RELATIVE WIDTH OF SUBLATTICES

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Introduction

Kendall [5] has given a geometrical characterization of vector lattices among the vector spaces with a generating cone. The characterizing property is that the cone should have a linearly compact simplex as a base.

Among the vector lattices we specialize to those which admit a lattice norm with a certain nice property, called *proximity*, which is defined below. All Abstract Lebesgue spaces have the proximity property.

In this note, we give (for normed lattices with the proximity property) a characterization of the norm-closed sublattices among the vector subspaces. The characterizing property is that of *thickness*, also defined below.

These results have application in the operational approach to the theory of statistical systems and elsewhere (see for example [4], [6, Ch. 1]).

Definitions. Let K be a convex set, K_o a convex subset of K. Say that $[x, y]$ is a *segment* in K_o *iff* $x, y \in K_o$ and $x \neq y$.

Say that K_o is a *thick* subset of K *iff* for all segments $[x, y]$ in K_o we have

$$\begin{aligned} &\sup\{t \in R^+ : \exists x', y' \in K_0 \ \ st \ \ x' - y' = t(x - y)\} \\ =&\sup\{t \in R^+ : \exists x', y' \in K \ \ st \ \ x' - y' = t(x - y)\} < \infty. \end{aligned}$$

Let Z be a normed vector lattice. Say that Z has the *proximity* property *iff* for all $x \geq 0$ with $\|x\| = 1$ and all $\varepsilon > 0$, there exists $t < 1$ such that for all $z \geq tx$ with $\|z\| \leq 1$ we have $\|z - x\| < \varepsilon$. This amounts to demanding that

$$\operatorname{diam}\big((tx + C) \cap B\big) \to 0 \quad \text{as } t \uparrow 1.$$

Theorem. *Let Z be a normed vector lattice, equipped with positive cone C and closed unit ball B. Let K denote $B \cap C = \{x \geq 0 \ \ st \ \ \|x\| \leq 1\}$.*

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Let Z_0 be a positively generated linear subspace of Z (so that $Z_0 = C_0 - C_0$ where $C_0 = Z_0 \cap C$) and let K_0 denote $Z_0 \cap K$.

If Z has the proximity property, then Z_0 is a norm closed sublattice of Z iff *K_0 is a norm closed thick subset of K.*

Proof. Let $[x, y]$ be a segment in K and define

$$k = \max\{\|(x-y)^+\|, \|(y-x)^+\|\} > 0; \quad \widehat{x} = \frac{(x-y)^+}{k}; \quad \widehat{y} = \frac{(y-x)^+}{k}$$

where $z^+ = z \vee 0$. Note that either $\widehat{x}$ or $\widehat{y}$ has norm one. We show that $[\widehat{x}, \widehat{y}]$ is the unique thick segment in K parallel to $[x, y]$.

Certainly $[\widehat{x}, \widehat{y}]$ is parallel to $[x, y]$. Now suppose $[x', y']$ another segment in K with $x' - y' = t(\widehat{x} - \widehat{y})$. Assume without loss of generality that $\|\widehat{x}\| = 1$. Then

$$x' \geq t(\widehat{x} - \widehat{y}) \vee 0 = t\widehat{x} \quad \text{since} \quad \widehat{x} - \widehat{y} = (x-y)/k.$$

Hence (taking norms) $t \leq 1$, so $[\widehat{x}, \widehat{y}]$ is thick in K. If $t = 1$ then $x' = \widehat{x}$ by proximity, whence $y' = \widehat{y}$ also.

Now suppose that Z_0 is a sublattice of Z. We show that K_0 is thick in K.

Let $[x, y]$ be a segment in K_0. Then $[\widehat{x}, \widehat{y}]$ is also in K_0 since Z_0 is a sublattice. But $[\widehat{x}, \widehat{y}]$ is thick in K. Hence K_0 is thick in K.

Now suppose that K_0 is thick in K. We show that for all $x, y \in Z_0$ we have $x \vee y \in Z_0$.

Since Z_0 is positively generated, we may assume without loss of generality that $[x, y]$ is a segment in K_0 and that $\|\widehat{x}\| = 1$. Since $x \vee y = (x-y)^+ + y$ it suffices to show that $\widehat{x} \in K_0$. Choose $t \in (0, 1)$. By thickness, there exist $x', y' \in K_0$ *st* $x' - y' = t(\widehat{x} - \widehat{y})$ for this t. But

$$x' \geq t(\widehat{x} - \widehat{y}) \vee 0 = t\widehat{x}$$

as before, so by proximity x' converges in norm to $\widehat{x}$ as t approaches one. Since K_0 is norm closed, we have $\widehat{x} \in K_0$.

Norm closure of Z_0 now follows from the norm continuity of the lattice operations. *qed*

Discussion. The term *thick* used here is equivalent to *relative width one* in the sense of P.D. Taylor [**8**]. This property was used by Anderson and Batty to characterize Choquet simplexes among compact convex sets by a facial extension property. See [**1**], [**2**, Lemma 3 Proof].

Clearly every Abstract Lebesgue space has the proximity property, for if $x, z > 0$ with $||x|| = 1$, $||z|| \leq 1$, $z \geq tx$ then

$$||z - x|| \leq ||z - tx|| + ||x - tx|| \leq 2(1 - t).$$

Various other characterizations of the closed sublattices of Lebesgue spaces are known, for example they are precisely the ranges of the contractive positive projections of the space. Generally these projections can be interpreted as conditional expectation operators applied to the integrable functions (or measures) making up the space (see for example [**7**].)

In the light of these results the characterization in our Theorem can be regarded as giving an intrinsic geometric criterion for a set of positive measures or density functions (representing physical states) to be the range of a conditional expectation defined on some larger set.

It is important to note that Z_0 may be a lattice in the order inherited from C without being a sublattice of Z. For further discussion and examples of the significance of this point see [**3**].

References

1. T.B. Anderson, *Relative Widths and Extension Constants for Compact Convex Sets*, J Lond. Math. Soc. **(2)22** (1980), 333–337.
2. C.J.K. Batty, *A Characterization of Choquet Simplexes by an Extension Property*, Quart J. Math. Oxford **(2)34** (1983), 391–397.
3. D.B. Christianson, *The Positive Fixed Points of Banach Lattices*, Proc. Amer. Math. Soc. **107(1)** (1989), 255–260.
4. E.B. Davies and J.T. Lewis, *An Operational Approach to Quantum Probability*, Comm. Math. Phys. **17** (1970), 239–260.
5. D.G. Kendall, *Simplexes and Vector Lattices*, J. Lond. Math. Soc. **37** (1962), 365–371.
6. L.M. LeCam, *Asymptotic Methods in Statistical Decision Theory*, Springer Series in Statistics, Springer-Verlag, 1986.

7. H. Miyadera, *A Characterization of Conditional Expectations for $L_\infty(X)$-valued Functions*, Osaka J. Math. **25(1)** (1988), 105–113.

8. P.D. Taylor, *The Extension Property for Compact Convex Sets*, Israel J. Math. **11** (1972), 159–163.

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