94.82% (51.12%) for Wald and Wilson CIs, respectively.

It is worth noting that the recommended sample sizes based on the Crosswise Model, Poisson ICT Model, and Negative Binomial ICT are greater than the number of actual participants (i.e., 826), as recruited in the study by Bogale & Seme [25]. In contrast, the recommended sample size based on the Parallel Model is smaller than 826. With a sample size of 826 in the study by Bogale & Seme [25], the actual Empirical Coverage Probabilities ECPs, ECWs, and EAPs of various CIs for π under the considered parameter settings in the aforementioned studies for each model are reported in Table 8.

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Insert Table 8 about here

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According to our results, the ECPs of CIs for π under all models are very close to the pre-assigned nominal confidence level (i.e., 95%). However, the probabilities of controlling the half width of the CI such that it is not larger than $\omega = 0.25\pi = 0.0475$ are 0.0 for the Crosswise Model, Poisson ICT Model, and Negative Binomial ICT Model. Under the Parallel Model, however, the probability of controlling the half width of a CI that is not larger than $\omega = 0.25\pi$ is 1.0. In fact, the actual half widths of CIs for the Crosswise Model, Poisson ICT Model, and Negative Binomial ICT Model with a sample size of 826 are much greater than $\omega = 0.25\pi = 0.0475$, but that of the CI for the Parallel Model is less than $\omega = 0.25\pi = 0.0475$. Specifically, our findings suggest that when the assurance probability is not incorporated into the sample size estimation, the width of CIs cannot be controlled within the specified width, even if the coverage probability is close to the nominal confidence level.

6 Summary and Discussion

The determination of sample size is a critical aspect of research, particularly when investigating the prevalence of sensitive attributes through surveys. Within the context of survey sampling, determining sample size based on interval estimation is a fundamental objective. This study focuses on sample size determination using interval width control, specifically considering two types of confidence intervals (CIs): Wald CIs and Wilson CIs. The analysis encompasses four distinct non-randomized response models, i.e., the Crosswise Model, Parallel Model, Poisson ICT Model, and Negative Binomial ICT Model. The derived sample size formulas aim to control the width of a confidence interval at a specified confidence level, with an assurance probability of achieving the predetermined precision. Simulation results demonstrate the accuracy and effectiveness of all formulated algorithms based on Wald and Wilson CIs, as evidenced by empirical coverage probability (ECP) and empirical assurance probability (EAP). Notably, sample size formulas/algorithms based on Wilson CIs outperform their Wald CI counterparts across various non-randomized response models, with the former exhibiting ECPs and EAPs closer to the pre-specified levels. The sample size formulas/algorithms presented in this study can assist researchers in determining a sample size that achieves a pre-specified precision with a given assurance probability in survey studies aimed at detecting meaningful prevalence rates. ~~The numerical examples are provided in Section 4, concerning premarital sexual practices among in-school youths, offer~~ clear illustrations of how to estimate the required sample size through interval width control in the preliminary stages of a survey before embarking on a full study.

In the domain of sample size determination, two predominant methodologies are commonly utilized: hypothesis testing and confidence interval estimation. The former involves considerations of both the Type I error rate and power, while the latter does not explicitly involve power. To ensure that sample size estimation based on expected confidence interval width provides high assurance in achieving the desired precision, we incorporate an assurance probability into the sample size determination process, aiming to control the width of a confidence interval. In other words, sample size can be estimated by controlling the width of a confidence interval at a specified assurance probability. While the four non-random response models addressed in this study have previously been examined in terms of sample size determination from a power perspective, sample size formulas based on confidence interval width are not currently available in the existing literature. It is noteworthy that the non-randomized response models covered in this paper exclusively focus on designs for a single dichotomous sensitive attribute. However, in many randomized response applications, more than one sensitive question is asked. For instance, Sayed *et al.*[32] developed a non-saturated multinomial model for the analysis of randomized response "ever" and "last year" questions. Determining sample size for more than one sensitive question could be an interesting area for future research. In this article, we have also developed R codes to compute the estimated sample sizes, which are made available to readers in the online supplementary material.

Appendix

Proof of the asymptotic distributions of $\hat{a}_\lambda$ and $\hat{a}_\lambda^2$

Let $\hat{a}_\lambda = f(\hat{\lambda}) = \sqrt{\hat{\lambda}(1-\hat{\lambda})}$, and we expand it at $\hat{\lambda} = \lambda$ by using Taylor expansion formula to obtain its first-order approximation as follows.

$$\hat{a}_\lambda = f(\hat{\lambda}) \approx \sqrt{\lambda(1-\lambda)} + \frac{1}{2}[\lambda(1-\lambda)]^{-1/2}(1-2\lambda)(\hat{\lambda}-\lambda) = a_\lambda + \frac{1}{2}a_\lambda^{-1}(1-2\lambda)(\hat{\lambda}-\lambda).$$

