Aspects of Quantum Integrable Systems

Tommaso Franzini

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Abstract

In classical mechanics, an integrable system is a model that admits a complete set of integrals of motion that are in involution. This notion can be extended to include classical integrable field theories. A natural question is how to lift these notions to the quantum setting. Despite the extensive literature on the subject, these models are still not very well understood. As an example, in the first section of this work, we present a new class of integrable quantum field theories which develop an unusual behaviour that may be interpreted as a Hagedorn transition.

The principal reason for the enigmatic nature of these models is the absence of a universal mathematical framework to describe them. One potential solution may be provided by quantum affine Gaudin models, which we examine in greater detail in the second part of this work. In particular, we introduce the first non-trivial Hamiltonian of quartic order for the affine $\mathfrak{sl}_2$ Gaudin model, as well as the next-to-leading order expression for all higher Hamiltonians. Furthermore, we provide new insights into the double-loop version of the Feigin-Frenkel homomorphism, which is expected to be a crucial component in the construction of the Bethe ansatz for these models.

Preface

This doctoral dissertation is the result of the individual work of the author. The research here presented was conducted within the School of Physics, Engineering and Computer Science at the University of Hertfordshire, under the supervision of Dr Charles Young.

The results in this thesis are based on the following articles the author has published together with collaborators

- T. Franzini and C. A. S. Young, *Quartic Hamiltonians, and higher Hamiltonians at next-to-leading order, for the affine* $\mathfrak{sl}_2$ *Gaudin model*, J. Phys. A: Math. Theor. **56** 105201 (2023); preprint available at [math.QA/2205.15815]
- C. Ahn, T. Franzini and F. Ravanini, *Hagedorn singularity in exact* $U_{\mathfrak{q}}(\mathfrak{su}(2))$ *S-matrix theories with arbitrary spins* (2024), J. High Energ. Phys. **06**, 157 (2024); preprint available at [hep-th/2402.15794]

and the following preprint

- T. Franzini, *Wakimoto construction for double loop algebras and* ζ*-function regularisation*, (2024); preprint available at [math-ph/2406.17855]

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Contents

Introduction

The importance of being integrable. Since the early days of classical mechanics, the problem of solving the equations of motion of a system has always been of central importance, and culminated in the 19th century with the remarkable theorem by Liouville [**Lio55**], later generalised by Arnold [**Arn74**], which clarifies under which conditions one can hope to find explicit solutions, *i.e.* when the system under consideration is *integrable*. This result, despite being extremely powerful and profound, is not of simple applicability. It was only one century later, prompted by the study of fluid mechanics, non-linear partial differential equations and classical field theory, in particular by the study of the Korteweg-de Vries equation [**GGKM67**], that people came up with new methods, now collectively known as the *classical inverse scattering program* [**Lax68, AKNS73, ZS79, AS81**], whose central object of study is the *classical r matrix*, which has to satisfy certain equations known as *classical Yang-Baxter equations*.

Quantum theories. With the advent of quantum mechanics, the question of whether it was possible to extend the concept of integrability to that setting arose. One of the first and most remarkable example was provided within the study of Ising-type magnets, with the work of Bethe on the exact solution of the Heisenberg spin chain [**Bet31**], described by the nearest-neighbour Hamiltonian

$$\mathcal{H} = -J \sum_{n=1}^{N} \boldsymbol{S}^{(n)} \cdot \boldsymbol{S}^{(n+1)},$$

where $\boldsymbol{S}^{(i)}$ are the $\mathfrak{sl}_2$ spin operators at site $i = 1, \dots, N$ and $\boldsymbol{S}^{(N+1)} = \boldsymbol{S}^{(1)}$, J being the coupling constant. In his work he devised a new method, now known as *coordinate Bethe ansatz*, by assuming that the wave function of the system could be expressed as the superposition of free waves by adding some phase shifts, which have to satisfy certain compatibility conditions known as *Bethe equations*.

As pointed out in [**Skl92**] the *quantum inverse scattering program* emerged as the synthesis of the two currents, one stemming from the work of Bethe and expanded by many others after him, and the aforementioned machinery from the classical setting. The idea goes as follows: one first introduces a certain associative algebra $\mathcal{A}$, where the quantum analogue of the r matrix, called the (quantum) R matrix, appears in the commutation relations of the generators of $\mathcal{A}$. This R matrix has to satisfy the (quantum) *Yang-Baxter equation*. Given this data, one can consider a certain representation of $\mathcal{A}$. Finally, one can address the central problem of diagonalising the commuting charges, finding their eigenvalues and eigenvectors. One possible way to do that is to use a generalisation of the idea proposed by Bethe, known as *algebraic Bethe ansatz*. It is worth mentioning that the development of the mathematical framework of the quantum inverse scattering led to the introduction of interesting algebraic objects such as Yangians and quantum groups [**Dri85**, **Jim85**].

Quantum integrable field theories. In the context of (1+1)-dimensional quantum field theories, the presence of an infinite number of conserved quantities introduces a series of constraints on the scattering matrix of the theory. As shown in the seminal work [**ZZ79**], it is indeed possible to compute it *exactly*, modulo some overall scalar factor. Moreover, from physical considerations, it follows that the scattering matrix has to satisfy the Yang-Baxter equation. For this reason, this represents a natural setting to apply the quantum inverse scattering program mentioned above. Another remarkable approach was proposed in [**Zam89**], based on the idea of regarding integrable quantum field theories as certain relevant deformation of conformal field theories, preserving integrability. These procedures gave rise to a plethora of examples of exact asymptotic scattering theories [**AFZ79**, **CM89**, **BCDS90**, **FOZ93**].

One can also proceed in the opposite way. Once the asymptotic scattering matrix is given, one can get information on the finite-size effects of the theory using the so-called *thermodynamic Bethe ansatz* [**YY69**, **Zam90**]. This allows, for example, to extract the central charge of the underlying conformal field theory, whose integrable deformations would give rise to the massive theory under study.

A question we raise in the first part of this work is the following: *are the exact S-matrices constructed as above valid at all energy scales?* Indeed, there exist well-known examples of higher dimensional theories which undergo a phase transition, resulting in certain divergencies of the thermodynamic quantities. An example is given by the Hagedorn transition within the standard model, which can be interpreted as the point at which the hadronic description of matter has to be replaced with the quark model [**Hag65**]. A similar behaviour has been described in string theories [**AW88**].

Phase transitions of the Hagedorn type have been recently described in integrable quantum field theories in two dimensions [**SZ17**, **CNST16**]. These theories are obtained by deforming integrable theories which are well-defined at all scales. The deforming irrelevant operators are constructed from the components T and $\overline{T}$ of the energy momentum tensor (and its descendants), and are known as $T\overline{T}$-deformed theories. At the level of the S-matrix, this results in the appearance of overall scalar factors, called CDD factors, which are ultimately responsible for the appearance of the singularity at a certain scale. It is worth noting that in these cases the singularity is somehow "apparent", since under a fine tuning of the deformation parameters the phase transition is completely removed and one can recover a UV-complete theory [**AL22**].

In this thesis, we present the first example of integrable scattering theories with Hagedorn-like singularity which are *minimal*, *i.e.* not obtained as a deformation of a UV-complete integrable quantum field theory. In this case, the divergence is more "fundamental", as it is a particular feature of the theory that cannot be removed.

These, apart from being an interesting class of theories *per se*, also show how rich, and still not very well understood, integrable models are in the quantum field theory setting. Indeed, a complete and universal description of such models is still missing.

Two frameworks for classical models. At this point it is useful to make a step back and consider once again classical integrable field theories. In recent years, two descriptions, which are ultimately related, have found numerous and fruitful applications.

The first one in given by 4D semi-holomorphic Chern-Simons theories with defects [**CY18**]. This interpretation has roots in the idea that the algebraic structure of integrable models can be recovered by considering 4-dimensional gauge theories [**Cos14, CWY18a, CWY18b**]. The idea is to consider a 4-dimensional space $\mathcal{M} = \Sigma \times \mathbb{CP}^1$, where Σ will ultimately represent the space-time of the two-dimensional theory with coordinates (x, t), while $\mathbb{CP}^1$ is the Riemann sphere with coordinate z, which will be the interpreted as the spectral parameter of the theory. The theory is described by the action

$$S(A) = \frac{i}{4\pi} \int_{\mathcal{M}} \omega \wedge \mathsf{CS}(A),$$

where $A(x, t; z, \bar{z})$ is the gauge field of the theory and $\mathsf{CS}(A) = \mathrm{tr}\big(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A\big)$ is the Chern-Simons term. Here $\omega = \varphi(z)dz$ is a meromorphic 1-form, depending on the function $\varphi(z)$, which is a rational function with certain poles $\boldsymbol{z} \subset \mathbb{CP}^1$ and zeros $\boldsymbol{\zeta} \subset \mathbb{CP}^1$, both with certain multiplicities. The advantage of this approach is that by imposing boundary conditions of the field at the poles of ω, the theory localises to a 2 dimensional integrable field theory on Σ. Using this procedure, many examples and deformations thereof were found [**DLMV19**]. More recently, it has been shown that this procedure can be obtained by a more general 6 dimensional holomorphic Chern-Simon theory on twistor space [**CCHL**$^+$**23**].

The other approach, that we explore in this thesis, is that of Gaudin models. Historically, they were first introduced to describe integrable quantum spin chains with long range interactions [**Gau76**], called the Gaudin magnets, described by the quadratic Hamiltonians

$$\mathcal{H}_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{I^{a,(i)} I_a^{(j)}}{z_i - z_j}.$$

where $\{z_1, \ldots, z_N\}$ are a set of points on the Riemann sphere and can be thought as the sites of the model. As in the Heisenberg chain, $I^{a,(i)}$ are the spin operators of $\mathfrak{sl}_2$ at site $i = 1, \ldots, N$ and $1/(z_i - z_j)$ is the coupling constant. It was later shown that this system is always integrable for any choice of finite-dimensional semi-simple Lie algebra $\mathfrak{g}$ [**Gau14**].

In the seminal paper [FFR94], it was shown that the Gaudin models admit higher conserved charges and that they can be diagonalised with an elegant form of Bethe ansatz, based on the construction of Wakimoto modules at the critical level. The study of quantum finite Gaudin models culminated with the remarkable correspondence between the space of commuting charges and the algebra of functions on the space of certain differential operators called ${}^L\mathfrak{g}$-*opers* with regular singularities [Fre05a], which initiated a new line of research in the context of the geometric Langlands correspondence applied to physical systems [Fre05b, CT06, FH18].

In more recent years, people have come up with several generalisations of the Gaudin models, for example by considering arbitrary multiplicities [FFTL10], by introducing certain automorphism of $\mathfrak{g}$, called *cyclotomic* Gaudin models [VY16, LV18] or by imposing reality conditions to obtain *dihedral* Gaudin models [Vic20].

Affine Gaudin models. Another possible direction, which is the main focus in this thesis, is obtained by replacing the finite-type Lie algebra with one of untwisted affine type.

To motivate the study of such generalisation, it is worth recalling the classical limit of these models, which are better understood. As before, consider a set of points on the Riemann sphere. One introduces the phase space, which has a Poisson structure defined by the so-called *Kirillov-Kostant* bracket. One can then define a Lax matrix and the corresponding r matrix. The quadratic Hamiltonians defining the models can be obtained as certain classical limits of those above, as well as the other higher conserved charges (for more details see [Lac18]). Also in this setting one can consider the generalisation to the affine setting. The remarkable result from [Vic20] is that dihedral, *i.e.* cyclotomic real, classical affine Gaudin models provide a universal language to describe a large class of classical integrable field theories with twist function, *i.e.* whose r matrix has the form

$$r(z_1, z_2) = r^0(z_1, z_2)\varphi(z_2)^{-1},$$

where $r^0(z_, z_2)$ can be thought as a "standard" non-twisted r matrix and $\varphi(z)$ is a rational function called the *twist function*. A series of explicit examples have been worked out explicitly for example in [Lac23]. This function is precisely the same rational function entering the definition of the 1-form ω in the 4 dimensional Chern-Simons formulation. Indeed, as mentioned above, it is known that these two languages are in fact related [Vic21].

For these reasons, the quantisation of classical affine Gaudin models is of central importance as the expectation is that it could provide a powerful tool to study a large class of integrable *quantum* field theories [FF07].

Unfortunately, unlike their finite-type counterparts, quantum affine Gaudin models are still not well understood. For example, there is still no explicit description of the space of commuting charges, but only some conjectures [FF07, LVY18] and there is no analogue of the Bethe ansatz construction with Wakimoto modules.

In the second part of this thesis, we touch some of these open problems. For example, we provide the explicit construction of the first non-trivial higher Hamiltonian for the $\widehat{\mathfrak{sl}}_2$ Gaudin model and concerning the Bethe ansatz construction, we introduce a novel generalisation of the Feigin-Frenkel homomorphism, which might play a central role in the construction of Wakimoto modules.

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This thesis is organised as follows.

In chapter 1 we introduce some ideas from the theory of classical and quantum integrability. First, in section 1.1 we state the Liouville-Arnold theorem and we recall the Lax formalism. We show how the classical r-matrix appears in this context and in particular how the classical Yang-Baxter equation follows by imposing certain conditions. In section 1.2 we move to the quantum setting. Following the approach by Zamolodchikov, we proceed to summarise the theory of the S-matrix with a particular focus on $1+1$ dimensional theories. The result of having a two-dimensional theory with an infinite number of conserved charges imposes a series of constraints on the scattering processes, as we describe in section 1.2.4. These give rise to the quantum Yang-Baxter equation for the S-matrix. We end the chapter by recalling additional properties that the scattering matrix has to satisfy, and we introduce the *bootstrap method*.

In chapter 2 we provide the construction of a new class of integrable scattering theories with quantum group $U_{\mathfrak{q}}(\mathfrak{su}_2)$ symmetry. This is done by constructing explicit minimal S-matrices, starting from the quantum group R-matrices in the spin s representation, which automatically satisfy the Yang-Baxter equation and imposing crossing symmetry and unitarity. These theories present two distinct regimes: an *attractive* one, where the theory admits bound states, and a *repulsive* one. In section 2.2 we focus on the latter, and we perform thermodynamic Bethe ansatz to obtain information on the finite-size effects of the theory. In section 2.3, we proceed to study the TBA equations with numerical methods. Quite remarkably and unexpectedly, we find that these new theories develop singularities at a certain energy scale.

In chapter 3, we recall the theory of quantum Gaudin models of finite-type. We give some general definition and we describe the Bethe ansatz construction. The second part of the chapter focuses on the Feigin-Frenkel-Reshetikhin construction introduced in [**FFR94**]. To do this, we first describe the appropriate language which is customarily used in these contexts, which is the theory of vertex algebras. We describe the space of commuting Hamiltonians, or Gaudin/Bethe subalgebra, in terms of coinvariants. In section 3.4, we end the chapter by recalling the Feigin-Frenkel homomorphism of vertex algebras and how this can be used to construct Wakimoto modules, and ultimately to describe a new way to reproduce the Bethe ansatz.

In chapter 4 we describe the explicit construction of the first non-trivial higher Hamiltonian for the $\widehat{\mathfrak{sl}}_2$ Gaudin model, of quartic order. To do this, we follow the conjecture proposed in [**LVY18**]. We do this by explicitly requiring that the new charge commutes with the generators of the Lie algebra and with the other known Hamiltonians of the model. In section 4.5 we also provide a next-to-leading order expression for all other higher charges of order $n \geq 5$.

In chapter 5 we focus on the Feigin-Frenkel homomorphism. For any finite-dimensional Lie algebra $\mathfrak{g}$, it provides a map of vertex algebras, relating the vacuum Verma module at critical level with the Fock module for the $\beta\gamma$-system of free fields. This map has a central role in the construction of the Wakimoto modules as described in section 3.4. Following [**You21**], we attempt a generalisation of this map to the case where $\mathfrak{g}$ is of untwisted affine type. By doing this, we observe that divergent quantities appear. To deal with this problem, in section 5.2 we introduce a new space with the structure of a vertex Lie algebra, whose higher products depend on a regulation parameter z. Inspired by standard techniques in physics, we proceed by introducing, in section 5.3, a regularisation procedure to "cure" the infinities. This allows us to show in section 5.4 the suggestive fact that first products are all vanishing on the nose.

CHAPTER 1

Quantum integrability and the S-matrix program

1.1. Classical integability

In section 1.1.1 we are going to briefly recall the notion of classical integrability, the fundamental Liouville-Arnold theorem and in section 1.1.2 the Lax formalism. We will describe how the latter can be utilised to study integrability in classical field theories. Using this formalism, in we will see how the classical r-matrix naturally arises and we proceed to study its main properties.

1.1.1. Poisson structure and Liouville theorem. In classical mechanics, a state in a n-dimensional system is described by a point on a $2n$-dimensional manifold $\mathcal{M}$, called the *classical phase space*. It has the structure of a Poisson manifold, meaning that the commutative algebra of smooth function $C^\infty(\mathcal{M})$ is a Poisson algebra, known as the *algebra of classical observables*. This implies the existence of a non-degenerate bilinear map $\{\cdot,\cdot\}$: $C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$, called the *Poisson bracket*, with the following properties

- *Skew-symmetry*: $\{f,g\} = -\{g,f\}$;
- *Jacobi identity*: $\{f,\{g,h\}\} = \{\{f,g\},h\} + \{g,\{f,h\}\}$;
- *Leibniz rule*: $\{fg,h\} = f\{g,h\} + \{f,h\}g$.

Time evolution is determined by the choice of a function $\mathcal{H} \in C^\infty(\mathcal{M})$, called the *Hamiltonian* of the system. The Hamiltonian flow parametrised by $t \in \mathbb{R}$ of an observable $f \in C^\infty(\mathcal{M})$ on the phase space is computed using the Poisson bracket, namely

$$\{f,\mathcal{H}\} = \partial_t f. \tag{1.1}$$

Locally, there always exists a set of coordinates $\{p^i, q_i\}_{i=1,\ldots,n}$ on $\mathcal{M}$, with $\{p^i, q_j\} = \delta^i_j$ and $\{p^i,p^j\} = \{q_i,q_j\} = 0$, such that one can express the Poisson bracket as

$$\{f,g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial p^i}\frac{\partial g}{\partial q_i} - \frac{\partial g}{\partial p^i}\frac{\partial f}{\partial q_i}, \tag{1.2}$$

for any two functions $f, g \in C^\infty(\mathcal{M})$. In this case, the Hamiltonian flow is described by the famous Hamilton's equations

$$\partial_t p^i = \frac{\partial \mathcal{H}}{\partial q_i}, \qquad \partial_t q_i = -\frac{\partial \mathcal{H}}{\partial p^i}. \tag{1.3}$$

for all $i = 1,\ldots,n$.

A quantity that Poisson-commutes with the Hamiltonian is an integral of motion, *i.e.* it is conserved along the flow.

We say that a n-dimensional system is *Liouville-integrable* if there are n independent[1] conserved quantities Q_i, $i = 1, \ldots, n$ that are in involution, *i.e.*

$$\{Q_i, Q_j\} = 0 \text{ for all } i, j = 1, \ldots, n. \tag{1.4}$$

Note that the Hamiltonian itself has to be a combination of the Q_i. These charges are called the *higher Hamiltonians* of the system, and each of them can be picked to generate a time flow on $\mathcal{M}$.

An important result, known as the *Liouville-Arnold theorem*, states that the equations of motion of a Liouville-integrable system can always be solved by quadratures, *i.e.* straightforward integration, by re-parametrising the system using the so-called *action-angle variables.* This method, while being very powerful when dealing with simple classical systems such as the harmonic oscillator, Kepler's problems, and many more (see e.g. [**BBT03**] for other examples), becomes impractical when dealing with classical field theories, which by nature are described by an infinite number of degrees of freedom.

1.1.2. Classical integrable field theories. Consider a 2-dimensional Minkowski space $\Sigma = \mathbb{R} \times \mathbb{R}$ or $\Sigma = \mathbb{R} \times S^1$. A field on Σ is a collection of functions $\phi_i : \Sigma \to \mathbb{C}$ for some $i \in \mathbb{Z}_{\geq 1}$. If the space coordinate is taken to be on the infinite line $\mathbb{R}$, they need to satisfy the asymptotic conditions $\phi_i(t, x) \to 0$ when $x \to \pm\infty$, while if we take it on the circle S^1 we need to specify periodicity, *i.e.* $\phi_i(t, x) = \phi_i(t, x + 2\pi)$. The reason we focus on two-dimensional theories is that, as a consequence of the Coleman-Mandula theorem [**CM67**], at the quantum level it is known that integrable field theories in more than two dimensions have trivial scattering, as we will describe in more detail in section 1.2.2.

By naively extending the notion of Liouville integrability to this setting, the expectation is that one would have to find an infinite number of conserved independent charges in involution. Doing this by direct inspection is clearly impossible, so new tools are needed.

1.1.2.1. *Lax pairs.* Consider two matrices $\mathcal{L}$ and $\mathcal{M}$ valued in the Lie subalgebra $\mathfrak{g}$ of the Poisson algebra of smooth functions $C^\infty(\Sigma)$, which depend on an additional parameter z, called the *spectral parameter.* They form a *Lax pair* for a classical system if the equations of motion can be written in the form [**Lax68, BBT03**]

$$\partial_t \mathcal{L}(t, x; z) - \partial_x \mathcal{M}(x, t; z) = [\mathcal{M}(x, t; z), \mathcal{L}(x, t; z)], \qquad \text{for all } z \in \mathbb{C}. \tag{1.5}$$

At this point one can introduce the path ordered exponential, sometimes called the *transfer matrix* from x to y

$$T(t, x, y; z) = \mathrm{P}\overleftarrow{\exp}\left(\int_x^y \mathcal{L}(t, u; z) \mathrm{d}u\right). \tag{1.6}$$

By some manipulations (see e.g. [**Tor16**]), one can find that eq. (1.5) can be rewritten as

$$\partial_t T(t, y, x; z) = T(t, y, x; z)\mathcal{M}(t, y; z) - \mathcal{M}(t, x; z)T(t, y, x; z). \tag{1.7}$$

[1] This means that the one forms dQ_i are linearly independent.

Suppose we are working with a field theory on the real line $\mathbb{R}$. One can define the *monodromy matrix* as

$$T(t,z) = T(t,\infty,-\infty;z). \tag{1.8}$$

If we impose that the Lax matrix satisfies the fall-off condition $\mathcal{M}(t,x;z) \to 0$ as $x \to \pm\infty$, from eq. (1.7) we obtain that

$$\partial_t T(t,z)^n = 0 \qquad \text{for any } n \in \mathbb{Z}_{\geq 1}. \tag{1.9}$$

This means that powers of the transfer matrix are conserved in time, *i.e.* they form a family of integrals of motions. In the case when the space direction is compactified on the circle S^1, one can repeat a similar argument finding that now the powers of the traces of the monodromy matrix are conserved charges,

$$\partial_t \operatorname{tr}(T(t,z)^n) = 0 \tag{1.10}$$

1.1.2.2. *Classical Yang-Baxter equation.* The Lax formulation gives a direct construction of a tower of conserved charges. In order to check Liouville integrability, one also has to ensure that they Poisson-commute.

