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Entanglement of topological phase factors

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Abstract. The topological phase factor induced on interfering electrons by external quantum electromagnetic fields has been studied. Two and three electron interference experiments inside distant cavities are considered and the influence of correlated photons on the phase factors is investigated. It is shown that the classical or quantum correlations of the irradiating photons are transferred to the topological phases. The effect is quantified in terms of Weyl functions for the density operators of the photons and illustrated with particular examples. The scheme employs the generalized phase factor as a mechanism for information transfer from the photons to the electric charges. In this sense, the scheme may be useful in the context of flying qubits (corresponding to photons) and stationary qubits (electrons), as well as the conversion from one type to the other.

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Contents

1. Introduction

The study of phase factors arising during quantum interference has been crucial for the understanding of a wide range of physical phenomena [1]. The Aharonov–Bohm phase factor [2], $\exp(\mathrm{i}q\Phi)$, is acquired by a particle with charge q in a looping trajectory that encloses a classical magnetostatic flux Φ. This is true even when the particle moves in entirely field-free regions. The effect has been investigated in relation to transport phenomena in solid-state physics [3] and electron coherence in mesoscopic devices [4]. The reciprocal phase factor [5] and the dual counterparts [6, 7] have also been studied and have recently found applications in different contexts, such as topological quantum information processing [8], the quantum Hall-effect analogue with neutral atoms [9] and ultra-cold atom technology [10].

The generalized phase factor $\exp(\mathrm{i}q\hat{\phi})$, which is induced on a charge q by a nonclassical electromagnetic field with magnetic flux $\hat{\phi}$, has also been studied in the literature [11]. In this case, the magnetic flux and the induced phase factor are quantum mechanical operators. Consequently, an important quantity in terms of interference properties is the expectation value of the phase factor $\langle\exp(\mathrm{i}q\hat{\phi})\rangle = \mathrm{Tr}\,[\rho\exp(\mathrm{i}q\hat{\phi})]$, with respect to the density matrix ρ that describes the external electromagnetic field. This phase factor is topological in the sense that it depends on the number of times an electron winds around the enclosed magnetic flux and is independent of the electron velocity. The $\langle\exp(\mathrm{i}q\hat{\phi})\rangle$ is a complex quantity, in general, and is known as the Weyl (or characteristic) function from quantum phase-space studies [12].

Clearly, the inherent fluctuations of the external quantum fields bring about the problem of decoherence of the interfering electrons. Solutions have been proposed in relation to this problem using various methods [13]. In this paper, it is assumed that, under certain conditions, the external photons do not interact with the interfering charges. In particular, it is assumed that the electromagnetic fields that are induced via Faraday's law by the circulating electrons are negligible in comparison to the external fields, and so there is no back reaction. The inherent noise of the external photons manifests itself as a decrease in the absolute value of the phase factor $|\langle \exp(\mathrm{i}q\hat{\phi})\rangle|$, which becomes slightly less than one [14].

Nonclassical electromagnetic fields in various quantum states [15], such as squeezed and number states, have been generated at both optical and microwave frequencies [16]. Quantum mechanically correlated [17] photons have also been produced in the laboratory [18]. It is therefore reasonable to ask whether we can use certain quantum interference devices, which are sensitive to the external radiation, as detectors of photon correlations. This has indeed been proposed recently using different techniques [19, 20]. In this paper, we study photon-induced correlations between electron phase factors, which is the precursory mechanism for the detection of photon entanglement in distant quantum interference devices. It is shown that the phase factors of the electrons in interference experiments, which are initially independent of one another, become correlated when the experiments are irradiated with correlated photons. The set-up considered here may also be useful in the general area of flying and stationary qubits [21] and their interaction.

The rest of the paper is organized as follows. A possible implementation is analysed and the background material is provided in section 2. The correlations induced by the photons on the phase factors are quantified for the bipartite case in section 3. The problem is approached through examples, which involve classically and quantum mechanically correlated photons in number states and coherent states, in section 4. This is subsequently generalized to the tripartite case in section 5, where examples are also provided. The results are discussed and conclusions are drawn in section 6.

2. Influence of entangled photons on distant interference experiments

We begin by introducing the set-up depicted in figure 1: two interference devices for charged particles, **A** and **B**, are placed inside cavities that are far from each other. A source S_{EM} of two-mode nonclassical microwaves sends one mode of frequency ω_1 into the cavity where **A** has been placed and sends the other mode of frequency ω_2 into the cavity where **B** has been placed. It has been shown that, in this case, the correlation between the two electromagnetic field modes is transferred to the distant quantum interference devices [20]. These devices could be, for example, nanoscale superconducting quantum interference devices (SQUIDs) [19], in which case the interfering particles are Cooper pairs, or simply two-path electron interference devices [20]. In either case, the value of the phase factor, which depends on the external electromagnetic fields, influences the measurable physical quantities (in the case of superconducting rings, the measurable variable is the current, while in electron interference one measures the intensity of electrons on the interference screen).

The external quantum fields are usually described by the vector potential $\hat{A}_i$ and the electric field $\hat{E}_i$, which are dual quantum variables. $\hat{A}_i$, $\hat{E}_i$ can be transformed into another pair of dual variables by integrating them around a small loop l (that is, 'small' in comparison to the wavelength so that the field strengths are the same locally). This operation yields the magnetic flux $\hat{\phi} = \oint_l \hat{A}_i \mathrm{d}x_i$ and the electromotive force $\hat{V}_{\mathrm{EMF}} = \oint_l \hat{E}_i \mathrm{d}x_i$, respectively. The boson creation and annihilation operators may now be introduced as

$$\hat{a}^{\dagger} = \frac{1}{\sqrt{2}\xi}(\hat{\phi} - \mathrm{i}\omega^{-1}\hat{V}_{\mathrm{EMF}}), \qquad \hat{a} = \frac{1}{\sqrt{2}\xi}(\hat{\phi} + \mathrm{i}\omega^{-1}\hat{V}_{\mathrm{EMF}}), \tag{1}$$