Thus, it's expectation is $E(\hat{a}_\lambda) \approx a_\lambda$ and it's variance is $\text{Var}[\hat{a}_\lambda] \approx \frac{(1-2\lambda)^2}{4a_\lambda^2}\text{Var}(\hat{\lambda})$. Since that $\text{Var}(\hat{\lambda}) = a_\lambda^2/n$, then $\text{Var}(\hat{a}_\lambda) = \frac{(1-2\lambda)^2}{4n} = \frac{b_\lambda^2}{n}$. According to the Large Sample Theory, we have

$$\hat{a}_\lambda \sim N\left(a_\lambda, \frac{b_\lambda^2}{n}\right).$$

Similarly, expanding $\hat{a}_\lambda^2$ at $\hat{\lambda} = \lambda$ by using Taylor expansion formula, we have $\hat{a}_\lambda^2 \approx a_\lambda^2 + (1-2\lambda)(\hat{\lambda}-\lambda)$. Thus, $E(\hat{a}_\lambda^2) \approx a_\lambda^2$, $\text{Var}(\hat{a}_\lambda^2) \approx (1-2\lambda)^2\,\text{Var}(\hat{\lambda}) = 4a_\lambda^2 b_\lambda^2/n$. Therefore, we have

$$\hat{a}_\lambda^2 \sim N\left(a_\lambda^2, 4a_\lambda^2 b_\lambda^2/n\right).$$

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Table 3. Performance of the sample size formula with $1-\beta$ assurance probability for Crosswise Model under (i) $\beta = 0.5$ and (ii) $\beta = 0.05$ with $p = 0.3, 0.6, 0.8$.

