DIVISION OF COMPUTER SCIENCE

Positive Fixed Points of Lattices under Semigroups of Positive Linear Operators

Bruce Christianson

Technical Report No.177

November 1993

POSITIVE FIXED POINTS OF LATTICES UNDER SEMIGROUPS OF POSITIVE LINEAR OPERATORS

BRUCE CHRISTIANSON

November 1993

ABSTRACT. Let Z be a Banach lattice endowed with positive cone C and an order-continuous norm $||.||$. Let G be a semigroup of positive linear endomorphisms of Z. We seek conditions on G sufficient to ensure that the positive fixed points C_0 of Z under G form a lattice cone, and that their linear span Z_0 is a Banach lattice under an order-continuous norm $||.||_0$ which agrees with $||.||$ on C_0, although we do not require that Z_0 contain all the fixed points of Z under G, nor that Z_0 be a sublattice of (Z, C). We give a simple embedding construction which allows such results to be read off directly from appropriate fixed point theorems. In particular, we show that left-reversibility of G (a weaker condition than left-amenability) suffices. Results of this kind find application in statistical physics and elsewhere.

Definition 1. A semigroup G is called *left-reversible* iff for all $T_1, T_2 \in G$ there exist $T_3, T_4 \in G$ such that $T_1T_2T_3 = T_2T_1T_4$.

A right ideal of a semigroup G is a set of the form TG where $T \in G$. Left-reversibility of G is equivalent to demanding that every pair of right ideals of G intersect non-trivially. Left-reversibility is a weaker condition than left-amenability for discrete semigroups since the support of any left-invariant mean must be contained in every right ideal. It is strictly weaker since (for example) the free group on two generators is left-reversible (because it is a group) but is not left-amenable (because it is not solvable.) For a survey of the relationships between left-reversibility and other properties of semigroups, see [6, §8].

Proposition 2. Let Z be an order-complete vector lattice with positive cone C, and let G be a semigroup of positive order-continuous linear operators from Z into Z. Let $C_0 = \{x \in C : Tx = x \text{ for all } T \in G\}, Z_0 = C_0 - C_0$.

If G is left reversible then (Z_0, C_0) is a vector lattice.

Proof. Choose $x, y \in C_0$. Let $A = \{T(x \vee y) : T \in G\}$. Clearly

$$x + y = T(x + y) \geq T(x \vee y) \geq Tx \vee Ty = x \vee y$$

1991 *Mathematics Subject Classification.* Primary 46B30, 52A43..

so A is order-bounded above by $x+y$, and hence has a least upper bound z. For $T_1, T_2 \in G$ we have by left-reversibility of G that

$$T_1(x \vee y) \leq T_1T_2T_3(x \vee y) = T_2T_1T_4(x \vee y) \geq T_2(x \vee y)$$

which shows that A is directed as a subset of C, and hence A (considered as a net) is order-convergent to z. The same argument shows that for each $T \in G$, TA is a subnet of A, whence $Tz = z$ and so $z \in C_0$. Clearly z is the least upper bound in C_0 of x and y. It follows that C_0 is a lattice cone and hence that Z_0 is a lattice. □

Under the conditions of Proposition 2, Z_0 need not contain all the fixed points of Z under G, and need not be a sublattice of (Z, C) [1, Examples 2,3].

Proposition 3. Let (Z_0, C_0) be a vector lattice. Let Z be a Banach lattice endowed with positive cone C and order-continuous norm $||.||$, and suppose that Z_0 can be embedded in Z in such a way that C_0 is a norm closed subset of C.

Then Z_0 is a Banach lattice with positive cone C_0 and order-continuous norm $||.||_0$ defined on Z_0 by $||x||_0 = ||\ |x|_0||$ where $|.|_0$ is the lattice modulus on (Z_0, C_0).

Proof. Straightforward, for details see the last part of the proof in [1, p 257]. □

Again, Z_0 may be a lattice in the order inherited from C but fail to be a sublattice of Z, in which case $||.||$ will generally differ from $||.||_0$ on non-positive elements of Z_0. Indeed, Z_0 need not even be closed in Z with respect to $||.||$ [1, Example 4]. Conditions under which Z_0 is a sublattice of (Z, C) in Proposition 3 are discussed in [2].

