DIVISION OF COMPUTER SCIENCE

Positive Fixed Points of Lattices under Semigroups of Positive Linear Operators

Bruce Christianson

Technical Report No.177

November 1993
POSITIVE FIXED POINTS OF LATTICES
UNDER SEMIGROUPS OF POSITIVE LINEAR OPERATORS

BRUCE CHRISTIANSON

November 1993

ABSTRACT. Let $Z$ be a Banach lattice endowed with positive cone $C$ and an
order-continuous norm $\|\|$. Let $G$ be a semigroup of positive linear endomor-
phisms of $Z$. We seek conditions on $G$ sufficient to ensure that the positive
fixed points $Z_0$ of $Z$ under $G$ form a lattice cone, and that their linear span
$Z_0$ is a Banach lattice under an order-continuous norm $\|\|_0$ which agrees with
$\|\|$ on $C_0$, although we do not require that $Z_0$ contain all the fixed points of
$Z$ under $G$, nor that $Z_0$ be a sublattice of $(Z,C)$. We give a simple embedding
construction which allows such results to be read off directly from appropri-
ate fixed point theorems. In particular, we show that left-reversibility of $G$
(a weaker condition than left-amenability) suffices. Results of this kind find
application in statistical physics and elsewhere.

Definition 1. A semigroup $G$ is called left-reversible iff for all $T_1, T_2 \in G$ there
exist $T_3, T_4 \in G$ such that $T_1T_2T_3 = T_2T_1T_4$.

A right ideal of a semigroup $G$ is a set of the form $TG$ where $T \in G$. Left-
reversibility of $G$ is equivalent to demanding that every pair of right ideals of $G$
intersect non-trivially. Left-reversibility is a weaker condition than left-amenability
for discrete semigroups since the support of any left-invariant mean must be con-
tained in every right ideal. It is strictly weaker since (for example) the free group on
two generators is left-reversible (because it is a group) but is not left-amenable (be-
cause it is not solvable.) For a survey of the relationships between left-reversibility
and other properties of semigroups, see [6, §8].

Proposition 2. Let $Z$ be an order-complete vector lattice with positive cone $C,$
and let $G$ be a semigroup of positive order-continuous linear operators from $Z$ into
$Z$. Let $C_0 = \{x \in C : Tx = x \text{ for all } T \in G\}, Z_0 = C_0 - C_0$.

If $G$ is left reversible then $(Z_0, C_0)$ is a vector lattice.

Proof. Choose $x, y \in C_0$. Let $A = \{T(x \vee y) : T \in G\}$. Clearly

$$x + y = T(x + y) \geq T(x \vee y) \geq Tx \vee Ty = x \vee y$$

1991 Mathematics Subject Classification. Primary 46B30, 52A43.
so $A$ is order-bounded above by $x + y$, and hence has a least upper bound $z$. For $T_1, T_2 \in G$ we have by left-reversibility of $G$ that

$$T_1(x \vee y) \leq T_1T_2T_3(x \vee y) = T_3T_1T_4(x \vee y) \geq T_2(x \vee y)$$

which shows that $A$ is directed as a subset of $C$, and hence $A$ (considered as a net) is order-convergent to $z$. The same argument shows that for each $T \in G$, $TA$ is a subset of $A$, whence $Tz = z$ and so $z \in C_0$. Clearly $z$ is the least upper bound in $C_0$ of $x$ and $y$. It follows that $C_0$ is a lattice cone and hence that $Z_0$ is a lattice. □

Under the conditions of Proposition 2, $Z_0$ need not contain all the fixed points of $Z$ under $G$, and need not be a sublattice of $(Z, C)$ [1, Examples 2,3].

**Proposition 3.** Let $(Z_0, C_0)$ be a vector lattice. Let $Z$ be a Banach lattice endowed with positive cone $C$ and order-continuous norm $||.||$, and suppose that $Z_0$ can be embedded in $Z$ in such a way that $C_0$ is a norm closed subset of $C$.

Then $Z_0$ is a Banach lattice with positive cone $C_0$ and order-continuous norm $||.||_0$ defined on $Z_0$ by $||x||_0 = ||x||_0$ where $||.||_0$ is the lattice modulus on $(Z_0, C_0)$.

**Proof.** Straightforward, for details see the last part of the proof in [1, p 257]. □

Again, $Z_0$ may be a lattice in the order inherited from $C$ but fail to be a sublattice of $Z$, in which case $||.||$ will generally differ from $||.||_0$ on non-positive elements of $Z_0$. Indeed, $Z_0$ need not even be closed in $Z$ with respect to $||.||$ [1, Example 4]. Conditions under which $Z_0$ is a sublattice of $(Z, C)$ in Proposition 3 are discussed in [2].

Propositions 2 and 3 combine to give us

**Proposition 4.** Let $Z$ be a Banach lattice endowed with positive cone $C$ and an order-continuous norm $||.||$. Let $G$ be a semigroup of positive linear endomorphisms of $Z$.

If $G$ is left-reversible then the positive fixed points $C_0$ of $Z$ under $G$ form a lattice cone, and their linear span $Z_0$ is a Banach lattice under an order-continuous norm $||.||_0$ which agrees with $||.||$ on $C_0$.

Proposition 4 may fail when $G$ is a projection semigroup [1, Example 1] so some condition is required on $G$. But frequently we can use a standard fixed point theorem to recover the conclusion of Proposition 4 for semigroups which are not left-reversible. As an illustration of this, we prove the following:

**Definition 5.** In the set up of Proposition 4 call $G$ norm-distal iff $Gu$ is norm bounded away from zero for all $u \in Z - \{0\}$.

**Proposition 6.** Proposition 4 remains true if $G$ is assumed norm-distal in place of left-reversible.

**Proof.** Adopting the notation of Proposition 3, pick $x, y$ in $C_0$ and let $A$ be the smallest subset of $C$ containing $x$ and $y$ and closed under join and orbit, so that for $u, v \in A$ and $T \in G$ we have $u \vee v, Tu \in A$. Now $A$ is directed as a subset of $C$, and hence convergent to $z = \sup A \leq x + y$. Setting $K$ to be the order interval $[x \vee y, z]$, we have (using order continuity of the norm on $Z$) that the elements of $G$ act as continuous affine maps from the weakly compact set $K$ into itself [8, §2.4]. Since $G$ is distal, $K$ must have a fixed point under $G$ by the Ryall-Nardzewski fixed
point theorem [10][9]. This fixed point must be \( z \), which is therefore the least upper bound of \( z \) and \( y \) in \( C_0 \). This is true for each choice of \( z \) and \( y \) in \( C_0 \), so \( C_0 \) is a lattice cone and the conclusion of Proposition 4 is recovered. \( \square \)

Different variations of Proposition 4 can be obtained by applying other fixed point theorems to the compact convex set \( K \) defined in the proof of Proposition 6. See [5] for a selection of suitable fixed point properties. As well as yielding the new results presented here, this approach also allows us to give simple transparent proofs for a wide range of known results. Properties of this kind find application in statistical mechanics [11], quantum physics [3], statistical decision theory [7, Chapter 1] and elsewhere. See [1] for further discussion of the significance of these and related propositions, and [4] for a range of recent related work.

REFERENCES

2. B. Christianson, 1991, Relative Width of Sublattices, Math Chronicle 20 75-78

SCHOOL OF INFORMATION SCIENCES, HATFIELD CAMPUS, UNIVERSITY OF HERTFORDSHIRE, ENGLAND, EUROPE

E-mail address: B.Christianson@herts.ac.uk