It turns out that there is a sufficient condition for this to be verified, which is determined by the particular form of the Poisson bracket of the Lax matrix $\mathcal{L}$ with itself, called the *non-ultralocal Maillet bracket* [**Mai85, Mai86**]

$$\begin{aligned}\{\mathcal{L}_1(t,x;z_1),&\mathcal{L}_2(t,y;z_2)\} = [r_{12}(z_1,z_2),\mathcal{L}_1(t,x;z_1)]\delta(x-y)\\ &- [r_{21}(z_2,z_1),\mathcal{L}_2(t,y;z_2)]\delta(x-y) - (r_{12}(z_1,z_2)+r_{12}(z_2,z_1))\partial_x\delta(x-y)\end{aligned} \tag{1.11}$$

where we introduced the notation $X_1 = X \otimes \mathrm{id}$ and $X_2 = \mathrm{id} \otimes X$, for any $X \in \mathfrak{g}$ and for some matrix $r(z_1,z_2) \in \mathfrak{g}\otimes\mathfrak{g}$ depending on two spectral parameters z_1, z_2.

When r is skew-symmetric the term proportional to the non-local term, *i.e.* the one involving the derivative of $\delta(x-y)$, vanishes and the rest reduces to a simpler expression, known as *ultralocal Sklyanin bracket* [**Skl82**].

It is straightforward to check that the bracket in eq. (1.11) is skew-symmetric if and only if r is. In order to satisfy the Jacobi identity one has to impose extra conditions on the matrix r. The constraint is known as the *classical Yang-Baxter equation* (CYBE)

$$[r_{12}(z_1,z_2), r_{13}(z_1,z_3)] + [r_{12}(z_1,z_2), r_{23}(z_2,z_3)] + [r_{32}(z_3,z_2), r_{13}(z_1,z_3)] = 0, \tag{1.12}$$

which is understood as an identity on a triple tensor product, *i.e.* $r_{12}(z_i,z_j) := \phi_{12}(r(z_1,z_2))$, where $\phi_{12} : \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$ is the map defined as follows $\phi_{12}(a\otimes b) = a\otimes b\otimes 1$, with $a, b \in \mathfrak{g}$ and similarly for the other terms $r_{13}(z_1,z_2)$ and $r_{23}(z_1,z_2)$. The matrices satisfying these relations are called classical r-matrices.

When $r(z_1,z_2)$ is meromorphic in the spectral parameters and $\det r(z_1,z_2) \neq 0$, it can be shown that the r matrix is a function of the difference $u := z_1 - z_2$, or it can be recast in such form [**BD83**]. Under these assumptions, Belavin and Drinfel'd [**BD82**] showed that the poles of $r(u)$ are all simple and form a lattice $\Gamma \subset \mathbb{C}$. This allows for a classification of such matrices: if $\operatorname{rank}\Gamma = 0$, $r(u)$ depends rationally on u, if $\operatorname{rank}\Gamma = 1$, $r(u)$ depends trigonometrically on u and if $\operatorname{rank}\Gamma = 2$, $r(u)$ depends elliptically on u.

1.1.2.3. *Towards quantisation.* The existence of the classical r matrix satisfying the above properties implies the existence of a (quasi)-triangular bialgebra structure on $\mathfrak{g}$ [**Dri83**]. The idea behind quantisation is replacing the commutative algebra of observables, with a non-commutative one. This is obtained by deforming the Lie bialgebra structure on $\mathfrak{g}$ underlying the classical theory. This structure is known as *quantum group*, which has the structure of a quasi-triangular Hopf algebra. A complex parameter $\mathsf{q} \in \mathbb{C}$ controls the deformation, in such a way that in the limit $\mathsf{q} \to 1$ one recovers the classical Poisson structure. To each case of the classification mentioned above, one can construct the corresponding deformation, obtaining the *Yangian* in the rational case [**Dri85**], the *quantum affine algebra* in the trigonometric case [**Dri85, Jim85**] and the *elliptic quantum group* in the elliptic case [**Skl82, Fel94**].

In this context, one can define the *universal R matrix* of the quantum group satisfying, the *quantum Yang-Baxter equation*

$$R_{12}(z_1, z_2)R_{13}(z_1, z_3)R_{23}(z_2, z_3) = R_{23}(z_2, z_3)R_{13}(z_1, z_3)R_{12}(z_1, z_2). \tag{1.13}$$

where we are using the same notation as above.

By defining $R = \mathrm{id} \otimes \mathrm{id} + i\hbar r + \mathcal{O}(\hbar^2)$, one finds that this relation at leading order reduces to the classical Yang-Baxter equation in (1.12). This particular algebraic structure can be recovered from physics arguments, as we will describe in the next section.

1.2. Quantum integrability

In this section, we are going to describe the general theory of the scattering matrix in quantum field theories. We will illustrate the fundamental no-go Coleman-Mandula theorem [**CM67**], which states that under certain mild physical assumptions, non-trivial scattering processes are not allowed in theories in $1+d$ dimensions, with $d > 1$. However, in the $1+1$ setting, the scattering can still be non-trivial. Moreover, we will describe how, by imposing certain properties and using the symmetries of the system, the S-matrix can be defined *exactly*, following the so-called *bootstrap program*.

1.2.1. The S-matrix. In order to describe scattering processes, we need to assume that the interactions happen in a small region of space, so that away from it the particles are essentially described by free theories. We introduce the vacuum state $|0\rangle \in \mathcal{H}$ where $\mathcal{H}$ is the Hilbert space of the theory, and we define the so-called *asymptotic states*

$$|A_{a_1}(p_1) \dots A_{a_n}(p_n)\rangle = A_{a_1}(p_1) \dots A_{a_n}(p_n) |0\rangle , \qquad n \in \mathbb{Z}_{\geq 1}, \tag{1.14}$$

where the symbols $A_{a_i}(p_i)$ can be thought as creation operators of particles with internal quantum numbers a_i and momenta p_i in the infinite past or in the infinite future.

In our conventions, the scattering matrix is defined as the operator on $\mathcal{H}$ transforming out states in the infinite future in incoming states in the infinite past

$$|A_{a_1}(p_1)\dots A_{a_n}(p_n)\rangle_{\text{in}}$$
$$= \sum_{m=1}^{\infty} \sum\!\!\!\!\!\!\int_{b,p'} S^{b_1,\dots,b_m}_{a_1,\dots,a_n}(p_1,\dots,p_n;p'_1,\dots,p'_m)\, |A_{b_1}(p'_1)\dots A_{b_m}(p'_m)\rangle_{\text{out}}\,, \tag{1.15}$$

where on the right-hand side one has to sum over the internal quantum numbers b_i and integrate over all outgoing momenta p'_i, with $i = 1,\dots,m$, m being the number of outgoing particles.

1.2.1.1. *General properties of the S-matrix.* Any initial state can be expressed in the basis of final states via the S-matrix, from which it follows that the probability for a superposition of initial states to evolve into a superposition of final states is 1. This implies that the S-matrix is a *unitary operator*

$$S^\dagger S = SS^\dagger = \text{id}\,. \tag{1.16}$$

In relativistic scattering theories we also need to impose *Lorentz invariance*, *i.e.* we require physical observables measured in different reference frames to be equal. This implies that the scattering matrix can depend on the momenta only through Lorentz scalars, *i.e.* combinations of the scalar products of momenta. Other properties are *marocausality* and *analyticity*, see [**Mus10**].

1.2.2. Coleman-Mandula theorem. Consider a unitary operator U on the Hilbert space $\mathcal{H}$ of the system. Assume that it transforms one-particle states into one-particle states

$$U\,|A_a(p)\rangle = |A_{a'}(p')\rangle\,, \tag{1.17}$$

and it acts on multi-particle states as if they were tensor products of one-particle states, as follows

$$U\,|A_{a_1}(p_1)A_{a_2}(p_2)\rangle = (\text{id}\otimes U + U\otimes\text{id})\,|A_{a_1}(p_1)\rangle\otimes|A_{a_2}(p_2)\rangle\,. \tag{1.18}$$

Moreover, in order to preserve the probabilities under the symmetry transformation, it has to commute with the scattering matrix

$$[U,S] = 0. \tag{1.19}$$

The question is what kind of symmetry groups, which include the Poincaré group as a subgroup, are allowed in this context. It was shown in [**CM67**] that under some mild physical assumption[2], the structure of the symmetry group of the theory becomes quite

[2]The assumptions are: 1) the particle content has finite non-zero masses; 2) elastic scattering amplitudes are analytic functions of the Mandelstam variables s and t; 3) except for a finite set of values of the center of mass energy s, the scattering always occurs.

trivial, as the only symmetry groups allowed are those in the form

$$G = P \times U, \tag{1.20}$$

where P is the Poincaré group and U a group of internal symmetries.

Famous workarounds to these assumptions are supersymmetric theories, where the Poincaré algebra is replaced with a supersymmetry algebra, and conformal field theories, where the particles are massless. Another possible way to relax the Coleman-Mandula theorem is to modify the rule of how generators of the symmetry act on multiparticle states, *i.e.* replacing the trivial co-multiplicative structure in eq. (1.18) with a non-trivial one. As we have mentioned in section 1.1.2.3, this feature is naturally present in two-dimensional quantum integrable models whose underlying symmetry structure is that of a quantum group, which has a non-trivial co-multiplication [**BL91**].

1.2.3. Scattering theories in $1 + 1$ dimensions. For the rest of the chapter, we focus on two-dimensional relativistic quantum field theories. A particle with mass m_a and momentum p_a has to satisfy the on-shell condition[3] $(p_a)^\mu (p_a)_\mu = \mathsf{m}_a^2$. It is customary to parametrise the momenta in terms of *rapidities*, by introducing the variable $\theta \in \mathbb{R}$

$$(p_a)^0 = \mathsf{m}_a \cosh \theta_a, \qquad (p_a)^1 = \mathsf{m}_a \sinh \theta_a. \tag{1.21}$$

In this setting, considering the scattering process $1 + 2 \to 3 + 4$, the Maldestam variables can be expressed as

$$\begin{aligned} s &= (p_1 + p_2)^2 = \mathsf{m}_1^2 + \mathsf{m}_2^2 + 2\mathsf{m}_1\mathsf{m}_2 \cosh(\theta_1 - \theta_2), \\ t &= (p_1 - p_4)^2 = \mathsf{m}_1^2 + \mathsf{m}_4^2 - 2\mathsf{m}_1\mathsf{m}_4 \cosh(\theta_1 - \theta_4), \\ u &= (p_1 - p_3)^2 = \mathsf{m}_1^2 + \mathsf{m}_3^2 - 2\mathsf{m}_1\mathsf{m}_3 \cosh(\theta_1 - \theta_3). \end{aligned} \tag{1.22}$$

It becomes evident that Lorentz invariance translates into the dependence of s, t and u only on the differences $\theta_i - \theta_j$. They are related by the following relation

$$s + t + u = \sum_{i=1}^{4} \mathsf{m}_i^2. \tag{1.23}$$

Recall that for an incoming state, there are no further interactions for $t \to -\infty$: for particles moving on a line, this means that the fastest particle is on the left while the slowest is on the right. For outgoing states, the situation is similar but reversed, as there are no more interactions for $t \to \infty$; therefore

$$\begin{aligned} &|A_1(\theta_1) \ldots A_n(\theta_n)\rangle_{\text{in}} \,, \qquad \theta_1 > \theta_2 > \cdots > \theta_n, \\ &|A_1(\theta_1) \ldots A_n(\theta_n)\rangle_{\text{out}} \,, \qquad \theta_1 < \theta_2 < \cdots < \theta_n. \end{aligned} \tag{1.24}$$

[3]We are using the following convention for the metric $(+, -)$.

1.2.4. Conserved charges and Parke's theorem. Having an infinite number of conserved charges inevitably imposes constraints on the theory, as observed in [**SW78**]. However, showing their existence is often a rather challenging task.

Quite remarkably, it was shown by Parke [**Par80**] that the existence of just two local conserved charges which transform as tensors of rank higher than 2 under Lorentz transformation (*i.e.* they are not scalars or vectors), is enough to show the following properties of the S-matrix

- there is *no particle production*: the number of initial particles is the same after the collision;
- the *momenta are preserved*: the set of incoming momenta is the same as the outgoing ones;
- the *scattering is factorised*: any scattering of $n \to n$ particles can be decomposed into $2 \to 2$ particle processes, *i.e.* it can be described in terms on the 2-body S-matrix

$$S : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}, \tag{1.25}$$

  where $\mathcal{H}$ denotes the Hilbert space of the asymptotic particles.

We will not present explicit and detailed proof of these results, which can be found in the original papers and in many reviews [**Mus10, Dor97, Bom16**]. The idea is to employ the conserved charges, whose action on a localised wave function is that of moving the particle in space, by an amount proportional to their momentum.[4]

1.2.4.1. *Yang-Baxter equation.* The first direct consequence is related to the scattering of more than two particles. The first non-trivial example is given by the $3 \to 3$ scattering process. In fig. 1.1 we see three different possible situations: the first two only differ by the chronological order of the collisions and are ultimately factorised in $2 \to 2$ successive scattering collisions; the third case instead cannot in general be expressed in terms of the two-body data. However, in the present case, the existence of higher conserved charges simplifies the calculations. Indeed, the action of the conserved charges on the initial asymptotic states has the effect of shifting the trajectories of the particles and therefore one can obtain any of the diagrams from the others. This feature is encoded in the famous *Yang-Baxter equation*

$$\sum_{\alpha,\beta,\gamma} S_{a_1 a_2}^{\alpha\beta}(\theta_{12}) S_{\alpha a_3}^{b_1 \gamma}(\theta_{13}) S_{\beta\gamma}^{b_2 b_3}(\theta_{23}) = \sum_{\alpha,\beta,\gamma} S_{a_2 a_3}^{\beta\alpha}(\theta_{23}) S_{a_1 \alpha}^{\gamma b_3}(\theta_{13}) S_{\gamma\beta}^{b_1 b_2}(\theta_{12}), \tag{1.26}$$

where we used the notation $\theta_{ij} = \theta_i - \theta_j$.

It is natural to ask what kind of constraints are imposed on a general $n \to n$ scattering process. It turns out that the Yang-Baxter equation (1.26) is a sufficient and necessary condition for factorisation.

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[4] In the case of the momentum operator, the particles are all shifted by the same amount, independently of their momenta. This is why higher rank charges are required.

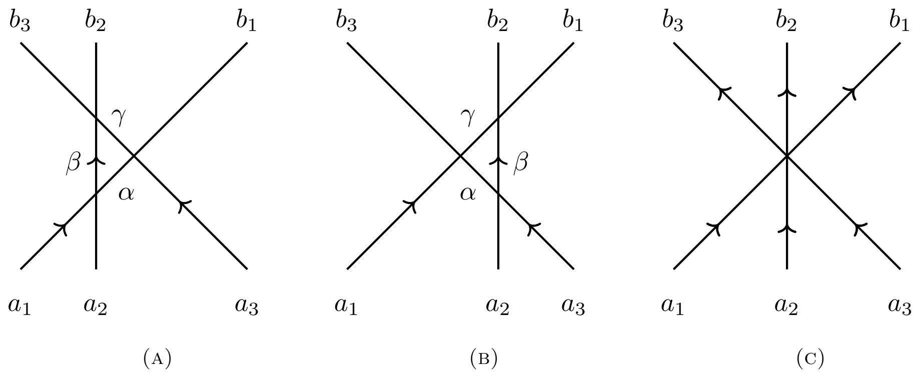


FIGURE 1.1. Equivalent amplitudes of the $3 \to 3$ scattering process in the presence of higher spin conserved charges.

1.2.5. Other properties of the 2-body S-matrix. The goal is to be able to completely fix, up to an overall factor, the structure of the scattering matrix. In order to do that, one has to impose additional constraints on the S-matrix, given by discrete symmetries, unitarity and crossing symmetry.

1.2.5.1. *Discrete symmetries.* If the theory is invariant under charge conjugation $\mathcal{C}$, it means that the scattering process involving the charge-conjugated particles is equivalent to the original process

$$S_{ab}^{cd}(\theta) = (\mathcal{C} \otimes \mathcal{C}) S_{ab}^{cd}(\theta) (\mathcal{C} \otimes \mathcal{C}) = S_{\bar{a}\bar{b}}^{\bar{c}\bar{d}}(\theta), \tag{1.27}$$

where we denoted with $\bar{\blacksquare}$ the charge-conjugated to $\blacksquare$.

Parity transformation $\mathcal{P}$ flips the direction of the space coordinate. If the S-matrix is parity invariant, it means that the process obtained by mirroring the Feynman diagram in fig. 1.2 along the vertical direction is physically equivalent to the original process, *i.e.*

$$S_{ab}^{cd}(\theta) = S_{ba}^{dc}(\theta), \tag{1.28}$$

Applying time reversal $\mathcal{T}$ to the scattering process in fig. 1.2 results in looking at the Feynman diagram in the opposite direction (top to bottom), which results in the following additional constraint

$$S_{ab}^{cd}(\theta) = S_{dc}^{ba}(\theta). \tag{1.29}$$

1.2.5.2. *Unitarity, crossing, and analytic considerations.* Recall from section 1.2.3 the definition of the Mandelstam variables. Since we have no particle production and the set of initial and final momenta are the same, $u = 0$ and we have

$$t = \mathsf{m}_1^2 + \mathsf{m}_2^2 - 2\mathsf{m}_1\mathsf{m}_2 \cosh(\theta_1 - \theta_2) = 2\mathsf{m}_1^2 + 2\mathsf{m}_2^2 - s, \tag{1.30}$$

which follows from the fact that $s + t + u = \sum_{i=1}^{4} \mathsf{m}_i^2$. Hence, there is only one independent variable and from now on we will just express all amplitudes in terms of s. Since the difference $\theta_1 - \theta_2$ must be real, from eq. (1.22) we conclude that $s \geq (\mathsf{m}_1 + \mathsf{m}_2)^2$, which is the energy threshold for production of particles in the s-channel.

In order to express one channel in terms of the other, one needs to consider an analytic continuation of the S-matrix by extending the Mandelstam variables to the whole complex plane. From eq. (1.30) it is clear that the t-channel is obtained by the transformation

$$t(\theta) = s(i\pi - \theta), \tag{1.31}$$

which can be interpreted geometrically as the rotation of the Feynman diagram on the plane, see fig. 1.2. In this channel, we have another threshold in the s plane, namely for $s \leq (\mathsf{m}_1 - \mathsf{m}_2)^2$. From eq. (1.31), it follows that the s-channel and the t-channel are related, and it is possible to continuously move on the s plane from one to the other, a property known as the *crossing symmetry*

$$S_{ab}^{cd}(\theta) = S_{\bar{d}a}^{c\bar{b}}(i\pi - \theta). \tag{1.32}$$

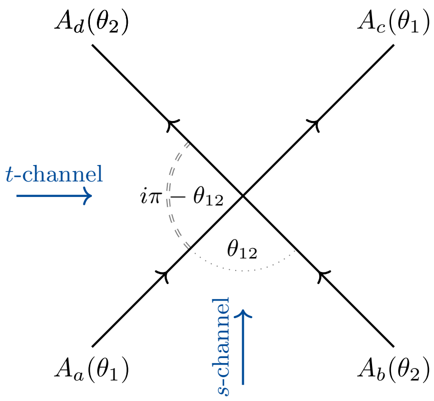


FIGURE 1.2. $2 \to 2$ scattering process. Time flows in the vertical direction. The s-channel and t-channel are obtained by a rotation.

As we will see below, there are branch cuts propagating from the particle production thresholds in the two channels. What we obtain is the the so-called *physical sheet*, depicted in fig. 1.3. Physical values are $s + i\epsilon$, for $\epsilon \to 0^+$, in the region right above the right cut: this is equivalent to the Feynman prescription for causal propagators.

At this point, one can impose that the amplitudes obtained from S and $S^\dagger$ are opposite boundary values of the *same* analytic function [**Oli62, Mir99**]. This property is known as *Hermitian analyticity* of the S-matrix and ultimately it can be expressed as

$$S_{ab}^{cd}(s^*) = [S_{dc}^{ba}(s)]^*. \tag{1.33}$$

If the theory has additional time-reversal symmetry then the S-matrix is *real analytic*, *i.e.* it takes complex-conjugate values at complex-conjugate points

$$S_{ab}^{cd}(s^*) = [S_{ab}^{cd}(s)]^*. \tag{1.34}$$

Using these properties, we can rewrite the unitarity condition in eq. (1.16) as follows

$$\lim_{\epsilon\to 0} S_{ab}^{ij}(s+i\epsilon)S_{ij}^{cd}(s-i\epsilon) = \delta_a^c\delta_b^d, \tag{1.35}$$

which signals the presence of the anticipated cut along the real axis.

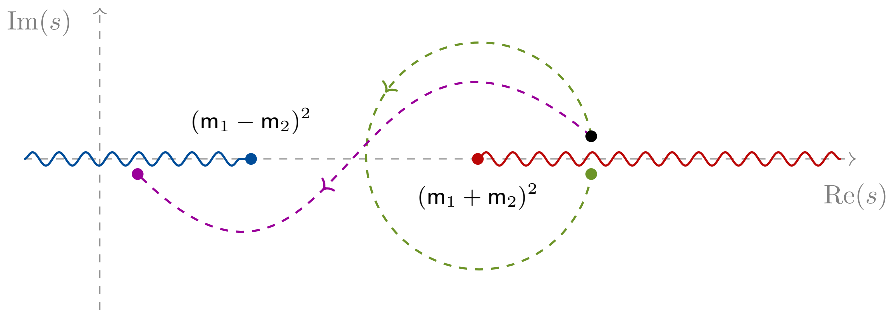


FIGURE 1.3. The complex s-plane and the physical sheet. Unitarity transformation (green) maps values above the cut to the corresponding ones below. Crossing symmetry (purple) identifies points above the right cut to points below the left cut.

Following [ZZ79], the relation $s = \mathsf{m}_a^2 + \mathsf{m}_b^2 + 2\mathsf{m}_1\mathsf{m}_2\cosh(\theta_a - \theta_b)$ can be inverted to obtain

$$\theta_a - \theta_b = \log\left(\frac{s - \mathsf{m}_a^2 - \mathsf{m}_b^2 + \sqrt{(s-(\mathsf{m}_a+\mathsf{m}_b)^2)(s-(\mathsf{m}_a-\mathsf{m}_b)^2)}}{2\mathsf{m}_a\mathsf{m}_b}\right), \tag{1.36}$$

which allows to transform the physical sheet into the so-called *physical strip* on the complex θ-plane, defined for $0 \leq \operatorname{Im}\theta \leq \pi$, where θ, as usual, stands for a difference of rapidities. The right branch point is mapped to the origin, while the left one to $i\pi$. Different Riemann sheets of the s-plane correspond to different strips in the θ plane.

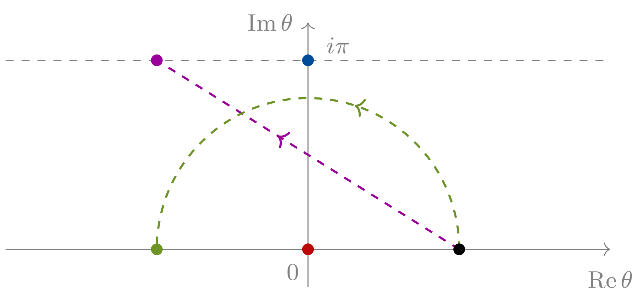


FIGURE 1.4. The complex θ-plane and the physical strip. Unitarity transformation (green) maps $\theta \to -\theta$. Crossing symmetry (purple) maps $\theta \to i\pi - \theta$.

