On the Complexity of Reachability and Mortality for Bounded Piecewise Affine Maps

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Abstract. Reachability is a fundamental decision problem that arises across various domains, including program analysis, computational models like cellular automata, and finite- and infinite-state concurrent systems. Mortality, closely related to reachability, is another critical decision problem.

This study focuses on the computational complexity of the reachability and mortality problems for two-dimensional hierarchical piecewise constant derivative systems (2-HPCD) and one-dimensional piecewise affine maps (1-PAM). Specifically, we consider the bounded variants of 2-HPCD and 1-PAM, as they are proven to be equivalent regarding their reachability and mortality properties [3].

The proofs leverage the encoding of the simultaneous incongruences problem, a known NP-complete problem, into the reachability (alternatively, mortality) problem for 2-HPCD. The simultaneous incongruences problem has a solution if and only if the corresponding reachability (alternatively, mortality) problem for 2-HPCD does not. This establishes that the reachability and mortality problems are co-NP-hard for both bounded 2-HPCD and bounded 1-PAM.

Keywords: Reachability problem · Mortality problem · Complexity.

1 Introduction

Reachability is a fundamental problem that arises in various contexts, including program analysis, computational models like cellular automata, and finite- and infinite-state concurrent systems. It seeks to answer the question: given a computational system with a set of rules, can a certain state (or set of states) be reached from a given initial state (or set of states)? [1].

Another fundamental problem in the analysis of hybrid dynamical systems is the mortality problem: given a computational system, decide whether the system halts when starting from any state [6]. The mortality problem relates closely to stability properties and the long-term behaviour of trajectories within the system dynamics.

Neither the reachability problem nor the mortality problem is strictly more general than the other. Both problems are generally undecidable and are only known to be decidable for specific classes [9, 10].

Studying the computational complexity of both the reachability and mortality problems is crucial for establishing theoretical boundaries on the efficiency of underlying algorithms. To date, this aspect has received less attention compared to the fundamental question of decidability, although some research has been conducted in this direction [5, 6, 16].

Asarin et al. explored systems that straddle the boundary between decidability and undecidability [3]. They investigated variants of two-dimensional piecewise constant derivative systems (2-PCD) and demonstrated that certain variants are equivalent, in terms of reachability, to subclasses of one-dimensional piecewise affine maps (1-PAM).

In particular, Asarin et al. considered an extension of the 2-PCD model, known as a two-dimensional hierarchical PCD (2-HPCD), where discrete locations are organised hierarchically, with each location defined by a 2-PCD. Affine reset rules govern transitions between these locations. When all locations are bounded, the 2-HPCD is referred to as a bounded 2-HPCD (2-BHPCD). Similarly, if all intervals are bounded, 1-PAM is called bounded (1-BPAM).

This study explores the computational complexity of the reachability and mortality problems for 2-HPCD and 1-PAM, aiming to elucidate the theoretical boundaries of their computational hardness. It also addresses in a broader context the open question posed in [6] regarding the complexity of mortality in dimension two, specifically focusing on restricted 2-HPCD.

Our proofs are based on encoding the NP-complete simultaneous incongruences problem [7, 14] into the reachability and mortality problems for 2-BHPCD. We then extend these results to 1-BPAM. Note that, in contrast to [6], one of the challenges in this study is simulating the simultaneous incongruences problem using a 2-BHPCD, that is, using a model of a lower dimension. In [6], verifying whether the current k is a solution to the given instance of the simultaneous incongruences problem is achieved through the z-coordinate, which represents the current value of k in a restricted 3-HPCD.

It is notable that piecewise-affine models, together with reachability analysis, are applied in various domains, including gene-regulatory networks [15], biochemical kinetics [8], and qualitative biological models that depict interactions involving protein promotion or inhibition [4].

The rest of the paper is organised as follows. In Section 2 we introduce preliminaries. Section 3 demonstrates how the simultaneous incongruences problem can be simulated by 2-BHPCD. In Section 4 we prove that the reachability problem for 2-BHPCD and 1-BPAM is co-NP-hard. Section 5 establishes that the mortality problem for 2-BHPCD and 1-BPAM is also co-NP-hard. Finally, Section 6 contains our concluding remarks.

2 Preliminaries

In this section, we define the notions of one-dimensional piecewise affine maps (1-PAM) and two-dimensional hierarchical piecewise constant derivative systems (2-HPCD), closely following the notations used in [2, 3, 11, 12].

2.1 Piecewise Affine Maps

A rational interval in $\mathbb{R}$ is defined as one of the following forms: $[x, y]$, $[x, y)$, $(x, y]$, (x, y), $(-\infty, y]$, $(-\infty, y)$, $[x, \infty)$, (x, ∞), where $x, y \in \mathbb{Q}$ with $x \leqslant y$.

