POSTIVE FIXED POINTS OF LATTICES UNDER SEMIGROUPS OF POSITIVE LINEAR OPERATORS

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Abstract. Let $Z$ be a Banach lattice endowed with positive cone $C$ and an order-continuous norm $\| \cdot \|$. Let $G$ be a semigroup of positive linear endomorphisms of $Z$. We show that if $G$ is left-reversible (a weaker condition than left-amenability) then the positive fixed points $C_0$ of $Z$ under $G$ form a lattice cone, and their linear span $Z_0$ is a Banach lattice under an order-continuous norm $\| \cdot \|_0$ which agrees with $\| \cdot \|$ on $C_0$. Simple counterexamples show that $Z_0$ need not contain all the fixed points of $Z$ under $G$, and need not be a sublattice of $(Z, C)$. Our proof is a simple embedding construction which allows other such results (with different conditions on $G$) to be read off directly from appropriate fixed point theorems. Results of this kind find application in statistical physics and elsewhere.

Definition 1. A semigroup $G$ is called left-reversible iff for all $T_1, T_2 \in G$ there exist $T_3, T_4 \in G$ such that $T_1 T_2 T_3 = T_2 T_1 T_4$.

A right ideal of a semigroup $G$ is a set of the form $TG$ where $T \in G$. Left-reversibility of $G$ is equivalent to demanding that every pair of right ideals of $G$ intersect non-trivially. Left-reversibility is a weaker condition than left-amenability for discrete semigroups since the support of any left-invariant mean must be contained in every right ideal. It is strictly weaker since (for example) the free group on two generators is left-reversible (because it is a group) but is not left-amenable (because it is not solvable.) For a survey of the relationships between left-reversibility and other properties of semigroups, see [6, §8].

Proposition 2. Let $Z$ be an order-complete vector lattice with positive cone $C$, and let $G$ be a semigroup of positive order-continuous linear operators from $Z$ into $Z$. Let $C_0 = \{ x \in C : Tx = x \text{ for all } T \in G \}, Z_0 = C_0 - C_0 = \{ x - y : x, y \in C_0 \}$.

If $G$ is left reversible then $(Z_0, C_0)$ is a vector lattice.

Proof. Choose $x, y \in C_0$. Write $x \lor y$ for the least upper bound of $x$ and $y$ in $C$, and let $A = \{ T(x \lor y) : T \in G \}$. Clearly

$$x + y = T(x + y) \geq T(x \lor y) \geq Tx \lor Ty = x \lor y$$

so $A$ is order-bounded above by $x + y$, and hence has a least upper bound $z$. For $T_1, T_2 \in G$ we have by left-reversibility of $G$ that

$$T_1(x \lor y) \leq T_1 T_2 T_3(x \lor y) = T_2 T_1 T_4(x \lor y) \geq T_2(x \lor y)$$

which shows that $A$ is directed as a subset of $C$, and hence $A$ (considered as a net) is order-convergent to $z$. The same argument shows that for each $T \in G$, $TA$ is a subnet of $A$, whence $Tz = z$ and so $z \in C_0$. Clearly $z = x \lor y$, the least upper bound in $C_0$ of $x$ and $y$. It follows that $C_0$ is a lattice cone and hence that $Z_0$ is a lattice.

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Under the conditions of Proposition 2, \( Z_0 \) need not contain all the fixed points of \( Z \) under \( G \), and need not be a sublattice of \((Z,C)\), as the following examples show.

**Example 3.** [1, Example 2] \( Z_0 \) need not equal the set of all fixed points of \( Z \) under \( G \), and this latter set need not be a lattice. Consider \( R^2 \) and let \( G \) be the semigroup \( \{T^n\} \) where \( T(x,y) = (2x + y, x + 2y) \). The fixed points are the line \( x + y = 0 \) but \( C_0 = \{0\} \).

