# Lakatos' Mathematical Hegelianism[1]

In the preface to the *Phenomenology of Spirit*, Hegel claims that the knowledge found in the various special disciplines is (at the time of his writing) in one way or another defective. History, mathematics and the natural sciences are all limited in their methods and therefore leave fundamental questions unanswered (and indeed unasked). Consequently, Hegel thought, these specialisms must be completed, explained and in some manner subsumed by philosophy. He was especially scathing about mathematics. He took the subject matter of pure mathematics to be space and number (not an unnatural assumption for the time) and described this subject matter as "inert and lifeless" (§45; p. 26). Mathematical thought, for Hegel, "moves forward along the line of *equality*" (*ibid.*). In other words, mathematics consists of equations. There is nothing unstable or incomplete about an equation so there is no dialectical impetus to conceptual revision. Therefore equations, for Hegel, do not get at the essence of anything. Philosophical thought, on the other hand, does consider essences. Any particular essence (i.e. the essence of anything smaller than the entire universe) is in some way incomplete and cries out to be included in some larger scheme. Thus philosophy is goaded onwards and upwards towards ever more comprehensive and self-subsistent conceptions of the world. In consequence, philosophers do not need ingenuity or creativity. All they need do is steep themselves in the subject matter and thus become efficient vehicles for its internal necessity. This does not happen with the "rigid, dead propositions" (*ibid.*) of mathematics. Of these Hegel says, "We can stop at any one of them; the next one starts afresh on its own account, without the first having moved itself on to the next, and without any necessary connection arising through the nature of the thing itself" (*ibid.*).

In short, mathematics does not have the power of self-movement. What this means is that, having got halfway through a proof, there is nothing in the nature of the case to determine what the next line should be. Hegel (in common with his contemporaries) regarded Euclid's *Elements* as the paradigmatic mathematical work. In Euclidian geometry it is normal, in the course of a proof, to draw lines in addition to the original figure. As the proof unfolds these lines turn out to have important roles to play, though a student seeing the proof for the first time may not be able to tell at the moment of their drawing what role would fall to which line. This is because nothing in the original figure prompts one to draw this line rather than that. In a modern algebraic proof one typically has to prove lemmas (intermediate results), the significance of which may only become apparent as the proof nears completion. Therefore, according to Hegel, a proof tells us more about the ingenuity of the mathematician than it does about the meaning of the theorem. In Hegel's words, "The movement of mathematical proof does not belong to the object, but rather is an activity external to the matter in hand" (§42; p. 24). One way of putting the point is to observe that a given step in a proof immediately entails infinitely many further steps. By what principle is the next line chosen out of this vast field of candidates? A special case of this lack of essential connection between one line of a proof and the next is the fact that one theorem may have many proofs.

[1] I am grateful to Dr. M. Inwood, Prof. K. Westphal and colleagues at the universities of Liverpool, Durham and Hertfordshire for their helpful comments on earlier drafts of this paper.

This is possible because (Hegel thought) a proof does not reveal an essence, but merely establishes a truth.[2]

I shall not here make very much of the philosophical side of Hegel's contrast between philosophical and mathematical thought. I shall argue instead that he radically misunderstood the nature of mathematics. Some of this misunderstanding has unremarkable origins. Some of his arguments were simply poor. For example, he complains that in proofs of Pythagoras' theorem, "the triangle is dismembered, and its parts consigned to other figures, whose origin is allowed by the construction upon the triangle. Only at the end is the triangle we are actually dealing with reinstated" (§43; p. 25). If the study of triangles were a discipline in its own right Hegel might have had a point. However, in studying triangles one is really making a study of space (as Hegel's own analysis of the subject matter of mathematics implies). Indeed, one can give this thought something of a Hegelian spin: the concept 'triangle' is not self-contained because triangles are related to other figures (there is, for example, a whole family of theorems concerning triangles inscribed within circles). Hence, a narrowly focused 'triangle-ology' would miss important truths about triangles. When these omissions became unbearable triangle-ology would be overcome-by-yet-preserved-in general plane geometry. This thought is perhaps more obvious to the modern eye than to Hegel because we know that the study of (Euclidian) space led to the study of (Euclidian and non-Euclidian) space*s*. Non-Euclidian geometry was developed (after Hegel's time) to address a problem internal to Euclidian geometry (the independence of the parallel postulate). To continue the Hegelian gloss, Euclidian geometry turned out to be incomplete and demanded its own sublation by a more general study.[3] If we look beyond geometry, the contrast between the condition of mathematics now and in Hegel's day is even more acute. Mathematics in general (and analysis in particular) re-invented itself in the century following Hegel's death. Moreover it did so partly as a consequence of applying its own standards of rigour to itself. It is hardly surprising, therefore, that the dialectical potential of mathematics is more obvious to us than to Hegel.[4]

The chief cause of Hegel's erroneous dismissal of pure mathematics as a 'dead' discipline was not the age in which he lived, however. The real problem was his assumption (a commonplace then as now) that the logic of mathematics is exhausted by the formal deduction of theorems from axioms. Given this assumption, it follows that mathematics can

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[2]Here I follow Inwood's (pp. 226-7) analysis of Hegel on proof. Hegel returns to the point in the *Science of Logic* (vol. one, book I, p. 72; vol. two, section 3, chapter 2 A (b) 'The Theorem' p. 812) and in his treatment of Spinoza in the *Lectures on the history of Philosophy* (1.g pp. 488-9). Hegel's philosophical dialectic employs the traditional metaphysical notion of *substance* i.e. that which is complete in itself. It turns out (as in Spinoza) that only the universe as a whole satisfies this condition absolutely. Hegel's criticism of Spinoza is that in his philosophy he tried to employ the formal (i.e. undialectical) methods of mathematics. This criticism has its root in Descartes' view that philosophy differs from geometry in that philosophers cannot begin with a set of self-evident axioms. Rather, the first and chief task of philosophy is (for Descartes) the analysis of concepts so that self-evident basic truths are made manifest (replies to second objections: AT VII 155-157).