Definition 1 (1-PAM, [3]). *Let I_i be a finite set of disjoint rational intervals. A function $f : \mathbb{R} \to \mathbb{R}$ is a one-dimensional piecewise affine map (1-PAM) if it can be expressed as $f(x) = a_i x + b_i$ for $x \in I_i$.*

Definition 2 (Trajectory). *A trajectory of a 1-PAM f is a sequence $x_1, x_2, \ldots$ such that $x_{i+1} = f(x_i)$ for all i. We say that y is reachable from x if there exists a finite trajectory starting at x and ending at y.*

Definition 3 (1-BPAM, [3]). *A 1-PAM f is called bounded (1-BPAM) if none of its intervals is infinite.*

The class of the bounded 1-PAM is particularly noteworthy because it represents a subset of piecewise affine functions that lie on the boundary between decidability and undecidability when it comes to analysing their trajectories and reachability properties. They pose interesting challenges in terms of determining the reachability of points under repeated applications of f.

2.2 Hierarchical Piecewise Constant Derivative Systems

A two-dimensional piecewise constant derivative system (2-PCD) is defined as a finite set of regions, where each region is associated with a constant vector field. In this context, the vector field within each region is characterised by a single vector in $\mathbb{R}^2$. Intuitively, the vector assigned to each region determines the direction a particle would follow upon entering the region from any of its boundaries.

Definition 4 (2-PCD, [11]). *A two-dimensional piecewise constant derivative system (2-PCD) is defined by a pair $\mathcal{H} = (\mathcal{P}, \varphi)$, where:*

1. *$\mathcal{P} = \{p_1, \ldots, p_k\}$ is a finite set of non-overlapping polygons with nonempty interiors, referred to as regions throughout this paper.*
2. *$\varphi : \mathcal{P} \to \mathbb{R}^2$ is a function that assigns a vector $\varphi(p) \in \mathbb{R}^2$ to each region $p \in \mathcal{P}$, defining the dynamics within p.*

The set of all border points of $\mathcal{P}$, denoted $\mathcal{B}(\mathcal{P}) = \bigcup_{p \in \mathcal{P}} \mathcal{B}(p)$, consists of the union of the boundaries of all regions p, where $\mathcal{B}(p)$ denotes the boundary of region p. Formal definitions can be found in [11] or [12].

Since such a system exhibits deterministic behaviour within each polygonal region, the reachability analysis primarily focuses on computing the discrete successors of points located on the boundaries of these regions.

Definition 5 (Step, [11]). *Let $\mathcal{H} = (\mathcal{P}, \varphi)$ be a 2-PCD, and let $\boldsymbol{x}$ and $\boldsymbol{x}'$ be two distinct points in $\mathbb{R}^2$. We say that the pair $(\boldsymbol{x}, \boldsymbol{x}')$ is a step if there exists a region $p \in \mathcal{P}$ and a $t > 0$ such that the following conditions hold:*

1. $\boldsymbol{x}' = \boldsymbol{x} + t \cdot \varphi(p)$*,*
2. $\boldsymbol{x}, \boldsymbol{x}' \in \mathcal{B}(p)$*, that is, both points lie on the boundary of region* p*,*
3. $\boldsymbol{x}'' = \boldsymbol{x} + t' \cdot \varphi(p) \in p$ *for every* $0 < t' < t$*.*

Intuitively, the pair $(\boldsymbol{x}, \boldsymbol{x}')$ is considered a step if both points $\boldsymbol{x}$ and $\boldsymbol{x}'$ lie on the boundary of a region p, and the straight line segment connecting $\boldsymbol{x}$ and $\boldsymbol{x}'$ is entirely contained within p.

Definition 6 (Trajectory). *Let* $\mathcal{H} = (\mathcal{P}, \varphi)$ *be a 2-PCD, and let* $\boldsymbol{x}_0$ *be a point in* $\mathcal{B}(\mathcal{P})$*. A trajectory rooted at* $\boldsymbol{x}_0$ *is a sequence* $\tau^{\ell}_{\boldsymbol{x}_0} = \boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_\ell$*, where each pair* $(\boldsymbol{x}_i, \boldsymbol{x}_{i+1})$ *for* $0 \leqslant i \leqslant \ell - 1$ *is a step. We say that* ℓ *is the length of* $\tau^{\ell}_{\boldsymbol{x}_0}$*.*

Note that for each point $\boldsymbol{x}_0$ and each ℓ, such a trajectory is unique. Furthermore, we say that a point $\boldsymbol{x}_f$ is reachable from $\boldsymbol{x}_0$ if $\boldsymbol{x}_f$ belongs to the trajectory $\tau^{\ell}_{\boldsymbol{x}_0}$ for some finite ℓ.