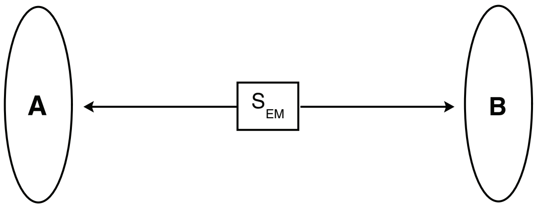

Figure 1. Two distant interference devices for charged particles **A** and **B** are irradiated with nonclassical electromagnetic fields of frequencies ω_1 and ω_2, respectively. The electromagnetic fields emanate from a single source S_{EM} and are correlated. It is required that the wavelengths of the fields are $\sim$1 mm (microwaves) and that the interference devices have mesoscopic dimensions ($\sim$0.1 μm) operating at low temperatures of 10–100 mK, such that $k_B T \ll \hbar\omega_1, \hbar\omega_2$.

where ξ is a constant proportional to the area enclosed by l. They obey the usual commutation relation $[a, a^\dagger] = 1$ (note that we employ units in which the Boltzmann constant, the Planck constant divided by 2π and the speed of light in vacuum are set equal to one, $k_B = \hbar = c = 1$). The flux operator is consequently written in the Heisenberg picture as

$$\hat{\phi}(t) = \exp(\mathrm{i}t\mathcal{H})\hat{\phi}(0)\exp(-\mathrm{i}t\mathcal{H}), \tag{2}$$

where

$$\mathcal{H} = H_{\mathrm{free}} + H_{\mathrm{int}}, \qquad H_{\mathrm{free}} = \omega(\hat{a}^\dagger\hat{a} + 1/2). \tag{3}$$

The full Hamiltonian $\mathcal{H}$ contains the free electromagnetic field Hamiltonian and an interaction term H_{int}, which includes the Hamiltonian of the interfering charges as well. In this paper, we assume that H_{int}, which describes the back reaction from the charges to the electromagnetic field, is neglected. In other words, it is assumed that the self-induced magnetic flux of the charges is negligible compared to the external flux $\langle\hat{\phi}(t)\rangle$. In this approximation [11, 20], we get

$$\hat{\phi}(t) = \frac{\xi}{\sqrt{2}}\left[\exp(\mathrm{i}\omega t)\hat{a}^\dagger + \exp(-\mathrm{i}\omega t)\hat{a}\right]. \tag{4}$$

Exponentiating, we obtain the phase factor for an electron of charge e:

$$\exp\left[\mathrm{i}e\hat{\phi}(t)\right] = D[\mathrm{i}q\exp(\mathrm{i}\omega t)], \qquad q = \frac{\xi e}{\sqrt{2}}, \tag{5}$$

where q is introduced as a scaled electric charge. $D(\lambda) \equiv \exp(\lambda\hat{a}^\dagger - \lambda^*\hat{a})$ is the displacement operator.

Let ρ_A be the density matrix describing the external nonclassical electromagnetic field mode in cavity **A**. The expectation value of the phase factor is given by the trace of the operator $\exp[\mathrm{i}e\hat{\phi}_A(t)]$ with respect to ρ_A. It is easily seen that taking the trace we obtain the single-mode Weyl function

$$\tilde{W}_A(\lambda_A) \equiv \mathrm{Tr}\,[\rho_A D(\lambda_A)], \qquad \lambda_A = \mathrm{i}q\exp(\mathrm{i}\omega_1 t). \tag{6}$$

Similarly, the expectation value of the electron phase factor in experiment **B** is given by the Weyl function

$$\tilde{W}_{\mathrm{B}}(\lambda_{\mathrm{B}}) \equiv \mathrm{Tr}\,[\rho_{\mathrm{B}} D(\lambda_{\mathrm{B}})], \qquad \lambda_{\mathrm{B}} = \mathrm{i}q \exp(\mathrm{i}\omega_2 t). \tag{7}$$

It is important to note that these 'expectation values' are, in general, complex numbers. The reason for this is that the operator $D(z)$ is not Hermitian, since $D^{\dagger}(z) = D(-z)$.

To provide a physical interpretation, consider that **A** is a two-path electron interference experiment. We associate with each path, a wavefunction for the electrons, for example, ψ_0 and ψ_1 (let us assume equal splitting among them, for simplicity). It has been shown elsewhere [20] that the intensity, or number density, of electrons at position $x \equiv \arg\psi_0 - \arg\psi_1$ on the interference screen of experiment **A** is given by

$$I_{\mathrm{A}}(x) = \mathrm{Tr}\left[\rho_{\mathrm{A}}|\psi_0 + \langle\exp(\mathrm{i}e\hat{\phi}_{\mathrm{A}})\rangle\psi_1|^2\right] = 1 + |\tilde{W}_{\mathrm{A}}(\lambda_{\mathrm{A}})| \cos\{x + \arg[\tilde{W}_{\mathrm{A}}(\lambda_{\mathrm{A}})]\}. \tag{8}$$

It is clearly seen that the absolute value of the expectation value of the phase factor, $|\tilde{W}_{\mathrm{A}}(\lambda_{\mathrm{A}})|$, is the visibility $\nu \equiv (I_{\max} - I_{\min})/(I_{\max} + I_{\min})$ of the interference. The $\arg[\tilde{W}_{\mathrm{A}}(\lambda_{\mathrm{A}})]$ is the phase shift induced on the electrons by the irradiating electromagnetic field.

3. Correlations between electron phase factors

In this section, we show how the electron phase factors in distant interference experiments become correlated when they are irradiated with correlated photons. The nature of the correlation between the external photons can be classical or quantum [17, 18] and the aim here is to compare and contrast the two cases. Firstly, the difference between the two cases is clarified.

The photons of frequencies ω_1 and ω_2 are described by density operators ρ_{A} and ρ_{B}, respectively. If they are completely independent of each other, then the density operator describing the bipartite state is factorizable, i.e., $\rho_{\mathrm{fac}} = \rho_{\mathrm{A}} \otimes \rho_{\mathrm{B}}$. If they are classically correlated, then the bipartite state is described by the separable density operator $\rho_{\mathrm{sep}} = \sum_k P_k \rho_{\mathrm{A},k} \otimes \rho_{\mathrm{B},k}$, where P_k are probabilities that sum up to unity. If the two photons are quantum mechanically correlated, then their density operator ρ_{ent} is entangled and cannot be cast in the above forms in any way.