π	ω^b	Wald			Wilson		
		$n^\dagger$	$ECP(L,R)\%^\ddagger$	EAP	n	$ECP(L,R)\%$	EAP
		(i) $\beta = 0.5$					
		$p = 0.3$					
0.04	25	51894	95.07 (2.36 , 2.57)	49.76	51891	95.26 (2.46 , 2.28)	50.07
	50	12974	95.65 (2.11 , 2.24)	49.94	12970	95.45 (2.32 , 2.23)	49.60
0.08	25	13312	94.74 (2.68 , 2.58)	50.93	13308	95.13 (2.56 , 2.31)	50.44
	50	3328	94.75 (2.44 , 2.81)	49.23	3325	94.97 (2.47 , 2.56)	51.29
0.12	25	6053	95.22 (2.50 , 2.28)	49.93	6049	95.07 (2.67 , 2.26)	49.57
	50	1513	94.84 (2.50 , 2.66)	50.02	1510	94.85 (2.59 , 2.56)	50.72
0.16	25	3474	95.20 (2.46 , 2.34)	50.59	3470	94.86 (2.58 , 2.56)	49.66
	50	868	95.23 (2.20 , 2.57)	48.68	865	94.94 (2.56 , 2.50)	49.01
		$p = 0.6$					
0.04	25	231963	94.75 (2.61 , 2.64)	49.96	231959	95.07 (2.42 , 2.51)	50.68
	50	57991	94.83 (2.42 , 2.75)	50.09	57987	94.65 (2.70 , 2.65)	49.44
0.08	25	58329	94.97 (2.42 , 2.61)	50.58	58325	95.00 (2.55 , 2.45)	50.33
	50	14582	95.38 (2.22 , 2.40)	49.73	14578	95.24 (2.41 , 2.35)	49.27
0.12	25	26060	95.17 (2.40 , 2.43)	49.29	26057	95.00 (2.60 , 2.40)	49.86
	50	6515	94.86 (2.17 , 2.97)	49.72	6511	94.91 (2.52 , 2.57)	50.11
0.16	25	14728	94.92 (2.61 , 2.47)	50.55	14724	94.99 (2.50 , 2.51)	50.16
	50	3682	94.74 (2.67 , 2.59)	50.39	3678	94.60 (2.49 , 2.91)	49.36
		$p = 0.8$					
0.04	25	18548	94.96 (2.43 , 2.61)	49.14	18546	95.02 (2.57 , 2.41)	49.67
	50	4637	95.18 (2.33 , 2.49)	49.00	4635	95.15 (2.39 , 2.46)	50.47
0.08	25	4975	95.04 (2.32 , 2.64)	49.72	4973	94.74 (2.81 , 2.45)	50.49
	50	1244	95.21 (2.02 , 2.77)	50.44	1241	95.32 (2.62 , 2.06)	48.98
0.12	25	2348	94.74 (2.51 , 2.75)	49.47	2345	95.30 (2.35 , 2.35)	49.66
	50	587	94.84 (2.33 , 2.83)	49.76	584	94.84 (2.86 , 2.30)	49.22
0.16	25	1390	95.12 (2.17 , 2.71)	51.14	1387	94.59 (3.01 , 2.40)	50.23
	50	347	95.48 (1.85 , 2.67)	49.82	344	94.93 (2.63 , 2.44)	47.98
		(ii) $\beta = 0.05$					
		$p = 0.3$					
0.04	25	52190	95.08 (2.50 , 2.42)	94.80	52187	94.87 (2.46 , 2.67)	95.18