Definition 7 (2-HPCD, [3]). *A two-dimensional hierarchical piecewise constant derivative system (2-HPCD) consists of a collection of locations, where each location is a 2-PCD system. Additionally:*

1. *Transition guards determine transitions between locations based on specified conditions.*
2. *Affine reset rules govern the behaviour of variables when transitioning between locations.*

A formal definition of a 2-HPCD system can be found in [3] and [6]. The definition of a 2-HPCD system emphasises that its trajectories largely resemble those of a 2-PCD system, but with occasional jumps induced by transition guards. We note that the notion of a trajectory in a 2-PCD can be straightforwardly extended to a 2-HPCD.

In this study we consider deterministic 2-HPCD systems: the transition guards for each location are mutually exclusive.

Definition 8 (2-BHPCD, [3]). *A 2-HPCD* $\mathcal{H}$ *is called bounded (abbreviated as 2-BHPCD) if none of its regions is infinite.*

2.3 Reachability and Mortality Problems

Reachability typically manifests in two forms: point-to-point reachability and interval-to-interval reachability.

Definition 9 (Reachability problem). *Given a 2-HPCD* $\mathcal{H}$*, the point-to-point and interval-to-interval reachability problems are defined as follows:*

– ***Point-to-point reachability:*** *given an initial point* $\boldsymbol{x}_0$ *and a final point* $\boldsymbol{x}_f$*, determine whether* $\boldsymbol{x}_f$ *is reachable from* $\boldsymbol{x}_0$*.*
– ***Interval-to-interval reachability:*** *given an initial interval* I_0 *and a final interval* I_f*, determine whether some point* $\boldsymbol{x}_f \in I_f$ *is reachable from some point* $\boldsymbol{x}_0 \in I_0$*.*

Definition 10 (Mortality problem). *Given a 2-HPCD $\mathcal{H}$, the mortality problem asks whether there exists an initial point x_0 such that the trajectory τ_{x_0} starting from x_0 is infinite. If such a point exists, $\mathcal{H}$ is called immortal. If for every point x_0, the trajectory τ_{x_0}, starting at x_0, eventually halts, then $\mathcal{H}$ is called mortal.*

The reachability problem, including interval-to-interval reachability, and the mortality problem for 1-PAM are defined analogously to those for 2-HPCD.

2.4 Notion of Simulation

The concept of simulation typically involves a relationship between two computational systems, A and B, where one system (the simulating system, in this case A) can reproduce the behaviour of the other system (the simulated system, in this case B). This implies that every computational step or action that B can perform can also be executed by A.

Simulation is particularly relevant when comparing different models of computation or analysing their complexity. It is formally defined below in terms of state transition systems:

Definition 11 (Simulation). *We assume deterministic transition systems $\Gamma = (S, \delta, s_0, s_f)$ and $\Gamma' = (S', \delta', s'_0, s'_f)$, where:*

- *S, S' are sets of states,*
- *$\delta : S \to S$, $\delta' : S' \to S'$ are transition functions,*
- *s_0, s'_0 are initial states,*
- *s_f, s'_f are final states.*

We say that Γ can be simulated by Γ' with respect to the reachability and mortality problems if there exists a function $f : S \to S'$ such that the following conditions hold:

1. *$s'_0 = f(s_0)$,*
2. *$s'_f = f(s_f)$,*
3. *$f(\delta(s)) = \delta'(f(s))$ for any $s \in S$.*

This definition specifies that simulation involves finding a function f that maps states of system Γ to states of system Γ', preserving initial and final states, and ensuring that the transitions of Γ are mirrored by Γ's transitions under f.

We note that it follows from Lemma 3.3 and Lemma 3.4 in [3] that bounded 1-PAM and bounded 2-HPCD can simulate each other, as summarised in the following theorem:

Theorem 1 ([3]). *Every bounded 1-PAM can be simulated by a bounded 2-HPCD. Conversely, every bounded 1-HPCD can be simulated by a bounded 1-PAM.*

Furthermore, the complexity of simulating bounded 2-HPCD by bounded 1-PAM (and vice versa) is polynomial in the size of the simulated instance, as demonstrated in the proofs provided in [3].