The expectation values of the electron phase factors $\langle\exp(\mathrm{i}e\hat{\phi}_{\mathrm{A}})\rangle$ and $\langle\exp(\mathrm{i}e\hat{\phi}_{\mathrm{B}})\rangle$ in the interference experiments **A** and **B** are given by the single-mode Weyl functions $\tilde{W}_{\mathrm{A}}(\lambda_{\mathrm{A}})$ and $\tilde{W}_{\mathrm{B}}(\lambda_{\mathrm{B}})$ of equations (6) and (7), respectively. It is also possible to measure the product of the electron phase factors in **A** and **B** (joint phase factor). The expectation value of this product, $\langle\exp(\mathrm{i}e\hat{\phi}_{\mathrm{A}})\exp(\mathrm{i}e\hat{\phi}_{\mathrm{B}})\rangle$, is given by the two-mode Weyl function

$$\tilde{W}_{\mathrm{AB}}(\lambda_{\mathrm{A}}, \lambda_{\mathrm{B}}) = \mathrm{Tr}\,[\rho D(\lambda_{\mathrm{A}}) D(\lambda_{\mathrm{B}})]. \tag{9}$$

In the case of independent subsystems, which are described by $\rho_{\mathrm{fac}} = \rho_{\mathrm{A}} \otimes \rho_{\mathrm{B}}$, the $\tilde{W}_{\mathrm{AB}}(\lambda_{\mathrm{A}}, \lambda_{\mathrm{B}})$ is equal to the product $\tilde{W}_{\mathrm{A}}(\lambda_{\mathrm{A}})\tilde{W}_{\mathrm{B}}(\lambda_{\mathrm{B}})$. However, for classically or quantum mechanically correlated subsystems, the two-mode Weyl function is not equal to this product of one-mode Weyl functions, in general. This implies that the electron phase factors in **A** and **B** are correlated with each other.

In order to quantify the induced correlations between the electron phase factors, we define

$$C \equiv \tilde{W}_{\mathrm{AB}}(\lambda_{\mathrm{A}}, \lambda_{\mathrm{B}}) - \tilde{W}_{\mathrm{A}}(\lambda_{\mathrm{A}})\tilde{W}_{\mathrm{B}}(\lambda_{\mathrm{B}}). \tag{10}$$

If the subsystems are not correlated with each other, then $C = 0$. If they are correlated, then $C \neq 0$ (i.e., the real and imaginary parts of C do not both vanish).

The experimentally measurable quantities are the visibilities of the electron interferences in **A** ($|\tilde{W}_A(\lambda_A)|$) and in **B** ($|\tilde{W}_B(\lambda_B)|$); and the corresponding shifts of the interference fringes $\arg(\tilde{W}_A)$, $\arg(\tilde{W}_B)$. The absolute value (joint visibility) and the argument (joint phase shift) of $\tilde{W}_{AB}(\lambda_A, \lambda_B)$ have to be measured simultaneously in the two experiments. Alternatively, one may use a SQUID ring with a single Josephson junction irradiated with nonclassical electromagnetic fields [19], in which case $\tilde{W}_A$, $\tilde{W}_B$ and $\tilde{W}_{AB}$ are calculated from the expectation values of the currents in **A** and **B** (and the product of the currents in both rings). For example, it is known that the current measured in **A** is given by $I_A = I_{cr}\mathrm{Im}[\tilde{W}_A(\lambda_A)]$, where I_{cr} is the critical current.

4. Examples for the bipartite case

In this section, we consider particular examples of classically and quantum mechanically correlated two-mode nonclassical electromagnetic fields in number states and coherent states. The fundamental relations that are necessary for the derivation of the following results are given in appendix A (for the number states) and appendix B (for the coherent states).

4.1. Photons in number states

Consider a two-mode electromagnetic field in the separable state

$$\rho_{sep} = \tfrac{1}{2}(|N_1N_2\rangle\langle N_1N_2| + |N_2N_1\rangle\langle N_2N_1|). \tag{11}$$

In this case, the difference C of equation (10) is

$$C_{sep} = \exp(-q^2)L_{N_1}(q^2)L_{N_2}(q^2) - \tfrac{1}{4}\exp(-q^2)[L_{N_1}(q^2) + L_{N_2}(q^2)]^2, \tag{12}$$

where L_N^α are Laguerre functions. C_{sep} is time-independent; it depends only on the number of photons N_1, N_2. It is clearly seen that $|C| > 0$ for any number of photons.

On the other hand, the entangled number state $|n\rangle = 2^{-1/2}(|N_1N_2\rangle + |N_2N_1\rangle)$, with a density operator

$$\rho_{ent} = \rho_{sep} + \tfrac{1}{2}(|N_1N_2\rangle\langle N_2N_1| + |N_2N_1\rangle\langle N_1N_2|) \tag{13}$$

yields

$$C_{ent} = C_{sep} + \exp(-q^2)L_{N_1}^{N_2-N_1}(q^2)L_{N_2}^{N_1-N_2}(q^2)\cos(\Omega t), \tag{14}$$

which is time-dependent and oscillates around C_{sep} with frequency

$$\Omega = (N_1 - N_2)(\omega_1 - \omega_2). \tag{15}$$

If there is no detuning between the external electromagnetic fields, in which case $\omega_1 = \omega_2$, then C_{ent} is constant in time but it is still different from C_{sep}. It is worth noting that, for this example, the difference C is purely real in both the separable and entangled cases.