[3] For a more detailed dialectical reconstruction of the development of non-Euclidian geometry, see Gaston Bachelard's *New Scientific Spirit*.

[4] Hegel was more competent in mathematics than most philosophers. He taught calculus at the Gymnasium and understood the state of the discipline well enough to support continental analysis against the geometrical methods of the Newtonian school (a debate in which he backed the right horse, though not necessarily for the right reasons). I am indebted to Prof. Westphal on this point.

have no role in choosing the axioms from which it proceeds. These axioms must either be somehow absolutely self-evident, or they must be supplied by some other discipline.[5] Hegel opts for the latter:

> [Axioms] are commonly but incorrectly taken as absolute firsts, as though in and for themselves they required no proof... If, however, axioms are more than tautologies, they are *propositions* from some *other science*... Hence they are, strictly speaking, *theorems*, and theorems taken mostly from logic.[6]

By 'logic', of course, Hegel means a philosophical reflection on essences. It is by thinking about the essence of the triangular, the round, *etc*. that the first principles of geometry are established. Thus, on Hegel's view, mathematics receives its axioms from philosophy.

In fact the formal deduction of theorems from axioms is only part of the story of mathematical thought. Mathematical results may be discovered empirically, groped for conceptually, modified in the light of proofs and counterexamples and subsumed under theories intended to solve some other problem entirely. The final, polished, deductive proof typically appears late in the day, after a lengthy period of informal to-ing and fro-ing during which definitions are refined and axiom-candidates selected. As the geometric example above suggests, mathematics looks far less dead, and far more dialectical, when we look beyond the formal, deductive proofs found in Euclid's *Elements* and consider the whole of mathematical thinking. Now, there have been a number of attempts to find dialectics in mathematics.[7] Indeed, Hegel himself held that mathematics blundered into something dialectical with the development of the differential calculus.[8] However, he thought that mathematics could neither understand its own success, nor repair its logical defects because mathematical thought is fundamentally undialectical. In Hegel's view, the logical difficulties associated with the calculus could only be solved if the requisite concepts were developed within a general dialectical logic. This Hegel tried to do in the section of the *Science of Logic* headed 'quantum'. His approach makes sense because the calculus is, after all, the

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[5]Hegel did not consider the view that mathematical results are all conditional statements: *if* these axioms are true *then* those theorems follow. The existence of this third option may be thought to dispose of the question of how we can know that our mathematical axioms are true. However, the selection problem remains, even if it is admitted that axioms need not be true of anything but need only be 'interesting'. Some version of Hegel's point will stand so long as it is conceded that some rational standard or goal guides the choice of axioms. For we can always ask a) how it is that the axioms chosen meet the standard or serve the goal in view? and b) how was this goal settled on?

[6]*The Science of Logic* vol. two, section 3, chapter 2 A (b) 'The Theorem' (p. 808). The logicist school (including Frege, Russell and the Tractatus-period Wittgenstein) tried to argue that the axioms of mathematics are *not* more than tautologies. The effort is generally held to have foundered on Russell's paradox and Gödel's incompleteness results.

[7]See *The Science of Logic* vol. one, book I, section 2, chapter 2 C (c) 'The Infinity of Quantum' (pp. 252-313). Engels offered a popular version of the same theme in *Anti-Dühring* part I §xii-xiii. For a wholly different attempt to read dialectics into mathematics see Peter Várdy 'On the Dialectics of Metamathematics' (**Graduate Faculty Philosophy Journal** vol. 17, 1994 pp. 191-216). Moving the opposite direction, there is an attempt to build a mathematical model of dialectical logic in R.S. Cohen (ed.) *Hegel and the Sciences* (1984).

[8] "[The infinitesimal] magnitudes have been defined as such that they *are* in their vanishing, not *before* their vanishing… or *after* their vanishing… Against this pure notion it is objected… that such magnitudes are *either* something *or* nothing; that there is no *intermediate state* between being and non-being… But against this it has been shown that… *there is nothing which is not an intermediate state between being and nothing*. It is to the adoption of the said determination, which understanding opposes, that mathematics owes its most brilliant successes." (*The Science of Logic* vol. one, book I, section 1, chapter 1 'Being' C 'Becoming' 1 Remark 4; pp. 104-5)

mathematics of change.[9] In fairness we should record that the calculus was in logical disarray until late in the nineteenth century. It was beyond argument that there was something about the calculus which mathematicians did not understand. Naturally, Hegel produced a diagnosis that emphasises the alleged superiority of his own logical system.[10] What matters for the present argument is that, for Hegel, the dialectical part of the job (the derivation of the requisite determinations of the notion) belongs exclusively to philosophy.