With this change of variable, the S-matrix results analytic on the images of the branch cuts. Since integrability ensures that there are no other cuts corresponding to the creation of other particles, we can conclude that the S-matrix is a meromorphic function in θ. In this variable, the real analyticity condition translates into

$$S_{ab}^{cd}(\theta) = [S_{ab}^{cd}(-\theta^*)]^*, \tag{1.37}$$

which implies that the S-matrix takes real values on the imaginary θ-axis. Similarly, unitarity takes the following form

$$S_{ab}^{jk}(\theta) S_{jk}^{cd}(-\theta) = \delta_a^c \delta_b^d, \tag{1.38}$$

where there is an implicit sum over the internal indices. All these conditions are enough to determine the functional structure of the S-matrix, up to an overall factor.

1.2.6. Poles and bootstrap principle. In the discussion above we have not mentioned bound states, which can arise whenever the total energy of the state is lower than the sum of the single energies of the colliding particles. Although we will not deal with them in the present work, we want to mention that they are identified with points along the imaginary θ direction, which in the s-plane correspond to the points that lie between the two branch points and they are the poles of the S-matrix[5]. Let iu_{ab}^n be a pole of the S-matrix, corresponding to the creation of a bound state or *breather* from the scattering process of two particles. The total energy of the process gives the mass of the bound state,

$$s = \mathsf{m}_n^2 = \mathsf{m}_a^2 + \mathsf{m}_b^2 + 2\mathsf{m}_a \mathsf{m}_b \cos(u_{ab}^n), \tag{1.39}$$

which has a nice geometrical interpretation using Carnot's theorem for triangles, and leads to the so-called *fusing angle relation* [**Zam89**]. The scattering matrix element can be written as

$$S_{ab}^{cd}(\theta) \simeq \frac{\Gamma_{ab}^n R_n \Gamma_n^{cd}}{\theta - iu_{ab}^n} \tag{1.40}$$

where R_n is the residue at iu_{ab}^n and Γ_{ab}^n are the projector of the single particles $A_a(\theta_a)$ and $A_b(\theta_b)$, with $\theta = \theta_1 - \theta_2$, onto the bound state $B_n(\theta)$, see fig. 1.5.

One can assume that the bound state particles are part of the spectrum of the theory, *i.e.* they must be treated as fundamental particles, on the same footing as all the asymptotic states.

As we did for Yang-Baxter equation, considering the action of higher charges on the wave packets describing these particles, leads to the equality of the processes in fig. 1.6. This identification can be described by to the so-called *bootstrap equation*.

$$\Gamma_{ab}^n S_{cn}^{dn'}(\theta) = S_{ca}^{c'a'}(\theta - i\bar{u}_{a\bar{n}}^{\bar{b}}) S_{c'b}^{db'}(\theta + i\bar{u}_{b\bar{n}}^{\bar{a}}) \Gamma_{a'b'}^{n'}, \tag{1.41}$$

where we introduced the notation $\bar{u} = \pi - u$.

[5]In complete generality, these only represent *stable* bound states. It is possible to have *unstable* bound states, that correspond to poles which do not appear on the physical sheet, but on different Riemann sheets.

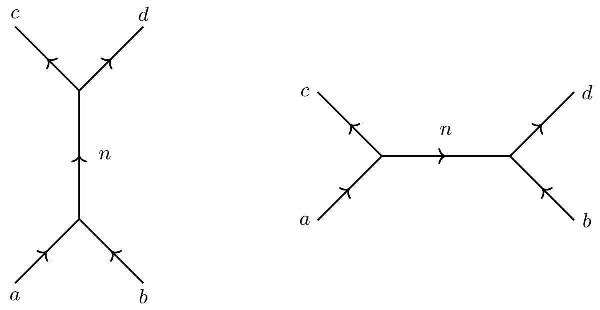


FIGURE 1.5. Production of a bound state in the t (left) and s channel (right).

For this equation to be satisfied, one might have to add extra poles to the S-matrix. These can be again interpreted as bound states, for which one can impose the bootstrap equations. When this iterative procedure ends, one "closes" the bootstrap, finding the complete spectrum of the theory [Zam89].

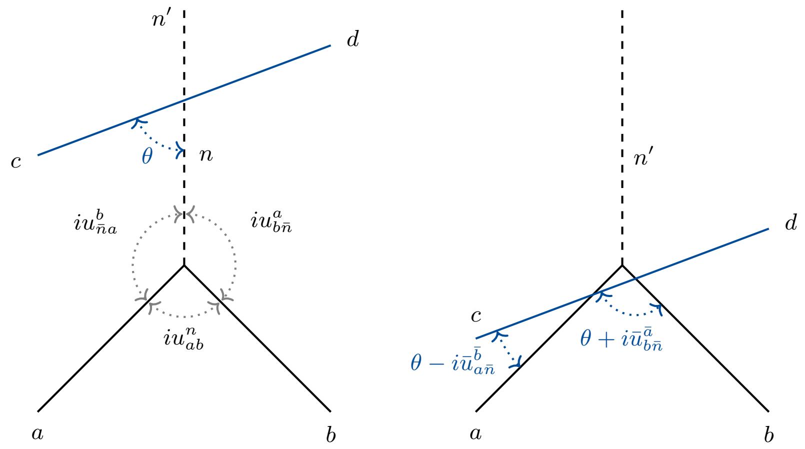


FIGURE 1.6. These two processes have to be considered equal if assuming the bootstrap principle. The dashed line represents the bound state. Here we introduced the notation $\bar{u} = \pi - u$.

CHAPTER 2

Scattering theories with Hagedorn singularities

In this chapter, we are going to see all the machinery described above in action. In section 2.1 we will explicitly construct new 2-body *minimal* S-matrices which are unitary, crossing symmetric and satisfy all additional properties described before. The finite size effects of these new theories are studied via thermodynamic Bethe ansatz in section 2.2. In the last section, we perform a numerical analysis of these models. We observe that they present singular behaviours, which might be of Hagedorn type.

This chapter is based on the article [**AFR24**], written in collaboration with Changrim Ahn and Francesco Ravanini.

2.1. Scattering theories with $U_{\mathfrak{q}}(\mathfrak{sl}_2)$ symmetry

The idea behind the S-matrix program is that of fixing the structure of the S-matrix by imposing the properties we described in the previous chapter. In 1+1 dimensional theories it often turns out that these conditions together with the bootstrap principle are enough to do it, modulo multiplicative factors which satisfy unitarity and crossing symmetry and which do not spoil the analyticity condition, called CDD factors [**CDD56**]. They can ultimately change the physics of the theory described by the scattering matrix since they can introduce additional poles, which must be regarded as additional bound states. They also appear as the result of certain integrable deformations [**SZ17, CNST16, CFLN$^+$21**].

In this section, we will proceed with the construction of a family of *minimal* S-matrices, by simply imposing all the defining relations and without adding extra CDD factors.

2.1.1. Exact S-matrices with $\mathfrak{sl}_2$ symmetry. To set the scene, we recall the construction for S-matrices with $\mathfrak{sl}(2,\mathbb{C})$ symmetry. In the next section we will generalise this to quantum group symmetry.

Consider the Lie algebra $\mathfrak{sl}_2$ with non-degenerate symmetric invariant bilinear form κ, generated by $\{J_\pm,\, J_3\}$ obeying the commutation relations

$$[J_3, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = 2J_3. \tag{2.1}$$

A particular element of the universal enveloping algebra is the quadratic Casimir element

$$\mathcal{C} = J_+J_- + J_-J_+ + 2J_3^2 \tag{2.2}$$

which is independent of the choice of basis, and commutes with all elements of the Lie algebra. It is known that finite dimensional representations $\mathcal{H}_s$ of this algebra are labelled by a positive integer or half-integer number $s \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, called the *spin* of the representation with dimension $2s+1$.

Using the notation introduced in section 1.2.1, we denote asymptotic on-shell multiplets of spin s as the collection of $2s+1$ particles $A_m(\theta)$ of same mass m, where $m = -s, -s+1, \ldots, s-1, s$ is the label of the internal quantum number, sometimes called the "magnetic" quantum number, in analogy with the theory of angular momentum. Here we are using the rapidity variables introduced in section 1.2.3.

If the scattering theory is integrable, the S-matrix factorises, cfr. section 1.2.4, *i.e.* multi-particle scattering amplitudes are decomposed into two-particle elastic S-matrix element $S_{m_1 m_2}^{m_1' m_2'}(\theta_1 - \theta_2)$, which can be graphically depicted as in fig. 1.2. This two-particle S-matrix has to satisfy the Yang-Baxter equations (1.26) together with unitarity (1.38) and crossing symmetry (1.32).

If we use the standard notation $|J, M\rangle$ for the basis of the total spin J representation with internal quantum numbers $M = -J, \ldots, J$, we can decompose the two-particle S-matrix as described in [KRS81] as follows

$$S(\theta) = \mathsf{P} \sum_{J=0}^{2s} f^{[J]}(\theta)\, \mathbb{P}^{[J]}, \tag{2.3}$$

where P is the permutation operator, $f^{[J]}(\theta)$ are some θ-dependent rational scalar functions and $\mathbb{P}^{[J]}$ are the projectors onto the spin-J representation,

$$\mathbb{P}^{[J]} = \sum_{M=-J}^{J} |J, M\rangle\langle J, M|, \qquad J = 0, \ldots, 2s, \tag{2.4}$$

which satisfies the usual properties

$$\sum_{J=0}^{2s} \mathbb{P}^{[J]} = \mathrm{id}, \qquad \text{and} \qquad \left(\mathbb{P}^{[J]}\right)^2 = \mathbb{P}^{[J]}. \tag{2.5}$$

Their matrix elements can be written in terms of the Clebsch-Gordan coefficients

$$\mathbb{P}^{[J]}{}_{m_1 m_2}^{m_1' m_2'} = \sum_{M=-J}^{J} \langle s, m_1'; s, m_2' | J, M\rangle\langle J, M | s, m_1; s, m_2\rangle. \tag{2.6}$$

where $|s, m_1'; s, m_2'\rangle$ is a basis for the tensor product of two particles of spin s.

Following the construction from [KRS81], the Yang-Baxter equation constrains the scalar functions to have the form

$$f^{[J]}(\theta) = S_0(\theta) \prod_{k=1}^{J} \frac{i\pi k - \theta}{i\pi k + \theta}, \tag{2.7}$$

up to an overall function $S_0(\theta)$ which can be fixed by imposing unitarity and crossing symmetry. These are the S-matrices constructed and studied in [AM94].

2.1.2. Exact S-matrices with $U_{\mathsf{q}}(\mathfrak{sl}_2)$ symmetry. We extend these rational S-matrices to the trigonometric ones, by introducing certain interactions expressed in terms of a coupling constant which is related to a deformation parameter $\mathsf{q} \in \mathbb{C}$ of the quantum symmetry algebra $U_{\mathsf{q}}(\mathfrak{sl}_2)$, generated by $\{J_\pm, \mathsf{q}^{\pm J_3}\}$ and satisfying the following relations

$$\mathsf{q}^{J_3} J_\pm \mathsf{q}^{-J_3} = \mathsf{q}^{\pm 1}\mathsf{q}^{J_3}, \qquad [J_+, J_-] = [2J_3]_{\mathsf{q}}, \tag{2.8}$$

where we introduced the following notation, called the q-number

$$[\lambda]_{\mathsf{q}} := \frac{\mathsf{q}^{\lambda/2} - \mathsf{q}^{-\lambda/2}}{\mathsf{q}^{1/2} - \mathsf{q}^{-1/2}}. \tag{2.9}$$

The Casimir operator can be found in [Kir91][1], and it is defined as follows

$$\mathcal{C}_{\mathsf{q}} = J_- J_+ + \frac{\mathsf{q}^{1/2}\mathsf{q}^{J_3} + \mathsf{q}^{-1/2}\mathsf{q}^{-J_3}}{(\mathsf{q}^{1/2} - \mathsf{q}^{-1/2})^2}. \tag{2.10}$$

For a generic value of q, *i.e.* not a root of unity, the Lusztig-Rosso theorem [Lus88, Ros88] states that the irreducible representations of the $U_{\mathsf{q}}(\mathfrak{su}_2)$ are in one to one correspondence to those of $\mathfrak{su}_2$, and labelled by integer or half-integer $s \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. As before, the asymptotic states of mass m form a spin-s representation of $U_{\mathsf{q}}(\mathfrak{su}_2)$.

By generalising the expression of the S-matrix from (2.3), we have

$$S(\theta) = \mathsf{P} \sum_{J=0}^{2s} f_{\mathsf{q}}^{[J]}(\theta)\, \mathbb{P}_{\mathsf{q}}^{[J]}, \tag{2.11}$$

for some trigonometric scalar functions $f_{\mathsf{q}}^{[J]}(\theta)$ and q-deformed projectors $\mathbb{P}_{\mathsf{q}}^{[J]}$ of $U_{\mathsf{q}}(\mathfrak{su}_2)$. Following the same ideas as above, all these ingredients of the S-matrices will be determined completely by imposing constraints such as the Yang-Baxter equation, unitarity, and crossing symmetry.

2.1.2.1. q-*deformed projectors.* The tensor products of two irreducible quantum group representations are decomposed into a direct sum of other irreducible ones in a similar way to the usual addition of two angular momenta in $\mathfrak{su}_2$ described in the previous section. As before, we have the change of basis

$$|J, M\rangle = \sum_{m_1, m_2 = -s}^{s} \langle s, m_1; s, m_2 | J, M\rangle_{\mathsf{q}}\, |s, m_1; s, m_2\rangle\,, \qquad J = 0, \ldots, 2s, \tag{2.12}$$

where the coefficients are now the *quantum Clebsch-Gordan coefficients*, from which it is possible to construct the quantum-deformed projectors:

$$\mathbb{P}_{\mathsf{q}}^{[J]}{}^{m_1' m_2'}_{m_1 m_2} = \sum_{M=-J}^{J} \langle s, m_1'; s, m_2' | J, M\rangle_{\mathsf{q}} \langle J, M | s, m_1; s, m_2\rangle_{\mathsf{q}}. \tag{2.13}$$

[1] In the rest of the chapter, we follow [Rue90]. To match notation and conventions with [Kir91], one should rescale q to q^4, and identify $\mathsf{q}^{\pm J_3/2}$ with $K^{\pm 1}$ and $J_\pm$ with $X_\pm$.

The explicit expression of the q-deformed Clebsh-Gordan coefficients were found in [Rue90, Kir91, AS94], and are defined as follows

$$\begin{aligned}\langle s, m_1; s, m_2 | J, M\rangle_{\mathsf{q}} = f(J) \cdot \mathsf{q}^{(2s-J)(2s+J+1)/4+s(m_2-m_1)/2} \\ \times \sqrt{[s+m_1]![s-m_1]![s+m_2]![s-m_2]![J+M]![J-M]!} \\ \times \sum_{\nu\geq 0}(-1)^{\nu}\frac{\mathsf{q}^{-\nu(2s+J+1)/2}}{\mathcal{D}_{\nu}},\end{aligned} \tag{2.14}$$

with

$$\mathcal{D}_{\nu} = [\nu]![2s-J-\nu]![s-m_1-\nu]![s+m_2-\nu]![J-s+m_1+\nu]![J-s-m_2+\nu]!,$$
$$f(J) = \left\{\frac{[2J+1]_{\mathsf{q}}([J]!)^2[2s-J]!}{[2s+J+1]!}\right\}^{1/2}, \tag{2.15}$$

where we are using the following convention for the q-factorial

$$[n]! = [n]_{\mathsf{q}}[n-1]_{\mathsf{q}}\cdots[1]_{\mathsf{q}} \quad \text{for } n\in\mathbb{Z}_{\geq 1}, \qquad [0]! = 1, \qquad [-n]! = \infty. \tag{2.16}$$

The infinite sum appearing in eq. (2.14) always truncates to a finite one. In fact, one can always find a $\bar{\nu}$ big enough so that at least one of the factors in the definition of $\mathcal{D}_{\nu}$ remains negative for $\nu \geq \bar{\nu}$, which implies that $\mathcal{D}_{\nu\geq\bar{\nu}} = \infty$.

From these expressions, one can construct the q-projectors straightforwardly. As an example, we present the q-projectors for the quantum group $U_{\mathsf{q}}(\mathfrak{su}_2)$ in the spin $s = 1/2$ representation

$$\mathbb{P}_{\mathsf{q}}^{[0]} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\mathsf{q}}{\mathsf{q}+1} & -\frac{\sqrt{\mathsf{q}}}{\mathsf{q}+1} & 0 \\ 0 & -\frac{\sqrt{\mathsf{q}}}{\mathsf{q}+1} & \frac{1}{\mathsf{q}+1} & 0 \\ 0 & 0 & 0 & 0\end{pmatrix}, \qquad \mathbb{P}_{\mathsf{q}}^{[1]} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\mathsf{q}+1} & \frac{\sqrt{\mathsf{q}}}{\mathsf{q}+1} & 0 \\ 0 & \frac{\sqrt{\mathsf{q}}}{\mathsf{q}+1} & \frac{\mathsf{q}}{\mathsf{q}+1} & 0 \\ 0 & 0 & 0 & 1\end{pmatrix}. \tag{2.17}$$

Remark. All these formulae are valid when q is a generic value. When instead it is a root of unity, *i.e.* $\mathsf{q} = e^{2\pi i m/n}$ for some $n \in \mathbb{Z}_{>0}$ and $m = 0, 1, 2, \ldots, n-1$, one can see that any multiple k of n is trivially zero $[k]_{\mathsf{q}} = 0$. This can lead to ill-defined quantities in formulae (2.14) and (2.15). In this case, it was shown in [HHM92] that the Clebsh-Gordan coefficients can still be defined by a careful choice of normalization of the states and they can be computed by introducing a limiting procedure from a generic to a root of unity value. Using this procedure, the expression for q root of unity can be obtained from that of generic q in a continuous manner and the formulae below are still valid for any value of q on the unit circle.

2.1.2.2. *Construction of the scattering matrices.* Similarly to the previous section, the Yang-Baxter equation forces the scalar functions $f_{\mathsf{q}}^{[J]}(\theta)$ in (2.11) to have the following form

$$f_{\mathsf{q}}^{[J]}(\theta) = S_0(\theta)\prod_{k=1}^{J}\frac{\mathsf{q}^k - \mathsf{q}^{\theta/2\pi i}}{\mathsf{q}^k\mathsf{q}^{\theta/2\pi i}-1}, \qquad J = 0, 1, \ldots, 2s, \tag{2.18}$$

where the function $S_0(\theta)$ can be found by imposing crossing symmetry and unitarity.

It was shown in [**BL90**] in the case of sine-Gordon that the definition in eq. (2.11) does not provide an S-matrix satisfying the crossing symmetry equation. To fix this issue, the authors introduced the gauge transformation

$$\sigma = \mathsf{q}^{\mathbb{J}_3\,\theta_1/2\pi i} \otimes \mathsf{q}^{\mathbb{J}_3\,\theta_2/2\pi i}. \tag{2.19}$$

where θ_1, θ_2 are the rapidities of the incoming particles. This generalises to higher spin S-matrices, therefore we have

$$S(\theta) = \sigma \left(\mathsf{P} \sum_{J=0}^{2s} f_{\mathsf{q}}^{[J]}(\theta)\, \mathbb{P}_{\mathsf{q}}^{[J]} \right) \sigma^{-1}. \tag{2.20}$$

In addition to crossing symmetry, this S-matrix is also invariant under charge conjugation $\mathcal{C}$, parity $\mathcal{P}$, and a time reversal $\mathcal{T}$ as in eqs. (1.27)–(1.29).

We can find the relation between the deformation parameter of the quantum group and the coupling constant γ of the theory by looking at the known cases of the sine-Gordon ($s = 1/2$) and sausage models ($s = 1$), and it is given by

$$\mathsf{q} = e^{2\pi i \gamma}, \qquad \gamma \in \mathbb{R}, \tag{2.21}$$

and with this choice, the scalar function can be expressed as

$$f_{\mathsf{q}}^{[J]}(\theta) = S_0(\theta) \prod_{k=1}^{J} \frac{\sinh[\gamma(ik\pi - \theta)]}{\sinh[\gamma(ik\pi + \theta)]}, \quad J = 0, 1, \cdots, 2s. \tag{2.22}$$

By requiring unitarity and crossing symmetry, this function has to satisfy

$$S_0(\theta) S_0(-\theta) = 1, \tag{2.23}$$

$$S_0(i\pi - \theta) = \prod_{k=1}^{2s} \frac{\sinh[\gamma(i(k+1)\pi - \theta)]}{\sinh[\gamma(ik\pi + \theta)]}\, S_0(\theta). \tag{2.24}$$

In order to fix the scalar function S_0, we follow the same steps of [**ZZ79**]. One starts with a first ansatz for $S_0(\theta)$ which solves the equation, depending on some new unknown function of θ that has to be fixed by the unitarity condition (2.23). To satisfy this, one needs to introduce a new function that must satisfy crossing and so on. This gives rise to a recursive definition of the overall factor, that eventually can be written as an infinite product of terms

$$S_0(\theta) = -\prod_{k=1}^{2s} \left[\frac{\sinh[\gamma(i\pi k + \theta)]}{\sinh[\gamma(i\pi k - \theta)]} \left(\prod_{\ell=1}^{\infty} \frac{\sinh[\gamma(i\pi(k+\ell) - \theta)]\sinh[\gamma(i\pi(k-\ell) - \theta)]}{\sinh[\gamma(i\pi(k+\ell) + \theta)]\sinh[\gamma(i\pi(k-\ell) + \theta)]} \right) \right]. \tag{2.25}$$

When s is an integer, *i.e.* even $2s$, many simplifications take place in this infinite product, and one ends up with the finite product

$$S_0(\theta) = -\prod_{m=1}^{s} \frac{\sinh[\gamma(i2m\pi + \theta)]}{\sinh[\gamma(i2m\pi - \theta)]}. \tag{2.26}$$

The S-matrix element S_{ss}^{ss} describing scattering between two particles of with internal quantum number s is

$$S_{ss}^{ss}(\theta) = -\prod_{m=1}^{s} \frac{\sinh\left[\gamma(i(2m-1)\pi - \theta)\right]}{\sinh\left[\gamma(i(2m-1)\pi + \theta)\right]}. \tag{2.27}$$

This correctly reproduces the S_{++}^{++} element of the sausage model for $s = 1$ obtained in [**FOZ93**].