4.2. Photons in coherent states

Consider two coherent states $|A_1\rangle$ and $|A_2\rangle$ in the classically correlated state

$$\rho_{sep} = \tfrac{1}{2}(|A_1A_2\rangle\langle A_1A_2| + |A_2A_1\rangle\langle A_2A_1|). \tag{16}$$

In this case, the reduced density operators that describe the coherent states propagating in cavities **A** and **B** are

$$\rho_{\mathrm{sep,A}} = \rho_{\mathrm{sep,B}} = \tfrac{1}{2}(|A_1\rangle\langle A_1| + |A_2\rangle\langle A_2|). \tag{17}$$

We also consider the entangled state $|S\rangle = \mathcal{N}(|A_1 A_2\rangle + |A_2 A_1\rangle)$ with density operator

$$\rho_{\mathrm{ent}} = 2\mathcal{N}^2 \rho_{\mathrm{sep}} + \mathcal{N}^2(|A_1 A_2\rangle\langle A_2 A_1| + |A_2 A_1\rangle\langle A_1 A_2|), \tag{18}$$

where the normalization constant, which is such that $\langle S|S\rangle = 1$, is given by

$$\mathcal{N} = \left[2 + 2\exp\left(-|A_1 - A_2|^2\right)\right]^{-1/2}. \tag{19}$$

In this case, the reduced density operators in **A** and **B** are

$$\rho_{\mathrm{ent,A}} = \rho_{\mathrm{ent,B}} = \mathcal{N}^2(|A_1\rangle\langle A_1| + |A_2\rangle\langle A_2| + \tau_{12}|A_1\rangle\langle A_2| + \tau_{12}^*|A_2\rangle\langle A_1|), \tag{20}$$

where

$$\tau_{12} = \langle A_1|A_2\rangle = \exp\left(-\frac{|A_1|^2}{2} - \frac{|A_2|^2}{2} + A_1^* A_2\right). \tag{21}$$

The quantity C of equation (10) has been studied numerically using the relations provided in appendix B. For the separable case, we have calculated numerically $C_{\mathrm{sep}} = \tilde{W}_{\mathrm{AB,sep}}(\lambda_{\mathrm{A}}, \lambda_{\mathrm{B}}) - \tilde{W}_{\mathrm{A,sep}}(\lambda_{\mathrm{A}})\tilde{W}_{\mathrm{B,sep}}(\lambda_{\mathrm{B}})$; and for the entangled case, we have calculated $C_{\mathrm{ent}} = \tilde{W}_{\mathrm{AB,ent}}(\lambda_{\mathrm{A}}, \lambda_{\mathrm{B}}) - \tilde{W}_{\mathrm{A,ent}}(\lambda_{\mathrm{A}})\tilde{W}_{\mathrm{B,ent}}(\lambda_{\mathrm{B}})$. These are complex quantities and therefore, in the following, we present the results in terms of their absolute values $|C_{\mathrm{sep}}|$, $|C_{\mathrm{ent}}|$ and their imaginary parts $\mathrm{Im}(C_{\mathrm{sep}})$, $\mathrm{Im}(C_{\mathrm{ent}})$.

4.3. Numerical results

For the numerical results in this section, the values of the microwave frequencies have been set at $\omega_1 = 1.2 \times 10^{-4}$ and $\omega_2 = 1.0 \times 10^{-4}$. We have used units in which $k_{\mathrm{B}} = \hbar = c = 1$. Other fixed parameters are $\xi = 1$ and the dimensionless electric charge $e = (4\pi/137)^{1/2}$.

We study the entangled number state $|\beta\rangle = 2^{-1/2}(|01\rangle + |10\rangle)$ and its closest separable state. We therefore let $N_1 = 1$, $N_2 = 0$ in the separable state of equation (11) and the entangled state of equation (13). The corresponding results for $|C_{\mathrm{sep}}|$ and $|C_{\mathrm{ent}}|$ as a function of Ωt are plotted in figure 2. We note that $|C_{\mathrm{sep}}|$, which is time-independent, is not zero, but it is very small in this case ($\simeq 5 \times 10^{-4}$).

In the case of coherent states, we study the separable state of equation (16) and the entangled state of equation (18) for the same average number of photons as in the number states, i.e., $|A_1|^2 = N_1$ and $|A_2|^2 = N_2$ (whereas $\arg A_1 = 0$, $\arg A_2 = 0$). The results for $|C_{\mathrm{sep}}|$ and $|C_{\mathrm{ent}}|$ have been plotted against Ωt in figures 3(a) and (c), respectively. We note that, in this case, C is complex and also C_{sep} is time-dependent (in contrast to number states). In figures 3(b) and (d) the imaginary parts $\mathrm{Im}(C_{\mathrm{sep}})$ and $\mathrm{Im}(C_{\mathrm{ent}})$ have been plotted against Ωt.

In figure 2, it is seen that both C_{sep} and C_{ent} are nonzero and that C_{ent} is time-dependent. In fact, this is true for any number of photons N_1, N_2 in the separable and entangled states ρ_{sep}, ρ_{ent} as can be seen from equations (12) and (14). Consequently, the electron phase factors become correlated when the interference devices are irradiated with classically correlated (ρ_{sep}) or quantum mechanically correlated (ρ_{ent}) photons in number states. Clearly the quantity C of equation (10) is different for the two cases, which implies that the nature of the correlation

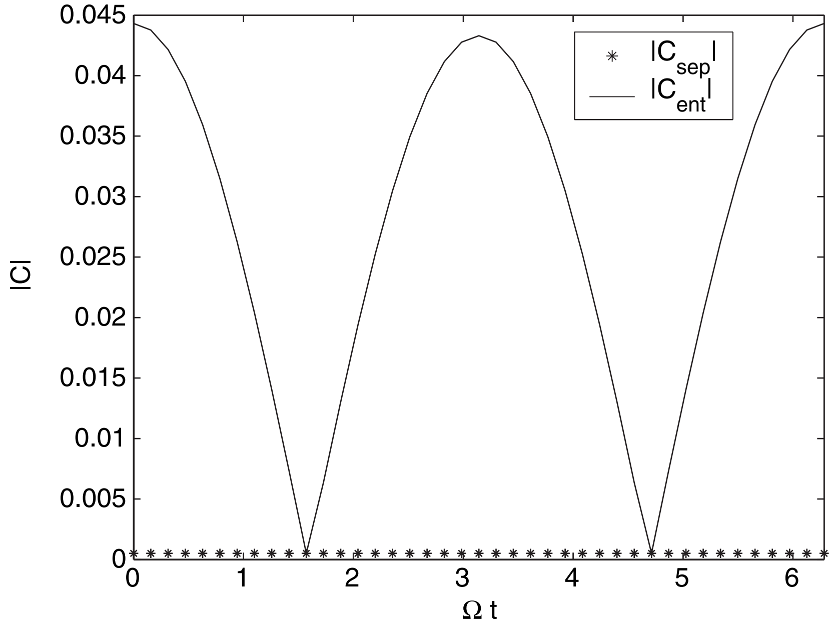

Figure 2. $|C_{\rm sep}|$ (∗∗∗) corresponding to the separable number states of equation (11) and $|C_{\rm ent}|$ (——) corresponding to the entangled number states of equation (13) for $N_1 = 1$, $N_2 = 0$ as a function of Ωt. The frequencies are $\omega_1 = 1.2 \times 10^{-4}$ and $\omega_2 = 10^{-4}$, in units where $k_{\rm B} = \hbar = c = 1$. Note that $|C_{\rm sep}|$ is not zero but equals 5×10^{-4}.

between the irradiating photons influences the induced correlation between the topological phase factors. In figure 3, we see that the same general result is true for the classically and quantum mechanically correlated photons in coherent states. It is also evident that the correlations between the phase factors are influenced by the quantum noise and statistics of the external photons, by comparison of figures 2 and 3, which correspond to photons in number states and coherent states.