	50	13121	95.07 (2.15 , 2.78)	95.67	13118	94.93 (2.79 , 2.28)	95.12
0.08	25	13447	94.82 (2.41 , 2.77)	95.52	13443	95.08 (2.48 , 2.44)	95.57
	50	3395	94.55 (2.49 , 2.96)	95.91	3392	94.85 (2.76 , 2.39)	95.59
0.12	25	6134	95.08 (2.24 , 2.68)	95.78	6131	95.10 (2.59 , 2.31)	95.35
	50	1554	95.43 (2.09 , 2.48)	96.56	1550	95.69 (2.11 , 2.20)	96.37
0.16	25	3528	95.13 (2.42 , 2.45)	95.92	3525	94.56 (2.50 , 2.94)	95.77
	50	896	94.79 (2.48 , 2.73)	97.04	892	94.73 (3.07 , 2.20)	96.51
		$p = 0.6$					
0.04	25	232259	95.16 (2.48 , 2.36)	95.35	232255	95.54 (2.08 , 2.38)	95.57
	50	58139	95.13 (2.51 , 2.36)	95.45	58135	95.20 (2.39 , 2.41)	95.27
0.08	25	58464	94.95 (2.75 , 2.30)	95.09	58460	95.09 (2.37 , 2.54)	95.52
	50	14650	95.03 (2.32 , 2.65)	96.09	14646	95.21 (2.20 , 2.59)	96.15
0.12	25	26142	95.62 (2.21 , 2.17)	95.93	26138	95.41 (2.31 , 2.28)	95.53
	50	6556	95.08 (2.53 , 2.39)	95.96	6552	94.80 (2.64 , 2.56)	96.12
0.16	25	14783	94.82 (2.69 , 2.49)	95.85	14779	95.19 (2.59 , 2.22)	95.70
	50	3709	95.03 (2.36 , 2.61)	96.81	3706	95.45 (2.19 , 2.36)	97.14
		$p = 0.8$					
0.04	25	18844	94.85 (2.37 , 2.78)	95.24	18840	95.44 (2.20 , 2.36)	95.50
	50	4784	94.92 (2.30 , 2.78)	95.69	4781	94.85 (2.52 , 2.63)	95.31
0.08	25	5110	95.13 (2.51 , 2.36)	95.25	5106	94.93 (2.74 , 2.33)	95.29
	50	1311	95.13 (2.07 , 2.80)	95.89	1307	95.32 (2.50 , 2.18)	95.46
0.12	25	2429	95.19 (2.24 , 2.57)	95.98	2425	94.72 (2.85 , 2.43)	94.90
	50	627	94.99 (2.09 , 2.92)	96.39	623	94.86 (2.72 , 2.42)	95.30
0.16	25	1444	95.07 (2.21 , 2.72)	95.73	1440	95.02 (2.41 , 2.57)	95.74
	50	374	94.36 (2.38 , 3.26)	96.99	371	95.27 (2.49 , 2.24)	96.83

^{b} Half width (i.e., ω) of a CI as given by the value of π, i.e., 25% and 50% of π.
$\dagger n$ denotes the estimated sample size; $\ddagger(L,R)$ denotes (LNCP, RNCP).

Table 4. Performance of the sample size formula with $1-\beta$ assurance probability for Parallel Model under (i) $\beta = 0.5$ and (ii) $\beta = 0.05$ with $q = 0.2, 0.5, 0.8$.