For a half-integer s, *i.e.* odd $2s$, one can convert the infinite products of hyperbolic functions into products of Γ-functions using the identity $\sinh(\pi x) = -i\pi[\Gamma(ix)\Gamma(1-ix)]^{-1}$ repeatedly, obtaining

$$S_0(\theta) = -\prod_{m=1}^{2s}\left\{\frac{1}{i\pi}\sinh\left[\gamma(\theta + im\pi)\right]\Gamma\left[1-\gamma(m-1)+\frac{i\gamma\theta}{\pi}\right]\Gamma\left[1-\gamma m - \frac{i\gamma\theta}{\pi}\right]\right.$$
$$\left.\times\prod_{n=1}^{\infty}\left[\frac{R_n^{[s,m]}(\theta)R_n^{[s,m]}(i\pi-\theta)}{R_n^{[s,m]}(0)R_n^{[s,m]}(i\pi)}\right]\right\}, \tag{2.28}$$

where we introduced

$$R_n^{[s,m]}(\theta) = \frac{\Gamma\left[\gamma(4sn-4s+2m-1)-\frac{i\gamma\theta}{\pi}\right]\Gamma\left[1+\gamma(4sn-2m+1)-\frac{i\gamma\theta}{\pi}\right]}{\Gamma\left[\gamma(4sn-2s+2m-1)-\frac{i\gamma\theta}{\pi}\right]\Gamma\left[1+\gamma(4sn-2s-2m+1)-\frac{i\gamma\theta}{\pi}\right]}. \tag{2.29}$$

The S-matrix element between top-spin s particles is

$$S_{ss}^{ss}(\theta) = -\prod_{m=1}^{2s}\left\{\frac{1}{i\pi}\sinh\left[\gamma(\theta - im\pi)\right]\Gamma\left[1-\gamma(m-1)+\frac{i\gamma\theta}{\pi}\right]\Gamma\left[1-\gamma m - \frac{i\gamma\theta}{\pi}\right]\right.$$
$$\left.\times\frac{\Gamma[\gamma m]}{\Gamma[1-\gamma(m-1)]}\prod_{n=1}^{\infty}\left[\frac{R_n^{[s,m]}(\theta)R_n^{[s,m]}(i\pi-\theta)}{R_n^{[s,m]}(0)R_n^{[s,m]}(i\pi)}\right]\right\}. \tag{2.30}$$

Remarkably, even if the two expressions for integer or half-integer spin look rather different, they can be recast in the *same* integral Fourier form, for any spin s,

$$\mathsf{S}_{ss}^{ss}(\theta) = -\exp\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{\omega}\frac{\sinh(\pi\omega s)\sinh\pi\omega(s-\frac{1}{2\gamma})}{\sinh\frac{\pi\omega}{2\gamma}\sinh\pi\omega}e^{i\omega\theta}. \tag{2.31}$$

Note that for $s = 1/2$, this integral expression reduces to the famous pre-factor of the sine-Gordon S-matrix found in [**ZZ79**]. From this representation, one can notice that

$$S_{ss}^{ss}(0) = -1 \qquad \text{when} \qquad \gamma = \frac{1}{2s}, \tag{2.32}$$

which can be thought of as a kind of free point.

By factoring out the top-spin component, the S-matrix can be written as

$$S(\theta) = S_{ss}^{ss}(\theta)\cdot S_{\mathrm{mat}}(\theta) \tag{2.33}$$

where

$$S_{\mathrm{mat}}(\theta) := \sigma\left(\mathsf{P}\sum_{J=0}^{2s}\left[\prod_{k=J+1}^{2s}\frac{\sinh\left[\gamma(ik\pi+\theta)\right]}{\sinh\left[\gamma(ik\pi-\theta)\right]}\right]\mathbb{P}_{\mathsf{q}}^{[J]}\right)\sigma^{-1}. \tag{2.34}$$

In the interval

$$0 \leq \gamma \leq \frac{1}{2s} \tag{2.35}$$

the S-matrix does not present any pole in the physical strip $0 \leq \mathrm{Im}\theta \leq \pi$ for any s, i.e. there are no bound states. We will refer to this range of the coupling constant as the *repulsive regime*. For $\gamma > 1/2s$ the S-matrix will develop poles in the physical strip, signalling the presence of bound states, or breathers: we call this the *attractive regime*. In this work, we focus on the former case.

We present the full expression of the $s = 3/2$ S-matrix. The asymptotic particles are denoted by the symbols A_m with $m = 3/2, 1/2, -1/2, -3/2$ with $\mathsf{C}(A_m) = \overline{A}_m = A_{-m}$. Denoting these particles with index $1, 2, 3, 4$, hence $\bar{1} = 4, \bar{2} = 3$, the non-vanishing S-matrix elements are given by the prefactor in (2.31) multiplied by the following terms

$$\begin{gathered} S_{11}^{11} = 1,\ S_{12}^{12} = \frac{(0)}{(3)},\ S_{12}^{21} = \frac{s_3}{(3)},\ S_{13}^{13} = \frac{(0)(-1)}{(2)(3)},\ S_{13}^{22} = \frac{s_2\sqrt{s_3/s_1}(0)}{(2)(3)}, \\ S_{13}^{31} = \frac{(s_1 s_4 + 2s_2)(0)}{(2)(3)},\ S_{22}^{22} = \frac{f_1}{(2)(3)},\ S_{14}^{14} = \frac{(0)(-1)(-2)}{(1)(2)(3)},\ S_{14}^{23} = \frac{s_3(0)(-1)}{(1)(2)(3)}, \\ S_{14}^{32} = \frac{s_2 s_3(0)}{(1)(2)(3)},\ S_{14}^{41} = \frac{s_1 s_2 s_3}{(1)(2)(3)},\ S_{23}^{23} = \frac{(0)f_1}{(1)(2)(3)},\ S_{23}^{32} = \frac{s_2 f_2}{(1)(2)(3)}, \end{gathered} \tag{2.36}$$

and those related by $\mathcal{C}, \mathcal{P}, \mathcal{T}$ transformations given in eqs. (1.27)–(1.29). Here we have introduced the following notations

$$\begin{gathered} (n) := 2\sinh\left[\gamma(\theta - i\pi n)\right], \quad s_n = 2\sinh(in\pi\gamma), \\ f_1 = 2\cosh\left[\gamma(2\theta - i\pi)\right] + \frac{s_{10}}{s_5} - 2\frac{s_2}{s_1},\ f_2 = 2\frac{s_2}{s_1}\cosh\left[\gamma(2\theta - i\pi)\right] + s_2^2 - 2s_1^2 - 4. \end{gathered}$$

2.2. Thermodynamic Bethe ansatz

In the previous section, we constructed the R matrix of the quantum group $U_{\mathfrak{q}}(\mathfrak{su}_2)$ for different spin s representations, which automatically satisfies the Yang-Baxter equation. We then imposed the other constraints, such as unitarity and crossing symmetry. We found a family of minimal matrices without poles in the physical strip satisfying all the axioms and therefore representing well-defined S-matrices describing the scattering of asymptotic particles belonging to multiplets of iso-spin s. For $s = 1/2$ and $s = 1$, they correctly reproduce the known cases of the sine-Gordon and sausage models, respectively.

In this section, we aim at exploring the ultraviolet behaviour of these theories. To do that we employ the thermodynamic Bethe ansatz (TBA) technique, which first appeared in [**YY69**] and was later generalised to the relativistic case by Al. Zamolodchikov in a series of papers [**Zam90**, **Zam91a**, **Zam91b**, **Zam91c**] and further studied by [**KM90**, **KM91**].

We will first recall the main ideas behind the TBA analysis, to then apply it to the family of theories introduced in the previous section.

2.2.1. TBA, in a nutshell. Consider a 1+1 relativistic quantum field theory in its Euclidean formulation on a torus generated by the two circles of radius R and L. There are two equivalent ways to quantise the system, one where the time direction is chosen to be along the circle of radius L and one where it is chosen to be along the circle of radius R.

If we send $L \to \infty$, in the first case we obtain a theory on a cylinder of radius R with the time flowing in the L direction, while in the second case, we obtain a theory on an infinite line, with compactified time R, which can be interpreted as a finite (inverse) temperature. The partition function of the first theory in the large L limit is dominated by the ground state energy while in the second theory, the limit $L \to \infty$ can be interpreted as the thermodynamic limit and the partition function can be expressed in terms of the free energy per unit length. They are expressed respectively as follows

$$Z \simeq e^{-LE_0(R)}, \qquad Z \simeq e^{-LRf(R)}. \tag{2.37}$$

Since these two quantisation procedures have to be equivalent, we obtain the following equivalence

$$E_0(R) = Rf(R). \tag{2.38}$$

Following [**BCN86**], the ground state energy of the theory can be expressed as follows

$$E_0(R) = -\frac{\pi \widetilde{c}(r)}{6R}, \tag{2.39}$$

with $r = R/R_c$, R_c being the largest correlation length of the theory, $R_c \sim 1/\mathsf{m}$, where m is the lightest mass of the theory. The function $\widetilde{c}(r)$ is called the scaling function of the theory and in the limit $r \to 0$, the *ultraviolet regime*, which corresponds to the conformal (massless) limit, it is related to the central charge of the underlying conformal field theory.

At this point, we can consider a gas of $\mathcal{N}$ particles on the infinite line, in a configuration such that they are far enough to prevent interactions: this is the typical setting where one can use Bethe ansatz techniques, as we will see below. In this configuration, the wave functions are essentially those of free particles. If two particles become close enough, they can scatter and all the information of the process is contained in the asymptotic S-matrix of the theory. For simplicity, consider a diagonal scattering theory with just one type of particle of mass m, so that the S-matrix is given by some scalar function $S(\theta)$. A particle doing a full revolution around the circle will scatter with all other particles of the gas. By imposing periodic boundary conditions one finds the so-called *Bethe-Yang* equations

$$e^{iL\mathsf{m}\sinh\theta_i} \prod_{\substack{j=1 \\ j\neq i}}^{\mathcal{N}} S(\theta_i - \theta_j) = 1, \qquad i = 1, \dots, N, \tag{2.40}$$

where we are using the rapidity variables introduced in section 1.2.3.

In the thermodynamic limit, one can compute the free energy of the gas of particles at temperature T, defined as $F = E - TS$ where E and S are the energy and entropy of the system, respectively. By minimising the free energy, keeping the relation (2.40) as a constraint on the rapidities, one can find the so-called TBA equations, whose solutions can be used to compute thermodynamical quantities at equilibrium, as well as the scaling function in the ultraviolet regime.

As we will show in the next section, the TBA equations are a system of non-linear integral equations whose exact solution is in general a quite challenging task. Only in special cases, a closed solution can be found, while in all other situations, one has to rely on numerical solutions.

2.2.2. Non-diagonal scattering theories. The S-matrices we constructed in the previous sections have a major difference from those used in the example above, which makes the TBA analysis technically more difficult, which is that they are not diagonal. This follows from the fact that particles of the same mass but different internal quantum numbers can both transmit or reflect. For this reason, we need to consider the following family of objects labelled by $j = 1, \dots, \mathcal{N}$, $\mathcal{N}$ being the number of particles in the gas, called the *colour transfer-matrices*,

$$\mathbb{T}_j(\theta_i, \dots, \theta_{\mathcal{N}})^{m'_1, \cdots, m'_{\mathcal{N}}}_{m_1, \cdots, m_{\mathcal{N}}} = S^{n_2 m'_1}_{n_1 m_1}(\theta_1 - \theta_j) S^{n_3 m'_2}_{n_2 m_2}(\theta_2 - \theta_j) \cdots S^{n_1 m'_{\mathcal{N}}}_{n_{\mathcal{N}} m_{\mathcal{N}}}(\theta_{\mathcal{N}} - \theta_j), \tag{2.41}$$

with an implicit sum over all internal quantum numbers. This object describes all the scattering processes of one of the particles of the gas with all the others. These operators can be diagonalised simultaneously, and the eigenvalues $\Lambda_j(\theta_i, \dots, \theta_{\mathcal{N}})$ are precisely those functions entering the Bethe-Young equation for the non-diagonal case

$$e^{iL\mathsf{m}\sinh\theta_j} \Lambda_j(\theta_1, \dots, \theta_{\mathcal{N}}) = 1. \tag{2.42}$$

The study of the eigenvalues and eigenvectors of these operators without an explicit form of the wave function of the system is the main object of interest of the inverse scattering program. To do that, the most common method is the *algebraic Bethe ansatz*, which has the effect of introducing a number of non-physical parameters λ_j, $j = 1, \dots, M$, called the *Bethe roots*, which in this context can be interpreted as fictitious massless particles called magnons. These additional particles have to satisfy auxiliary equations called *Bethe equations*.

In our setting, the matrix part of the colour transfer matrices is formally equivalent to the transfer matrices of the higher spin XXZ spin chain models studied in [**KR87**], with the addition of inhomogeneities at each site which can be understood as the rapidities of the particles.

Taking into account also the prefactor from eq. (2.33), the resulting Bethe-Yang equations with the eigenvalues of $\mathbb{T}_j$ for $j = 1, \dots, \mathcal{N}$ are given by

$$e^{iL\mathsf{m}\sinh\theta_j} \prod_{\substack{k=1 \\ k\neq j}}^{\mathcal{N}} S^{ss}_{ss}(\theta_j - \theta_k) \prod_{\ell=1}^{M} \frac{\sinh\gamma(\lambda_\ell - \theta_j + i\pi s)}{\sinh\gamma(\lambda_\ell - \theta_j - i\pi s)} = 1. \tag{2.43}$$

As anticipated, the Bethe roots λ_ℓ, $\ell = 1, \ldots, M$, must satisfy the Bethe equations

$$\prod_{j=1}^{\mathcal{N}} \frac{\sinh \gamma(\theta_j - \lambda_\ell - i\pi s)}{\sinh \gamma(\theta_j - \lambda_\ell + i\pi s)} \prod_{k=1, k\neq \ell}^{M} \frac{\sinh \gamma(\lambda_\ell - \lambda_k - \pi i)}{\sinh \gamma(\lambda_\ell - \lambda_k + \pi i)} = 1. \tag{2.44}$$

2.2.3. Bethe strings. A remarkable fact is that in the thermodynamic limit, defined by sending $L \to \infty$ and $\mathcal{N}$, $M \to \infty$ keeping their ratios M/L and $\mathcal{N}/L$ finite, the Bethe roots start to populate the complex plane and organise into *strings* of length $n \in \mathbb{Z}_{\geq 1}$ and parity $v = +, -$. They repeat in the imaginary direction with a certain periodicity, which ultimately follows from the periodicity of eq. (2.44), given by $i\pi p_0$ with $p_0 = 1/\gamma$. They are defined as follows

$$\lambda_{j,\alpha}^{(n),+} = \lambda_j^{(n)} + \frac{i\pi}{2}(n + 1 - 2\alpha), \quad \alpha = 1, 2, \ldots, n, \tag{2.45}$$

$$\lambda_{j,\alpha}^{(n),-} = \lambda_j^{(n)} + \frac{i\pi}{2}(n + 1 - 2\alpha) + \frac{i\pi}{2}p_0, \quad \alpha = 1, 2, \ldots, n. \tag{2.46}$$

It turns out that in order to ensure the normalisability of the wave function of the system, one has to impose some constraining relations on the orders n and parities v of the strings. The general procedure to define which strings are allowed after imposing these relations was first introduced by Takahashi and Suzuki [TS72], and we now proceed to recall it. One starts by introducing the following series of numbers

$$p_1 = 1, \qquad b_i = \lfloor p_i/p_{i+1} \rfloor, \qquad p_{i+1} = p_{i-1} - b_{i-1}p_i, \qquad i \geq 1, \tag{2.47}$$

$$y_{-1} = 0, \qquad y_0 = 1, \qquad y_1 = b_0, \qquad y_{i+1} = y_{i-1} + b_i y_i, \qquad i \geq 0, \tag{2.48}$$

$$m_0 = 0, \qquad m_1 = b_0, \qquad m_{i+1} = m_i + b_i, \qquad i \geq 0, \tag{2.49}$$

where the values b_i arise from the continued fraction decomposition of p_0

$$p_0 := [b_0, b_1, b_2, \ldots] = b_0 + \frac{1}{b_1 + \frac{1}{b_2 + \ldots}}, \qquad p_i/p_{i+1} := [b_i, b_{i+1}, \ldots]. \tag{2.50}$$

One can then introduce the so-called Takahashi-Suzuki variables, which describe the allowed length n_a and parities v_a for the Bethe strings, with a taking values in the Takahashi zones defined by the $\{m_i\}$ series, as follows

$$n_a = y_{i-1} + (a - m_i)y_i, \qquad m_i \leq a < m_{i+1}, \tag{2.51}$$

$$v_a := v_{n_a} = \exp\left\{ i\pi \left\lfloor \frac{n_a - 1}{p_0} \right\rfloor \right\}, \qquad a \neq m_1, v_{m_1} := -1. \tag{2.52}$$

It is clear that the structure of these strings deeply depends on the fraction decomposition of p_0. If p_0 is an irrational number, the b-series is infinite and it will lead to strings of any length. It $p_0 = p/q$ with $p, q \in \mathbb{Z}_{\geq 1}$ the series becomes finite, but it will lead to Bethe strings whose components are not equispaced and therefore still quite difficult to study (see e.g. [Tat95] for an example of this situation in the sine-Gordon model). When $p_0 \in \mathbb{Z}_{\geq 1}$, the number of Bethe strings is finite, and their internal structure is very regular.

For this reason from now on we will focus on the latter case, by setting

$$\gamma = \frac{1}{N}, \quad \text{which implies} \quad p_0 = N, \qquad \text{with } N \in \mathbb{Z}_{\geq 2s+1}, \tag{2.53}$$

which corresponds to the repulsive regime of the theory, since, as explained above, the S-matrix has no poles in the physical strip in this range of values of the coupling constant.[2] In this setting, the continued fraction decomposition (2.50) truncate at $b_0 = N$ and $m_0 = 0$, $m_1 = N$ are the only non-zero values of the m-series. The y series is given by $y_{-1} = 0$, $y_0 = 1$, $y_1 = N$. This leads to two Takahashi zones, namely $0 \leq a < N$ and $a = N$. The Takahashi numbers then read

$$\begin{aligned} &n_a = y_{-1} + (a - m_0)y_0 = a, \quad v_a = (-1)^{\lfloor (a-1)/N \rfloor} = 1, \qquad \text{for } 0 \leq a < N, \\ &n_N = y_0 = 1, \quad v_N = -1. \end{aligned} \tag{2.54}$$

The strings of type $+$ can be of length $n = 1, \dots, N-1$ while the string of type $-$ are only of length 1, see e.g. fig. 2.1 for an explicit example with $p_0 = 5$.

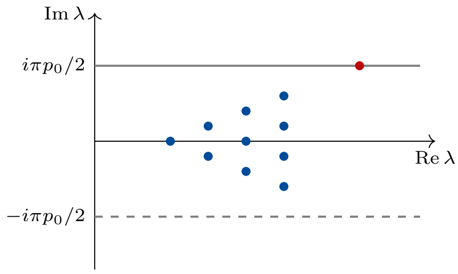


FIGURE 2.1. Allowed strings in the complex λ palne for $\gamma = 1/5$, $p_0 = 5$. We see that only $(n_a, v_a) = (1, +)$, $(2, +)$, $(3, +)$, $(4, +)$ and $(1, -)$ are allowed by Takahashi-Suzuki constraints.

As a consequence, in the string limit, all products over the number of auxiliary particles can be expressed as products over the string components

$$\prod_{\ell=1}^{M} \longrightarrow \prod_{a=1}^{N} \prod_{\ell=1}^{M_a} \prod_{\alpha=1}^{n_a} \tag{2.55}$$

where the first product is over the N allowed strings, the second one over the multiplicity M_a of a given string of length n_a (as defined in eq. (2.54)) and the last one over its n_a internal components.

Substituting the expressions for the strings into eqs. (2.43) and (2.44), will lead to a system of equations which remarkably only depends on the centres. To see that, we substitute the expression of the string in eq. (2.44), then take the product over the n_b

[2] In the case of the sine-Gordon model one should consider $N > 2$.

elements of the string

$$\prod_{j=1}^{\mathcal{N}} \underbrace{\prod_{\beta=1}^{n_b} f_{2s}(\theta_j - \lambda_{\ell,v_b}^{(n_b),\beta})}_{S_{0b}} \prod_{a=1}^{N}\prod_{k=1}^{M_a} \underbrace{\prod_{\beta=1}^{n_b}\prod_{\alpha=1}^{n_a} f_2(\lambda_{\ell,v_b}^{(n_b),\beta} - \lambda_{k,v_a}^{(n_a),\alpha})}_{S_{ab}} = 1, \tag{2.56}$$

for $b = 1, \ldots, N$, $\ell = 1, \ldots, M_b$ and where we have introduced the functions

$$f_\alpha(x) = \frac{\sinh\gamma(x - i\pi\alpha/2)}{\sinh\gamma(x + i\pi\alpha/2)}. \tag{2.57}$$

These products can be now further simplified as follows

$$\begin{aligned} S_{ab}(\lambda_{\ell,v_b}^{(n_b),\beta} - \lambda_{k,v_a}^{(n_a),\alpha}) =& g_{v_a v_b, |n_b - n_a|}(\lambda_\ell^{(n_b)} - \lambda_k^{(n_a)}) g_{v_a v_b, n_a + n_b}(\lambda_\ell^{(n_b)} - \lambda_k^{(n_a)}) \\ & \times \prod_{i=1}^{\min(n_a,n_b)-1} g_{v_a v_b, |n_b - n_a| + 2i}(\lambda_\ell^{(n_b)} - \lambda_k^{(n_a)})^2 \end{aligned} \tag{2.58}$$

and

$$S_{0b}(\theta_j - \lambda_{\ell,v_b}^{(n_b)}) = \prod_{i=1}^{\min(n_b,2s)} g_{v_b, |n_b - 2s| + 2i - 1}(\theta_j - \lambda_\ell^{(n_b)})^{-1} \tag{2.59}$$

where

$$g_{v,\alpha}(x) = \begin{cases} \dfrac{\sinh[\gamma(x - i\pi\alpha/2)]}{\sinh[\gamma(x + i\pi\alpha/2)]} = f_\alpha(x) & \text{when } v = + \\ \dfrac{\cosh[\gamma(x - i\pi\alpha/2)]}{\cosh[\gamma(x + i\pi\alpha/2)]} & \text{when } v = - \end{cases} \tag{2.60}$$

In a similar way, one can use eq. (2.55) in Bethe-Yang equations (2.43), obtaining

$$e^{iL\mathsf{m}\sinh\theta_j} \prod_{\substack{k=1 \\ k\neq j}}^{\mathcal{N}} S_{ss}^{ss}(\theta_j - \theta_k) \prod_{a=1}^{N}\prod_{\ell=1}^{M_a} \underbrace{\prod_{\alpha=1}^{n_a} f_{-2s}(\lambda_{\ell,v_a}^{(n_a),\alpha} - \theta_j)}_{S_{0a}} = 1 \tag{2.61}$$

and use eq. (2.59) to simplify the product further.