*Lakatos*

In his *Proofs and Refutations* the Hungarian philosopher Imre Lakatos showed, using detailed examples, that there is a lot more to mathematical thought than the formal deduction of theorems from axioms. What is more, his account of the additional, non-deductive part of mathematical thinking shares many of the features attributed by Hegel to dialectical (or 'philosophical') cognition. Indeed, Lakatos agreed with Hegel that the assumptions and methods of deductive mathematics stand in need of justification, and that this grounding must be dialectical in character. However, he maintained that mathematicians can (and do) supply this need themselves. He also differed from Hegel in thinking that perfect certainty cannot be achieved in mathematics any more than in physical science.

It is no secret that Lakatos' analysis of mathematical thought owes much to Hegel.[11] Lakatos cited Hegel's dialectic as one of the three 'ideological' sources of the Ph.D. thesis which eventually became *Proofs and Refutations* (the other two being Pólya's mathematical heuristic and Popper's critical philosophy). He followed Hegel in using biological metaphors to express the idea that living systems of thought are constantly in flux, and therefore our grasp of such a system must have a corresponding dynamism. Knowledge is said to *grow*; theories must survive *childhood illnesses* if they are to reach *maturity*. In particular he argued that formal logic can only dissect dead theories (see *Proofs and Refutations* p. 3n3).

*Proofs and Refutations* is primarily a polemic against 'formalism' in the philosophy of mathematics. 'Formalism' identifies mathematics with its 'formal shadow', that is, with the formal systems of axioms, proofs and theorems which are often, nowadays, its end-product.

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[9] Here as everywhere Hegel aims to find order in the concrete rather than impose order upon it: "*continuous magnitude, becoming, flow*, etc.,… are *formal* in that they are only general categories which do not indicate just what is the *specific nature* of the subject matter [of calculus], this having to be learned and abstracted from the concrete theory, that is, the applications." (*The Science of Logic* vol. one, book I, section 2, chapter 2, C, (c) Remark 2 'The purpose of the Differential Calculus Deduced from its Application'; p. 302n).

[10] The problem was to make sense of $dy/dx$ given that $dy$ and $dx$ are each equal to zero. Hegel's solution was to regard $dy$ and $dx$ as 'moments' whose nature is to vanish. The solution settled on by mainstream mathematics was to abandon the $dy/dx$ notation altogether in favour of Cauchy's ε-δ formulation. This solution contains no dynamic elements of the sort envisioned by Hegel.

[11] Ian Hacking has already suggested a Hegelian influence in the metaphysics of Lakatos's philosophy of science and his use of history in philosophical argument ("Imre Lakatos's Philosophy of Science"). John Kadvany ("A Mathematical *Bildungsroman*") cites Hegel's *Phenomenology* to explain the literary form of *Proofs and Refutations*. All efforts to trace a line from Lakatos back to Hegel suffer from the difficulty that Lakatos learned his Hegel at a time when he was a convinced Marxist. Consequently, the Hegel in Lakatos has been filtered through the Hungarian Marxist tradition of which Lukács is perhaps the most famous representative. Moreover, it is difficult to judge the extent to which Lakatos was aware of the Hegelian-Marxist influence on his work. While citing Hegel's dialectic as an 'ideological source' of his essay Lakatos made no attempt to identify precisely what the contribution of this Hegelian tributary might have been. See also Larvor (1998) pp. 23-29, 65-71, 102.

This leads (according to Lakatos) to the philosophical neglect of everything about mathematics not caught by formalisation. It may also in time distort mathematical practice. Central to his critique of 'formalism' is his distinction between (in his own terminology) the 'heuristic' and 'deductivist' styles of mathematics.[12]

*Heuristic Vs. Deductivism*

Lakatos spelled out this distinction in what appears as the second appendix to *Proofs and Refutations*, entitled "The Heuristic Approach" (drawn from chapter three of his thesis). The prevailing 'Euclidian' or 'deductivist' methodology, according to Lakatos, requires mathematical work to be presented in a very particular manner:

> This style starts with a painstakingly stated list of *axioms, lemmas* and/or *definitions*. The axioms and definitions frequently look artificial and mystifyingly complicated. One is never told how these complications arose. The list of axioms and definitions is followed by the carefully worded *theorems*. These are loaded with heavy-going conditions; it seems unlikely that anyone should ever have guessed them. The theorem is followed by the proof. (p. 142)

This 'deductivist' style is pernicious, according to Lakatos, because it hides the struggle through which the finely tuned theorem and its definitions were achieved. The original problem, the first naïve conjecture and the critical process of its refinement are banished to history. Meanwhile, the end-product is regarded as an infallible truth, hedged about as it is with proof-generated monster-barring definitions ('monster-barring' is Lakatos' term for the trick of fending off putative counterexamples to a theorem by restricting the scope of its central terms).[13]

Part of the trouble is that students are required to take the repertoire of definitions and theorems on trust. This authoritarianism is repugnant to Lakatos—it is not surprising that someone with Popperian sympathies should find fault with an intellectual culture that refuses to acknowledge the importance of criticism. His main claim, however, is that the 'deductivist' style permits a kind of degeneration, because it allows authors to 'atomise' mathematics, that is, to present proof-generated definitions separately and ahead of the proofs from which they were born. This tearing apart of heuristic connections obscures the 'problem situation' from which the theorem and proof emerged. The 'deductivist' style permits this because its authoritarianism relieves authors of the responsibility of motivating their work. In 'Euclidian' methodology a theorem need not have a point; all it needs is a proof. Lakatos clearly suspected that some contemporary mathematics is indeed pointless:

> Stating the primitive conjecture, showing the proof, the counterexamples, and following the heuristic order up to the theorem and to the proof-generated definition would dispel the authoritarian mysticism of abstract mathematics, and would act as a brake on degeneration. A couple of case-studies in this

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[12]The quotation marks around 'formalism' mark the fact that Lakatos' use of the term is broader than is usual in the philosophy of mathematics. In his sense, Frege and Russell were formalists.