		Wald			Wilson		
π	ω	n	$ECP(L,R)\%$	EAP	n	$ECP(L,R)\%$	EAP
		(i) $\beta = 0.5$					
		$q = 0.2$					
0.04	25	5026	95.22 (2.10 , 2.68)	51.82	5032	94.96 (2.74 , 2.30)	50.58
	50	1257	94.67 (1.67 , 3.66)	49.90	1262	94.77 (2.96 , 2.27)	48.82
0.08	25	1671	94.62 (2.03 , 3.35)	49.30	1674	95.02 (2.42 , 2.56)	50.34
	50	418	95.08 (1.36 , 3.56)	47.79	420	94.91 (2.89 , 2.20)	53.74
0.12	25	914	94.81 (2.23 , 2.96)	52.10	914	94.92 (2.66 , 2.42)	49.35
	50	228	93.66 (1.71 , 4.63)	47.51	229	95.61 (2.49 , 1.90)	54.78
0.16	25	602	95.05 (1.92 , 3.03)	51.10	601	94.69 (2.56 , 2.75)	51.73
	50	151	94.30 (2.04 , 3.66)	49.40	150	95.21 (2.87 , 1.92)	51.31
		$q = 0.5$					
0.04	25	8945	94.70 (2.75 , 2.55)	49.27	8944	94.64 (2.78 , 2.58)	50.16
	50	2236	94.73 (1.97 , 3.30)	49.96	2236	95.09 (2.26 , 2.65)	50.99
0.08	25	2574	94.93 (2.52 , 2.55)	51.15	2573	95.09 (2.58 , 2.33)	50.75
	50	644	94.20 (2.21 , 3.59)	51.32	642	95.04 (2.64 , 2.32)	49.30
0.12	25	1281	95.29 (2.15 , 2.56)	50.69	1279	94.54 (3.02 , 2.44)	50.97
	50	320	95.02 (2.02 , 2.96)	48.63	318	94.44 (2.76 , 2.80)	51.54
0.16	25	790	95.50 (1.93 , 2.57)	50.34	787	94.66 (2.59 , 2.75)	49.00
	50	197	94.67 (1.95 , 3.38)	51.94	195	94.71 (2.48 , 2.81)	49.34
		$q = 0.8$					
0.04	25	12095	95.19 (2.35 , 2.46)	49.07	12092	95.14 (2.30 , 2.56)	50.88
	50	3024	94.61 (2.40 , 2.99)	50.26	3021	95.24 (2.46 , 2.30)	49.02
0.08	25	3285	95.19 (2.21 , 2.60)	50.46	3282	94.86 (2.42 , 2.72)	51.11
	50	821	94.69 (2.56 , 2.75)	50.04	819	94.83 (2.61 , 2.56)	51.92
0.12	25	1562	95.06 (2.41 , 2.53)	49.27	1559	94.97 (2.60 , 2.43)	48.91
	50	391	94.96 (2.14 , 2.90)	50.89	388	94.99 (2.32 , 2.69)	50.31
0.16	25	929	94.44 (2.56 , 3.00)	51.36	926	95.15 (2.42 , 2.43)	51.57
	50	232	94.27 (2.48 , 3.25)	51.81	229	95.16 (2.31 , 2.53)	51.91
		(ii) $\beta = 0.05$					
		$q = 0.2$					
0.04	25	5381	95.19 (1.85 , 2.96)	95.32	5380	95.07 (2.43 , 2.50)	95.10
	50	1431	95.61 (1.52 , 2.87)	96.16	1430	94.84 (2.88 , 2.28)	94.85
0.08	25	1835	94.34 (2.42 , 3.24)	94.95	1833	94.51 (3.04 , 2.45)	95.14
	50	498	94.55 (2.03 , 3.42)	96.23	496	95.35 (2.55 , 2.10)	95.32
0.12	25	1014	94.72 (2.23 , 3.05)	95.89	1011	95.64 (2.44 , 1.92)	95.51
	50	278	95.29 (1.67 , 3.04)	96.39	275	94.30 (3.21 , 2.49)	95.59
0.16	25	671	94.69 (1.98 , 3.33)	96.15	668	94.74 (2.89 , 2.37)	95.52
	50	184	94.53 (1.52 , 3.95)	96.25	181	95.19 (3.33 , 1.48)	95.22
		$q = 0.5$					
0.04	25	9239	94.84 (2.50 , 2.66)	95.09	9236	94.70 (2.65 , 2.65)	95.06
	50	2382	95.22 (1.92 , 2.86)	95.53	2379	94.66 (2.68 , 2.66)	95.38
0.08	25	2708	94.77 (2.39 , 2.84)	95.35	2705	95.28 (2.39 , 2.33)	95.48
	50	710	94.76 (2.17 , 3.07)	95.86	707	95.45 (2.44 , 2.11)	95.46
0.12	25	1361	94.79 (2.11 , 3.10)	95.76	1358	95.05 (2.53 , 2.42)	95.32
	50	360	93.54 (2.50 , 3.96)	95.87	357	94.73 (2.91 , 2.36)	96.10
0.16	25	843	94.48 (2.22 , 3.30)	96.14	840	95.02 (2.61 , 2.37)	95.70
	50	224	94.11 (2.01 , 3.88)	97.10	221	95.14 (2.65 , 2.21)	97.35
		$q = 0.8$					
0.04	25	12326	95.48 (2.14 , 2.38)	95.44	12322	94.91 (2.52 , 2.57)	95.27
	50	3139	94.95 (2.11 , 2.94)	95.76	3135	95.15 (2.47 , 2.38)	95.38
0.08	25	3387	94.85 (2.33 , 2.82)	95.18	3384	94.96 (2.75 , 2.29)	95.62
	50	872	95.02 (2.29 , 2.69)	96.09	869	94.88 (2.61 , 2.51)	95.77
0.12	25	1622	94.62 (2.45 , 2.93)	95.82	1618	94.89 (2.85 , 2.26)	95.23
	50	420	94.93 (2.28 , 2.79)	96.51	417	95.02 (2.44 , 2.54)	96.91
0.16	25	967	95.53 (2.00 , 2.47)	96.59	964	94.91 (2.65 , 2.44)	96.31
	50	251	95.06 (2.04 , 2.90)	97.22	248	94.64 (3.21 , 2.15)	97.69