5. Examples for the tripartite case

In this section, we consider three electron-interference devices of mesoscopic dimensions that are placed inside distant microwave cavities. The interference experiments **A**, **B** and **C** are irradiated with nonclassical electromagnetic fields of frequencies ω_1, ω_2 and ω_3, respectively. The three electromagnetic field modes are described by density operators $\rho_{\rm A}$, $\rho_{\rm B}$ and $\rho_{\rm C}$. If they are completely independent of each other, then the density operator describing the tripartite state is factorizable, i.e., $\rho_{\rm fac} = \rho_{\rm A} \otimes \rho_{\rm B} \otimes \rho_{\rm C}$. If they are classically correlated, then the tripartite state is described by the separable density operator $\rho_{\rm sep} = \sum_k P_k \rho_{{\rm A},k} \otimes \rho_{{\rm B},k} \otimes \rho_{{\rm C},k}$. If the three field modes are quantum mechanically correlated, then their density operator $\rho_{\rm ent}$ is entangled and cannot be written in a separable form.

The phase factor acquired by the interfering electrons in **A** is given by $\tilde{W}_{\rm A}(\lambda_{\rm A})$ of equation (6) and the phase factor in **B** is given by $\tilde{W}_{\rm B}(\lambda_{\rm B})$ of equation (7). Similarly, the phase factor in **C** is obtained from $\tilde{W}_{\rm C}(\lambda_{\rm C}) = {\rm Tr}\,[\rho_{\rm C} D(\lambda_{\rm C})]$, where $\lambda_{\rm C} = {\rm i}q\exp({\rm i}\omega_3 t)$. We can also measure the

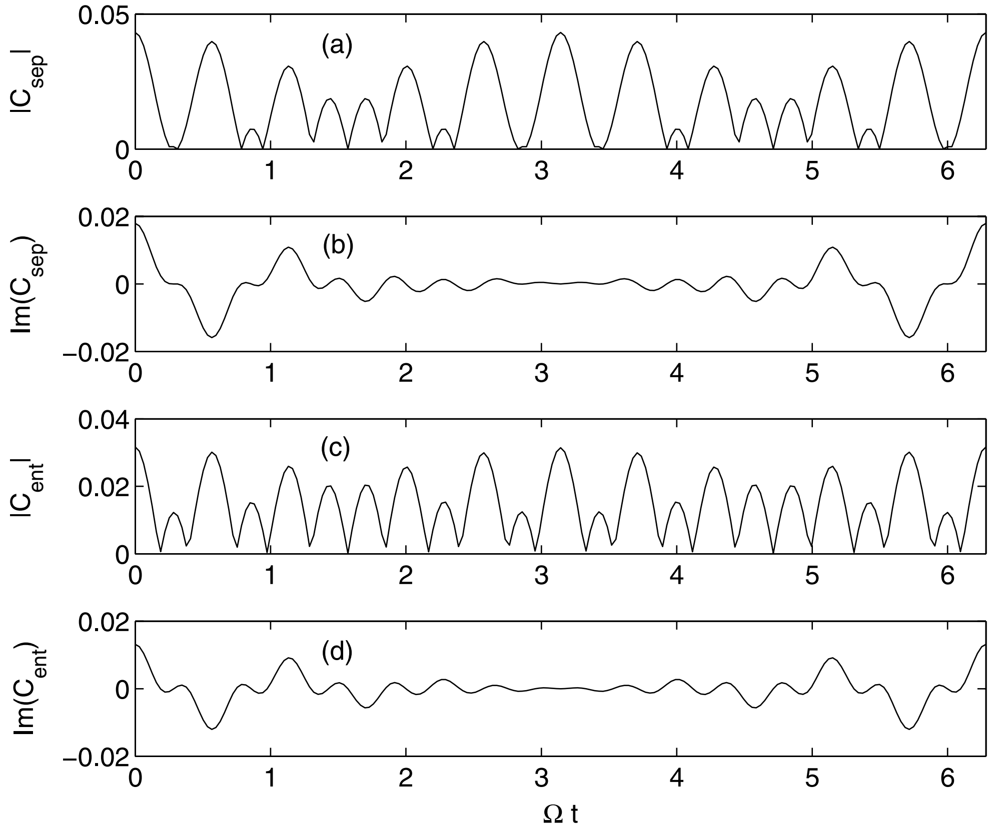

Figure 3. (a) $|C_{\rm sep}|$ and (b) $\mathrm{Im}(C_{\rm sep})$ corresponding to the separable coherent states of equation (16). (c) $|C_{\rm ent}|$ and (d) $\mathrm{Im}(C_{\rm ent})$ corresponding to the entangled coherent states of equation (18) for $A_1 = 1$, $A_2 = 0$ as a function of Ωt. The frequencies are $\omega_1 = 1.2 \times 10^{-4}$ and $\omega_2 = 10^{-4}$, in units where $k_B = \hbar = c = 1$.

product of the three-phase factors, which is given by the three-mode Weyl function

$$\tilde{W}_{\rm ABC}(\lambda_{\rm A}, \lambda_{\rm B}, \lambda_{\rm C}) = \mathrm{Tr}\,[\rho D(\lambda_{\rm A}) D(\lambda_{\rm B}) D(\lambda_{\rm C})]. \tag{22}$$

The tripartite correlations between the electron phase factors can be quantified with a straightforward generalization of the quantity C of equation (10); i.e., in this case, we define

$$C \equiv \tilde{W}_{\rm ABC}(\lambda_{\rm A}, \lambda_{\rm B}, \lambda_{\rm C}) - \tilde{W}_{\rm A}(\lambda_{\rm A}) \tilde{W}_{\rm B}(\lambda_{\rm B}) \tilde{W}_{\rm C}(\lambda_{\rm C}). \tag{23}$$

If the phase factors are not correlated then $C = 0$. If they are correlated then $|C| > 0$ (and also possibly $\mathrm{Im}(C) \neq 0$).

