Covariant interacting fractons

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We show that there exists a natural analog of the Yang-Mills equations using the Frölicher-Nijenhuis bracket between vector-valued differential forms. The gauge field is a rank-2 tensor, and when one constrains it to be symmetric, then the system exhibits fractonic behaviors. In the linearized limit, the constrained equations of motion reduce to those of the covariant fracton model [Phys. Rev. D **106**, 125008 (2022), Phys. Lett. B **833**, 137304 (2022), and Notes from the bulk, Ph.D. thesis, Università di Genova, 2024.].

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I. INTRODUCTION AND SUMMARY

In recent years, a new kind of quasiparticle has emerged from the physics literature: *fractons* [1–4]. Originating in lattice models in the context of spin glasses [5] and quantum information [6,7], fractons have quickly attracted the interest of a wide variety of physicists from condensed matter to mathematical physics and have been influential in gauge theories and other quantum field theories [8–17]. The main characteristic of fracton quasiparticles is immobility, which is also the reason for the name. Indeed a fracton is defined as a fraction of a mobile quasiparticle, and in isolation it cannot move at all. Only dipolelike excitations are free to displace, or, in general have fewer constraints on their motion [3,4,8,9]. These additional constraints define other fracton-related quasiparticles such as *lineons* and *planons*, which can move in a one- or two-dimensional subspace respectively. The restricted-motion feature, which unites all fracton theories, is shared by many physical systems and models, and it is one of the reasons for which fractons are so popular nowadays. For instance limited mobility, or complete immobility, can be harnessed for developing quantum memories [6,7], or used as a mapping/duality to study topological defects in elastic media [18–25]. It is also a characteristic found in Carrollian theories, which thus seem to display some fractonic behavior [26,27]. The other reason for attracting so much interest is found in the tensorial nature of fractonic theories. In gauge theory [8,9] these models are indeed typically described in terms of a rank-2 symmetric tensor $A_{ij}(x)$ (with i, j spatial indices), which transforms under the gauge transformation

$$\delta_\epsilon A_{ij} = \partial_i \partial_j \epsilon \tag{1}$$

and shares strong similarities with the electromagnetic Maxwell theory, of which they represent higher-rank generalizations. A generalized Gauss law is typically postulated as

$$\partial_i \partial_j E^{ij} = \rho, \tag{2}$$

where $E^{ij}(x)$ is a symmetric electric tensor field, implying dipole moment conservation through

$$D^i = \int \mathrm{d}^d x x^i \rho = -\int \mathrm{d}^d x \partial_j E^{ij} = \oint \mathrm{d}^{d-1} x(\cdots), \tag{3}$$

that is, the dipole moment cannot change except through a nonzero flux at the boundary. This encodes the immobility of the fractonic charge $\rho(x)$ [3,4] since, if a single charge were to move, it would change the total dipole moment of the system. The tensorial nature of the gauge field also hints toward connections with the theory of linearized gravity [28–32], which emerges naturally when the covariant fracton theory is taken into account [33–36]. The covariant extension

$$\delta A_{\mu\nu} = \partial_\mu \partial_\nu \epsilon \tag{4}$$

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of the fractonic transformation (1) is indeed a particular case of the infinitesimal diffeomorphisms that define linearized gravity, sometimes called longitudinal diffeomorphisms [37]. Thus an action invariant under (4) would naturally carry a linearized gravity term. This covariant formulation [33] gives rise to the definition of an invariant rank-3 field strength $F_{\mu\nu\rho}(x)$, through which the Maxwell analogy and fracton phenomenology of [8,9,38] is reproduced and expanded from first principles. The covariant theory of fractons [33] is free (or at least quantum-electrodynamics-like) in the sense that the gauge field $A_{\mu\nu}(x)$ does not interact directly with itself. However, the existence of a generalized invariant field strength $F_{\mu\nu\rho}(x)$ suggests a mathematically natural "non-Abelianization" of the above theory through the use of the Frölicher–Nijenhuis bracket [39,40] (reviewed in [41] Sec. 8), which is the focus of this paper.

In an interacting theory of fractons where the gauge field interacts with itself, the gauge field modes themselves are fractonic, so that there may be constraints on asymptotic in states and out states. This may be interesting from an amplitude-theoretic point of view and may possibly signal new loopholes to Weinberg–Witten-type no-go theorems [42].