Table 5. Performance of the sample size algorithm with $1-\beta$ assurance probability for Poisson ICT under (i) $\beta = 0.5$ and (ii) $\beta = 0.05$ with $\tau = 2,3,4$.

		Wald			Wilson		
π	ω	n	$ECP(L,R)\%$	EAP	n	$ECP(L,R)\%$	EAP
		(i) $\beta = 0.5$					
		$\tau = 2$					
0.04	25	310266	95.58 (1.93 , 2.50)	51.18	310280	95.13 (2.53 , 2.35)	50.80
	50	77560	94.78 (2.40 , 2.83)	49.83	77556	94.78 (2.53 , 2.70)	49.83
0.08	25	78234	95.35 (2.30 , 2.35)	49.48	78234	94.80 (2.33 , 2.88)	50.67
	50	19565	94.88 (2.58 , 2.55)	52.48	19548	94.85 (2.48 , 2.68)	49.48
0.12	25	35053	95.18 (2.43 , 2.40)	50.60	35039	94.78 (2.95 , 2.28)	49.40
	50	8763	95.63 (2.18 , 2.20)	52.80	8750	95.38 (2.18 , 2.45)	48.88
0.16	25	19851	94.95 (2.60 , 2.45)	50.10	19841	95.40 (2.35 , 2.25)	49.15
	50	4963	94.70 (2.50 , 2.80)	51.85	4955	94.28 (2.98 , 2.75)	52.38
		$\tau = 3$					
0.04	25	463912	94.33 (3.13 , 2.55)	49.73	463923	94.40 (2.90 , 2.70)	49.85
	50	115988	95.83 (1.75 , 2.43)	50.65	115984	94.33 (2.75 , 2.93)	51.79
0.08	25	116653	95.38 (2.48 , 2.15)	50.68	116658	94.73 (2.70 , 2.58)	51.48
	50	29158	95.40 (2.23 , 2.38)	49.83	29158	95.50 (2.43 , 2.08)	52.10
0.12	25	52114	94.68 (2.70 , 2.63)	49.05	52120	95.35 (2.30 , 2.35)	49.53
	50	13028	95.20 (2.53 , 2.28)	50.99	13018	95.33 (2.30 , 2.38)	49.48
0.16	25	29455	95.33 (2.48 , 2.20)	50.43	29447	95.15 (2.45 , 2.40)	50.50
	50	7361	95.10 (2.33 , 2.58)	49.45	7355	95.48 (2.30 , 2.23)	49.30
		$\tau = 4$					
0.04	25	617596	94.68 (2.88 , 2.45)	52.55	617570	95.08 (2.55 , 2.38)	49.28
	50	154404	95.43 (2.33 , 2.25)	51.83	154386	94.68 (2.93 , 2.40)	49.43
0.08	25	155070	94.90 (2.63 , 2.48)	50.70	155059	94.73 (2.70 , 2.58)	50.05
	50	38765	95.28 (2.23 , 2.50)	49.93	38753	95.53 (2.15 , 2.33)	48.60
0.12	25	69192	95.10 (2.63 , 2.28)	51.25	69185	95.08 (2.58 , 2.35)	49.48
	50	17297	95.40 (2.50 , 2.10)	51.88	17291	95.35 (2.35 , 2.30)	51.63
0.16	25	39062	95.05 (2.45 , 2.50)	51.70	39059	95.43 (2.73 , 1.85)	52.60
	50	9763	95.03 (2.63 , 2.35)	49.93	9755	95.23 (2.30 , 2.48)	50.65
		(ii) $\beta = 0.05$					
		$\tau = 2$					
0.04	25	310947	95.00 (2.85 , 2.15)	95.50	310884	94.80 (2.88 , 2.33)	94.45
	50	77886	95.00 (2.28 , 2.73)	94.85	77893	94.85 (2.93 , 2.23)	95.63
0.08	25	78604	94.73 (2.70 , 2.58)	95.85	78541	94.33 (2.50 , 3.18)	94.43
	50	19722	94.75 (2.73 , 2.53)	95.70	19703	95.03 (2.53 , 2.45)	94.10
0.12	25	35277	94.38 (2.80 , 2.83)	95.98	35252	95.05 (2.68 , 2.28)	95.20
	50	8872	94.88 (2.40 , 2.73)	95.70	8860	95.93 (2.18 , 1.90)	95.38
0.16	25	20019	94.05 (3.05 , 2.90)	95.40	20014	94.45 (2.85 , 2.70)	95.23
	50	5043	95.55 (2.05 , 2.40)	94.57	5032	95.30 (2.73 , 1.98)	94.33
		$\tau = 3$					
0.04	25	464611	94.85 (2.65 , 2.50)	96.02	464561	94.88 (2.43 , 2.70)	95.38
	50	116296	94.83 (2.63 , 2.55)	94.74	116310	95.15 (2.33 , 2.53)	96.45
0.08	25	116987	95.60 (2.00 , 2.40)	95.35	116976	94.83 (2.68 , 2.50)	95.30
	50	29313	95.38 (2.30 , 2.33)	94.15	29321	94.95 (2.63 , 2.43)	95.23
0.12	25	52335	95.23 (2.63 , 2.15)	94.18	52325	95.28 (2.40 , 2.33)	94.70
	50	13139	94.68 (2.70 , 2.63)	95.07	13134	95.33 (2.55 , 2.13)	95.74
0.16	25	29615	94.75 (2.73 , 2.53)	94.33	29619	95.38 (2.45 , 2.18)	95.73
	50	7445	94.70 (2.93 , 2.38)	95.73	7441	94.93 (2.25 , 2.83)	95.70
		$\tau = 4$					
0.04	25	618247	95.40 (2.33 , 2.28)	95.74	618222	94.50 (2.43 , 3.08)	94.93
	50	154708	94.65 (2.63 , 2.73)	94.54	154704	94.85 (2.58 , 2.58)	95.00
0.08	25	155388	94.65 (2.90 , 2.45)	94.43	155392	95.33 (2.33 , 2.35)	95.53
	50	38931	94.43 (2.63 , 2.95)	95.47	38922	94.58 (2.93 , 2.50)	95.18
0.12	25	69408	94.90 (2.70 , 2.40)	95.03	69404	95.50 (2.45 , 2.05)	95.28
	50	17400	94.83 (2.63 , 2.55)	94.13	17399	94.95 (2.68 , 2.38)	94.68
0.16	25	39219	95.03 (2.83 , 2.15)	94.80	39219	94.45 (2.88 , 2.68)	94.90
	50	9848	94.70 (2.70 , 2.60)	95.73	9837	95.68 (2.20 , 2.13)	95.18