3 Simulation of Simultaneous Incongruences Problem by Bounded 2-HPCD

Our proof of the co-NP-hardness of reachability for bounded 2-HPCD, and subsequently for bounded 1-PAM, is based on the simulation of the simultaneous incongruences problem by 2-HPCD as outlined below:

1. In Section 3.1, we define the simultaneous incongruences problem and discuss its feature used in our proofs.
2. In Section 3.2, we illustrate how to simulate an instance of the simultaneous incongruences problem using the reachability problem for 2-BHPCD. By construction, the instance of the simultaneous incongruences problem has a solution if and only if the corresponding reachability problem does not.
3. In Section 3.3, we discuss the complexity of the provided simulation.

3.1 Simultaneous Incongruences Problem

In this section, we define the simultaneous incongruences problem, which is known to be NP-complete [7]. We will use this problem to show that the reachability and mortality problems for 2-HPCD and 1-PAM are co-NP-hard.

Definition 12 (Simultaneous incongruences problem). *Given a set* $\mathcal{S} = \{(\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)\}$*, where* $k \geqslant 1$*, of ordered pairs of positive numbers such that* $\alpha_i \leqslant \beta_i$ *for every* $1 \leqslant i \leqslant k$*, the simultaneous incongruences problem asks: does there exist an integer* x *such that* $x \not\equiv \alpha_i \pmod{\beta_i}$ *for every* $1 \leqslant i \leqslant k$*?*

In the following, we use the notation $\mathsf{LCM}(\beta_1, \ldots, \beta_k)$ to denote the least common multiple of the numbers $\beta_1, \ldots, \beta_n$. This is the smallest positive integer that is evenly divisible by $\beta_1, \ldots, \beta_k$. We will also use the technical lemma provided below.

Lemma 1 ([6]). *There exists a solution for the simultaneous incongruences problem for* $S = \{(\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)\}$ *if and only if there exists a solution* x *such that* $1 \leqslant x \leqslant \Lambda$*, where* $\Lambda = \mathsf{LCM}(\beta_1, \ldots, \beta_k)$*, the least common multiple of the numbers* $\beta_1, \ldots, \beta_k$*.*

3.2 Simulation

In this section, we demonstrate the construction of a 2-HPCD that simulates the simultaneous incongruences problem. Unlike the approach taken by Bell et al. in [6], our study has a challenge of simulating this problem using a lower-dimensional system. In contrast, Bell et al. determine whether the current k is a solution to the given simultaneous incongruences problem using the z-coordinate, which tracks the current value of k in the restricted 3-HPCD.

Let $\mathcal{S} = \{(\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)\}$ be a set of ordered pairs of positive integers such that $\alpha_i \leqslant \beta_i$ for $1 \leqslant i \leqslant k$. In the following, we will use $\mathcal{H}^{\mathsf{sip}}(\mathcal{S})$ to denote the 2-HPCD that simulates the simultaneous incongruences problem for $\mathcal{S}$.

The reachability problem for 2-HPCD simulates the simultaneous incongruences problem for $\mathcal{S}$, incorporating the insights from Lemma 1.

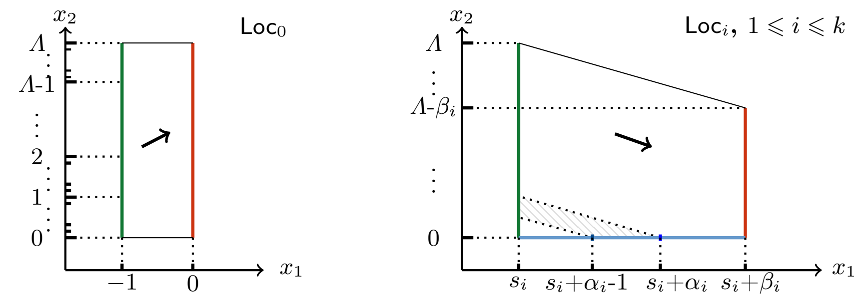

Fig. 1. Encoding of the simultaneous incongruences problem by the reachability problem for 2-HPCD, where: (a) location Loc_0 with the flow given by the vector $\boldsymbol{x} = (1, 1 - 1/(1 + \Lambda))$; (b) location Loc_i with the flow given by the vector $\boldsymbol{x} = (1, -1)$.

1. For the current value of x, where $1 \leqslant x \leqslant \Lambda$ and $\Lambda = \mathsf{LCM}(\beta_1, \ldots, \beta_k)$, we check whether $x \not\equiv \alpha_i \ (mod\ \beta_i)$ holds for each pair (α_i, β_i), where $1 \leqslant i \leqslant k$. If for every i, x is a solution to the given simultaneous incongruences problem, then the reachability problem for the 2-HPCD $\mathcal{H}^{\mathsf{sip}}(\mathcal{S})$ will not have a solution. That is, the trajectory will not reach the final state.
2. If for any x, where $1 \leqslant x \leqslant \Lambda$, there is an i, where $1 \leqslant i \leqslant k$, such that $x \equiv \alpha_i \ (mod\ \beta_i)$, then x is not a solution to the simultaneous incongruences problem for $\mathcal{S}$ and the trajectory will reach the final state.