[13]The production of sophisticated definitions through criticism and proof-analysis is described in the main text of *Proofs and Refutations*, especially pp. 88-91.

Felix Klein distinguishes "Euclidian presentation" from that of certain French mathematicians, whose works "read just like a well written gripping novel". For Klein, as for Lakatos, this distinction reflects methodological differences (*Elementary Mathematics from an Advanced Standpoint* p. 84).

degeneration would do much good for mathematics. Unfortunately the deductivist style and the atomization of mathematical knowledge protect 'degenerate' papers to a very considerable degree. (*Proofs and Refutations* p. 154; see also p. 98n2)

In Lakatos' alternative style of mathematics—the 'heuristic approach'—a 'distilled' history of the theorem and its proof is the chief part of the exposition. This compressed and streamlined history starts not with definitions but with a problem or a question. A naïve answer ('primitive conjecture') is offered, and criticised. Through criticism the solution is improved and eventually the final result emerges. Thus the result is motivated by the initial question and its technicalities are explained by the narrative of conjectures and criticism.

Lakatos offered as an example Seidel's uniform convergence theorem (which states that a *uniformly convergent* sequence of continuous functions converges to a continuous function)[14]. A normal, 'Euclidian' presentation of Seidel's theorem starts by stating the definitions of continuity and uniform convergence:

(1) A function $f(x)$ is *continuous* at a point $x_0$ if and only if for every real number $\varepsilon > 0$ there is a real number $\delta > 0$ such that $\varepsilon > | f(x) - f(x_0) |$ whenever $\delta > | x - x_0 |$.

(2) A sequence of functions $f_0, f_1, f_2, \ldots f_n \ldots$ *converges uniformly* to a limit function $f$ on an interval $\boldsymbol{I}$ if and only if for every real number $\varepsilon > 0$ there is a natural number $N$ such that for every $x$ in $\boldsymbol{I}$, $| f_n(x) - f(x) | \leq \varepsilon$ whenever $n \geq N$.

Students have to try to understand these definitions without knowing what if anything their point is, (readers of this paper who find these definitions impenetrable may take solace in the fact that this is the present point). The proof follows and the definitions turn out to be exactly what one would need to prove just that theorem, but one is not told how those definitions were found, nor why this theorem is an important result.

An 'heuristic' presentation, by contrast, would tell a compressed version of the historical path from the earliest speculations about continuity through to Seidel's theorem. Thus the definitions would lose their *ex cathedra* quality and would appear instead as natural solutions to real problems. According to Lakatos, this theorem started life as an instance of the 'Leibnizian principle of continuity'. In this context, Leibniz's principle suggests that the limit function of a convergent sequence of continuous functions is itself continuous. This was accepted without proof, for it seemed obvious that what is true up to the limit must be true in the limit (*Proofs and Refutations* p. 128). The Leibniz principle only came to be doubted when Fourier's work on heat produced a family of counterexamples. E.g.:

$$\sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \ldots + \tfrac{(-1)^{n-1}}{n}\sin nx + \ldots$$

In this case, the partial sums are continuous but the limit function is not. At least, that is the case given the modern ε-δ definition of continuity. That definition is usually credited to Cauchy, who also produced a purported proof of the Leibniz continuity-principle. Thus,

[14]Lakatos used the emergence of this theorem several times to illustrate his approach to mathematics and its history: twice in the appendices to *Proofs and Refutations* and once in "Cauchy and the Continuum" (collected papers volume II).

Cauchy's attempt to rigorise analysis left mathematicians with a problem. On the one hand, they had Cauchy's proof of the Leibniz principle. On the other hand, they had the Fourier counterexamples to it. Seidel resolved the difficulty in his 1847 paper. By analyzing Cauchy's proof in the light of the counterexamples, Seidel was able to identify a hidden assumption. He made this assumption explicit and gave it a name: *uniform convergence*.

It might be argued that Lakatos overstates the case regarding the dangers of the deductivist style to mathematics. There may be some degenerate papers published, but mathematics does not seem to be in a general decline. However, if mathematics is in good shape, it is because the referees of mathematical journals know enough about the state of the discipline to distinguish useful, interesting contributions from trite or pointless proofs. In other words, referees carry an understanding of the heuristic background to formal proofs in their heads, and they use this knowledge to sift significant papers from degenerate ones. Lakatos' recommendation, then, is that this part of mathematical rationality should be set down in print along with the formal proofs. Until such heuristic reasoning is made explicit, it cannot be properly subjected to scrutiny, criticism and philosophical reflection.[15]

*Thesis-Antithesis-Synthesis*

Once, in *Proofs and Refutations* (pp. 144-5), Lakatos couched this argument in (what he took to be) 'Hegelian' language: Seidel's theorem is presented as the *synthesis* in a dialectical triad. The *thesis* is the Leibniz continuity-principle. The *antithesis* is the family of Fourier counterexamples. The dialectical opposition was heightened and clarified by Cauchy's ε-δ definition of continuity, which excluded the various available compromises, thereby preventing the thesis and antithesis from coexisting. Seidel produced the synthesis by analysing the proof of the thesis in the light of the counterexamples. In order to express the improved theorem and proof, he had to define a new concept, *uniform convergence*. Thus, the process of proof and criticism is creative (*pace* Popper), giving rise to new conjectures and new, proof-generated concepts. This is possible because the counterexamples do not only show that the naïve conjecture is false: they point to a specific problem, the solution of which leads to a new conjecture. In 'Hegelian' jargon, the counterexamples do not stand in 'bare opposition' to the Leibniz principle, but rather offer a 'determinate (i.e. specific) negation' of it. The synthesis in this three-step does not simply unite the best of the thesis and the antithesis. Rather, the synthesis solves the problem posed by the antithesis for the thesis.