5.1. Photons in number states

As an example of tripartite number states, consider the separable state

$$\rho_{\rm sep} = \tfrac{1}{2}(|N_1 N_2 N_3\rangle\langle N_1 N_2 N_3| + |N_2 N_3 N_1\rangle\langle N_2 N_3 N_1|) \tag{24}$$

and the entangled state $|n_{\rm tri}\rangle = 2^{-1/2}(|N_1 N_2 N_3\rangle + |N_2 N_3 N_1\rangle)$ with density operator

$$\rho_{\rm ent} = \rho_{\rm sep} + \tfrac{1}{2}(|N_1 N_2 N_3\rangle\langle N_2 N_3 N_1| + |N_2 N_3 N_1\rangle\langle N_1 N_2 N_3|). \tag{25}$$

The results for the three-mode Weyl function of equation (22) corresponding to the separable and entangled number states are straightforward, albeit lengthy. Only the numerical calculations are presented in terms of time $\Omega' t$, where Ω' has replaced Ω of equation (15), which was valid for the bipartite case. In particular, it is not hard to show that the difference between the separable and entangled Weyl functions includes a time-dependent term of frequency Ω', which is given by

$$\tilde{W}_{\mathrm{ABC,ent}} - \tilde{W}_{\mathrm{ABC,sep}} \propto \mathrm{Re}(\langle N_1|D(\lambda_{\mathrm{A}})|N_2\rangle\langle N_2|D(\lambda_{\mathrm{B}})|N_3\rangle\langle N_3|D(\lambda_{\mathrm{C}})|N_1\rangle). \quad (26)$$

From this term, we obtain the appropriate frequency for the tripartite case, namely

$$\Omega' = N_1(\omega_3 - \omega_1) + N_2(\omega_1 - \omega_2) + N_3(\omega_2 - \omega_3). \quad (27)$$

5.2. *Photons in coherent states*

We consider the separable coherent state

$$\rho_{\mathrm{sep}} = \tfrac{1}{2}(|A_1A_2A_3\rangle\langle A_1A_2A_3| + |A_2A_3A_1\rangle\langle A_2A_3A_1|). \quad (28)$$

In this case, the reduced density operators are

$$\begin{aligned}
\rho_{\mathrm{sep,A}} &= 2^{-1}(|A_1\rangle\langle A_1| + |A_2\rangle\langle A_2|),\\
\rho_{\mathrm{sep,B}} &= 2^{-1}(|A_2\rangle\langle A_2| + |A_3\rangle\langle A_3|),\\
\rho_{\mathrm{sep,C}} &= 2^{-1}(|A_3\rangle\langle A_3| + |A_1\rangle\langle A_1|).
\end{aligned} \quad (29)$$

We also consider the entangled state $|S_{\mathrm{tri}}\rangle = \mathcal{N}'(|A_1A_2A_3\rangle + |A_2A_3A_1\rangle)$ with density operator

$$\rho_{\mathrm{ent}} = 2\mathcal{N}'^2\rho_{\mathrm{sep}} + \mathcal{N}'^2(|A_1A_2A_3\rangle\langle A_2A_3A_1| + |A_2A_3A_1\rangle\langle A_1A_2A_3|), \quad (30)$$

where the normalization constant is given by

$$\mathcal{N}' = [2 + 2\mathrm{Re}(\tau_{12} + \tau_{23} + \tau_{31})]^{-1/2} \quad (31)$$

for $\tau_{ij} = \langle A_i|A_j\rangle = \exp(-|A_i|^2/2 - |A_j|^2/2 + A_i^* A_j)$ as in equation (21), for example. In this case, the reduced density operators are

$$\begin{aligned}
\rho_{\mathrm{ent,A}} &= \mathcal{N}'^2(2\rho_{\mathrm{sep,A}} + \tau_{13}\tau_{32}|A_1\rangle\langle A_2| + \tau_{13}^*\tau_{32}^*|A_2\rangle\langle A_1|),\\
\rho_{\mathrm{ent,B}} &= \mathcal{N}'^2(2\rho_{\mathrm{sep,B}} + \tau_{21}\tau_{13}|A_2\rangle\langle A_3| + \tau_{21}^*\tau_{13}^*|A_3\rangle\langle A_2|),\\
\rho_{\mathrm{ent,C}} &= \mathcal{N}'^2(2\rho_{\mathrm{sep,C}} + \tau_{12}\tau_{23}|A_3\rangle\langle A_1| + \tau_{12}^*\tau_{23}^*|A_1\rangle\langle A_2|).
\end{aligned} \quad (32)$$

5.3. *Numerical results*

For the numerical results in this section, the photon frequencies are $\omega_1 = 1.2 \times 10^{-4}$, $\omega_2 = 1.1 \times 10^{-4}$ and $\omega_3 = 1.0 \times 10^{-4}$, in units where $k_{\mathrm{B}} = \hbar = c = 1$ and $\xi = 1$.

We study the entangled tripartite state $|\beta_{\mathrm{tri}}\rangle = 2^{-1/2}(|012\rangle + |120\rangle)$ and its closest separable state. We therefore let $N_1 = 0$, $N_2 = 1$ and $N_3 = 2$ in the separable number state of equation (24) and the entangled number state of equation (25). The corresponding results for $|C_{\mathrm{sep}}|$ and $|C_{\mathrm{ent}}|$ as a function of $\Omega' t$ in the case of tripartite number states are plotted in figure 4. In this case, both $|C_{\mathrm{sep}}|$ and $|C_{\mathrm{ent}}|$ are very small but, in principle, measurable.