The locations of $\mathcal{H}^{\mathsf{sip}}(\mathcal{S})$ are utilised in two primary ways: firstly, to perform modulo operations, and secondly, to increment the value being tested as needed. These functionalities are schematically depicted in Figure 1.

We assume that $s_0 = -1$ and $s_i = i + \sum_{j=1}^{i-1} \beta_j$, where $1 \leqslant i \leqslant k$. Now, we specify details of the construction of each Loc_i, where $0 \leqslant i \leqslant k$:

1. Location Loc_0:
   (a) Loc_0 is the convex polygon bounded by the straight lines $x_1 = s_0$, $x_1 = 0$, $x_2 = 0$ and $x_2 = \Lambda$.
   (b) The flow in Loc_0 is given by the vector $\boldsymbol{x} = (1, 1 - 1/(1 + \Lambda))$.
2. Location Loc_i, $1 \leqslant i \leqslant k$:
   (a) Each Loc_i is the convex polygon bounded by the straight lines $x_1 = s_i$, $x_1 = s_i + \beta_i$, $x_2 = 0$ and $x_2 = -x_1 + (\Lambda + s_i)$.
   (b) The flow in Loc_i is given by the vector $\boldsymbol{x} = (1, -1)$.

To determine if the current value of x, $1 \leqslant x \leqslant \Lambda$, is a solution to the given instance of the simultaneous incongruences problem, we simulate this check using the trajectory starting at the point $(1, x - x/(1 + \Lambda))$ within location Loc_1. Thus, x is encoded as $(1, x - x/(1 + \Lambda))$.

In the following, $I \times \{y\}$ will be used to denote the set $\{(x, y) \mid x \in I\}$, where I is an open or half-open or closed bounded rational interval. We will use $\{x\} \times I$ likewise. Now, each location Loc_i and its flow are used to perform the modulo operation for the pair (α_i, β_i) as follows:

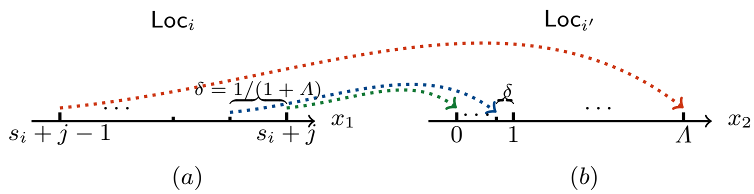

Fig. 2. Mapping the line interval $(s_i + j - 1, s_i + j) \times \{0\} \subset \mathsf{Loc}_i$, where $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant \beta_i$, onto the line interval $\{s_{i'}\} \times (0, \Lambda) \subset \mathsf{Loc}_{i'}$, where either $0 \leqslant i+1 = i' \leqslant k$ or $i' = 0$: (a) a schematic representation of the line segment $(s_i + j - 1, s_i + j) \times \{0\}$; (b) a schematic representation of the line segment $\{s_{i'}\} \times (0, \Lambda)$.

– Whenever the trajectory transitions from a point in the line interval $\{s_i\} \times (0, \Lambda)$ to a point in the line interval $\{s_i + \beta_i\} \times (0, \Lambda - \beta_i)$, the value of the variable x_2 decreases by β_i.
– If the trajectory reaches a point $(x', 0) \in (s_i + j - 1, s_i + j) \times \{0\}$, where $1 \leqslant j \leqslant \beta_i$, it implies that $x \equiv j\ (mod\ \beta_i)$, where $x = [(s_i + j) - x'](1 + \Lambda)$.

A summary of the details of the 2-HPCD $\mathcal{H}^{\mathsf{sip}}(\mathcal{S})$ such as its transition guards and reset relations are provided in Table 3.2.

Guard	Transition to	Reset relation
$\mathsf{G}_0 = \{0\} \times (0, \Lambda)$	$\{1\} \times (0, \Lambda) \subset \mathsf{Loc}_1$	$x_1 \to 1$ $x_2 \to x_2$
$\mathsf{G}_i^1 = \{s_i + \beta_i\} \times (0, \Lambda - b_i)$	$\{s_i\} \times (0, \Lambda - \beta_i) \subset \mathsf{Loc}_i$	$x_1 \to s_i$ $x_2 \to x_2$
$\mathsf{G}_i^2 = (s_i + \alpha_i - 1, s_i + \alpha_i) \times \{0\}$	$\{-1\} \times (0, \Lambda) \subset \mathsf{Loc}_0$	$x_1 \to -1$ $x_2 \to -\Lambda x_1 + \Lambda(s_i + a_i)$
$\mathsf{G}_{i,j}^3 = (s_i + j - 1, s_i + j) \times \{0\}$	$\{s_{i+1}\} \times (0, \Lambda) \subset \mathsf{Loc}_{i+1}$	$x_1 \to s_{i+1}$ $x_2 \to -\Lambda x_1 + \Lambda(s_i + j)$