Lakatos acknowledged that even in this paradigmatic case, the 'Hegelian' vocabulary is just a way of talking about the episode that has drawbacks as well as advantages. Lakatos' mathematical 'Hegelianism' would be of little interest if it consisted only in his having once employed the three-step model of knowledge-growth. It is a familiar fact that this pattern can be 'found' in almost any intellectual field if it is searched for with sufficient ingenuity. Indeed it would hardly count as Hegelianism, since Hegel never described his own logic in these terms.[16] Any rigid, formal model of dialectical thinking would be unacceptable to

[15] One kind of degeneration identified by Lakatos is *cheap generalisation*: it is often the case that generalising a formula requires no insight and offers no illumination (*Proofs and Refutations* pp 80-81, 97-98). Hegel makes exactly this complaint about generalisations of the binomial formula (*The Science of Logic* vol. one, book I, section 2, chapter 2, C, (c) Remark 2 'The purpose of the Differential Calculus Deduced from its Application'; p. 280n1).

[16] Hegel used the three terms together only once, and then to castigate Schelling's 'schemetising formalism' (here I am grateful to Prof. Westphal). On the other hand, Hegel's efforts to transcribe the music of reason were

Hegel and Lakatos alike ("One cannot describe the growth of knowledge… in 'exact' terms, one cannot put it into formulae"[17]). However, Lakatos' distinction between the deductivist and heuristic styles owes much more to Hegel than that.

To see the connection, we must move beyond the crude thesis-antithesis-synthesis formula. Recall Hegel's central criticism of mathematical thought as he understood it:

> [Mathematical] proof,... follows a path that begins somewhere or other without indicating as yet what relation such a beginning will have to the result that will emerge. In its progress it takes up *these* particular determinations and relations, and lets others alone, without its being immediately clear what the controlling necessity is; an external purpose governs this procedure. (*Phenomenology of Spirit* §44, p. 25)

In fact, what Hegel describes here is mathematics *in the deductivist style* (indeed, it is clear from the surrounding text that Hegel has Euclid in mind). To use Lakatos' example, deductivist presentations of Seidel's theorem and proof begin with the definitions of continuity and uniform convergence 'without indicating as yet what relation such a beginning will have to the result that will emerge'. The 'controlling necessity' is the heuristic background to the proof. That is what explains the choice of 'determinations' (definitions and lemmas). This 'purpose' is only 'external' because Euclidian methodology requires that the heuristic background be banished from the scene. In other words, Hegel's distinction between dialectical and so-called 'mathematical' reasoning is a direct ancestor of Lakatos' distinction between the heuristic and deductive styles. Hegel's claim is that in dialectical reasoning, each stage grows out of and is explained by what came before. There is no need to take definitions or 'problems to solve' on trust, in the hope that they will turn out to be just right for the job. Lakatos' claim is that mathematics done in the heuristic style has the same virtue.

Hence, Lakatos' use of 'Hegelian' jargon was not casual. Hegel's dialectical logic is an attempt to represent the emergence of new concepts as a rational process, which is exactly what Lakatos wanted to achieve for mathematics. Hegel's method was to write a special kind of 'distilled' or 'philosophically comprehended' history ('distilled' is Lakatos' word; 'philosophically comprehended' is the Hegelian expression). Such histories explain the advent of a new concept by portraying it as the solution to a logical problem. These narratives can serve several purposes: they offer a reading of intellectual history; they can be didactically useful; and they can work as philosophical argument. Lakatos saw his own essay in this idiom, "The dialogue form[18] should reflect the dialectic of the story; it is meant to contain a sort of *rationally reconstructed or 'distilled' history*" (*Proofs and Refutations* p. 5, Lakatos' emphasis). As noted above, he intended it primarily as a philosophical argument against 'formalism' ("the school of mathematical philosophy which tends to identify

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usually scored in waltz time. Consequently, the thesis-antithesis-synthesis model probably comes as close to capturing Hegel's logic as any formal model could. But the main point remains the inadequacy of *all* formal models.

[17] Collected papers volume II pp. 136-7. In a letter to Feyerabend Lakatos hotly denied that his rationalism was 'mechanical'. In other words, Lakatos like Hegel thought that scientific rationality should find order in the subject matter rather than imposing some rigid methodology on it from without. See Larvor (1998) pp. 80-88.

[18]Hegel of course did not present his own dialectical works in dialogue form. In his view, human characters are superfluous to the development of the subject-matter.

mathematics its formal abstraction" *Proofs and Refutations* p. 1). However, he also thought that recognition of dialectical patterns in mathematics would lead to better history and better pedagogy.