Table 6. Performance of the sample size algorithm with $1-\beta$ assurance probability for Negative Binomial ICT under (i) $\beta = 0.5$ and (ii) $\beta = 0.05$ with $p = 0.6, 0.7, 0.8$.

		Wald			Wilson		
π	ω	n	$ECP(L,R)\%$	EAP	n	$ECP(L,R)\%$	EAP
		(i) $\beta = 0.5$					
		$p = 0.6$					
0.04	25	344424	94.90 (2.45 , 2.65)	50.45	344397	95.25 (2.25 , 2.50)	50.83
	50	86115	95.55 (2.10 , 2.35)	51.21	86097	95.25 (2.23 , 2.53)	50.03
0.08	25	86771	94.85 (2.85 , 2.30)	49.45	86793	95.00 (2.33 , 2.68)	52.35
	50	21702	95.10 (2.15 , 2.75)	51.30	21677	94.98 (2.73 , 2.30)	50.25
0.12	25	38847	95.13 (1.95 , 2.93)	51.40	38828	95.95 (2.08 , 1.98)	49.70
	50	9714	94.23 (2.48 , 3.30)	51.93	9708	95.23 (2.50 , 2.28)	50.83
0.16	25	21985	94.95 (2.35 , 2.70)	51.50	21996	95.40 (2.23 , 2.38)	51.48
	50	5500	95.23 (2.33 , 2.45)	51.60	5494	95.18 (2.33 , 2.50)	51.48
		$p = 0.7$					
0.04	25	191114	94.95 (2.65 , 2.40)	50.78	191090	95.40 (2.25 , 2.35)	51.25
	50	47785	95.10 (2.35 , 2.55)	51.75	47796	94.80 (2.53 , 2.68)	51.35
0.08	25	48461	94.75 (2.70 , 2.55)	50.53	48451	95.75 (2.48 , 1.78)	50.13
	50	12103	95.48 (2.25 , 2.28)	48.78	12105	95.43 (2.68 , 1.90)	49.13
0.12	25	21794	95.25 (2.75 , 2.00)	48.03	21795	94.80 (2.40 , 2.80)	49.08
	50	5450	95.10 (2.20 , 2.70)	49.50	5451	94.88 (2.63 , 2.50)	51.43
0.16	25	12402	94.68 (2.80 , 2.53)	49.60	12393	95.00 (2.25 , 2.75)	49.33
	50	3106	95.05 (2.25 , 2.70)	51.28	3089	95.03 (2.50 , 2.48)	49.76
		$p = 0.8$					
0.04	25	99009	94.50 (2.38 , 3.13)	48.83	98971	95.55 (2.18 , 2.28)	48.98
	50	24747	94.90 (2.55 , 2.55)	50.83	24745	95.10 (2.68 , 2.23)	50.06
0.08	25	25425	94.60 (2.78 , 2.63)	49.75	25408	94.55 (2.70 , 2.75)	49.45
	50	6360	95.10 (2.58 , 2.33)	51.03	6351	95.13 (2.55 , 2.33)	50.57
0.12	25	11584	95.35 (2.60 , 2.05)	51.88	11554	94.88 (2.45 , 2.68)	48.08
	50	2894	94.73 (2.40 , 2.88)	51.73	2882	95.15 (2.70 , 2.15)	49.20
0.16	25	6648	95.03 (2.48 , 2.50)	50.90	6638	94.20 (3.15 , 2.65)	50.76
	50	1657	94.75 (2.40 , 2.85)	49.75	1654	94.98 (2.70 , 2.33)	50.41
		(ii) $\beta = 0.05$					
		$p = 0.6$					
0.04	25	346752	95.05 (2.50 , 2.45)	94.48	346700	94.55 (2.88 , 2.58)	95.81
	50	87125	94.80 (2.63 , 2.58)	95.38	87214	94.95 (2.25 , 2.80)	95.37
0.08	25	87841	94.65 (2.78 , 2.58)	95.08	87943	94.58 (2.55 , 2.88)	95.40
	50	22221	95.48 (2.15 , 2.38)	95.28	22201	95.35 (2.43 , 2.23)	95.43
0.12	25	39521	94.90 (2.65 , 2.45)	95.15	39622	95.45 (2.15 , 2.40)	94.98
	50	10084	95.58 (2.28 , 2.15)	95.18	10077	95.73 (2.00 , 2.28)	95.56
0.16	25	22551	94.98 (2.38 , 2.65)	95.93	22535	94.53 (2.80 , 2.68)	95.47
	50	5779	95.38 (2.45 , 2.18)	95.85	5773	95.23 (2.55 , 2.23)	95.98
		$p = 0.7$					
0.04	25	192699	94.35 (2.78 , 2.88)	96.38	192892	95.10 (2.33 , 2.58)	95.09
	50	48570	95.25 (2.40 , 2.35)	95.90	48718	95.30 (2.35 , 2.35)	94.28
0.08	25	49332	95.03 (2.50 , 2.48)	95.13	49381	95.18 (2.55 , 2.28)	95.43
	50	12562	94.85 (2.68 , 2.48)	95.30	12584	95.05 (2.53 , 2.43)	95.80
0.12	25	22370	95.10 (2.50 , 2.40)	95.04	22330	95.38 (2.33 , 2.30)	94.95
	50	5769	94.40 (2.83 , 2.78)	94.20	5748	94.73 (2.68 , 2.60)	96.22
0.16	25	12868	94.50 (2.55 , 2.95)	95.10	12851	95.43 (2.38 , 2.20)	94.98
	50	3310	94.93 (2.50 , 2.58)	95.65	3298	94.50 (2.98 , 2.53)	95.01
		$p = 0.8$					
0.04	25	100552	94.65 (2.70 , 2.65)	95.88	100509	95.08 (2.20 , 2.73)	95.53
	50	25488	95.45 (2.10 , 2.45)	94.93	25541	95.25 (2.15 , 2.60)	94.50
0.08	25	26148	94.88 (2.63 , 2.50)	94.43	25999	95.23 (2.55 , 2.23)	94.83
	50	6722	95.55 (2.00 , 2.45)	95.87	6697	94.65 (2.70 , 2.65)	95.01
0.12	25	12017	95.38 (2.65 , 1.98)	95.13	12042	95.05 (2.65 , 2.30)	95.36
	50	3089	94.80 (2.28 , 2.93)	94.53	3123	95.08 (2.63 , 2.30)	96.00
0.16	25	7017	95.63 (2.13 , 2.25)	94.43	6999	94.80 (2.38 , 2.83)	95.82
	50	1816	94.95 (2.40 , 2.65)	95.78	1817	95.05 (2.45 , 2.50)	94.76

Table 7. Performance of the sample size formula with 95% assurance probability for various models when the expected prevalence (i.e., π_e) differs from the true prevalence (i.e., π).