This allows us to write the equations in the following more suggestive form

$$e^{i\mathsf{m}L\sinh\theta_j} \prod_{\substack{k=1 \\ k\neq j}}^{\mathcal{N}} S_{00}(\theta_j - \theta_k) \prod_{a=1}^{N}\prod_{\ell=1}^{M_a} S_{0a}(\theta_j - \lambda_\ell^{(a)}) = 1, \tag{2.62}$$

$$\prod_{k=1}^{\mathcal{N}} S_{a0}(\lambda_j^{(a)} - \theta_k) \prod_{b=1}^{N}\prod_{\substack{i=1 \\ i\neq j}}^{M_b} S_{ab}(\lambda_j^{(a)} - \lambda_i^{(b)}) = 1, \quad a = 1, \ldots, N, \tag{2.63}$$

which crucially only depend on the centres of the strings. The various functions appearing in the products can be thought as particle-particle scattering (S_{00}), particle-magnon scattering (S_{0a}) and magnon-magnon scattering (S_{ab}), where we introduced the notation

$$S_{00}(\theta) = \mathsf{S}_{ss}^{ss}(\theta). \tag{2.64}$$

2.2.4. The TBA equations. Having simplified the products in the equations, we can now proceed in the standard way to obtain the TBA equations. The first step is to consider the thermodynamic limit, defined by sending the volume $L \to \infty$, and the particle numbers $\mathcal{N}, M \to \infty$, keeping their ratios $\mathcal{N}/L$ and M/L finite. In this limit, one can introduce the density of particles or magnons

$$\sigma_a(\theta) = \frac{1}{L}\frac{\mathrm{d}\mathsf{n}_a}{\mathrm{d}\theta}, \qquad a = 0, 1, \ldots, N, \tag{2.65}$$

where by $\mathrm{d}\mathsf{n}_a$ we denote the number of particles ($a = 0$) or magnons ($1 \leq a \leq N$) which carry a rapidity between θ and $\theta + \mathrm{d}\theta$. Similarly, one can introduce the densities of *holes* $\widetilde{\sigma}_a$, which are defined in a similar way and describe unoccupied states.

Therefore, taking the logarithm on both sides of eqs. (2.62) and (2.63) and differentiating, one finds the *raw TBA equations* for the densities

$$\sigma_a(\theta) + \widetilde{\sigma}_a(\theta) = \delta_{a,0}\frac{\mathsf{m}}{2\pi}\cosh\theta + \nu_a \sum_{b=0}^{N}(K_{ab} \star \sigma_b)(\theta), \qquad a = 0, 1, \cdots, N, \tag{2.66}$$

where we introduced

$$\nu_a = \begin{cases} 1, & a = 0, N \\ -1, & a = 1, \cdots, N-1 \end{cases} \tag{2.67}$$

and a standard convolution notation

$$(f \star g)(\theta) = \int_{-\infty}^{\infty} f(\theta')g(\theta - \theta')\mathrm{d}\theta', \tag{2.68}$$

along with the kernels defined by

$$K_{ab}(\theta) = \frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}\theta}\ln S_{ab}(\theta), \qquad a, b = 0, \ldots, N. \tag{2.69}$$

Equations (2.66) are a system of non-linear integral equations for the densities of particles and holes. They are particularly intricate since for each particle type a, there appear "interaction terms" with every other particle of the system.

A remarkable fact is that this system of equations can be drastically simplified by taking into consideration certain identities of the Fourier transforms of the kernels.

2.2.5. Kernels' identities. We will now show that there are highly non-trivial identities among the kernels appearing in the TBA equations. For this reason, we proceed to explain the derivation of such identities in detail.

We introduce

$$\phi_{v,a}(x) = \frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}\theta}\log g_{v,\alpha}(x) = v\frac{\gamma}{\pi}\frac{\sin(n\pi\gamma)}{\cosh(2\gamma\theta) - v\cos(n\pi\gamma)} \tag{2.70}$$

Using these functions, the identities (2.58) and (2.59) for the kernels K_{ab} read

$$K_{0a}(x) = K_{a0}(x) = -\sum_{i=1}^{\min(n_a, 2s)} \phi_{v_a, |n_a - 2s| + 2i - 1}(x), \qquad a = 1, \ldots, N \tag{2.71}$$

$$K_{ab}(x) = K_{ba}(x) = \phi_{v_a v_b, |n_a - n_b|}(x) + \phi_{v_a v_b, n_a + n_b}(x)$$

$$+ 2 \sum_{i=1}^{\min(n_a,n_b)-1} \phi_{v_a v_b,|n_a-n_b|+2i}(x), \qquad a,b = 1,\dots,N \tag{2.72}$$

We now introduce the following notation for the Fourier transform,

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} e^{i\omega\theta} f(\theta)\mathrm{d}\theta, \quad f(\theta) = \int_{-\infty}^{\infty} e^{-i\omega\theta} \widehat{f}(\omega)\frac{\mathrm{d}\omega}{2\pi}. \tag{2.73}$$

For the function $\phi(x)$, with $\gamma = 1/N$, we obtain

$$\widehat{\phi}_{v,\alpha}(\omega) = \begin{cases} \dfrac{\sinh(\pi\omega(N-\alpha)/2)}{\sinh(\pi\omega N/2)} & v = +1 \\ \dfrac{\sinh(\pi\omega\alpha/2)}{\sinh(\pi\omega N/2)} & v = -1 \end{cases} \tag{2.74}$$

Using these, we can compute the Fourier transforms of the kernels defined in eq. (2.69), for all possible values of a and b, namely

$$\widehat{K}_{00}(\omega) = \frac{\sinh\left((N-2s)\pi\omega/2\right)\sinh\left(s\pi\omega\right)}{\sinh(\pi\omega)\,\sinh\left(N\pi\omega/2\right)}, \tag{2.75}$$

$$\widehat{K}_{NN}(\omega) = \frac{\sinh\left((N-2)\pi\omega/2\right)}{\sinh\left(N\pi\omega/2\right)}, \tag{2.76}$$

$$\widehat{K}_{0N}(\omega) = \widehat{K}_{N0}(\omega) = \frac{\sinh\left(s\pi\omega\right)}{\sinh\left(N\pi\omega/2\right)}, \tag{2.77}$$

$$\widehat{K}_{N-1,N}(\omega) = \widehat{K}_{N,N-1}(\omega) = -\frac{\sinh\left((N-2)\pi\omega/2\right)}{\sinh\left(N\pi\omega/2\right)}, \tag{2.78}$$

$$\widehat{K}_{0a}(\omega) = \widehat{K}_{a0}(\omega) = -\frac{\sinh\left(a\pi\omega/2\right)\sinh\left((N-2s)\pi\omega/2\right)}{\sinh\left(N\pi\omega/2\right)\sinh\left(\pi\omega/2\right)}, \qquad 1 \le a < 2s, \tag{2.79}$$

$$\widehat{K}_{0a}(\omega) = \widehat{K}_{a0}(\omega) = -\frac{\sinh\left(s\pi\omega\right)\sinh\left((N-a)\pi\omega/2\right)}{\sinh\left(N\pi\omega/2\right)\sinh\left(\pi\omega/2\right)}, \qquad 2s \le a \le N-2, \tag{2.80}$$

$$\widehat{K}_{aN}(\omega) = \widehat{K}_{Na}(\omega) = -\frac{2\sinh\left(a\pi\omega/2\right)\cosh\left(\pi\omega/2\right)}{\sinh\left(N\pi\omega/2\right)}, \qquad 1 \le a \le N-2, \tag{2.81}$$

$$\widehat{K}_{ab}(\omega) = \widehat{K}_{ba}(\omega) = \frac{\sinh\left((N-a)\pi\omega/2\right)\sinh\left(b\pi\omega/2\right)\sinh(\pi\omega)}{\sinh^2\left(\pi\omega/2\right)\sinh\left(N\pi\omega/2\right)} - \delta_{ab}, \qquad 1 \le b \le a \le N-1. \tag{2.82}$$

The kernel $\widehat{K}_{00}(\omega)$ can be directly read off from (2.31). We introduce the following function called the *universal kernel*, which plays a central role in the derivation of the final form of the TBA equations,

$$p(\theta) = \frac{1}{2\pi\cosh\theta}, \tag{2.83}$$

whose Fourier transform is

$$\widehat{p}(\omega) = \frac{1}{2\cosh\left(\frac{1}{2}\pi\omega\right)}. \tag{2.84}$$

We found that the following functional relations are satisfied for all $1 \le b \le N$,

$$\widehat{K}_{ab}(\omega) = \widehat{p}(\omega)(\eta_{a1}\widehat{K}_{a-1,b}(\omega) + \eta_{a,N-1}\widehat{K}_{a+1,b}(\omega))$$

$$+\,\widehat{p}(\omega)(\eta_{a,1}\delta_{a-1,b}+\eta_{a,N-1}\delta_{a+1,n}-\delta_{a,N-2}\delta_{b,N}),\quad 1\le n\le N-1, \tag{2.85}$$

$$\widehat{K}_{0b}(\omega)=-\widehat{p}(\omega)\widehat{K}_{2s,b}(\omega)+\widehat{p}(\omega)(-\delta_{2s,b}+\delta_{2s,N-1}\delta_{b,N}), \tag{2.86}$$

$$\widehat{K}_{Nb}(\omega)=-\widehat{p}(\omega)\widehat{K}_{N-2,b}(\omega)-\widehat{p}(\omega)\delta_{N-2,b}, \tag{2.87}$$

$$\widehat{K}_{00}(\omega)=\widehat{p}(\omega)^2(\widehat{K}_{2s,2s}(\omega)+1), \tag{2.88}$$

where we have used the short notation $\eta_{ab}=1-\delta_{ab}$.

2.2.6. Universal TBA equations. We can simplify the TBA equations (2.66) using the identities found above. Consider for example the equation for $a=1$. If we substitute in the simplified expression for the kernels, we obtain

$$\begin{aligned}\sigma_1(\theta)+\widetilde{\sigma}_1(\theta)&=-\sum_{b=0}^{N}(K_{1b}\star\sigma_b)(\theta)=-(K_{10}\star\sigma_0)(\theta)-\sum_{b=1}^{N}(K_{1b}\star\sigma_b)(\theta)\\&=-(p\star(-K_{2s,1}-\delta_{2s,1})\star\sigma_0)(\theta)-\sum_{b=1}^{N}(p\star(K_{2,b}+\delta_{2,b}-\delta_{3,N}\delta_{b,N})\star\sigma_b)(\theta).\end{aligned} \tag{2.89}$$

At this point, using again the kernel identities, we can express $-K_{2s,1}=-p\star(K_{2s,2}+\delta_{2,2s})$, which is nothing but $K_{0,2}$. Therefore, we have

$$\sigma_1(\theta)+\widetilde{\sigma}_1(\theta)=-p\star(K_{2,0}\star\sigma_0+\sum_{b=1}^{N}K_{2,b}\star\sigma_b)(\theta)+p\star(\delta_{2s,1}\sigma_0-\sigma_2-\delta_{3,N}\sigma_N)(\theta). \tag{2.90}$$

We recognise the term in the first bracket on the right-hand side to be precisely $\sigma_2(\theta)+\widetilde{\sigma}_2(\theta)$. Simplifying the various terms leads to

$$\sigma_1(\theta)+\widetilde{\sigma}_1(\theta)=(p\star\widetilde{\sigma}_2)(\theta)+\delta_{1,2s}(p\star\sigma_0)(\theta)+\delta_{3,N}(p\star\sigma_N)(\theta). \tag{2.91}$$

Following similar steps, one can simplify all raw TBA equations, obtaining for $1\le a\le N-1$

$$\sigma_a(\theta)+\widetilde{\sigma}_a(\theta)=p\star\left(\eta_{a1}\widetilde{\sigma}_{a-1}+\eta_{a,N-1}\widetilde{\sigma}_{a+1}+\delta_{a,N-2}\sigma_N+\delta_{a,2s}\sigma_0\right)(\theta) \tag{2.92}$$

while for $a=N$, we get

$$\sigma_N(\theta)+\widetilde{\sigma}_N(\theta)=p\star\widetilde{\sigma}_{N-2}(\theta)+\delta_{N-1,2s}p\star\sigma_0(\theta), \tag{2.93}$$

and from (2.86) and (2.88), for $a=0$, we get

$$\sigma_0(\theta)+\tilde{\sigma}_0(\theta)=\frac{\mathsf{m}}{2\pi}\cosh\theta+p\star(\widetilde{\sigma}_{2s}-\delta_{2s,N-2}\sigma_N)(\theta), \tag{2.94}$$

We introduce the following functional, describing the free energy of the gas of particles at temperature $1/R$ with the equations for the densities (2.92)– (2.94) as constraints,

$$\begin{aligned}\Phi[\sigma_i,\widetilde{\sigma}_i,\xi,\mu_i]=&\int \mathrm{d}\theta\mathsf{m}R\cosh\theta\sigma_0(\theta)-\sum_{n=0}^{N}\int\mathrm{d}x\Big([\sigma_n+\tilde{\sigma}_n]\log(\sigma_n+\tilde{\sigma}_n)-\sigma_n\log(\sigma_n)-\tilde{\sigma}_n\log(\tilde{\sigma}_n)\Big)\\&+\xi\Big[-\sigma_0(\theta)-\tilde{\sigma}_0(\theta)+\frac{\mathsf{m}}{2\pi}\cosh\theta+p\star(\widetilde{\sigma}_{2s}-\delta_{2s,N-2}\sigma_N)(\theta)\Big]\end{aligned}$$

$$+ \sum_{m=1}^{N} \mu_m \Big[-\sigma_a(\theta) - \widetilde{\sigma}_a(\theta) + p \star (\eta_{a1}\widetilde{\sigma}_{a-1} + \eta_{a,N-1}\widetilde{\sigma}_{a+1} + \delta_{a,N-2}\sigma_N + \delta_{a,2s}\sigma_0)(\theta) \Big], \tag{2.95}$$

where we introduced the Lagrange multipliers ξ, μ_a with $1 \le a \le N$. Here the first integral represents the total energy of the system while the second one is the entropy, where we are using the Fermi-Dirac statistics since $\mathsf{S}_{ss}^{ss}(0) = -1$.

The equilibrium condition is obtained by minimising this functional. This procedure leads to the *universal TBA equations*,

$$\epsilon_a(\theta) = \delta_{a,0}\mathsf{m}R\cosh\theta - \sum_{b=0}^{N} \mathbb{I}_{ab}\, p \star \log\left(1 + e^{-\epsilon_b}\right)(\theta), \qquad a = 0, 1, \ldots, N, \tag{2.96}$$

where we have introduced the pseudo-energies

$$\epsilon_0(\theta) = \log\frac{\tilde{\sigma}_0}{\sigma_0}, \qquad \epsilon_a(\theta) = \log\frac{\sigma_a}{\tilde{\sigma}_a}, \quad a = 1, \ldots, N-1, \qquad \epsilon_N(\theta) = \log\frac{\tilde{\sigma}_N}{\sigma_N}, \tag{2.97}$$

and p is the universal kernel introduced in eq. (2.83). Here $\mathbb{I}_{ab}$ are the matrix elements of the incidence matrix[3] of the graphs in figs. 2.2 and 2.3 when $2s < N-1$ and $2s = N-1$, respectively.

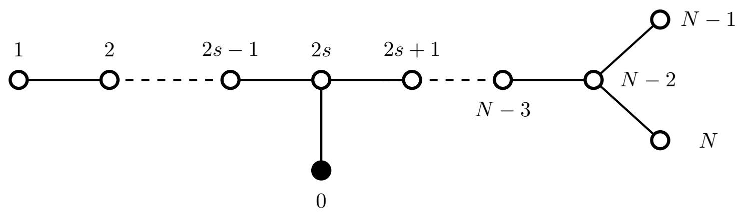


FIGURE 2.2. Dynkin-like structure of the TBA equations for $2s < N-1$. Note that the graph is a proper D_{N+1} Dynkin diagram only for $2s = 1$ and an extended one $\hat{D}_{N+1}$ for $2s = 2$.

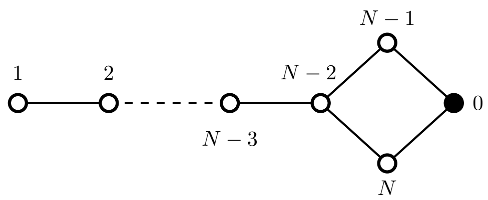


FIGURE 2.3. Structure of the TBA equations of $2s = N-1$.

[3]Consider a graph described by nodes and link between them. The incidence matrix is the matrix whose element a, b is 1 when the nodes a and b are connected and 0 otherwise.

At finite temperature $T = 1/R$, the free energy per unit length is obtained by the pseudo-energy ϵ_0 using

$$\frac{f(T)}{T} = -\int_{-\infty}^{\infty} \frac{\mathsf{m}}{2\pi} \cosh\theta \ln\left(1 + e^{-\epsilon_0(\theta)}\right) d\theta. \tag{2.98}$$

It is important to notice that this universal TBA is possible thanks to a remarkable relation (2.88) between the minimal scalar factor S_0 and other scattering amplitudes of the magnons. If a CDD factor is added, this relation is not valid anymore and the TBA cannot be written in the universal fashion.

2.3. Numerical analysis

Finding a solution to the TBA equations is a renowned difficult task. Some analytic results can be found in very particular situations, but in general, it is more common to rely on numerical solutions. In this section, we are going to address this problem. In particular, we will see that, even if the TBA equations do not present any particular new feature for higher spin theories, they actually hide some divergencies in the thermodynamical quantities, signalling a potential (second order) phase transition.

2.3.1. Sine-Gordon and sausage central charges. Recall the relation between the ground state energy and the free energy of the system from eq. (2.38). Using eq. (2.98) and the parametrisation with the scaling function eq. (2.39), we obtain

$$\widetilde{c}(r) = \frac{3}{\pi^2} \int_{-\infty}^{\infty} r \cosh(\theta) L_0(\theta) d\theta \tag{2.99}$$

where $L_0(\theta) = \log\left(1 + e^{-\epsilon_0(\theta)}\right)$ and $r = \mathsf{m}R$ is the dimensionless scale. In the limit $r \to 0$, the ultraviolet (UV) limit, this function encodes all the relevant data of the underlying conformal field theory, since

$$\lim_{r\to 0} \widetilde{c}(r) = c - 24\Delta_{\min}, \tag{2.100}$$

where c is the central charge and $\Delta_{\min}$ is the lowest eigenvalue of the zero-th Virasoro generator. As we stressed in the previous sections, the TBA equations (2.96) are a system of non-linear integral equations for which is in general very difficult to find an analytical closed solution.

Sometimes, however, it is possible to do so. For example, it is possible that as r approaches 0, the functions $\log\left(1 + e^{-\epsilon(\theta)}\right)$, develop a plateau region. In this case, it is possible to express the central charge in terms of dilogarithmic functions, depending on the plateau values of the pseudo energies of the system (2.96) when $r \to 0$, which can be obtained by solving the algebraic equations

$$x_i = \prod_{j=0}^{N} (1 + x_j^{-1})^{-\mathbb{I}_{ij}/2}, \qquad i = 0, \ldots, N, \tag{2.101}$$

where we have introduced $x_i = e^{\epsilon(0)}$ and $\mathbb{I}$ is the incidence matrix of the graphs of figs. 2.2 and 2.3.

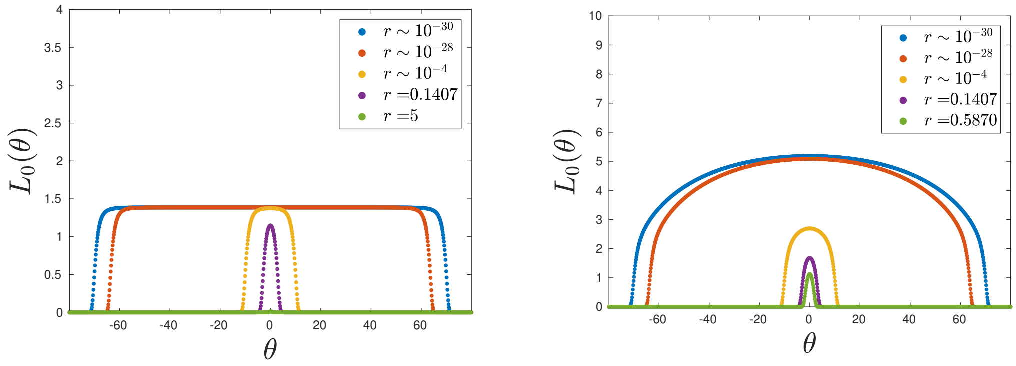


FIGURE 2.4. The functions $L_0(\theta)$ for spin $s = 1/2$ (left) and $s = 1$ (right) with $\gamma = 1/7$, for different values of r. One can see that for smaller values of r the plateau starts to form.

In the family of scattering theories we have introduced above, we have two well-known examples of this behaviour: the sine-Gordon model, corresponding to spin $s = 1/2$, and the sausage model for $s = 1$. In these cases, the plateaus start to form for small values of r, see e.g. fig. 2.4, and therefore one can explicitly compute the value of the central charge using dilogarithms obtaining $c = 1$ and $c = 2$, respectively, independently from the value of $\gamma = 1/N$.

2.3.2. Higher spin theories and Hagedorn singularity. As pointed out above, it is usually difficult to find a closed solution for generic r: for this reason, it becomes very useful to perform numerical analysis to study the ultraviolet behaviour of these theories. The method that has proven to be more effective is by solving the system of equations via successive iterations. The idea is to start from the initial guess $\epsilon_n^{(0)} = (r\cosh\theta, 0, \dots, 0)$ for $n = 0, \dots, N$ and then define the k-th iterative solution, with $k \geq 0$, as

$$\epsilon_n^{(k+1)}(\theta) = \delta_{n,0} r\cosh(\theta) - \sum_{m=0}^{N} \mathbb{I}_{nm}(p * L_m^{(k)})(\theta), \qquad n = 0, 1, \dots, N, \tag{2.102}$$

where $L_n^{(k)}(\theta) = \log\left(1 + e^{-\epsilon_m^{(k)}(\theta)}\right)$. This process allows us to find with arbitrarily high accuracy the values of the pseudo-energies and the corresponding $L_n(\theta)$ for any given r. An extensive study of this convergence problem has been done in [**HR20**]. Ultimately, these results can be used to compute numerically the integral (2.99) at different values of r, finding the value of the scaling function and, possibly, the rough value of the central charge of the underlying conformal theory. The cases of $s = 1/2$ and $s = 1$ are shown in fig. 2.5.

Having a natural generalisation of the S-matrix for higher values of the spin and of the corresponding TBA equations, it is natural to ask what kind of theories they describe in the ultraviolet regime. Performing the same iterative procedure as above, we observe an unexpected behaviour as the ground state energy $E_0(r)$ diverges at a positive finite value r^*

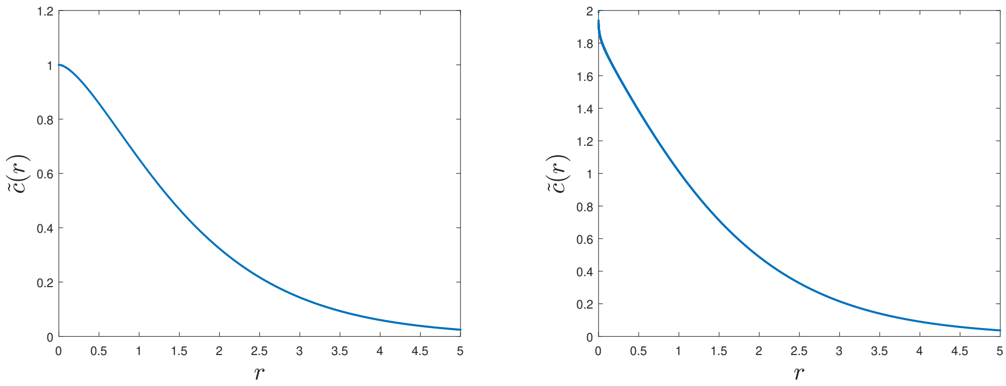


FIGURE 2.5. Scaling functions for spin $s = 1/2$ (left) and $s = 1$ (right).

and, correspondingly, that the functions $L_n(\theta)$ do not develop a plateau, but rather become more peaked around $\theta = 0$ as they approach the singular value, as shown in fig. 2.6.