*Theorem and Proof*

Part of Hegel's criticism of mathematical thought was that theorems are not essentially related to their proofs. The same theorem may have several different proofs. It would seem that we can understand a mathematical hypothesis in advance of any attempt to prove it. Another version of the same thought is Hegel's complaint that the premises of a mathematical proof are not altered by its conclusion. As above, Hegel is right on these points so long as we restrict our attention to the formal chains of reasoning found in deductivist expositions of mathematics. However, Lakatos showed (following Pólya) that proofs and theorems may evolve together. Starting with a hunch, a hypothesis or a problem one formulates a primitive conjecture. In trying to prove this naive conjecture one may find some exceptions to it. Now, a proof cannot be sound if its conclusion suffers exceptions. Therefore the proof must be examined, the error found and the premises adjusted accordingly. This was the pattern in the example of uniform convergence mentioned above. There the adjustment took the form of a *proof-generated definition*: the definition of uniform convergence emerged from Seidel's analysis of Cauchy's attempt to prove Leibniz's principle. In other words, the dialectical argument from Leibniz through to Seidel *did* bring about a change in the premises of the proof, namely, the introduction of the concept of uniform convergence.

The bulk of *Proofs and Refutations* is taken up with a dialogue in a fictional mathematics class. The pupils discuss the Descartes-Euler formula $V - E + F = 2$ concerning polyhedra. The hypothesis is that for any polyhedron the number of vertices (***V***) minus the number of edges (***E***) plus the number of faces (***F***) will always equal two. This result is easy to check for the regular Platonic solids, but does it hold generally? The first attempted proofs worked by showing that the formula is not affected by sawing corners off polyhedra for which it holds, nor by adding 'roofs' to their faces. In other words, these proofs treated the formula as a theorem about solids that can be sliced like cheese. The drawback with this approach is that it is difficult to check that all polyhedra can be generated out of the known solids by roofing and slicing. Later proofs took a different route. In these later thought-experiments the polyhedron is pumped up until it is spherical so that its edges form a map on a globe; or one side is replaced by a camera lens pointing inside; or one side is removed and the remaining sides stretched flat to form a plane network.

Crucially, these manipulations cannot be carried out on solids. In introducing these new proofs, mathematicians quietly shifted from treating polyhedra as solids to treating them as closed surfaces. This shift generated problems and questions of its own. For example, it was now possible to speak intelligibly of polyhedral faces intersecting one another. Thus a whole new class of polyhedra came to the attention of mathematicians. It turned out that the Euler formula was true of some of these new shapes, but not of others. In order to explain this fact it was necessary to develop the theory further still. To cut a long story short,

> The 'theory of solids', the original 'naive' realm of the Euler conjecture, dissolves, and the remodelled conjecture reappears in projective geometry if proved by Gergonne, in analytical topology if proved by Cauchy, in algebraic topology if proved by Poincaré. (*Proofs and Refutations* p. 90)

In short, there may be an intimate relation between a theorem and its proof (*pace* Hegel).

In order to see such a relation one must look for theorems that employ proof-generated concepts, for it is the process of conceptual refinement that ties theorem and proof together. A mature mathematical concept is (for Lakatos) produced through the diagnostic analysis of failed proofs and rejected theorem-candidates. The theorems of Euclidean geometry do not qualify because their central concepts (circle, triangle, etc.) are not the results of any developed mathematical dialectic. Rather they are simple idealisations of empirical forms.[19] If Hegel had Euclid in mind then it is not surprising that he saw no dialectical movement in mathematical thought. In contrast, Seidel's theorem cited above does illustrate the point. The theorem and its proof are the joint product of the emergence of the concept of uniform convergence.[20] Now, the fact that a theorem and its proof may share a history does not amount to the comprehensive identity of process and product promised by Hegelian logic. We can still detach the theorem and employ it independently of its proof. However, no special discipline meets this Hegelian standard, and none can. The absolute unity of process and product is only to be expected of absolute knowledge. The mere fact that the product of mathematical thinking is dead and formal does not show that the same is true of the thinking itself. All living things excrete dead matter and leave corpses. For the present purpose, it is enough to see that mathematics is not limited to the formal manipulation of concepts drawn from elsewhere. Mathematics is able to generate new concepts of its own. It has the power of self-movement.

*Understanding and Reason*

So far, then, Lakatos argues that there is a lot more to mathematical rationality than the mechanical deduction of theorems from axioms. This informal, non-mechanical part of mathematical rationality turns out, on Lakatos' account, to have much in common with 'philosophical' or 'dialectical' thought as Hegel understood it. That is why Hegel's (largely well-taken) criticisms of 'mathematical' (i.e. deductivist) proof do not tell against proofs in Lakatos' 'heuristic' style. It is, then, hardly surprising that when the fictional students turn their attention away from the Euler formula in favour of methodology in general, they find themselves articulating a familiar Hegelian distinction.

Lakatos' students learn that the word 'polyhedron' no longer means what it did at the outset. Euler's polyhedra were solids while Poincaré's polyhedra are rather abstract algebraic objects (indeed Descartes and Euler counted solid angles where today we would count vertices[21]). Once their attention has been directed to the topic of conceptual evolution, they come to see that the concept *polyhedron* has been in flux all along. Originally, it contained only the Platonic solids and similarly simple and well-behaved geometrical objects. Then (goes Lakatos' story) mathematicians began to consider less straightforward objects, such as

[19] Or so it must have seemed to Hegel. More recent research tells a more interesting story. See *Proofs and Refutations* p. 49n1. David Reed offers a reading of the *Elements* in which "The subject matter drives the argument. Euclid is not free to select a set of postulates according to philosophical predisposition, pedagogical efficiency or a subjective sense of beauty in mathematics." (*Figures of Thought* p. 19).