The analysis presented in this paper is also of independent mathematical interest. The Frölicher–Nijenhuis bracket is a fundamental geometric structure, which has recently appeared in the context of integrable models such as the self-dual Yang-Mills theory [43]. In fact, the Frölicher–Nijenhuis bracket generalizes to arbitrary Lie algebroids [44], which appear in gauge theory in many contexts [45–51], and both the fractonic case at hand and ordinary Yang-Mills theory may be seen as special cases (for the tangent Lie algebroid and a Lie algebroid bundle, respectively) of a more general construction associated to the general Lie algebroid. We thus see, again, that strong analogies manifest themselves between fractons and Maxwell/Yang-Mills theories.

Our interacting fractonic model is defined on flat space, as gauge invariance breaks on curved space: the field strength F only transforms tensorially if one assumes that the metric $g_{\mu\nu}$ is flat and also transforms under diffeomorphism. Thus, to write an action principle one cannot have $g_{\mu\nu}$ as a background (since it must transform), and the equations of motion for $g_{\mu\nu}$ must ensure that it remains flat (since otherwise gauge invariance fails). As a consequence, for the scope of the analysis presented here, we only postulate an equation of motion. For efforts at fractonic behaviors on curved spaces, see [52–54]. We also do not discuss issues regarding the classical or quantum stability of our model.

II. MATHEMATICAL BACKGROUND

In the following, we will need to make use of the Frölicher-Nijenhuis bracket on vector-valued differential forms and the language of twisting, which we review briefly.

A. Frölicher-Nijenhuis bracket

Let M be a smooth manifold. The graded vector space of vector-valued differential forms $\Omega^\bullet(M;\mathrm{T}M)=\bigoplus_{i=0}^{d}\Omega^i(M)$ becomes a graded Lie algebra with respect to the *Frölicher–Nijenhuis bracket* [39,40] (reviewed in [41] Sec. 8):

$$\begin{aligned}[\phi\otimes X,\psi\otimes Y] = {} & (\phi\wedge\psi)\otimes\mathcal{L}_XY+(\phi\wedge\mathcal{L}_X\psi)\otimes Y\\ & -(\mathcal{L}_Y\phi\wedge\psi)\otimes X\\ & +(-1)^p(\mathrm{d}\phi\wedge i_X\psi)\otimes Y\\ & +(-1)^p(i_Y\phi\wedge\mathrm{d}\psi)\otimes X\end{aligned}\tag{5}$$

for vector fields $X,Y\in\Gamma(\mathrm{T}M)$ [where $\Gamma(-)$ denotes the space of sections of a vector bundle] and homogeneous differential forms $\phi\in\Omega^p(M)$, $\psi\in\Omega^q(M)$, where $\mathcal{L}_X(-)$ is the Lie derivative of a tensor field along a vector field, and $i_X(-)$ is the interior derivative of a differential form along a vector field.

In particular, between two (1,1) tensors K^μ_ν and L^μ_ν, we have

$$\begin{aligned}[K,L]^\rho_{\mu\nu} = {} & 2K^\sigma_{[\mu|}\partial_\sigma L^\rho_{|\nu]}+2L^\sigma_{[\mu|}\partial_\sigma K^\rho_{|\nu]}-2K^\rho_\sigma\partial_{[\mu}L^\sigma_{\nu]}\\ & -2L^\rho_\sigma\partial_{[\mu}K^\sigma_{\nu]},\end{aligned}\tag{6}$$

where antisymmetrizations are normalized.

Notice that, when one of the arguments is a (1,0) tensor (i.e., a vector field), it reduces to the usual Lie derivative:

$$[X,-]=\mathcal{L}_X\qquad (X\in\Gamma(\mathrm{T}M)).\tag{7}$$

Suppose that M is equipped with a Riemannian metric g whose Riemann curvature vanishes. Then, using the induced Levi-Civita connection ∇, we may define the covariant exterior derivative:

$$\mathrm{d}^\nabla:\Omega^\bullet(M;\mathrm{T}M)\to\Omega^{\bullet+1}(M;\mathrm{T}M),\tag{8}$$

which squares to zero, and then $\Omega^\bullet(M;\mathrm{T}M)$ forms a differential graded Lie algebra. Note that, when the curvature of g does not vanish, then d^∇ need not square to zero.