Table 1. 2-HPCD $\mathcal{H}^{\mathsf{sip}}(\mathcal{S})$: a summary of the guards, transitions and reset relations, where $1 \leqslant i \leqslant k$, and $1 \leqslant j \leqslant \alpha_i - 1$ or $\alpha_i + 1 \leqslant j \leqslant \beta_i$.

The reset relation for variable x_1 determines the transition to the next location, and the reset relation for variable x_2 is defined as follows:

1. If $(x_1, x_2) \in \mathsf{G}_0 \cup \mathsf{G}_i^1$, then $x_2 \to x_2$. In this case, the reset relation for x_2 is the identity relation.
2. If $(x_1, x_2) \in \mathsf{G}_i^2 \cup \mathsf{G}_{i,j}^3$, then $x_2 \to -\Lambda x_1 + \Lambda(s_i + j)$ (see Figure 2). In this case, the reset relation for x_2 is computed by solving the system of equations $a(s_i + j - 1) + b = \Lambda$ and $a(s_i + j) + b = 0$, where a and b are unknowns. This yields $a = -\Lambda$ and $b = \Lambda(s_i + j)$.

The reachability problem for $\mathcal{H}^{\mathsf{sip}}(\mathcal{S})$ is defined as follows: is there a trajectory starting at the initial point $\boldsymbol{x}_0$ and reaching the final point $\boldsymbol{x}_f$, where

- $\boldsymbol{x}_0 = (1, 1 - 1/(1 + \Lambda)) \in \mathsf{Loc}_1$, and
- $\boldsymbol{x}_f = (s_0, \Lambda) \in \mathsf{Loc}_0$.

Now, we verify if the current x, where $0 \leqslant x \leqslant \Lambda$, is a solution to the simultaneous incongruences problem for the given set $\mathcal{S}$. We assume that the trajectory has reached a point $(x', 0) \in (s_i, s_i+\beta_i)\times\{0\}$ in Loc_i, where $1 \leqslant i \leqslant k$.

1. If $x \equiv \alpha_i \ (mod\ \beta_i)$, then the current value of x is not a solution to the simultaneous incongruences problem for $\mathcal{S}$. In this scenario, G_i^2 forces the transition from Loc_i to Loc_0 in order to increase the value of x by one.
2. If $x \not\equiv \alpha_i \ (mod\ \beta_i)$, then the current value of x is a potential solution to the simultaneous incongruences problem for $\mathcal{S}$. Here, $\mathsf{G}_{i,j}^3$, where $1 \leqslant j \leqslant \alpha_i - 1$ or $\alpha_i + 1 \leqslant j \leqslant \beta_i$, forces a transition from Loc_i to Loc_{i+1} to increment the value of i by one.

We assume that all $G_{k,j}^3$, where $1 \leqslant j \leqslant \alpha_k - 1$ or $\alpha_k + 1 \leqslant j \leqslant \beta_k$, have no outgoing transitions. If there exists a solution to the simultaneous incongruences problem for $\mathcal{S}$, then starting at the point $x_0 = (1, 1 - 1/(1 + \Lambda))$ in Loc_1, the trajectory will eventually reach a point in $G_{k,j}^3$, meaning the trajectory will halt without reaching the final state.

If there is no solution to the simultaneous incongruences problem for $\mathcal{S}$, then the trajectory will eventually reach the point $x_f = (s_0, \Lambda)$ in Loc_0, which represents the final state.

3.3 Complexity of Simulation

We adhere to the standard complexity theory convention, where both a set $\mathcal{S} = \{(\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)\}$ and the corresponding 2-HPCD $\mathcal{H}^{\mathsf{sip}}(\mathcal{S})$ are encoded in a "reasonable" manner, typically in binary. It is straightforward that the representation of $\mathcal{S}$ requires at least k bits. We also assume that the values of β_i's in $\mathcal{S}$ are polynomial in k.[1]

From a computational perspective, $\mathcal{H}^{\mathsf{sip}}(\mathcal{S})$ can be specified by lists of locations (vertices), guards, and flows. By construction, the number of locations in $\mathcal{H}^{\mathsf{sip}}(\mathcal{S})$ is $k + 1$. The number of guards in $\mathcal{H}^{\mathsf{sip}}(\mathcal{S})$ totals $g = 1 + \sum_{i=1}^{k}(\beta_i + 1)$, which is polynomial in k provided that all β_i are polynomial in k.