**Example 4.** [1, Example 3] \( Z_0 \) need not be a sublattice of \((Z,C)\). Consider \( R^3 \) and let \( G \) be the semigroup \( \{T^n\} \) where \( T(x,y,z) = (x,y,x + y) \). The points \( a = (1,0,1), b = (0,1,1) \in C_0 \) have \( a \vee b = (1,1,1), a \vee_0 b = (1,1,2) \).

The conclusion of Proposition 2 may fail if \( G \) is not left-reversible, as the following example shows.

**Example 5.** [1, Example 1] \( Z_0 \) need not be a lattice for an arbitrary semigroup \( G \). Consider \( R^5 \) and define two projections \( P, Q \) by

\[
P(v,w,x,y,z) = (v,w,x,y,v+w), \quad Q(v,w,x,y,z) = (v,w,x,y,x+y).
\]

Let \( G \) be the semigroup \( \{P,Q\} \), then \( C_0 \) is the cone with square base \( \{(s,1-s,t,1-t,1) : s,t \in [0,1]\} \) so \((Z_0,C_0)\) is not a lattice. For example \((1,1,2,0,2)\) and \((1,1,1,1,2)\) are minimal upper bounds in \( Z_0 \) for \((1,0,1,0,1)\) and \((0,1,1,0,1)\).

**Proposition 6.** Let \((Z_0,C_0)\) be a vector lattice. Let \( Z \) be a Banach lattice endowed with positive cone \( C \) and order-continuous norm \( \| \cdot \| \), and suppose that \( Z_0 \) can be embedded in \( Z \) in such a way that \( C_0 \) is a norm closed subset of \( C \).

Then \( Z_0 \) is a Banach lattice with positive cone \( C_0 \) and order-continuous norm \( \| \cdot \|_0 \) defined on \( Z_0 \) by \( \|x\|_0 = \| |x|_0 \| \) where \( | \cdot |_0 \) is the lattice modulus on \((Z_0,C_0)\).

**Proof.** Straightforward, for details see the last part of the proof in [1, p. 257]. \( \square \)

Again, \( Z_0 \) may be a lattice in the order inherited from \( C \) but fail to be a sublattice of \( Z \). Conditions under which \( Z_0 \) is a sublattice of \((Z,C)\) in Proposition 6 are investigated in [2]. Although we always have \( \| \cdot \| \leq \| \cdot \|_0 \) on \( Z_0 \), the two norms may differ on non-positive elements of \( Z_0 \) such as the element \( a - b \) in Example 4, or the more drastic example following.

**Example 7.** [1, Example 4] \( Z_0 \) need not be norm closed in \((Z,\| \cdot \|)\), consequently \( \| \cdot \| \) and \( \| \cdot \|_0 \) need not be equivalent on \( Z_0 \). Let \( Z = l^1 \) and define \( T \) by

\[
(Tw)_{3k} = w_{3k-2} + w_{3k-1} + (1 - 2^{-k})w_{3k} \quad (Tw)_{3k+1} = w_{3k+1} \quad (Tw)_{3k+2} = w_{3k+2}
\]

for \( w = (w_k) \in Z \). Let \( G = \{T^n\} \) and define \( v \in Z \) by

\[
v_{3k} = 0; \quad v_{3k+1} = -v_{3k+2} = 1/2^k.
\]

Define \( Z_0 \) as in Proposition 2. Then \( v \) is in the closure of \( Z_0 \) under \( \| \cdot \| \) but is not in \( Z_0 \). This example also satisfies the assertions of Examples 3 and 4.
Proposition 8. Let $Z$ be a Banach lattice endowed with positive cone $C$ and an order-continuous norm $\| \cdot \|$. Let $G$ be a semigroup of positive linear endomorphisms of $Z$.

If $G$ is left-reversible then the positive fixed points $C_0$ of $Z$ under $G$ form a lattice cone, and their linear span $Z_0$ is a Banach lattice under an order-continuous norm $\| \cdot \|_0$ which agrees with $\| \cdot \|$ on $C_0$.