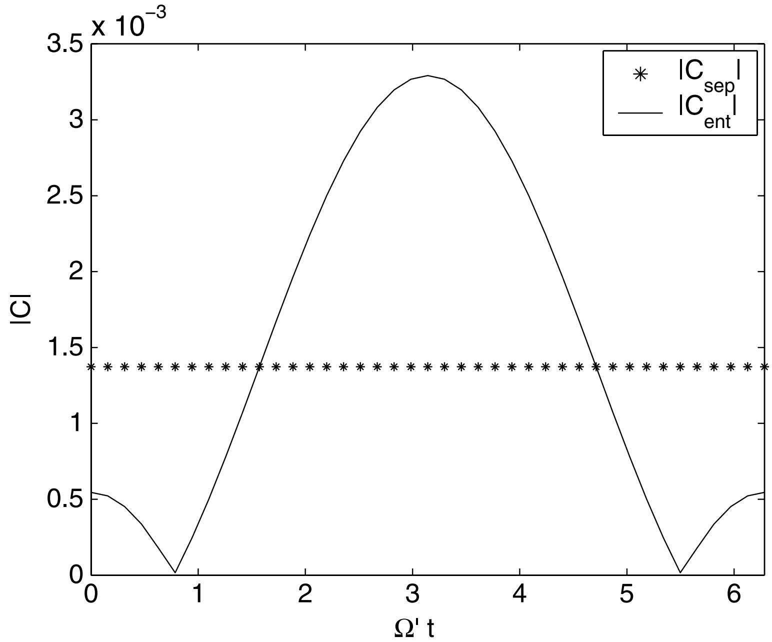

Figure 4. $|C_{\rm sep}|$ (∗∗∗) corresponding to the separable number states of equation (24) and $|C_{\rm ent}|$ (——) corresponding to the entangled number states of equation (25) for $N_1 = 0, N_2 = 1$ and $N_3 = 2$ are plotted as a function of $\Omega' t$. The frequencies are $\omega_1 = 1.2 \times 10^{-4}, \omega_2 = 1.1 \times 10^{-4}$ and $\omega_3 = 1.0 \times 10^{-4}$, in units where $k_{\rm B} = \hbar = c = 1$.

In the case of coherent states, we study the separable state of equation (28) and the entangled state of equation (30) for the same average number of photons as in the number states, therefore we let $A_1 = 0,\ A_2 = 1$ and $A_3 = \sqrt{2}$. The $|C_{\rm sep}|$ and $|C_{\rm ent}|$ have been plotted against $\Omega' t$ in figures 5(a) and (c), respectively. In figures 5(b) and (d) the corresponding imaginary parts, $\mathrm{Im}(C_{\rm sep})$ and $\mathrm{Im}(C_{\rm ent})$, have been plotted against $\Omega' t$.

In figure 6, we show $|C_{\rm ent} - C_{\rm sep}|$ for coherent states with $A_1 = 0,\ A_2 = 1$ and $A_3 = \sqrt{2}$ as a function of $\Omega' t$. It is clearly seen that, in both the tripartite case and the bipartite case, there is a significant difference between the $C_{\rm sep}$ and $C_{\rm ent}$. It is also seen that the absolute value of $C_{\rm ent} - C_{\rm sep}$ for the tripartite case is an order of magnitude greater than in the bipartite case. Therefore the quantum part of C does not diminish as the photon correlations are distributed to more than two electron-interference devices.

6. Discussion

It has been recognized that geometrical and topological phases [1, 2, 5, 6] could be harnessed for the purposes of inherently fault-tolerant quantum computation [8, 22]. It has also been known for some time that the quantum mechanical correlations of physical states are a useful resource for quantum information processing [17]. The aim of this paper has been to study the photon-induced correlations of topological phase factors for charged particles in distant interference experiments. It has been shown that the classical or quantum correlations of the irradiating

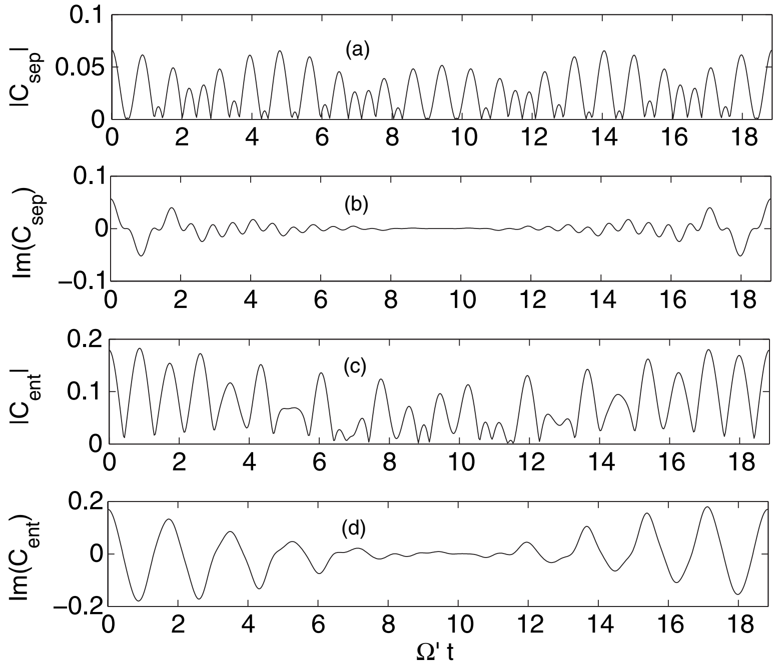

Figure 5. (a) $|C_{\rm sep}|$ and (b) $\text{Im}(C_{\rm sep})$ corresponding to the separable coherent states of equation (28). (c) $|C_{\rm ent}|$ and (d) $\text{Im}(C_{\rm ent})$ corresponding to the entangled coherent states of equation (30) for $A_1 = 0$, $A_2 = 1$ and $A_3 = \sqrt{2}$ as a function of $\Omega' t$. The frequencies are $\omega_1 = 1.2 \times 10^{-4}$, $\omega_2 = 1.1 \times 10^{-4}$ and $\omega_3 = 1.0 \times 10^{-4}$, in units where $k_{\rm B} = \hbar = c = 1$.