B. Twisting

A curved [55] differential graded Lie algebra $(\mathfrak{g},r,d,[-,-])$ is a $\mathbb{Z}$-graded Lie algebra $(\mathfrak{g},[-,-])$ together with a linear map,

$$d:\mathfrak{g}\to\mathfrak{g},\tag{9}$$

of degree one and an element $r\in\mathfrak{g}$ of degree two, called the *curvature*, such that d is a graded derivation with respect to the Lie bracket $[-,-]$ and

$$d(d(x)) = [r, x] \tag{10}$$

for any $x \in \mathfrak{g}$. This is the special case of the notion of a curved L_∞ algebra [56–58] $(\mathfrak{g}, \mu_0, \mu_1, \mu_2, \ldots)$, which is a graded vector space $\mathfrak{g}$ equipped with totally graded-anti-symmetric multilinear maps $\mu_i : \mathfrak{g}^{\otimes i} \to \mathfrak{g}$ that satisfy the Jacobi identity up to homotopy. Then a curved differential graded Lie algebra is the same as a curved L_∞ algebra in which $\mu_i = 0$ except for $i \in \{0, 1, 2\}$; then μ_0, μ_1, and μ_2 correspond to r, d, and $[-,-]$, respectively.

Given a differential graded Lie algebra $(\mathfrak{g}, d_\mathfrak{g}, [-,-]_\mathfrak{g})$ and an element $Q \in \mathfrak{g}^1$ of degree one, then the *twist* [59] of $\mathfrak{g}$ by Q is the curved differential graded Lie algebra $\mathfrak{g}_Q = (\mathfrak{g}, dQ + \frac{1}{2}[Q,Q]_\mathfrak{g}, d_\mathfrak{g} + [Q,-]_\mathfrak{g}, [-,-]_\mathfrak{g})$. When $[Q,Q]_\mathfrak{g} = 0$, then this is a differential graded Lie algebra.

III. MOTIVATION AND IDEA

A. A review of Yang-Mills theory

Linearized Yang-Mills theory (that is, a direct sum of copies of Maxwell theory) admits a natural non-Abelianization in the form of Yang-Mills theory. Let us recall how this works. The field strength in linearized Yang-Mills theory is

$$F^a_{\mu\nu} = 2\partial_{[\mu} A^a_{\nu]}. \tag{11}$$

In the non-Abelian theory, this is modified to

$$F^a_{\mu\nu} = 2\partial_{[\mu} A^a_{\nu]} + f^a{}_{bc} A^b_\mu A^c_\nu. \tag{12}$$

It is convenient to use the notation of Lie-algebra-valued differential forms, in terms of which we have

$$F = \mathrm{d}A + \frac{1}{2}[A, A]. \tag{13}$$

That is, the field strength is fixed by the structure of a differential graded Lie algebra on the space of $\mathfrak{g}$-valued differential forms $\Omega^\bullet(M) \otimes \mathfrak{g} = \bigoplus_{i=0}^{\dim M} \Omega^i(M) \otimes \mathfrak{g}$, where $\mathfrak{g}$ is the color Lie algebra, M is spacetime, and $^\bullet$ is a placeholder for the form degree. Furthermore, this fixes the structure of gauge transformations and Bianchi identities:

$$\delta_\alpha A = \mathrm{d}_A \alpha \qquad \mathrm{d}_A F = 0 \qquad \mathrm{d}_A = \mathrm{d} + \frac{1}{2}[A, -] \qquad \delta_\alpha F = [\alpha, F]. \tag{14}$$

The procedure of replacing d with d_A goes by the name of *twisting* [56–58] as discussed in Sec. II B. After this, we no longer have a differential graded Lie algebra in the usual sense since

$$\mathrm{d}_A^2 = [F, -] \neq 0, \tag{15}$$

but we speak of a *curved* differential graded Lie algebra. Given this, the equation of motion for the theory is fixed to be

$$\mathrm{d}_A \star F = 0. \tag{16}$$

B. Non-Abelianizing the covariant fracton model: The idea

The covariant fracton model [33] is a free theory whose fundamental field is a symmetric tensor $A_{\mu\nu}(x)$. An invariant field strength with one derivative can be defined, which is of the form [[60], Eq. (7.2.16)]

$$F_{\mu\nu\rho} = a_1 \partial_\mu A_{\nu\rho} + a_2 \partial_\rho A_{\mu\nu} - (a_1 + a_2)\partial_\nu A_{\mu\rho} \tag{17}$$

for some suitable parameters $a_1, a_2 \in \mathbb{R}$. For any value of a_1, a_2, the field strength $F_{\mu\nu\rho}(x)$ satisfies a Bianchi identity [[60] p. 83],

$$\begin{aligned} 0 &= \partial_\mu(F_{\beta\nu\rho} - F_{\beta\rho\nu}) + \partial_\nu(F_{\beta\rho\mu} - F_{\beta\mu\rho}) + \partial_\rho(F_{\beta\mu\nu} - F_{\beta\nu\mu}) \\ &= 6\partial_{[\mu|} F_{\beta|\nu\rho]}. \end{aligned} \tag{18}$$