Example 5 shows that some condition on $G$ is required. However, we can often use a standard fixed point theorem to recover the conclusion of Proposition 8 for semigroups which are not left-reversible. As an illustration of this, we prove the following:

Definition 9. In the set-up of Proposition 8 call $G$ norm-distal iff $Gu$ is norm bounded away from zero for all $u \in Z - \{0\}$.

Proposition 10. Proposition 8 remains true if $G$ is assumed norm-distal in place of left-reversible.

Proof. Adopting the notation of Proposition 3, pick $x, y$ in $C_0$ and let $A$ be the smallest subset of $C$ containing $x$ and $y$ and closed under join and orbit, so that for $u, v \in A$ and $T \in G$ we have $u \vee v, Tu \in A$. Now $A$ is directed as a subset of $C$, and hence convergent to $z = \sup A \leq x + y$. Setting $K$ to be the order interval $[x \vee y, z]$, we have (using order continuity of the norm on $Z$) that the elements of $G$ act as continuous affine maps from the weakly compact set $K$ into itself [8, §2.4]. Since $G$ is distal, $K$ must have a fixed point under $G$ by the Ryall-Nardzewski fixed point theorem [11] [10]. This fixed point must be $z$, which is therefore the least upper bound of $x$ and $y$ in $C_0$. This is true for each choice of $x$ and $y$ in $C_0$, so $C_0$ is a lattice cone and the conclusion of Proposition 8 is recovered.

Different variations of Proposition 8 can be obtained by applying other fixed point theorems to the compact convex set $K$ defined in the proof of Proposition 10. See [5] for a selection of suitable fixed point properties and [4] for a range of recent related work. As well as yielding the new results presented here, our approach also gives simple transparent proofs for a wide range of known results. Properties of this kind find application in statistical mechanics [9] [12], quantum physics [3], statistical decision theory [7, Chapter 1] and elsewhere. We conclude this paper with a brief outline of the route to these applications.

Call a convex set $S$ a simplex if $S$ can be embedded in a vector lattice $Z$ as a base of the positive cone $C$. Classical Choquet theory says that if $S$ is a compact metrizable simplex then each point $x \in S$ is the barycentre of exactly one probability measure $\mu_x$ supported on $\partial_c S$, the set of extreme points of $S$. A great deal of work has been devoted to proving similar unique representation results for classes of non-compact or non-metrizable simplices. Lifting the measure norm to $Z$ then makes $Z$ a Banach lattice with $S = \{ x \in C : \| x \| = 1 \}$. 
If the cone $C$ represents the set of states of some process or system (with the convex base $S$ corresponding to the normalized states) then it is frequently desirable to know that $S$ is a simplex (equivalently, that $C$ is a lattice cone) of a type for which a unique representation result is known. For example if the elements of $S$ are the (normalized) Gibbs states of some physical or statistical process, and it is known that $S$ is a simplex of an appropriate kind, then each Gibbs state is uniquely expressible as an average over the set $\partial_c S$ whose elements now correspond to the observables at infinity.

Often we have some simplex $S$ of states or measures, but are interested only in the elements $S_0$ of $S$ which are fixed under some semigroup $G$ of linear endomorphisms of $C$, invariance conditions which correspond to physical or observational constraints. The elements of $G$ may be non-conservative or non-stochastic, so need not map $S$ into $S$ even if they have unit spectral radius.

Our results give a simple geometric (rather than measure-theoretic) approach to proving $S_0$ to be a simplex, for various sets of conditions on $G$. Provided that the appropriate conditions on $S_0$ are inherited from $S$, the fact that $S_0$ is a simplex then suffices to recover a unique representation theorem for elements of $S_0$ in terms of extreme points of $S_0$, i.e. observables of the right kind. Dynkin’s entrance boundary can be constructed along similar lines.

The case where $Z_0$ is not a sublattice of $Z$ perhaps merits more attention than it has received. Here conditional expectations with respect to the tail-field correspond to non-contractive projections.

References


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