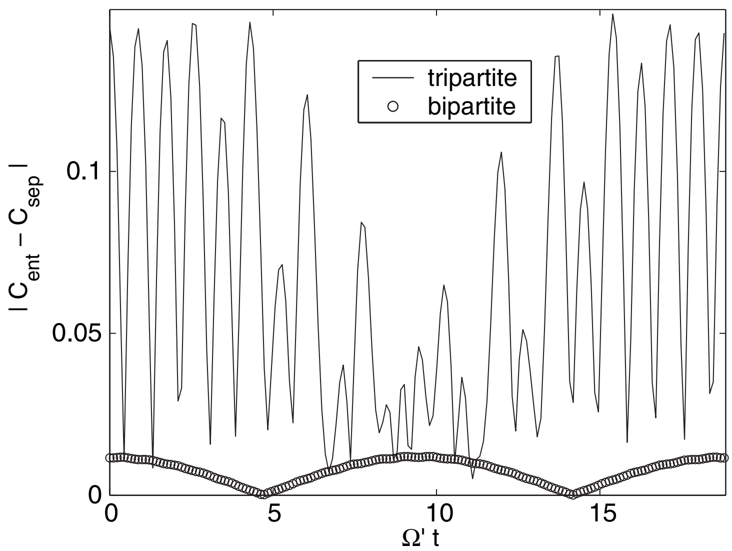

Figure 6. $|C_{\rm ent} - C_{\rm sep}|$ for coherent states with $A_1 = 0$, $A_2 = 1$ and $A_3 = \sqrt{2}$ as a function of $\Omega' t$. The solid line corresponds to the tripartite case of the separable and entangled states of equations (28) and (30). The line of circles corresponds to the bipartite case of the separable and entangled states of equations (16) and (18). The frequencies are $\omega_1 = 1.2 \times 10^{-4}$, $\omega_2 = 1.1 \times 10^{-4}$ and $\omega_3 = 1.0 \times 10^{-4}$, in units where $k_{\rm B} = \hbar = c = 1$.

photons are transferred to the phase factors of the circulating electrons. This mechanism may allow for the detection of photon entanglement using nanoscale electronic devices [19, 20].

In particular, we have considered the one-mode Weyl functions of equations (6) and (7) for the density operators ρ_A and ρ_B of the photons propagating in the distant cavities **A** and **B**. They yield the expectation values of the electron phase factors in the two interference experiments. These can be measured experimentally through the visibility and the phase shift of the interference fringes. We have also considered the two-mode Weyl function of equation (9) for the bipartite state ρ. This yields the joint phase factor in both experiments. Using these Weyl functions, we have defined the difference C of equation (10), which vanishes only for independent subsystems. Considering suitable examples of classically and quantum mechanically correlated photons in number states and coherent states, we have shown that C does not vanish and, therefore, that the electron phase factors are correlated. We have also shown that the value of C depends on the quantum noise and statistics of the external photons (figures 2 and 3). Further work is required in order to distinguish between classical and quantum mechanical correlations using the proposed set-up. One possibility is to derive a Bell-type inequality for the two-mode Weyl function, which is obeyed in the separable case, but is violated in the entangled case.

It has also been shown that the same general result applies to the tripartite case. In this case, the joint phase factor is measured in three distant electron interference experiments and its expectation value is given by the three-mode Weyl function of equation (22). The difference C is in this case replaced by that of equation (23). Numerical results have been presented in figures 4–6 for several examples of classically and quantum mechanically correlated number states and coherent states.

In conclusion, we have shown that it is possible to entangle the topological phase factors of interfering electrons that are irradiated with nonclassical electromagnetic fields. In future work, it would be very interesting to derive similar results on the photon-induced entanglement of geometric phases acquired by spin-1/2 particles [23], or Cooper pairs in mesoscopic Josephson junctions [24], for example. In the last few years, there has been a lot of work on the role of entanglement in mesoscopic devices [25]. The set-up discussed in this paper may be useful in the production of entangled electric charges in a normal conductor or a superconductor using topological phases that are induced by external photons. This is within the realm of current experimental techniques, whereby a nanoscale Josephson device can be controlled with a single microwave photon [26].

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Appendix A. Relations for number states

The following relation yields the matrix elements of the displacement operator in the number state basis [27]:

$$\langle m|D(z)|n\rangle = \left(\frac{n!}{m!}\right)^{1/2} z^{m-n} \exp\left(\frac{-|z|^2}{2}\right) L_n^{m-n}(|z|^2). \tag{A.1}$$

Using this, it can easily be shown that

$$\tilde{W}_{\mathrm{A}}(\lambda_{\mathrm{A}}) = \tilde{W}_{\mathrm{B}}(\lambda_{\mathrm{B}}) = 2^{-1}\exp\left(\frac{-q^2}{2}\right)\left[L_{N_1}(q^2) + L_{N_2}(q^2)\right] \tag{A.2}$$

for the ρ_{sep} of equation (11) and the ρ_{ent} of equation (13). The two-mode Weyl function of equation (9) for the ρ_{sep} is

$$\tilde{W}_{\mathrm{AB,sep}}(\lambda_{\mathrm{A}}, \lambda_{\mathrm{B}}) = \exp(-q^2)L_{N_1}(q^2)L_{N_2}(q^2). \tag{A.3}$$

However, for ρ_{ent}, we have

$$\tilde{W}_{\mathrm{AB,ent}}(\lambda_{\mathrm{A}}, \lambda_{\mathrm{B}}) = \tilde{W}_{\mathrm{AB,sep}}(\lambda_{\mathrm{A}}, \lambda_{\mathrm{B}}) + \exp(-q^2)L_{N_1}^{N_2-N_1}(q^2)L_{N_2}^{N_1-N_2}(q^2)\cos(\Omega t). \tag{A.4}$$

Appendix B. Relations for coherent states

In the coherent states basis, we have

$$\langle A|D(z)|B\rangle = \langle 0|D(-A+z+B)|0\rangle \exp(\chi), \tag{B.1}$$

where $\langle 0|D(-A+z+B)|0\rangle$ can be calculated with the help of equation (A.1) and the phase χ is given by

$$\chi = \tfrac{1}{2}(-Az^* + A^*z - AB^* + A^*B - z^*B + zB^*) \tag{B.2}$$

for any complex numbers A, B and z.

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