This has the form of an exterior derivative, except that the index β does not participate in the antisymmetrization. Thus, it is natural to regard F as a T^*M-valued two form, similar to how the Yang-Mills field strength is a Lie-algebra-valued two form. This then means that $A_{\mu\nu}(x)$ should also be regarded as a T^*M-valued one form.

There are however three problems that arise in this case, which are related.

(1) There is no obvious Lie bracket for T^*M-valued differential forms (unlike Lie-algebra-valued differential forms).
(2) A T^*M-valued one form will *not* generally be symmetric between its two indices.
(3) The gauge parameter should naturally be a T^*M-valued zero form, i.e., an ordinary one form, which is bigger than the scalar field gauge parameter of the covariant fracton model.

We resolve these interrelated problems as follows.

(1) Unlike T^*M-valued forms, there *does* exist a natural Lie bracket on $\mathrm{T}M$-valued forms: the Frölicher–Nijenhuis bracket [39,40] (reviewed in [41] Sec. 8). Thus, we work with $\mathrm{T}M$-valued forms, and initially ignore the symmetry property of the $\mathrm{T}M$-valued one-form gauge field $A^\mu{}_\nu(x)$. Therefore, $\Omega^\bullet(M; \mathrm{T}M)$ is a graded Lie algebra. For this to be a *differential* graded Lie algebra, we fix a flat connection on $\mathrm{T}M$.
(2) Having formulated this theory, then we will impose the symmetry requirement with respect to a pseudo-Riemannian metric:

$$g_{\mu\nu} A^\nu{}_\rho = g_{\rho\nu} A^\nu{}_\mu. \tag{19}$$

(3) The constraint (19) will then naturally reduce the gauge symmetry from $\Omega^0(M;\mathrm{T}M)$ to $\Omega^0(M)$, i.e., it will require the gauge parameter to be a scalar as for the covariant fracton theory.

IV. COVARIANT INTERACTING FRACTONIC GAUGE THEORY

As mentioned in Sec. III, we first construct a nonlinear equation of motion for a (1,1)-tensor field $A^\mu{}_\nu$ in Sec. IV A. Then we constrain it to be symmetric in Sec. IV B.

A. Nonsymmetric theory

Let M be a smooth manifold equipped with a flat Riemannian metric $g_{\mu\nu}$ (such as Minkowski space). Then $(\Omega^\bullet(M;\mathrm{T}M), \mathrm{d}^\nabla, [-,-])$ is a differential graded Lie algebra, where $[-,-]$ is the Frölicher-Nijenhuis bracket (5).

Consider a (1,1) tensor,

$$A^\mu{}_\nu \in \Omega^1(M;\mathrm{T}M). \tag{20}$$

Then we may twist, as discussed in Sec. II B, to obtain the curved differential graded Lie algebra

$$(\Omega^\bullet(M;\mathrm{T}M), F_A, \mathrm{d}^\nabla_A, [-\wedge -]), \tag{21}$$

where

$$\mathrm{d}^\nabla_A = \mathrm{d}^\nabla + [A,-] \tag{22}$$

and

$$F_A = \mathrm{d}^\nabla A + \frac{1}{2}[A,A] \tag{23}$$

is the curvature [61]. In particular, we have

$$(\mathrm{d}^\nabla_A)^2 = [F_A,-]. \tag{24}$$

We postulate the infinitesimal gauge symmetry

$$\delta_\epsilon A = \mathrm{d}_A \epsilon \tag{25}$$

for a vector gauge parameter $\epsilon \in \Omega^0(M;\mathrm{T}M) = \Gamma(\mathrm{T}M)$ which, in explicit component notation, is

$$\begin{aligned}(\delta_\epsilon A)^\mu{}_\nu &= \nabla_\nu \epsilon^\mu - \mathcal{L}_\epsilon A^\nu \\ &= \partial_\nu \epsilon^\mu - \epsilon^\rho \partial_\rho A^\mu{}_\nu + \partial_\rho \epsilon^\mu A^\rho{}_\nu - \partial_\nu \epsilon^\rho A^\mu{}_\rho,\end{aligned} \tag{26}$$