		Wald			Wilson		
ω	π_e^c	n	$ECP(L,R)\%$	EAP	n	$ECP(L,R)\%$	EAP
		Crosswise Model					
0.05	0.40	2167	95.28 (2.38 , 2.35)	1.38	2163	94.60 (2.73 , 2.68)	1.60
	0.60	2205	94.53 (2.50 , 2.98)	19.63	2201	95.05 (2.83 , 2.13)	17.80
	0.80	2240	95.25 (2.53 , 2.23)	69.15	2236	95.23 (2.43 , 2.35)	67.53
	1.20	2300	94.33 (2.68 , 3.00)	99.98	2296	94.63 (2.48 , 2.90)	99.93
0.10	0.40	556	95.03 (2.28 , 2.70)	46.65	552	95.68 (2.03 , 2.30)	45.43
	0.60	564	95.73 (1.80 , 2.48)	71.35	560	94.95 (2.80 , 2.25)	68.68
	0.80	572	94.50 (2.48 , 3.03)	89.13	568	95.25 (2.40 , 2.35)	89.63
	1.20	585	95.10 (2.13 , 2.78)	99.65	581	95.20 (2.43 , 2.38)	99.55
		Parallel Model					
0.05	0.40	549	94.40 (2.18 , 3.43)	2.08	546	95.03 (2.58 , 2.40)	1.88
	0.60	587	94.33 (2.80 , 2.88)	19.03	584	95.05 (2.70 , 2.25)	18.25
	0.80	622	94.70 (2.38 , 2.93)	65.33	619	95.35 (2.70 , 1.95)	64.15
	1.20	682	94.73 (2.05 , 3.23)	99.88	679	95.53 (2.55 , 1.93)	99.90
0.10	0.40	150	94.65 (1.58 , 3.78)	41.38	147	94.70 (2.98 , 2.33)	41.25
	0.60	159	94.63 (1.93 , 3.45)	66.70	156	94.88 (2.83 , 2.30)	71.73
	0.80	167	94.30 (1.88 , 3.83)	88.95	164	94.75 (2.80 , 2.45)	89.43
	1.20	180	94.38 (1.83 , 3.80)	99.70	177	95.08 (2.65 , 2.28)	99.75
		Poisson ICT					
0.05	0.40	12612	94.75 (2.85 , 2.40)	10.78	12599	95.43 (2.50 , 2.08)	9.58
	0.60	12697	95.00 (2.68 , 2.33)	40.63	12678	95.10 (2.65 , 2.25)	37.78
	0.80	12772	96.10 (2.18 , 1.73)	76.75	12771	94.70 (2.55 , 2.75)	78.53
	1.20	12907	94.38 (3.10 , 2.53)	99.03	12907	94.95 (2.60 , 2.45)	99.23
0.10	0.40	3184	94.80 (2.70 , 2.50)	56.83	3177	95.03 (2.83 , 2.15)	57.75
	0.60	3204	94.95 (2.65 , 2.40)	75.05	3202	95.48 (2.33 , 2.20)	79.88
	0.80	3224	94.95 (2.48 , 2.58)	89.68	3214	94.58 (3.10 , 2.33)	87.70
	1.20	3260	94.55 (3.05 , 2.40)	98.23	3245	95.28 (2.40 , 2.33)	96.48
		Negative Binomial ICT					
0.05	0.40	8056	95.33 (2.20 , 2.48)	71.65	8042	94.88 (2.63 , 2.50)	69.75
	0.60	8113	94.98 (2.55 , 2.48)	79.05	8139	95.30 (2.18 , 2.53)	83.75
	0.80	8222	95.28 (2.35 , 2.38)	92.35	8188	94.85 (2.38 , 2.78)	88.58
	1.20	8359	95.50 (2.13 , 2.38)	97.55	8337	95.23 (2.40 , 2.38)	97.30
0.10	0.40	2104	94.60 (2.45 , 2.95)	87.50	2097	95.38 (2.30 , 2.33)	87.75
	0.60	2110	94.83 (2.28 , 2.90)	89.08	2110	95.30 (2.08 , 2.63)	90.48
	0.80	2154	94.88 (2.50 , 2.63)	94.45	2134	95.03 (2.65 , 2.33)	93.15
	1.20	2179	95.10 (2.15 , 2.75)	97.18	2166	95.28 (2.50 , 2.23)	96.90

^{c} Expected prevalence π_e is given by the values of π, i.e., $\pi_e = (0.4, 0.6, 0.8, 1.2)\pi$.

Table 8. ECPs(%), ECWs and EAPs(%) of CIs for π under various models with the actual sample size (i.e.,826 participants) for the study of pre-marital sexual practices among adolescents.

	Wald		Wilson	
Model	$ECP(ECW)$	EAP	$ECP(ECW)$	EAP
Crosswise Model	95.58(0.129)	0	94.95(0.128)	0
Parallel Model	94.82(0.083)	1	94.96(0.081)	1
Poisson ICT	95.03(0.279)	0	94.48(0.277)	0
Negative Binomial ICT	95.58(0.223)	0	95.40(0.223)	0