Moreover, for any vertex $\boldsymbol{v} = (x, y)$ in $\mathcal{H}^{\mathsf{sip}}(\mathcal{S})$, by construction, we have $-1 \leqslant x \leqslant k + \sum_{i=1}^{k} \beta_k$ and $0 \leqslant y \leqslant \Lambda$, where Λ is the least common multiplier of $\beta_1, \ldots, \beta_k$. Since $\Lambda \leqslant \beta_1 \times \beta_2 \times \ldots \times \beta_k$, its binary representation requires

[1] A 3-SAT problem φ with n variables can be transformed into the simultaneous incongruences problem for some set $\mathcal{S}$, where the number of pairs in $\mathcal{S}$ is polynomial in the size of φ, and all β_i are integers polynomial in n [7, 14]. Therefore, in this context, it suffices to consider a set $\mathcal{S}$ containing k pairs, where all β_i are integers polynomial in k.

$O(\sum_{i=1}^{k} \log \beta_i)$ bits. This implies that representing all vertices in $\mathcal{H}^{\mathsf{sip}}(\mathcal{S})$ is also polynomial in k.

The complexity of representing the flow in location Loc_0, given by the vector $\boldsymbol{x} = (1, 1 - 1/(1 + \Lambda))$, is also polynomial in k assuming Λ is represented in binary. Therefore, we conclude that the complexity of representing $\mathcal{H}^{\mathsf{sip}}(\mathcal{S})$, in terms of the number of bits required, is polynomial in k.

4 Complexity of the Reachability Problem

In this section, we establish our main result, demonstrating the co-NP-hardness of the reachability problem for 2-HPCD and 1-PAM. Our approach is outlined as follows:

1. We use the encoding provided in Section 3, which has the property that the given instance of the simultaneous incongruences problem has a solution if and only if the reachability problem does not.
2. The complexity of our simulation is polynomial in the description size of the given simultaneous incongruences problem, as discussed in Section 3.3. Consequently, we demonstrate that the reachability problem for 2-BHPCD is co-NP-hard in Section 4.1.
3. In Section 4.2, we extend our results to show that the reachability problem for 1-BPAM is also co-NP-hard.

4.1 Complexity of Reachability for Bounded 2-HPCD

In this section, we prove that the complexity of the reachability problem for 2-HPCD is co-NP-hard.

Theorem 2. *The reachability problem for 2-BHPCD is co-NP-hard.*

Proof. Assume a set of pairs $\mathcal{S} = \{(\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)\}$, $k \geqslant 1$. As discussed in Section 3.3, we can assume without loss of generality that all β_i are of size polynomial in k.

We encode the simultaneous incongruences problem for $\mathcal{S}$ into the bounded 2-HPCD $\mathcal{H}^{\mathsf{sip}}(\mathcal{S})$ as described in Section 3.2. The complexity of the construction is polynomial in k, as outlined in Section 3.3.

By construction, the reachability problem for the 2-HPCD $\mathcal{H}^{\mathsf{sip}}(\mathcal{S})$ has a solution if and only if there is no solution to the simultaneous incongruences problem for $\mathcal{S}$. Since the simultaneous incongruences problem is NP-complete, this implies that the reachability problem for 2-HPCD is co-NP-hard. □

4.2 Complexity of Reachability for Bounded 1-PAM

Now, we show that the complexity of the reachability problem for a bounded 1-PAM is also co-NP-hard.

Corollary 1. *The reachability problem for 1-BPAM is co-NP-hard.*

Proof. The complexity of simulating 2-BHPCD by 1-BPAM (or vice versa) is polynomial in the instance size, as demonstrated in the proofs provided in [3]. Now, it follows from Theorem 1 and Theorem 2 that the reachability problem for 1-BPAM is co-NP-hard. □

5 Complexity of the Mortality Problem

Next, we establish that the mortality problem for bounded 2-HPCD and 1-PAM is also co-NP-hard. Unlike the reachability problem, which focuses on determining whether a trajectory reaches a final state, the mortality problem examines whether all trajectories halt. More specifically:

1. In Section 5.1, we demonstrate how to encode an instance of the simultaneous incongruences problem using the mortality problem for 2-BHPCD. Specifically, the system is mortal if and only if there is no solution for the corresponding simultaneous incongruences problem, otherwise the system is immortal.
2. The complexity of our simulation is polynomial in the description size of the given simultaneous incongruences problem. Consequently, we show that the mortality problem for 2-BHPCD is co-NP-hard in Section 5.1.
3. In Section 5.2 we extend our results to show that the mortality problem for 1-BPAM is also co-NP-hard.