and define the field strength

$$F_A = \mathrm{d}A + \frac{1}{2}[A,A] \in \Omega^2(X;E). \tag{27}$$

Explicitly,

$$F^\rho{}_{\mu\nu} = \nabla_\mu A^\rho{}_\nu - \nabla_\nu A^\rho{}_\mu + \mathcal{O}(A^2), \tag{28}$$

where ∇_μ is the (Riemannian) covariant derivative of a tensor field. This is, to linear order, similar to the field strength in [[60], (7.2.16)] with $(a_1, a_2) = (1, 0)$, which however depends on a symmetric rank-2 tensor, while here $A^\mu{}_\rho$ is an arbitrary rank (1,1) tensor. We shall discuss the symmetric case in Sec. IV B.

Under a gauge transformation, the field strength F then transforms covariantly:

$$\delta_\epsilon F = -\mathcal{L}_\epsilon F. \tag{29}$$

The fact that it is not invariant reminds us of the field strength in Yang-Mills theory (14). If we interpret the gauge parameter ϵ^μ as an infinitesimal diffeomorphism, then this implies that $F^\mu{}_{\nu\rho}$ transforms tensorially.

Now, we may postulate the equation of motion,

$$\nabla^\nu F^\mu{}_{\nu\rho} = 0. \tag{30}$$

This is a diffeomorphism-invariant equation as long as we also transform $g_{\mu\nu}$ under diffeomorphisms, i.e.,

$$\delta_\epsilon g_{\mu\nu} = -\mathcal{L}_\epsilon g_{\mu\nu} = -\nabla_\mu \epsilon_\nu - \nabla_\nu \epsilon_\mu. \tag{31}$$

B. Symmetric theory

To make contact with the covariant fracton model, we now constrain $A^\mu{}_\nu$ to be symmetric. That is, we impose the following constraint:

$$g_{\mu\nu} A^\nu{}_\rho = g_{\rho\nu} A^\nu{}_\mu. \tag{32}$$

This constraint is not gauge invariant for arbitrary ϵ^μ since the right-hand side of (26) need not be symmetric. However, it *is* gauge invariant if we restrict to "diffeomorphisms" of the form

$$\epsilon^\mu = \partial^\mu \lambda \tag{33}$$

(known as the longitudinal diffeomorphisms [37]) for some smooth function $\lambda \in \mathcal{C}^\infty(M)$, so that the resulting gauge transformation is

$$\delta_\epsilon A_{\mu\nu} = \partial_\mu \partial_\nu \lambda + \cdots, \tag{34}$$

which to linearized order agrees with the covariant fracton gauge transformation (4). Perturbatively and locally, (33) is the most general gauge transformation that preserves the constraint (32). This is because the general gauge transformation (26) is (after lowering indices) of the form $\delta_\epsilon A_{\mu\nu} = \partial_\nu \epsilon_\mu + \mathcal{O}(A)$, so we must have $\partial_\nu \epsilon_\mu - \partial_\nu \epsilon_\mu = 0$ (up to possible higher-order corrections); this is the exterior derivative of the one form ϵ, so by the Poincaré lemma, there exists locally a function λ such that $\epsilon_\mu = \partial_\mu \lambda$. This is to leading order, but it can then be easily checked that the ansatz (33) respects the constraint (32) to all orders.

Now, we have the symmetrized equation of motion,

$$\nabla^\mu F_{(\rho|\mu|\nu)} = 0. \tag{35}$$

The linearized equation of motion (with $g_{\mu\nu} = \eta_{\mu\nu}$ the Minkowski metric) is

$$\begin{aligned} 0 &= \partial^\mu(\partial_\mu A_{\rho\nu} - \partial_\nu A_{\rho\mu}) + \partial^\mu(\partial_\mu A_{\nu\rho} - \partial_\rho A_{\nu\mu}) \\ &= 2\partial^2 A_{\rho\nu} - \partial_\nu \partial^\mu A_{\rho\mu} - \partial^\mu \partial_\rho A_{\nu\mu}, \end{aligned} \tag{36}$$

which is the equation of motion found in the covariant fracton theory of fractons [33,34,60]. In d spacetime dimensions, this corresponds to $d(d-1)/2$ degrees of freedom, corresponding to the purely spatial components of $A_{\mu\nu}$ [34,35].

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DATA AVAILABILITY

No data were created or analyzed in this study.

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