[20] There are some clues in the oral folklore of contemporary mathematics. Jean-Pierre Marquis reports that "There are very often many different proofs of one and the same result. Many mathematicians feel that often one of these proofs provides 'the real reason' for a result..." ('Tools and Machines' p. 270n22).

[21]See *Proofs and Refutations* p. 6n1.

solids with tunnels through them, or objects hinged together along an edge. The Euler formula does not hold for many of these items. Therefore, if they count as polyhedra then they are counterexamples to the claim that the Euler formula holds for *all* polyhedra. In the event, the class produced a new theorem: for *simple polyhedra with simply-connected faces*, the Euler formula holds (where 'simple' is a defined technical term). Lakatos argues (p. 88) that this new theorem has the effect of tacitly expanding (or 'stretching') the meaning of 'polyhedron'. Whereas the old polyhedron-concept only included simple solids, the new one embraces whole families of Swiss cheeses, step pyramids and other topological curiosities.

The point is that these complex objects with ring-shaped faces are only counterexamples *after* the new, expanded polyhedron-concept has displaced the old, narrow one. Now, if it were the sole aim of mathematics to produce true theorems, it would seem that this stretching of the concept was a mistake. For what was a true theorem has now acquired counterexamples and is therefore, strictly, false. However, we are not satisfied with merely true theorems. We want *deep* theorems. The expansion of the polyhedron-concept to include objects with holes, steps and hinges does not only yield greater generality. It exposes the fact that the phenomenon at hand is more properly topological than geometrical. In general:

> You cannot separate refutations and proofs on the one hand and changes in the conceptual, taxonomical, linguistic framework on the other. Usually, when a 'counterexample' is presented, you have a choice: either you refuse to bother with it, since it is not a counterexample in your *given* language $L_1$, or you agree to change your language by concept-stretching and accept the counterexample in your new language $L_2$... According to traditional static rationality, you should make the first choice. Science teaches you to make the second. (p. 93)

The thought that theoretical advances are inseparable from shifts of meaning is not, by itself, distinctively Hegelian. It can be found in Quine, for example. What is Hegelian is the contrast between 'traditional static rationality' (which insists that the meanings of terms should remain fixed for fear of committing the fallacy of equivocation), and the rationality of science (which accepts conceptual change as a concomitant of progress). This distinction between a rationality of fixed concepts and a rationality of concepts in flux is a mathematical version of Hegel's contrast between 'understanding' (*Verstand*) which "determines, and holds the determinations fixed"[22] and 'reason' (*Vernunft*) which unpicks and reconfigures these determinations.

*Differences*

Lakatos' philosophy of mathematics never stabilised into a settled view. One issue in particular seems unresolved: the question of mathematical ontology. In *Proofs and Refutations* he describes, and seems to endorse, a 'Hegelian' view:

> Mathematics, this product of human activity, 'alienates itself' from the human activity which has been producing it. It becomes a living, growing organism, that *acquires a certain autonomy* from the activity which has produced it; it develops its own autonomous laws of growth, its own dialectic. (p. 146)

[22] *Science of Logic* preface to the first edition (p. 28).

The suggestion seems to be that the dialectical development of mathematical knowledge is coupled with (if not identical to) the dialectical evolution of mathematical reality. This view has something in common with Popper's emergentism. For Popper, the content of thoughts, theories and propositions has an objective existence in the 'third realm'. Here, logical problems exist independently of our awareness of them. This third realm (the other two being the material and the mental) supervenes upon the activities of humans and our machines. Crucially, Popper thought that numbers and mathematical structures also exist in this third realm. However, Popper and Lakatos were both resolutely opposed to the crassly determinist Hegel found in the pages of *The Open Society and Its Enemies*.[23] In a footnote, Lakatos adds:

> My concept of the mathematician as an imperfect personification of Mathematics is closely analogous to Marx's concept of the capitalist as the personification of Capital. Unfortunately Marx did not qualify his conception by stressing the imperfect character of this personification, and that there is nothing inexorable about the realisation of this process. On the contrary, human activity can always suppress or distort the autonomy of alienated processes and can give rise to new ones. (p. 146n1)

Lakatos did not believe progress in any area to be historically inevitable. It is not only that mathematical work can be interrupted by inconvenient wars and neuroses. Rather, the point is that intellectual problems do not determine their own solutions. There may be several ways out of any given logical maze and no guarantee that the best will be chosen. Indeed, there may be no criterion by which one solution can be identified as the best *post hoc*. Moreover, Lakatos was a fallibilist regarding mathematics, science and philosophy. The distant prospect of absolute knowledge was, for him, a mirage.