5.1 Complexity of Mortality for Bounded 2-HPCD

Now we can state that the complexity of the mortality problem for 2-HPCD is co-NP-hard.

Theorem 3. *The mortality problem for 2-BHPCD is co-NP-hard.*

Proof. Assume a set of pairs $\mathcal{S} = \{(\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)\}$, where $k \geq 1$. As discussed in Section 3.3, we can assume without loss of generality that all β_i are of size polynomial in k.

We construct the 2-HPCD $\mathcal{H}_m^{\mathsf{sip}}(\mathcal{S})$ such that the system is mortal if and only if there is no solution to the simultaneous incongruences problem for $\mathcal{S}$.

The construction follows the approach detailed in Section 3.2. Additionally, we assume that $G^3_{k,j}$, where $1 \leqslant j \leqslant \alpha_k - 1$ or $\alpha_k + 1 \leqslant j \leqslant \beta_k$, is reset to the interval $[x_0, x_0]$ (or, alternatively, to a sufficiently small interval around $\boldsymbol{x}_0$), where $\boldsymbol{x}_0 = (1, 1 - 1/(1 + \Lambda))$, which is the initial point in location Loc_1, as defined in Section 3.2. Further specifics are as follows:

1. Assuming that there is a solution to the simultaneous incongruences problem for $\mathcal{S}$:
   If there exists a solution to the simultaneous incongruences problem for $\mathcal{S}$, then starting from $\boldsymbol{x}_0 = (1, 1 - 1/(1 + \Lambda))$ in Loc_1, the trajectory will

eventually reach some point $\boldsymbol{x} \in G^3_{k,j}$, where $1 \leqslant j \leqslant \alpha_k - 1$ or $\alpha_k + 1 \leqslant j \leqslant \beta_k$, and it will be reset to the initial point $\boldsymbol{x}_0 = (1, 1 - 1/(1 + \Lambda))$ in Loc_1. That is, there is at least one infinite trajectory. Therefore, $\mathcal{H}^{\mathsf{sip}}_m(\mathcal{S})$ is immortal.

2. Assuming that there is no solution to the simultaneous incongruences problem for $\mathcal{S}$:
   If the simultaneous incongruences problem for $\mathcal{S}$ has no solution, then starting from any point, the trajectory will eventually reach the final point $\boldsymbol{x}_f = (s_0, \Lambda) \in \mathsf{Loc}_0$. We assume that there is no outgoing transition from $\boldsymbol{x}_f$. That is, if there is no solution to the simultaneous incongruences problem for $\mathcal{S}$, then regardless of where the trajectory starts, it will eventually halt. Therefore, $\mathcal{H}^{\mathsf{sip}}_m(\mathcal{S})$ is mortal.

Since the complexity of this simulation is polynomial in the description size of an instance of the simultaneous incongruences problem, following the reasoning provided in Section 3.3, we conclude that the mortality problem for 2-BHPCD is co-NP-hard. □

5.2 Complexity of Mortality for Bounded 1-PAM

Extending the results provided in Section 5.1, we can conclude that the complexity of the mortality problem for 1-PAM is also co-NP-hard.

Corollary 2. *The mortality problem for 1-BPAM is co-NP-hard.*

Proof. The complexity of simulating 2-BHPCD by 1-BPAM (or vice versa) is polynomial in the instance size, as demonstrated in the proofs provided in [3]. Now, it follows from Theorem 1 and Theorem 3 that the mortality problem for 1-BPAM is co-NP-hard. □

6 Conclusion

In this work, we have demonstrated that the reachability and mortality problems for bounded 2-HPCD and 1-PAM are co-NP-hard. Our proofs are based on encoding the NP-complete simultaneous incongruences problem into the reachability and mortality problems for bounded 2-HPCD, and consequently, for bounded 1-PAM.

The interval-to-interval version of the reachability problem in our proofs can be formulated by considering an ε_0-interval around the initial point to represent the initial interval, and an ε_f-interval around the final point to represent the final interval for sufficiently small ε_0 and ε_f. Furthermore, we anticipate that interval-to-interval reachability and mortality are decidable for bounded 1-PAM and 2-HPCD, which can be demonstrated by extending the results found in [12, 13].

This work addresses, in a broader context, the open question regarding the complexity of mortality in dimension two posed for restricted HPCD in [6]. It also leaves several immediate questions for future research, including providing an upper bound on the computational complexity of the reachability and mortality problems in the context of bounded 2-HPCD and 1-PAM.

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