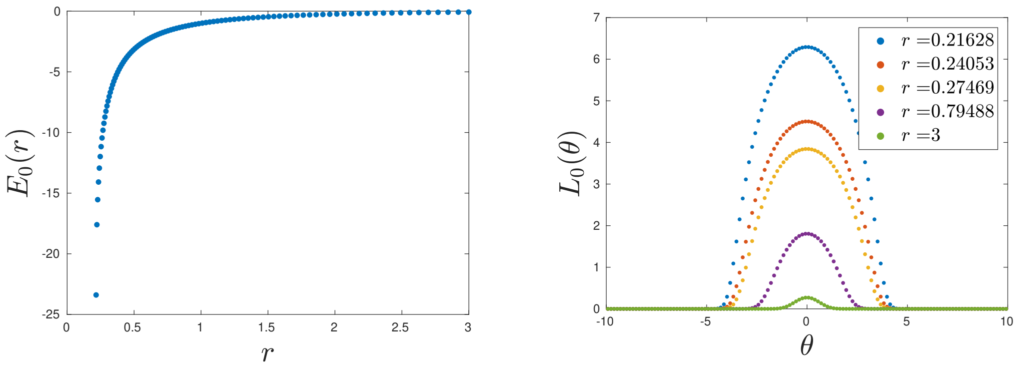


FIGURE 2.6. Left: the vacuum energy $E_0(r)$ as it approaches the singular point $r^* = 0.21628(2)$; right: the kernel $L_0(\theta)$ at different values of r. Both were obtained for $s = 5/2$ and $N = 12$.

Extending the numerical analysis to different values of the spin and of the coupling constant, we observe that the critical value r^* is a function of both s and $N = 1/\gamma$. Some values of r^* are listed in table 2.1.

Moreover, as can be seen from fig. 2.7, the critical values r^* seem to converge to non-zero values even for vanishing coupling constant $\gamma = 1/N \to 0$ for $s \geq 3/2$. This means that the singularity does occur also at the $\mathfrak{su}(2)$ symmetric points, obtained by sending $\gamma \to 0$, *i.e.* $\mathsf{q} \to 1$, and the UV limit does not exist even in those cases.

A similar behaviour has been recently studied in $T\overline{T}$-deformed theories with detailed numerical analysis [CNST16]. In this case, the vacuum energy develops a square root behaviour,

$$E_0(r) \sim_{r \to r^*} c_0 + c_{1/2}\sqrt{r - r^*}, \tag{2.103}$$

	$s = 3/2$	$s = 2$	$s = 5/2$	$s = 3$
$N = 4$	0.06024(4)	-	-	-
$N = 5$	0.01683(2)	0.22505(9)	-	-
$N = 6$	0.00722(5)	0.09996(5)	0.40380(3)	-
$N = 7$	0.00392(8)	0.05976(6)	0.21628(2)	0.57301(7)
$N = 8$	0.00248(7)	0.04195(5)	0.14665(8)	0.34110(6)
$N = 9$	0.00174(9)	0.03255(2)	0.11269(7)	0.24773(3)
$N = 10$	0.00132(7)	0.02699(9)	0.09349(6)	0.19958(2)
$N = 11$	0.00106(6)	0.0234(5)	0.08157(4)	0.17123(0)
$N = 12$	0.00089(4)	0.02106(7)	0.07367(8)	0.15307(2)

TABLE 2.1. Selected values of the critical scale r^* for different values of s and $\gamma = 1/N$.

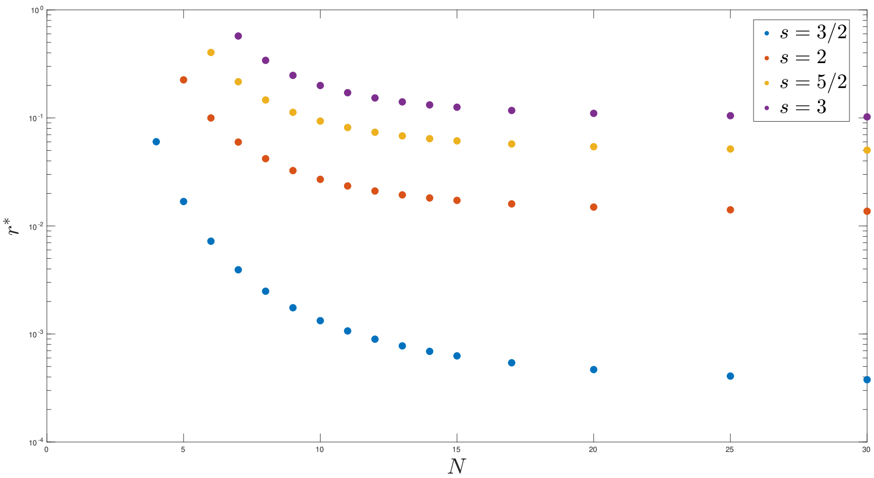


FIGURE 2.7. Value of the singular point r^*, for different values of spin and coupling constant $\gamma = 1/N$. The values are computed with precision to the 6th decimal digit. The r^*-axis is log-scaled.

so that its first derivative diverges at the critical value, signalling a phase transition. In this setting, the singularity ultimately appears as a consequence of the presence of a CDD factor and it has been regarded as the appearance of a Hagedorn-like phase transition. Remarkably, it has been shown that by finely tuning the parameters of the deformation, one can ultimately remove it [**AL22**].

The theories we have introduced in this work present some similar aspects, but they are crucially different. Indeed, the S-matrices we consider are not obtained as deformations of some known theory but are genuinely obtained by imposing the defining properties of a scattering theory in two dimensions, as explained in section 2.1.2. As a result, this singularity is in a sense more "fundamental", as it cannot be removed by a fine-tuning of the parameters.

We analysed the behaviour of these models close to the singularity, for different values of the spin, at different values of the coupling constant. More explicitly, we have generated several points in a close neighbourhood of width $\sim 1\%$ of the singular points of table 2.1. Using these data we fitted the curves, as shown in fig. 2.8 with a fitting function given by

$$E_0^{\texttt{fit}}(r) = b(r - r^*)^a + c_0. \tag{2.104}$$

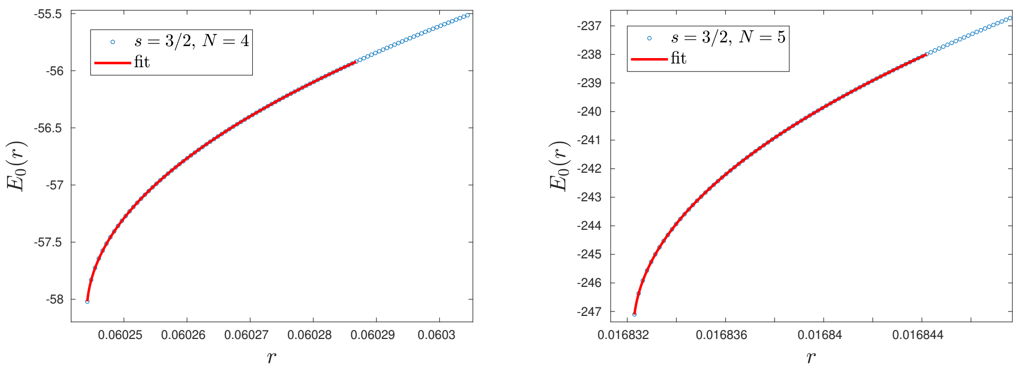


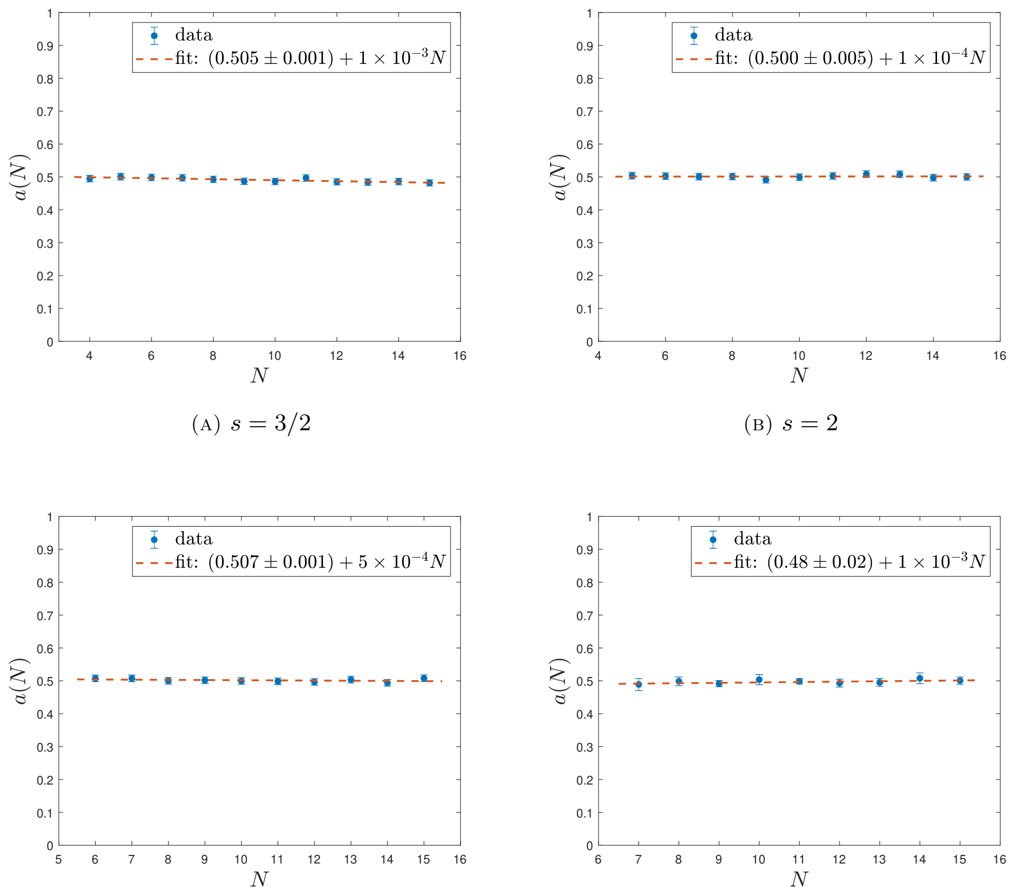
FIGURE 2.8. Examples of fitting for $s = 3/2$ and $N = 4$ (left) and $N = 5$ (right).

In table 2.2 we present the results of the critical exponent a obtained from the numerical analysis, and in fig. 2.9 we show a fit of these points. The main source of error in the exponent a is a sensitive dependence on the initial guesses used in the fitting algorithm. We performed the fit for different initial guesses to estimate the average value of a and the associated error which is around 2%.

	N											
	4	5	6	7	8	9	10	11	12	13	14	15
$s = 3/2$	0.495	0.501	0.498	0.497	0.492	0.487	0.486	0.497	0.486	0.485	0.488	0.482
$s = 2$	—	0.504	0.502	0.501	0.501	0.491	0.499	0.504	0.509	0.508	0.497	0.500
$s = 5/2$	—	—	0.507	0.507	0.500	0.502	0.499	0.499	0.497	0.493	0.494	0.508
$s = 3$	—	—	—	0.489	0.498	0.491	0.504	0.499	0.493	0.495	0.508	0.501

TABLE 2.2. Values of the fitted exponent a, for different values of s and N.

Remarkably, we observe that it becomes independent of the value of the coupling constant and the spin, approaching a universal value of $\sim 1/2$, compatible with a square root behaviour. Similarly, in table 2.3 we present the fitted values of the parameters b and c_0, only for the case $s = 3$.



(A) $s = 3/2$

(B) $s = 2$

(C) $s = 5/2$

(D) $s = 3$

FIGURE 2.9. Fitted value of the exponent a, for different values of the spin.

	N											
	4	5	6	7	8	9	10	11	12	13	14	15
b	—	—	—	17.81	44.09	82.45	127.59	167.71	219.89	264.01	298.47	341.50
c_0	—	—	—	-7.58	-15.57	-24.23	-32.92	-41.24	-48.94	-55.95	-62.25	-67.85

TABLE 2.3. Values of the fitted parameters b and c_0, for spin $s = 3$.

These features, however, need a more careful analysis since it is extremely difficult and computationally challenging to study the data in the close vicinity of the critical value r^*, as the iterative procedure becomes extremely slow. Clearly, the best way to overcome this problem is to find clever ways to solve the TBA equations (2.96) analytically, so that one would be able to make a more quantitative analysis of these models.

2.3.3. Simplified TBA equations with a toy kernel. Since we do not have a closed analytical solution of the TBA system (2.96), we now present a similar system that is highly simplified but develops the same singularity. We know that the main difference with other theories that are known to converge up to the conformal point is that the $L_n(\theta)$ functions become more and more peaked instead of developing a plateau. In order to implement this feature, we consider a different integral kernel, given by[4]

$$p(\theta) = \frac{1}{2}\delta(\theta). \tag{2.105}$$

We now consider a system of TBA equations similar to the one we obtained above, *i.e.* whose structure is described by the same graphs. The particular form of the kernel allows us to greatly simplify the equations, namely

$$\epsilon_n(\theta) = \delta_{n,0} r\cosh\theta - \frac{1}{2}\sum_{m=0}^{N} \mathbb{I}_{nm} \log\left[1 + e^{-\epsilon_m(\theta)}\right], \qquad n = 0, 1, \cdots, N, \tag{2.106}$$

and has the advantage that it can be expressed as a set of algebraic equations

$$x_n(\theta) = e^{-r\cosh\theta\,\delta_{n,0}} \prod_{m=0}^{N} \left[1 + x_m(\theta)\right]^{\mathbb{I}_{nm}/2}, \qquad x_n(\theta) := e^{-\epsilon_n(\theta)}, \tag{2.107}$$

for each value of θ. These are now algebraic equations, which can be analysed analytically. In particular, those TBA equations whose graph is shown above in fig. 2.3 are exactly solvable with a computer[5] for $N = 2s+1$ with $s = 3/2$ and $s = 2$ in terms of $a = e^{-r\cosh\theta}$,

$$x_0^{[2s=3]}(a) = -\frac{5}{2} + \frac{1}{2a^2} - a - \frac{(1+a)^2}{2a^2}\sqrt{1-4a}, \tag{2.108}$$

$$x_0^{[2s=4]}(a) = -\frac{3}{2} + \frac{1}{2a} - a - \frac{(1+a)}{2a}\sqrt{1-8a}. \tag{2.109}$$

All other exact expressions for x_n's are also found but we will not put them here since they are much more complicated and not relevant in further discussions.

These results show that the pseudo-energies can be real if

$$e^{-r\cosh\theta} \leq \frac{1}{4} \quad \text{for } 2s = 3; \qquad e^{-r\cosh\theta} \leq \frac{1}{8} \quad \text{for } 2s = 4, \tag{2.110}$$

for any value of θ. Therefore, the critical values r^*, which are the maximum values of r for them to remain all real, are found by considering $\theta = 0$, namely,

$$r^*_{s=3/2} = \log 4 = 1.3862943\ldots, \qquad r^*_{s=2} = \log 8 = 2.0794414\ldots. \tag{2.111}$$

For other values of s and N, we can solve only numerically to find r^*. We list them for different values of N in table 2.4. It is interesting to notice that the critical values r^* where the solutions turn into complex numbers, depend only on the spin s and not on the coupling constant $N = 1/\gamma$. This feature is definitely due to the exceptionally simplified kernel. For $s = 1/2, 1$ no singularity occurs, as the numerical result is $< 10^{-4}$.

[4] The normalisation is chosen to match the same normalisation of the (integrated) universal kernel eq. (2.83).

[5] Exact solutions for higher values of s are beyond our computational ability.

	N				
	4	5	6	7	8
$s = 3/2$	1.386294	1.386294	1.386294	1.386294	1.386294
$s = 2$	—	2.079441	2.079441	2.079441	2.079441
$s = 5/2$	—	—	—	2.574781	2.574781

TABLE 2.4. r^* for various s and N. r^* depends on s only and it is independent of N.

The above exact solutions of the simplified TBA can be used to analyse the free energy using (2.98). For $2s = 3$, one finds

$$\frac{f(T)}{T} = -\int_0^\infty \frac{\mathsf{m}}{\pi}\cosh\theta \,\ln\left[-\frac{3}{2} + \frac{1}{2a^2} - a - \frac{(1+a)^2}{2a^2}\sqrt{1-4a}\right] d\theta, \quad a = e^{-r\cosh\theta}, \tag{2.112}$$

Although this expression is given in terms of relatively simple integrals, it can not be expressed analytically. Instead, we perform this numerically and the plot in fig. 2.10 which show qualitatively similar behaviours as fig. 2.8.

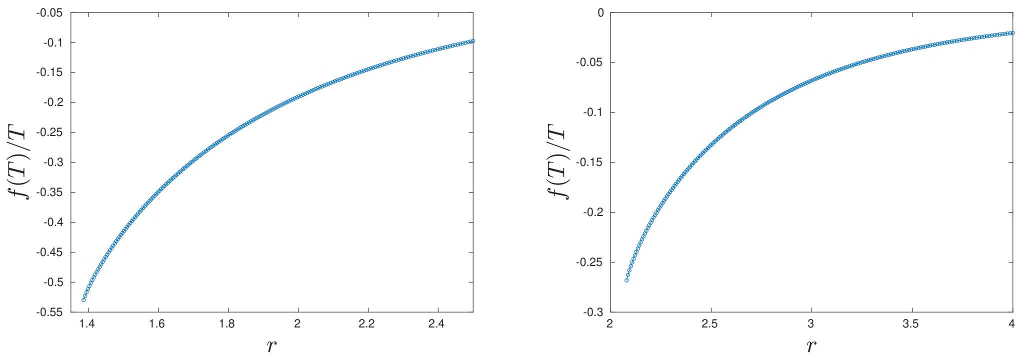


FIGURE 2.10. Toy TBA with $s = 3/2, N = 4$ (left) and $s = 2, N = 5$ (right).

CHAPTER 3

Gaudin models of finite type

Gaudin models were first introduced in [**Gau76**] to describe $\mathfrak{sl}_2$ integrable spin chains with long-range interactions and later generalised to any semi-simple finite Lie algebra [**Gau14**]. They are a very powerful tool to describe a large class of integrable systems and they provide a mathematical framework to study their properties. A complete description of the space of commuting charges of the model was first proposed in [**FFR94**], in terms of singular vectors in the vacuum Verma module over the affine algebra at critical level. A central question is the diagonalisation problem, *i.e.* finding the joint spectrum of eigenvalues and eigenvectors for these charges. This has been done by performing different types of Bethe ansatz [**Gau76, BF94**], but perhaps the most elegant one is the one proposed by Feigin, Frenkel and Reshetikhin in [**FFR94**], based on the construction of Wakimoto modules over the affine algebra at critical level.

This chapter is a short review of some aspects of Gaudin models of finite type; it mainly serves as a motivation for the various generalisations that we are going to discuss in the next chapters.

3.1. Generalities

Consider a simple finite-dimensional Lie algebra $\mathring{\mathfrak{g}} =_{\mathbb{C}} \mathring{\mathfrak{n}}_- \oplus \mathring{\mathfrak{h}} \oplus \mathring{\mathfrak{n}}_+$, where $\mathring{\mathfrak{h}}$ is the Cartan subalgebra and $\mathring{\mathfrak{n}}_+$, $\mathring{\mathfrak{n}}_-$ are the subalgebras corresponding to the positive and negarive root spaces. Let $U(\mathring{\mathfrak{g}})$ be the corresponding universal enveloping algebra. We introduce the one-dimensional $U(\mathring{\mathfrak{b}}_+)$ representation $\mathbb{C}\mathsf{v}_\lambda$ for any choice of weight $\lambda \in \mathring{\mathfrak{h}}^*$, where $\mathring{\mathfrak{b}} = \mathring{\mathfrak{h}} \oplus \mathring{\mathfrak{n}}_+$ is the positive Borel subalgebra, defined as follows

$$X\mathsf{v}_\lambda = 0, \qquad \text{for all } X \in \mathring{\mathfrak{n}}_+, \tag{3.1}$$

$$X\mathsf{v}_\lambda = \lambda(X)\mathsf{v}_\lambda, \qquad \text{for all } X \in \mathring{\mathfrak{h}}, \tag{3.2}$$

where $\lambda(X) = \langle X, \lambda \rangle$ is the canonical pairing between $\mathring{\mathfrak{h}}$ and its dual. One can introduce a non-degenerate bilinear form on $\mathring{\mathfrak{h}}$ which can be extended to a non-degenerate symmetric invariant bilinear form $\kappa : \mathring{\mathfrak{g}} \times \mathring{\mathfrak{g}} \to \mathbb{C}$, normalised as in [**Kac90**]. By restricting κ to $\mathring{\mathfrak{h}}$, one introduces the map $\nu : \mathring{\mathfrak{h}}^* \to \mathring{\mathfrak{h}}$, defined as follows: for each $\lambda \in \mathring{\mathfrak{h}}^*$, $\nu(\lambda)$ is the unique element in $\mathring{\mathfrak{h}}$ such that $\lambda(X) = \kappa(X, \nu(\lambda))$, for all $X \in \mathring{\mathfrak{h}}$. This can be used to define a non-degenerate inner product on the space of roots, denoted by $(\cdot|\cdot)$, as follows $(\alpha|\beta) = \kappa(\nu(\alpha), \nu(\beta))$, for all $\alpha, \beta \in \mathring{\mathfrak{h}}^*$.

The Verma module of highest weight λ is defined as the induced module

$$M_\lambda := U(\mathring{\mathfrak{g}}) \otimes_{U(\mathring{\mathfrak{b}}_+)} \mathbb{C}\mathsf{v}_\lambda \simeq_{\mathbb{C}} U(\mathfrak{n}_-)\mathbb{C}\mathsf{v}_\lambda, \tag{3.3}$$

where the last identification follows from Poincaré-Birkhoff-Witt (PBW) theorem (see e.g. [**Dix74**]). Moreover, if λ is an integral dominant weight, *i.e.* for any positive root α, $(\lambda|\alpha)$ is a positive integer number, it is always possible to identify a submodule $\overline{M} \subset M_\lambda$ such that the quotient $M_\lambda/\overline{M}$ is finite-dimensional.