Still, the above passage suggests that Lakatos adopted some sort of emergentism. However, in another paper he opts for a kind of Platonism:

> I have very strong feelings against Popper's linguistic conventionalist theory of mathematics and logic. I think with Kneale that logical necessity is a sort of natural necessity; I think that the bulk of logic and mathematics is God's doing and not human convention. (*Philosophical Papers* vol. 2 p. 127)[24]

On this evidence it would appear that Lakatos' mathematical dialectic is purely epistemological. For him, mathematical knowledge grows and reinvents itself in order to improve our apprehension of a fixed mathematical reality:

> As far as naive classification is concerned, nominalists are close to the truth when claiming that the only thing that polyhedra have in common is their name. But after a few centuries of proofs and refutations, as the theory of polyhedra develops, and theoretical classification replaces naive classification, the balance changes in favour of the realist. (*Proofs and Refutations* p. 92n1)

In short, the little that Lakatos wrote on ontology was unclear and unremarkable. It is unsurprising that he had little to say about mathematical ontology because the method of rational reconstruction gives no purchase on such questions. The rational reconstruction of thought within thought cannot tell us anything about any reality that may lie outside our

[23]Most Hegel scholars have clear views regarding the accuracy of the account of Hegel's philosophy in *The Open Society and Its Enemies*. I shall, therefore, resist the temptation to comment on the quality of Popper's exegesis.

[24]This paper dates from 1960 and is therefore contemporaneous with the 'Hegelian' passages quoted earlier.

thinking. Thus Lakatos was embarrassed by metaphysical questions because he combined realist doctrine with idealist philosophical method.

For Lakatos dialectical progress in mathematics is ‘quasi-empirical’. By this expression he meant that general mathematical theories bear the same relationship to small mathematical facts as general physical theories do to small material facts. Mathematical concepts develop through a cycle of conjecture, attempted proof, refutation and proof-analysis similar to that found in natural science. It is for this reason that Lakatos is a fallibilist in mathematics as in natural science. He was contemptuous of the ‘speculative proofs’ found in Hegel’s philosophy of nature because they seemed to suggest that infallible knowledge could be had, and without the pain of refutations too. Hegel seems to have thought that dialectical logic supplies mathematics with concepts (determinations of the notion) which mathematics then spins (formally) into theorems. Or rather, mathematics takes up forms found in nature and operates on them, but only as the understanding operates. It can rearrange but cannot transform. Hence mathematics needs philosophy to show that these forms are not arbitrary but are in fact grounded in the deepest categories of reason.[25] Lakatos thought that mathematics had to develop (and thereby ground) its own concepts using a ‘quasi-empirical’ dialectic because speculative proofs are not reliable.[26] Moreover if taken seriously they freeze thought at whatever stage it happens to have reached. In the case of the calculus, history seems to be on Lakatos’ side. Hegel’s approach was to give a logical grounding to the calculus as it stood in his day. Had his solution been adopted by mathematics we might never have had the great flowering of analysis and topology that has taken place since then.

An immediate consequence of Lakatos’ view is that formal and dialectical logic cannot be kept separate. In Hegel’s logic a concept develops out of itself. It does not need the formal machinery of theorem and counterexample because it contains its own opposite. There is thus a smooth progress from the seminal form of the concept to its final articulated state. Formal logic plays no part in this development. Equally, for Hegel, dialectical logic plays no part in mathematics proper. Since Lakatos rejected speculative proofs, he had to locate the dialectical process in mathematics itself. For him the development of mathematical thought is a perpetual struggle between the conservative tendency of formal logic and the revolutionary potential of dialectic. That is why, for Lakatos, mathematical thought has the power of self-movement (and indeed, self-restraint). Nevertheless, some of Hegel’s view of mathematics remains in good standing. Mathematical thought may be alive, but mathematical propositions are tenseless[27] and concerned with formal structure. On the other hand, mathematical thought is a) temporal and b) directed by its content as well as by its form. Consequently (in Lakatos’ view as in Hegel’s) mathematical thought cannot understand itself, and needs to turn to philosophy for illumination of its own workings.[28]

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[25] Thus for Hegel it is unsurprising that mathematics should discover but fail to understand the ephemeral $dy$ and $dx$ because change (or more precisely, becoming) is everywhere in nature.

[26] “Hegel and Popper represent the only fallibilist traditions in modern philosophy, but even they both made the mistake of reserving a privileged infallible status for mathematics” (*Proofs and Refutations* p. 139n1).

[27] This is one reason why Hegel’s solution to the calculus problem could not stand. Vanishing quantities are, presumably, tenselessly vanishing *now*. As we have learned since, the mathematics of change need not itself have any elements in flux.

[28] From what has been said here it may seem that mathematical thought is like a cat: a living organism, capable of self-movement, but incapable of self-understanding. In fact, mathematical logic provides mathematics with a self-*mis*understanding. Mathematical thought is thus like a mechanist philosopher who has a model of himself, but one that fails to explain how he is able to produce such models.

Hegel's view that mathematics always was and always will be simply the science of abstract space and number sits oddly given that, in Hegel, every other kind of human thinking or activity eventually transforms itself into something else. Only mathematics is excluded from the great ripening of the universe. There are historical reasons why Hegel should think this: the capacity of mathematics for self-transformation is more obvious now than it was then. Galileo famously observed that Aristotle would have changed his cosmology had he looked through a telescope, and that it is a better Aristotelianism to keep to Aristotle's general methods than it is to cling to his detailed doctrines.[29] In a similar spirit, I would suggest that present-day Hegelians abandon Hegel's dogmatically undialectical philosophy of mathematics. Then, they could recognise mathematics as a living body of thought, and its contribution to the development of human thinking in general could be properly understood.

[29]Galileo *Dialogue Concerning the Two Chief World Systems*, trans. S. Drake (Berkeley, Calif.: University of California Press, 1953), 56. Quoted in Losee, p. 57.

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