Propositions 2 and 3 combine to give us

Proposition 4. Let Z be a Banach lattice endowed with positive cone C and an order-continuous norm $||.||$. Let G be a semigroup of positive linear endomorphisms of Z.

If G is left-reversible then the positive fixed points C_0 of Z under G form a lattice cone, and their linear span Z_0 is a Banach lattice under an order-continuous norm $||.||_0$ which agrees with $||.||$ on C_0.

Proposition 4 may fail when G is a projection semigroup [1, Example 1] so some condition is required on G. But frequently we can use a standard fixed point theorem to recover the conclusion of Proposition 4 for semigroups which are not left-reversible. As an illustration of this, we prove the following:

Definition 5. In the set up of Proposition 4 call G *norm-distal* iff Gu is norm bounded away from zero for all $u \in Z - \{0\}$.

Proposition 6. Proposition 4 remains true if G is assumed norm-distal in place of left-reversible.

Proof. Adopting the notation of Proposition 3, pick x, y in C_0 and let A be the smallest subset of C containing x and y and closed under join and orbit, so that for $u, v \in A$ and $T \in G$ we have $u \vee v, Tu \in A$. Now A is directed as a subset of C, and hence convergent to $z = \sup A \leq x + y$. Setting K to be the order interval $[x \vee y, z]$, we have (using order continuity of the norm on Z) that the elements of G act as continuous affine maps from the weakly compact set K into itself [8, §2.4]. Since G is distal, K must have a fixed point under G by the Ryall-Nardzewski fixed point theorem [10][9]. This fixed point must be z, which is therefore the least upper bound of x and y in C_0. This is true for each choice of x and y in C_0, so C_0 is a lattice cone and the conclusion of Proposition 4 is recovered. □

Different variations of Proposition 4 can be obtained by applying other fixed point theorems to the compact convex set K defined in the proof of Proposition 6. See [5] for a selection of suitable fixed point properties. As well as yielding the new results presented here, this approach also allows us to give simple transparent proofs for a wide range of known results. Properties of this kind find application in statistical mechanics [11], quantum physics [3], statistical decision theory [7, Chapter 1] and elsewhere. See [1] for further discussion of the significance of these and related propositions, and [4] for a range of recent related work.

References

1. B. Christianson, 1989, The Positive Fixed Points of Banach Lattices, Proc Amer Math Soc **107**(1) 255-260
2. B. Christianson, 1991, Relative Width of Sublattices, Math Chronicle **20** 75–78
3. E.B. Davies and J.T. Lewis, 1970, An Operational Approach to Quantum Probability, Comm Math Phys **17** 239-260
4. C.B. Huijsmans *and* W.A.J. Luxemburg *eds*, 1992, Positive Operators and Semigroups on Banach Lattices, Acta Appl Math **27**(1&2).
5. A.T. Lau, 1976, Some Fixed Point Theorems and their Applications to W^*-algebras, pp 121-129 *in* Fixed Point Theory and its Applications, *ed* S. Swaninithan, Academic Press, New York
6. A.T. Lau, 1990, Amenability of Semigroups, pp 314–334 *in* The Analytical and Topological Theory of Semigroups, *ed* K.H. Hofmann *et al*, Walter de Gruyter & Co, Berlin
7. L.M. LeCam, 1986, Asymptotic Methods in Statistical Decision Theory, Springer Series in Statistics, Springer-Verlag
8. P. Meyer-Nieberg, 1991, Banach Lattices, Springer-Verlag, Berlin
9. I. Namioka *and* E. Asplund, 1967, A Geometric Proof of Ryll-Nardzewski's Fixed Point Theorem, Bull Amer Math Soc **73** 443–445
10. C. Ryll-Nardzewski, 1967, On Fixed Points of Endomorphisms of Linear Spaces, Proc Fifth Berkeley Symposium on Mathematics, Statistics and Probability **2**(1) 55–61, University of California Press, Berkeley
11. G. Winkler, 1985, Choquet Order and Simplices with Applications in Probabilistic Models, Lecture Notes in Mathematics **1145**, Springer Verlag, Berlin

School of Information Sciences, Hatfield Campus, University of Hertfordshire, England, Europe

E-mail address: B.Christianson@herts.ac.uk