We introduce a set of points on the Riemann sphere $\{z_1, \dots, z_N\} \subset \mathbb{CP}^1$, which can be thought of as the *sites* of the model. To each of these points, we attach one of the modules M_{λ_i} with $i = 1, \dots, N$. The tensor product of modules

$$M_{\boldsymbol{\lambda}} := M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_N}, \tag{3.4}$$

plays the role of the Hilbert space of the Gaudin model. Note that since any Verma module M_{λ_i} admits a highest weight vector v_{λ_i}, one can introduce a well-defined ground state, given by

$$\mathsf{v}_{\boldsymbol{\lambda}} := \mathsf{v}_{\lambda_1} \otimes \cdots \otimes \mathsf{v}_{\lambda_N}. \tag{3.5}$$

The algebra of the observables of the quantum Gaudin model is obtained by considering the N-fold tensor product of copies of the universal enveloping algebra of $\mathring{\mathfrak{g}}$, one for each site,

$$\mathrm{Obs}_{\boldsymbol{z}}(\mathring{\mathfrak{g}}) = U(\mathring{\mathfrak{g}})^{\otimes N}. \tag{3.6}$$

We introduce the following notation for the operator acting on the i-th site with $X \in U(\mathring{\mathfrak{g}})$ and with the identity elsewhere,

$$X^{(i)} = 1 \otimes \cdots 1 \otimes X \otimes 1 \otimes \cdots \otimes 1. \tag{3.7}$$

By denoting with $\{I^a\}_{a=1,\dots,\dim \mathring{\mathfrak{g}}}$ a basis for the Lie algebra $\mathring{\mathfrak{g}}$, the quantum $\mathring{\mathfrak{g}}$-Gaudin model is defined by the Gaudin Hamiltonians

$$\mathcal{H}_i = \sum_{\substack{j=1 \\ j\neq i}}^{N} \kappa_{ab} \frac{I^{a,(i)} I^{b,(j)}}{z_i - z_j}, \tag{3.8}$$

where there is an implicit sum over Lie algebra indices and κ_{ab} is the non-degenerate symmetric invariant bilinear form on $\mathring{\mathfrak{g}}$. From the point of view of spin chains, $I^{a,(i)}$ can be thought as the spin degree of freedom at site i and the factor $1/(z_i - z_j)$ represents the *interaction term* between the sites i and j.

This operator can be seen as descending from a more general object, which somehow has a more natural interpretation when working with vertex algebras and coinvariants as we will see shortly, which is the *quadratic Hamiltonian*

$$\mathcal{H}(z) = \frac{1}{2} \kappa_{ab} I^a(z) I^b(z), \tag{3.9}$$

where we introduced the following rational function

$$X(z) = \sum_{i=1}^{N} \frac{X^{(i)}}{z - z_i}. \tag{3.10}$$

depending on the auxiliary complex parameter z, called the *spectral parameter*. In fact, by substituting this into eq. (3.9), it follows by partial fraction decomposition that

$$\mathcal{H}(z) = \sum_{i=1}^{N} \left(\frac{1}{2} \frac{\mathcal{C}^{(i)}}{(z - z_i)^2} + \frac{\mathcal{H}_i}{z - z_i} \right), \tag{3.11}$$

where $\mathcal{C}^{(i)} \in U(\mathring{\mathfrak{g}})^{\otimes N}$ is the quadratic Casimir operator at site i defined as follows

$$\mathcal{C} = \sum_{a,b=1}^{\dim \mathring{\mathfrak{g}}} \kappa_{ab} I^a I^b. \tag{3.12}$$

It is an element of the centre $Z(\mathrm{Obs}_{\boldsymbol{z}}(\mathring{\mathfrak{g}}))$ of the algebra of observables and therefore by definition it trivially commutes with the generators of $\mathring{\mathfrak{g}}$ and with all other Hamiltonians.

3.1.1. Diagonal action. There is a map $\Delta : \mathring{\mathfrak{g}} \hookrightarrow \mathring{\mathfrak{g}}^{\oplus N}$, which is the diagonal embedding of $\mathring{\mathfrak{g}}$ into the direct sum of N copies of $\mathring{\mathfrak{g}}$, defined as follows

$$\Delta x = \sum_{i=1}^{N} x^{(i)}, \qquad \text{for all } x \in \mathring{\mathfrak{g}}. \tag{3.13}$$

It can be extended to an embedding of the corresponding enveloping algebras $\Delta : U(\mathring{\mathfrak{g}}) \hookrightarrow U(\mathring{\mathfrak{g}}^{\oplus N}) \simeq U(\mathring{\mathfrak{g}})^{\otimes N}$. It is a Lie algebra homomorphism, since for all $x, y \in \mathring{\mathfrak{g}}$

$$[\Delta x, \Delta y] = \sum_{j=1}^{N} \sum_{\ell=1}^{N} [x^{(j)}, y^{(\ell)}] = \sum_{j=1}^{N} [x, y]^{(j)} = \Delta[x, y], \tag{3.14}$$

where in the second-to-last step we used the fact that operators at different sites always commute, *i.e.* $[x^{(j)}, y^{(\ell)}] = \delta^{j\ell}[x, y]^{(j)}$. Given any state $\mathsf{w}_1 \otimes \cdots \otimes \mathsf{w}_N \in M_{\boldsymbol{\lambda}}$, there is a well-defined action of (3.13) on it,

$$\Delta x(\mathsf{w}_1 \otimes \cdots \otimes \mathsf{w}_N) = (x \cdot \mathsf{w}_1 \otimes \cdots \otimes \mathsf{w}_N) + \cdots + (\mathsf{w}_1 \otimes \cdots \otimes x \cdot \mathsf{w}_N), \tag{3.15}$$

called the *diagonal action* of the Lie algebra. The $\mathring{\mathfrak{g}}$-symmetry of the Gaudin model is ensured by requiring that the Hamiltonians commute with the diagonal action of $\mathring{\mathfrak{g}}$. For example, for the Gaudin Hamiltonians in eq. (3.8),

$$\begin{aligned}
[\Delta I^x, \mathcal{H}_i] &= \sum_{\ell=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \kappa_{ab} \frac{[I^{x,(\ell)}, I^{a,(i)} I^{b,(j)}]}{z_i - z_j} \\
&= \sum_{\ell=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \kappa_{ab} \frac{\delta_i^\ell f^{xa}_c I^{c,(i)} I^{b,(j)} + \delta_j^\ell f^{xb}_c I^{a,(i)} I^{c,(j)}}{z_i - z_j} \\
&= \sum_{\substack{j=1 \\ j \neq i}}^{N} (\kappa_{ab} f^{xa}_c + \kappa_{ca} f^{xa}_b) \frac{I^{c,(i)} I^{b,(j)}}{z_i - z_j} = 0.
\end{aligned} \tag{3.16}$$

In the last line, the term in brackets vanishes identically by the invariance of the bilinear form.

3.1.2. Bethe ansatz: case of $\mathfrak{sl}_2$. As we have anticipated in the introduction to this chapter, one of the central questions is the joint diagonalisation of the conserved charges of the model. For the case of $\mathfrak{sl}_2$, algebraic Bethe ansatz was used to construct eijgenvalues and eigenvectors [**Gau76, Gau14, Skl89**]. Consider the standard basis $\{E, F, H\}$ for this algebra. The vacuum state (3.5) is an eigenvector of the quadratic Hamiltonians. Indeed, one can explicitly write the Casimir operator and Gaudin Hamiltonian as follows

$$\mathcal{C}^{(i)} = (H^{(i)}H^{(i)} + E^{(i)}F^{(i)} + F^{(i)}E^{(i)}) = H^{(i)}H^{(i)} + 2H^{(i)} + 2F^{(i)}E^{(i)},$$

$$\mathcal{H}_i = \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{1}{z_i - z_j}(H^{(i)}H^{(j)} + F^{(i)}E^{(j)} + E^{(i)}F^{(j)}). \tag{3.17}$$

Recalling from eqs. (3.1) and (3.2) the action of the generators on a highest weight vector, we obtain the following vacuum eigenvalue relation

$$\mathcal{H}(z)\mathsf{v}_{\boldsymbol{\lambda}} = \left(\sum_{i=1}^{N} \frac{1}{2} \frac{\lambda_i(H)^2 + 2\lambda_i(H)}{(z - z_i)^2} + \sum_{i=1}^{N} \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{\lambda_i(H)\lambda_j(H)}{(z_i - z_j)(z - z_i)} \right) \mathsf{v}_{\boldsymbol{\lambda}} \tag{3.18}$$

3.1.2.1. *Higher excitations.* The idea behind algebraic Bethe ansatz is to create *excitations* by applying the operator F on the ground state a certain number of times. In the spin chain picture, if the ground state is a ferromagnet, *i.e.* all spins are up, this is equivalent to "flipping arrows", creating excitations of the spin chain. One can introduce

$$F(w) = \sum_{i=1}^{N} \frac{F^{(i)}}{w - z_i}, \tag{3.19}$$

where we introduced a new auxiliary parameter w, not equal to $z_1, \ldots, z_N$. The *Bethe vector* is constructed by acting with this operator on the ground state vector [**Gau76**],

$$|w_1, \ldots, w_M\rangle := F(w_1)\cdots F(w_M)\mathsf{v}_{\boldsymbol{\lambda}} \tag{3.20}$$

One can explicitly compute the action of the quadratic Hamiltonians (3.9) on these states, obtaining

$$\begin{aligned}\mathcal{H}(z)\,|w_1, \ldots, w_M\rangle = s(z)\,|w_1, \ldots, w_M\rangle \\ + \sum_{j=1}^{M} \frac{f^{(j)}}{z - w_j}\,|w_1, \ldots, w_{j-1}, z, w_{j+1}, \ldots, w_M\rangle\,,\end{aligned} \tag{3.21}$$

where $s(z)$ is some rational function defined as follows

$$s_M(z) = \frac{1}{4}\chi_M(z)^2 - \frac{1}{2}\partial_z \chi_M(z), \tag{3.22}$$

where

$$\chi_M(z) = \sum_{i=1}^{N} \frac{\lambda_i}{z - z_i} - \sum_{j=1}^{M} \frac{2}{z - w_j}, \tag{3.23}$$

while the functions $f^{(j)}$, for $j = 1, \dots, M$ depend on the marked points and the levels λ_j. Clearly, to obtain a genuine eigenvalue relation from eq. (3.21) with $s_M(z)$ as the eigenvalue, it has to happen that all factors $f^{(j)}$ vanish. These constraints are the *Bethe equations*,

$$f^{(j)} := \sum_{i=1}^{N} \frac{\lambda_i}{w_j - z_i} - \sum_{\substack{k=1 \\ k\neq j}}^{M} \frac{2}{w_j - w_k} = 0, \qquad j = 1, \dots, M, \tag{3.24}$$

which can be seen as a set of consistency equations for the auxiliary parameters, called Bethe roots.

It is easy to imagine that finding the explicit expression for the eigenvalues for an arbitrary Lie algebra $\mathring{\mathfrak{g}}$ can become extremely complicated. A possible generalisation has been proposed in [**BF94**], but it can be practically used for the quadratic Hamiltonians only; applying this construction to the higher Hamiltonians, which we have not described yet, is extremely complex.

3.2. Vertex algebras

Before moving to the description of the higher Hamiltonians and their diagonalisation, in this section we introduce the theory of vertex algebras. They provide a natural language to describe conformal field theories and, as we will see below, they turn out to be an extremely valuable tool in the study of Gaudin models. This section is mainly based on the references [**Kac01**, **FBZ04**].

3.2.1. Definition and properties.

Definition 3.2.1. *Given a graded vector space $\mathcal{V} = \sum_{k\in\mathbb{Z}} \mathcal{V}^{(k)}$ over a field $\mathbb{K}$, a vector $|0\rangle \in \mathcal{V}$ called the* vacuum, *a map $T \in \operatorname{End}(\mathcal{V})$ called* translation, *and a map called the* state-field correspondence $Y : \mathcal{V} \to \operatorname{End}\mathcal{V}[[x, x^{-1}]]$, *$a \mapsto Y(a, x) := \sum_{k\in\mathbb{Z}} a_{(k)} x^{-k-1}$, with $a_{(k)} \in \operatorname{End}\mathcal{V}$ of degree* $\deg(a) - k - 1$, i.e. $a_{(k)}\mathcal{V}^{(n)} \subset \mathcal{V}^{(n+\deg(a)-k-1)}$, *a* vertex algebra *is the quadruple $(\mathcal{V}, |0\rangle, T, Y(\bullet, x))$ satisfying the following axioms*

i) Vacuum axiom: $Y(|0\rangle, x) = \mathrm{id}_{\mathcal{V}}$;
ii) Creation axiom: for all $a \in \mathcal{V}$, $Y(a, x)|0\rangle \in \mathcal{V}[[x]]$ and $Y(a, x)|0\rangle\Big|_{x=0} = a$;
iii) Translation axiom: $T|0\rangle = 0$ and for all $a \in \mathcal{V}$, $[T, Y(a, x)] = \partial_x Y(a, x)$;
iv) Borcherds' identities: for all $a, b \in \mathcal{V}$,

$$\operatorname{res}_{x-y} \iota_{y,x-y} f(x,y) Y(Y(a, x-y)b, y) = \operatorname{res}_x \iota_{x,y} f(x,y) Y(a,x) Y(b,y) - \operatorname{res}_x \iota_{y,x} f(x,y) Y(b,y) Y(a,x), \tag{3.25}$$

where $f(x, y)$ is a rational function with poles at most at $x = 0$, $y = 0$ or $x - y = 0$ and $\iota_{x,y}$ denote the formal power series expansion in the domain $|x| > |y|$.

The state-field correspondence can be interpreted as the generating function of an infinite number of products, called the *n-th products*, $\mathcal{V} \times \mathcal{V} \to \mathcal{V}$, $(a, b) \mapsto a_{(n)}b$ for all $n \in \mathbb{Z}$, such that $a_{(n)}b = 0$ for sufficiently large n, where $a_{(n)} \in \operatorname{End} \mathcal{V}$ is called the *n-th mode of a*.

It is possible to rephrase the axioms in terms of modes as follows

i') Vacuum axiom: for all $n \in \mathbb{Z}$, $|0\rangle_{(n)}\, a = \delta_{n,-1}a$;

ii') Creation axiom: for all $a \in \mathcal{V}$, $n \in \mathbb{Z}_{\geq 0}$, $a_{(n)}\,|0\rangle = 0$ *and* $a_{(-1)}\,|0\rangle = a$;

iii') Translation axiom: for all $a \in \mathcal{V}$, $n \in \mathbb{Z}$, $[T, a_{(n)}] = -na_{(n-1)}$;

iv') Borcherds' identities: for all $a, b \in \mathcal{V}$, $n, m \in \mathbb{Z}$,

$$\sum_{j=0}^{\infty} \binom{m}{j} (a_{(n+j)}b)_{(m+k-j)} = \sum_{j=0}^{\infty} \binom{n}{j} \Big((-1)^j a_{(m+n-j)}(b_{(k+j)}) - (-1)^{j+n} b_{(n+k-j)}(a_{(m+j)}) \Big). \tag{3.26}$$

where for any $m \in \mathbb{Z}$, $\binom{m}{k} = \frac{1}{k!}m(m-1)\dots(m-k+1)$ *for all* $k > 0$ *and* $\binom{m}{0} = 1$.

Let us now discuss some particular cases of Borcherds' identities. Setting $n = 0$ in the last identity, we find the so-called *commutator formula* for the modes,

$$[a_{(m)}, b_{(k)}] = \sum_{j=0}^{\infty} \binom{m}{j} (a_{(j)}b)_{(m+k-j)}. \tag{3.27}$$

If we express this in terms of state-field map, we obtain the *locality formula*,

$$(x-y)^N [Y(a,x), Y(b,y)] = 0, \qquad \text{for } N \text{ big enough.} \tag{3.28}$$

This relation is often used as one of the vertex algebra axioms, replacing Borcherds' identities.

Instead, for $m = 0$ we obtain the *associativity formula*, which can be seen as a recursive formula for the composition of modes,

$$(a_{(n)}b)_{(k)} = \sum_{j=0}^{\infty} \binom{n}{j} \Big((-1)^j a_{(n-j)}(b_{(k+j)}) - (-1)^{j+n} b_{(n+k-j)}(a_{(j)}) \Big). \tag{3.29}$$

In terms of state-field maps, this identity reads

$$Y(a_{(n)}b, y) = \operatorname{res}_x \iota_{x,y} Y(a,x)Y(b,y)(y-x)^n - \operatorname{res}_x \iota_{y,x} Y(b,y)Y(a,x)(y-x)^n. \tag{3.30}$$

In particular, setting $n = -1$ one gets

$$Y(a_{(-1)}b, y) = Y(a,y)_+ Y(b,y) + Y(b,y)Y(a,y)_-, \tag{3.31}$$

where

$$Y(a,x)_+ := \sum_{k<0} a_{(k)}x^{-k-1}, \qquad Y(a,x)_- := \sum_{k\geq 0} a_{(k)}x^{-k-1}. \tag{3.32}$$

This can be generalised to define the so-called *normal ordering of fields*,

$$\mathopen{:} Y(a,x)Y(b,y) \mathclose{:} := \sum_{n\in\mathbb{Z}} \left(\sum_{m<0} a_{(m)}b_{(n)}x^{-m-1} + \sum_{m\geq 0} b_{(n)}a_{(m)}x^{-m-1} \right) y^{-n-1}. \tag{3.33}$$

where the effect of the ordering is to move positive modes to the right, which in a field theory context can be interpreted as "annihilation operators act first". This operation is right-associative, *i.e.* $\mathopen{:}Y(a,x)Y(b,y)Y(c,z)\mathclose{:} = \mathopen{:}Y(a,x)\mathopen{:}Y(b,y)Y(b,z)\mathclose{:}\mathclose{:}$.

From these axioms, it is possible to show that $Y(a,x)b = e^{xT}Y(b,-x)a$ for all $a,b \in \mathcal{V}$. In the language of modes, this gives rise to the so-called *skew-symmetry formula* for the products,

$$a_{(n)}b = \sum_{j=0}^{\infty} \frac{(-1)^{n+j}}{k!} T^j(b_{(n+j)}a). \tag{3.34}$$

3.2.1.1. *Operator product expansion.* One of the main applications of vertex algebras in theoretical physics is in the context of conformal field theories. The reason is that from the associativity properties of the fields, one can rigorously introduce the notion of *operator product expansion* (OPE) of fields.

Indeed, the locality property can be rephrased by requiring that for any $a,b,c \in \mathcal{V}$,

$$\begin{gathered} Y(a,x)Y(b,y)c \in \mathcal{V}((x))((y)), \qquad Y(b,y)Y(a,x)c \in \mathcal{V}((y))((x)), \\ Y(Y(a,x-y)b,y)c \in \mathcal{V}((y))((x-y)) \end{gathered} \tag{3.35}$$

are the expansions in their domains of the *same* function in $\mathcal{V}[[x,y]][x^{-1},y^{-1},(x-y)^{-1}]$. In particular, this implies that

$$Y(a,x)Y(b,y)c = \sum_{n\in\mathbb{Z}} Y(a_{(n)}b,y)(x-y)^{-n-1}c, \tag{3.36}$$

where each side has to be expanded as above.

Moreover, we have the following fundamental result, due to Kac (cfr. [**Kac01**, Theorem 2.3]), that given two state-field maps, we have

$$[Y(a,x),Y(b,y)] = \sum_{k=0}^{N-1} \frac{C_k(y)}{k!} \partial_y^k \delta(x-y). \tag{3.37}$$

for some fields $C_k(y)$, $k = 0,\dots,N-1$. In particular, this implies that

$$Y(a,x)Y(b,y) = \sum_{k=0}^{N-1} \iota_{x,y} \frac{1}{(x-y)^{k+1}} C_k(y) + \mathopen{:}Y(a,x)Y(b,y)\mathclose{:}, \tag{3.38}$$

$$Y(b,y)Y(a,x) = \sum_{k=0}^{N-1} \iota_{y,x} \frac{1}{(x-y)^{k+1}} C_k(y) + \mathopen{:}Y(a,x)Y(b,y)\mathclose{:}. \tag{3.39}$$

Putting together eqs. (3.36) and (3.38), we obtain the operator product expansion formula for a vertex algebra,

$$Y(a,x)Y(b,y) = \sum_{k=0}^{N-1} \iota_{x,y} \frac{Y(a_{(k)}b,y)}{(x-y)^{k+1}} + \mathopen{:}Y(a,x)Y(b,y)\mathclose{:} \tag{3.40}$$

and similarly for eq. (3.39). In particular, the singular behaviour of this product is completely determined by the non-negative k-products only. The terms $\mathopen{:}Y(a,x)Y(b,y)\mathclose{:}$ are regular on the diagonal $x = y$, and are sometimes just referred to as the "regular terms".

3.2.1.2. *Conformal vertex algebras.* We now recall a special class of vertex algebras that play an important role in conformal field theories, called *conformal vertex algebras.*

DEFINITION 3.2.2. *A vertex algebra $\mathcal{V} = \sum_{k\in\mathbb{Z}} V^{(k)}$ is called conformal of central charge c, if there exists a non-zero vector $\omega \in \mathcal{V}^{(2)}$, whose corresponding vertex operator is*

$$Y(\omega, x) := \omega(x) = \sum_{n\in\mathbb{Z}} L_n x^{-n-2}. \tag{3.41}$$

In particular, the modes L_n must generate a copy of the Virasoro algebra with central charge c,

$$[L_m, L_n] = (m-n)L_{m+n} + c\frac{m^3-m}{12}\delta_{m+n,0}, \tag{3.42}$$

and $L_{-1} = T$ and $L_0\Big|_{\mathcal{V}^{(n)}} = n\,\mathrm{id}_{\mathcal{V}}$.

Alternatively, this implies that the OPE of the field associated with the conformal vector with itself is given by

$$\omega(x)\omega(y) = \frac{c/12}{(x-y)^4} + \frac{2\omega(y)}{(x-y)^2} + \frac{\partial_y\omega(y)}{x-y} + \text{regular terms}. \tag{3.43}$$

3.2.2. An example: the Heisenberg vertex algebra. We now review an example of vertex algebra, which will appear in the construction of Wakimoto modules in section 3.4.

The Heisenberg vertex algebra is an example of a vertex algebra associated with an infinite dimensional Lie algebra. First, define the Heisenberg Lie algebra as the central extension of the commutative algebra of formal Laurent series,

$$0 \longrightarrow \mathbb{C}\mathbf{1} \longrightarrow \mathsf{H} \longrightarrow \mathbb{C}((t)) \longrightarrow 0, \tag{3.44}$$

where for any $f, g \in \mathbb{C}((t))$ the cocycle is defined as follows

$$\omega(f,g) = -\operatorname{res}_{t=0} f dg. \tag{3.45}$$

This algebra has a topological basis given by $b_n = t^n$, $n \in \mathbb{Z}$ with central extension given by $\mathbf{1}$. Explicitly, the cocycle reads

$$\omega(b_m, b_n) = -\operatorname{res}_{t=0} t^m dt^n = -\frac{1}{2\pi i}\oint_{t=0} nt^{m+n-1}dt = -n\delta_{m+n,0}. \tag{3.46}$$

Therefore, for any $m, n \in \mathbb{Z}$ the commutation relations are given by

$$[b_m, b_n] = m\delta_{m+n,0}\mathbf{1}, \qquad [\mathbf{1}, b_m] = 0. \tag{3.47}$$

We can consider a formal completion $\widetilde{U}(\mathsf{H})$, whose elements are possibly infinite sums $\sum_{k\geq 0} b_k$, where $b_k \in U(\mathsf{H})$, which truncate to finite ones when working modulo the left ideals generated by $t^N\mathbb{C}[t]$, $N \in \mathbb{Z}$. Any representation $\mathcal{V}$ of H is automatically a representation of $\widetilde{U}(\mathsf{H})$ if we require *smoothness*, *i.e.* for any $\mathsf{v} \in \mathcal{V}$, $b_N\mathsf{v} = 0$ for some $N \in \mathbb{Z}$ big enough. The quotient of $\widetilde{U}(\mathsf{H})$ by the two-sided ideal generated by $(\mathbf{1} - 1)$, where we identify the action of the central element with the unit in $\widetilde{U}(\mathsf{H})$, is called the *Weyl algebra* $\mathrm{Weyl}(\mathsf{H})$ or more concisely $\widetilde{\mathsf{H}}$.

Let $\widetilde{U}(\mathsf{H})_+$ be the subalgebra generated by non-negative modes $b_j \in \widetilde{U}(\mathsf{H})$. We introduce the one-dimensional representation defined by imposing that $b_j \mathsf{v}_0 = 0$ for all $j \in \mathbb{Z}_{\geq 0}$ and where we identify $\mathbf{1}$ with $k \in \mathbb{C}$, called the *level*. The *Fock representation* at level k is defined as the induced module

$$\pi_k = \widetilde{U}(\mathsf{H}) \otimes_{\widetilde{U}(\mathsf{H})_+} \mathbb{C}\mathsf{v}_0 \simeq_{\mathbb{C}} \widetilde{U}(\mathsf{H})_- \mathsf{v}_0. \tag{3.48}$$

where $\widetilde{U}(\mathsf{H})_-$ is generated by strictly negative modes b_{-j}, $j \in \mathbb{Z}_{>0}$. For clear reasons, the operators $b_{j\geq 0}$ are called "annihilation operators" while $b_{j<0}$ "creation operators". This vector space has a natural gradation $\pi_k = \bigoplus_{n\geq 0} \pi_k[n]$, where the state $b_{k_1} \cdots b_{k_n} \mathsf{v}_0$ has degree $-\sum_{i=1}^n k_i$. Here we are using the square bracket notation to denote the grading. It can be graphically depicted as in fig. 3.1.

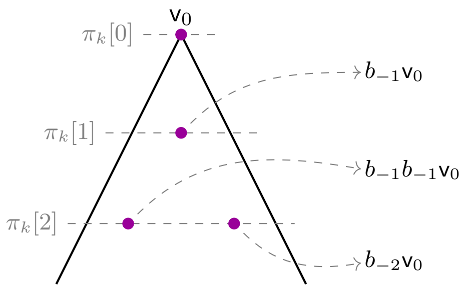


FIGURE 3.1. Graphical representation of the vector space $\pi_k = \sum_{n\geq 0} \pi_k[n]$.

We can introduce the translation map $T \in \operatorname{End} \mathcal{V}$, whose action on the generators is $[T, b_k] = -k b_{k-1}$ and $T\mathsf{v}_0 = 0$. This automatically satisfies axiom (*iii*).

We now introduce the state-field map. We assign $Y(\mathsf{v}_0, x) = \mathrm{id}_{\pi_k}$, so that axiom (*i*) is satisfied. We define

$$Y(b_{-1}\mathsf{v}_0, x) := b(x) = \sum_{k\in\mathbb{Z}} b_k x^{-k-1}. \tag{3.49}$$

which satisfies the creation axiom (*ii*). By acting on this state with the translation map, we find

$$Y(b_{-k}\mathsf{v}_0, x) = \frac{1}{(k-1)!} \partial_x^{k-1} b(x). \tag{3.50}$$

For arbitrary states, one has to introduce the normal ordering of fields which allows to avoid divergent sums (see e.g. [**FBZ04**, Section 2.2.1]). The final general formula is

$$Y(b_{-k_1} b_{-k_2} \cdots b_{-k_n} \mathsf{v}_0, x) = \frac{1}{(k_1-1)! \cdots (k_n-1)!} \, {}^{\circ}_{\circ}\, \partial_x^{k_1-1} b(z) \cdots \partial_x^{k_n-1} b(z) \,{}^{\circ}_{\circ}\, . \tag{3.51}$$

The only remaining axiom that has to be checked is locality. Explicitly, one has

$$\begin{aligned} [b(x), b(y)] &= \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} [b_m, b_n] x^{-m-1} y^{-n-1} \\ &= \sum_{m\in\mathbb{Z}} [b_m, b_{-m}] x^{-m-1} y^{m-1} = k \sum_{m\in\mathbb{Z}} m x^{-m-1} y^{m-1} = k \partial_y \delta(x-y) \end{aligned} \tag{3.52}$$

where in the last step we have used properties of the δ distribution

$$\delta(x-y)=\sum_{m\in\mathbb{Z}}x^{-m-1}y^m. \tag{3.53}$$

In particular, we have the following property

$$(x-y)^{k+1}\partial_y^k\delta(x-y)=0, \qquad \text{for any } k\in\mathbb{Z}_{>0}. \tag{3.54}$$

This means that by multiplying eq. (3.52) by $(x-y)^2$ we obtain

$$(x-y)^2[b(x),b(y)]=0. \tag{3.55}$$

Therefore, we can conclude that π_k has the structure of a vertex algebra. The operator product expansion of two fields is given by

$$b(x)b(y)=\frac{k}{(x-y)^2}+\text{regular terms}. \tag{3.56}$$

These vertex algebras admit a family of conformal vectors parametrised by $\lambda\in\mathbb{C}$,

$$\omega=\left(\frac{1}{2}b_{-1}b_{-1}+\lambda b_{-2}\right)\mathsf{v}_0\in\pi_k[2]. \tag{3.57}$$

The central charge of the corresponding Virasoro algebra is $c=1-12\lambda^2$.

A special case is given by π_0, which has the structure of a *commutative* vertex algebra. Indeed, setting $k=0$ in eq. (3.52), locality is always satisfied, *i.e.* formula (3.28) holds for any N,

$$[b(x),b(y)]=0. \tag{3.58}$$

More generally, for a commutative vertex algebra, using the second part of axiom (*ii*) we find that

$$Y(a,x)b=Y(a,x)Y(b,y)\mathsf{v}_0\big|_{y=0}=Y(b,y)Y(a,x)\mathsf{v}_0\big|_{y=0}. \tag{3.59}$$

Using now the first part of axiom (*ii*), we know that $Y(a,x)\mathsf{v}_0$ has only positive powers in x. This implies that $Y(a,x)b\in\pi_0[[x]]$ for all $a,b\in\pi_0$. Conversely, if the fields of a vertex algebra only have positive powers, it follows that $[Y(a,x)Y(b,y)]=0$. This gives an alternative definition of commutative vertex algebra, as the vertex algebra whose fields only have positive powers in the formal variable x, *i.e.* they are regular at $x=0$.

3.2.3. Another example: Kac-Moody vertex algebras. Consider a finite-type simple Lie algebra $\mathring{\mathfrak{g}}$. The affine Kac-Moody algebra $\mathfrak{g}$ is the central extension by the one-dimensional centre $\mathbb{C}\mathsf{k}$ of the corresponding loop algebra $L\mathring{\mathfrak{g}}=\mathring{\mathfrak{g}}\otimes\mathbb{C}((t))$. As a vector space $\mathfrak{g}\simeq_{\mathbb{C}}L\mathring{\mathfrak{g}}\oplus\mathbb{C}\mathsf{k}$. Introducing a basis $I^a_n=I^a\otimes t^n$, $a=1,\dots,\dim\mathring{\mathfrak{g}}$, $n\in\mathbb{Z}$, we have the following commutation relations

$$[I^a_m,I^b_n]=[I^a,I^b]_{n+m}-n\mathsf{k}\delta_{m+n,0}\kappa(I^a,I^b), \tag{3.60}$$

where κ is the non-degenerate symmetric invariant bilinear form.

We define the one-dimensional representation $\mathbb{C}\mathsf{v}_k$ where $\mathring{\mathfrak{g}}\otimes\mathbb{C}[[t]]\mathsf{v}_k = 0$ and $\mathsf{k}\mathsf{v}_k = k\mathsf{v}_k$, where $k \in \mathbb{C}$ is called the *level* of the representation. The vacuum Verma module is the induced module

$$\mathbb{V}_0^{\mathring{\mathfrak{g}},k} = U(\mathfrak{g}) \otimes_{U(\mathring{\mathfrak{g}}\otimes\mathbb{C}[[t]]\oplus\mathbb{C}\mathsf{k})} \mathbb{C}\mathsf{v}_k. \tag{3.61}$$

As a vector space, this is isomorphic to $U(\mathring{\mathfrak{g}}\otimes t^{-1}\mathbb{C}[t^{-1}])\mathsf{v}_k$. It has a natural depth gradation, that in the following we will denote as follows

$$\mathbb{V}_0^{\mathring{\mathfrak{g}},k} = \bigoplus_{k\geq 0} \mathbb{V}_0^{\mathring{\mathfrak{g}},k}[k]. \tag{3.62}$$

By construction, $\mathbb{V}_0^{\mathring{\mathfrak{g}},k}[0] \simeq \mathbb{C}$ and $\mathbb{V}_0^{\mathring{\mathfrak{g}},k}[1] \simeq \mathring{\mathfrak{g}}$.

One can introduce the translation map $[T, I_n^a] = -nI_{n-1}^a$ and $T\mathsf{v}_k = 0$, as well as the fields

$$Y(\mathsf{v}_k, x) = \mathrm{id}_{\mathbb{V}_0^{\mathring{\mathfrak{g}},k}} \tag{3.63}$$

$$Y(I_{-1}^a\mathsf{v}_k, x) := I^a(x) = \sum_{n\in\mathbb{Z}} I_n^a x^{-n-1} \tag{3.64}$$

$$Y(I_{-k_1}^{a_1}\cdots I_{-k_n}^{a_n}\mathsf{v}_k, x) = \frac{1}{(k_1-1)!\cdots(k_n-1)!} \,\mathring{\circ}\, \partial_x^{k_1-1}I^{a_1}(z)\cdots\partial_x^{k_n-1}I^{a_n}(z) \,\mathring{\circ}\,. \tag{3.65}$$

This space has the structure of a vertex algebra [**FBZ04**, Theorem 2.4.5]. The OPE of this vertex algebra is

$$I^a(x)I^b(y) = \frac{k\kappa(I^a, I^b)}{(x-y)^2} + \frac{[I^a, I^b](y)}{x-y} + \text{regular terms}. \tag{3.66}$$

This class of vertex algebras admits a natural conformal vector when $k \neq -h^\vee$, *i.e.* away from the critical level,

$$\omega = \frac{1}{(k+h^\vee)}\mathsf{S} \tag{3.67}$$

where $\mathfrak{h}^\vee$ is the dual Coxeter number of $\mathring{\mathfrak{g}}$ and state S is called the *Segal-Sugawara vector*,

$$\mathsf{S} = \frac{1}{2}\kappa_{ab}I_{-1}^a I_{-1}^b\mathsf{v}_k. \tag{3.68}$$

The conformal vector ω defines a copy of the Virasoro algebra with central charge

$$c_k = \frac{k\dim\mathring{\mathfrak{g}}}{k+h^\vee}. \tag{3.69}$$

3.2.4. Vertex Lie algebras. We now proceed to define vertex Lie algebras, which will be the main object of interest in chapter 5.

DEFINITION 3.2.3. *Given a graded vector space $\mathcal{L} = \sum_{k\in\mathbb{Z}}\mathcal{L}^{(k)}$ over a field $\mathbb{K}$, a vector $|0\rangle \in \mathcal{L}$ called the* vacuum, *a map $T \in \mathrm{End}(\mathcal{L})$ called* translation, *and a map $Y_- : \mathcal{L} \to \mathrm{Hom}(\mathcal{L}, x^{-1}\mathcal{L}[[x^{-1}]])$, $a \mapsto Y(a,x) := \sum_{k\geq 0} a_{(k)}x^{-k-1}$, with $a_{(k)} \in \mathrm{End}(\mathcal{L})$ of degree* $\deg(a) - k - 1$, i.e. $a_{(k)}\mathcal{L}^{(n)} \subset \mathcal{L}^{(n+\deg(a)-k-1)}$, *a* vertex Lie algebra *is the quadruple $(\mathcal{L}, |0\rangle, T, Y_-(\bullet, x))$ satisfying the following axioms*

i) $a_{(n)}|0\rangle = 0$ for n large enough;

ii) Translation axiom: $(Ta)_{(n)}b = -na_{(n-1)}b$*;*

iii) Skew-symmetry axioms: $a_{(n)}b = -\sum_{k\geq 0} \frac{(-1)^{n+k}}{k!} T^k(b_{(n+k)}a)$*;*

iv) Borcherds' identities: for all $a, b \in \mathcal{L}$*,*

$$\sum_{j=0}^{\infty} \binom{m}{j} (a_{(j)}b)_{(m+n-j)}c = a_{(m)}b_{(n)}c - b_{(n)}a_{(m)}c \tag{3.70}$$

It is clear that the polar part of a vertex algebra, obtained by forgetting about all terms with positive powers x^n, $n \geq 0$, in the state-field map $Y(\bullet, x)$, gives rise to a vertex Lie algebra.

To any vertex Lie algebra, one can associate a Lie algebra which is the Lie algebra of its modes, namely

$$\mathrm{Lie}(\mathcal{L}) = \mathcal{L} \otimes \mathbb{C}((t))/\,\mathrm{Im}(T \otimes 1 + \mathrm{id} \otimes \partial_t). \tag{3.71}$$

with commutation relations given by

$$[A_{[m]}, B_{[n]}] = \sum_{k\geq 0} \binom{m}{k} (A_{(k)}B)_{[m+n-k]}, \tag{3.72}$$

where by $A_{[n]}$ we identify the image of $A \in \mathcal{L}$ in $\mathrm{Lie}(\mathcal{L})$. It is also possible to define the left-adjoint functor as the one sending a vertex algebra to its polar part (for more detail cfr. [**FBZ04**]). It consists of defining the *universal enveloping vertex algebra*

$$\mathbb{V}(\mathcal{L}) := U(\mathrm{Lie}(\mathcal{L})) \otimes_{U(\mathrm{Lie}(\mathcal{L}))_+} \mathbb{C}, \tag{3.73}$$

which has the structure of a vertex algebra.

3.3. Local higher Hamiltonians and coinvariants

In this section, we are going to recall the coinvariant construction through which it is possible to identify the higher Hamiltonians with the space of singular vectors of the vacuum Verma module at critical level. This section is mainly a summary of the results from the first part of [**FFR94**].

3.3.1. Local action. Consider the affine Lie algebra $\mathfrak{g}$ associated with $\mathring{\mathfrak{g}}$. It can be realised as the one-dimensional central extension $\mathbb{C}\mathsf{k}$ of the formal loop algebra $\mathring{\mathfrak{g}} \otimes \mathbb{C}((t))$,

$$\mathfrak{g} \simeq_{\mathbb{C}} \mathring{\mathfrak{g}} \otimes \mathbb{C}((t)) \oplus \mathbb{C}\mathsf{k}. \tag{3.74}$$

As a vector space, $\mathfrak{g} =_{\mathbb{C}} \mathfrak{g}_- \oplus \mathfrak{g}_+ \oplus \mathbb{C}\mathsf{k}$, where $\mathfrak{g}_- = \mathring{\mathfrak{g}} \otimes t^{-1}\mathbb{C}[t^{-1}]$ is the *polar*, or singular, part and $\mathfrak{g}_+ = \mathfrak{g} \otimes \mathbb{C}[[t]]$ is the *positive*, or regular, part.

Consider the finite set of the $N \in \mathbb{Z}_{\geq 1}$ sites on the model $\{z_1, \dots, z_N\}$ on the Riemann sphere $\mathbb{CP}^1$ with global coordinate t, such that for any neighbourhood of z_i we have the local coordinate[1] $t - z_i$. To each of these points, we can define a $\mathring{\mathfrak{g}}$-module M_{λ_i} as we did in the previous section (cfr. eq. (3.3)), and define $M_{\boldsymbol{\lambda}}$ as in eq. (3.4).

For each site i, we can introduce a copy of the formal loop algebra in the local coordinate $(t - z_i)$, $\mathring{\mathfrak{g}}_{z_i} := \mathring{\mathfrak{g}} \otimes \mathbb{C}((t - z_i))$ and its central extension $\mathring{\mathfrak{g}} \otimes \mathbb{C}((t - z_i)) \oplus \mathbb{C}\mathsf{k}_i$. This can be thought as the *local algebra of functions* attached to the point z_i. For any two functions $f, g \in \mathring{\mathfrak{g}}_{z_i}$, the commutation relations are

$$[f + \lambda \mathsf{k}_i, g + \mu \mathsf{k}_i]_{\mathring{\mathfrak{g}}_{z_i} \oplus \mathbb{C}\mathsf{k}_i} = [f, g]_{\mathring{\mathfrak{g}}_{z_i}} + \omega(f, g)\mathsf{k}_i, \tag{3.75}$$

for some complex coefficients λ, μ. The cocycle ω is defined as follows

$$\omega_i(f, g) = \frac{1}{2\pi i} \int_{t - z_i} f dg = \mathrm{res}_{t - z_i} f dg. \tag{3.76}$$

The module M_{λ_i} can be regarded as a module over this algebra, denoted by $M^{k_i}_{\lambda_i}$, by declaring

$$\mathring{\mathfrak{g}} \otimes (t - z_i)\mathbb{C}[[t - z_i]] M^k_{\lambda} = 0, \tag{3.77}$$

$$\mathsf{k}_i M^{k_i}_{\lambda_i} = k_i M^{k_i}_{\lambda_i}, \tag{3.78}$$

where $k_i \in \mathbb{C}$ is the level of the representation. One then defines the Verma module as the induced module

$$\mathbb{M}^{k_i}_{\lambda_i} = U(\mathring{\mathfrak{g}}_{z_i} \oplus \mathbb{C}\mathsf{k}_i) \otimes_{U(\mathring{\mathfrak{g}} \otimes \mathbb{C}[[t - z_i]]) \oplus \mathbb{C}\mathsf{k}_i)} M^{k_i}_{\lambda_i}. \tag{3.79}$$

One can consider the direct sum of these algebras $\mathring{\mathfrak{g}}_N := \bigoplus_{i=1}^N \mathring{\mathfrak{g}}_{z_i}$ and extend it by $\bigoplus_{i=1}^N \mathbb{C}\mathsf{k}_i$. We introduce the following quotient,

$$\left(\mathring{\mathfrak{g}}_N \oplus \bigoplus_{i=1}^N \mathbb{C}\mathsf{k}_i \right) \Big/ \langle \mathsf{k}_i - \mathsf{k}_j \rangle_{i,j=1,\ldots,N}, \tag{3.80}$$

where we identify all the central extensions. Denoting this single central extension by K, we have

$$[f + \lambda \mathsf{K}, g + \mu \mathsf{K}]_{\mathring{\mathfrak{g}}_N \oplus \mathbb{C}\mathsf{K}} = [f, g]_{\mathring{\mathfrak{g}}_N} + \omega(f, g)\mathsf{K}, \tag{3.81}$$

This algebra naturally acts on the tensor products of $\mathring{\mathfrak{g}}_{z_i}$-modules defined in eq. (3.79),

$$\mathbb{M}^k_{\boldsymbol{\lambda}} = \mathbb{M}^k_{\lambda_1} \otimes \cdots \otimes \mathbb{M}^k_{\lambda_N}, \tag{3.82}$$

where we assigned a copy $\mathbb{M}^k_{\lambda_i}$ at each point.

[1]The point at infinity could be chosen as one of the marked points. In this case, the local coordinate is t^{-1} and we attach a copy of $\mathring{\mathfrak{g}} \otimes \mathbb{C}((t^{-1}))$ to it. In what follows, to keep the notation lighter, we will not specify this every time.

3.3.2. Global action. We now proceed to define the algebra of *global* functions on the Riemann sphere, with poles at most at the marked points. This will provide a good choice of complement to the local algebra of "positive modes", and it will be of fundamental importance in the coinvariant construction.

We denote by $\mathbb{C}_{\boldsymbol{z}}(t)$, the space of rational function in t with poles at most at the marked points $\{z_1, \dots, z_N\}$. It has the structure of an algebra over $\mathbb{C}$. We introduce the space of global functions vanishing at infinity,

$$\mathbb{C}^{\infty}_{\boldsymbol{z}}(t) := \{f(t) \in \mathbb{C}_{\boldsymbol{z}}(t) \text{ s.t. } f \text{ vanishes at infinity}\}\,. \tag{3.83}$$

We have a global-to-local embedding, defined by taking the Laurent expansion of the global function at each point

$$\begin{aligned} \iota : \mathbb{C}^{\infty}_{\boldsymbol{z}}(t) &\longhookrightarrow \oplus_{i=1}^{N} \mathbb{C}((t-z_i)), \\ f(t) &\longmapsto (\iota_{t-z_i} f(t), \dots, \iota_{t-z_N} f(t)). \end{aligned} \tag{3.84}$$

Tensoring these spaces with the Lie algebra $\mathring{\mathfrak{g}}$, one defines the following map

$$\mathrm{id} \otimes \iota : \mathring{\mathfrak{g}} \otimes \mathbb{C}^{\infty}_{\boldsymbol{z}}(t) \longhookrightarrow \bigoplus_{i=1}^{N} \mathring{\mathfrak{g}} \otimes \mathbb{C}((t-z_i)). \tag{3.85}$$

We have the following important lemma,

Lemma 3.3.1. *The cocycle in eq.* (3.81) *vanishes on the image of the map (3.85). Therefore, the map can be lifted to an embedding*

$$\mathring{\mathfrak{g}} \otimes \mathbb{C}^{\infty}_{\boldsymbol{z}}(t) \longhookrightarrow \mathring{\mathfrak{g}}_N \oplus_{\mathbb{C}} \mathbb{C}\mathsf{K}. \tag{3.86}$$

This follows from the fact that any collection of local functions obtained by Laurent-expanding a global one, have to satisfy the *strong residue theorem* (see e.g. [VY16]), *i.e.*

$$\sum_{i=1}^{N} \mathrm{res}_{t-z_i} f_i \iota_{t-z_i} g = 0, \qquad \text{for every } g \in \mathbb{C}^{\infty}_{\boldsymbol{z}}(t), \tag{3.87}$$

and as a consequence, the cocycle in eq. (3.81) vanishes.

Consider now the local function $f \in \mathbb{C}((t-z_i))$. By denoting by $f_- \in (t-z_i)\mathbb{C}[(t-z_i)^{-1}]$ its polar part, which in particular can be regarded as a function in $\mathbb{C}^{\infty}_{z_i}(t)$, we can always decompose $f = \iota_{t-z_i} f_- + (f - \iota_{t-z_i} f_-)$. The combination $f - \iota_{t-z_i} f_-$ is the Taylor part of the expansion, as all poles have been removed. This can be extended to a collection of local functions, obtaining the following isomorphism

$$\bigoplus_{i=1}^{N} \mathbb{C}((t-z_i)) \simeq_{\mathbb{C}} \iota(\mathbb{C}^{\infty}_{\boldsymbol{z}}(t)) \oplus \bigoplus_{i=1}^{N} \mathbb{C}[[t-z_i]]. \tag{3.88}$$

By considering the tensor product of this relation with $\mathring{\mathfrak{g}}$ and adding the central extension, we obtain

$$\mathring{\mathfrak{g}}_N \oplus \mathbb{C}\mathsf{K} \simeq_{\mathbb{C}} \mathring{\mathfrak{g}} \otimes \iota(\mathbb{C}^{\infty}_{\boldsymbol{z}}(t)) \oplus \left(\mathring{\mathfrak{g}} \otimes \bigoplus_{i=1}^{N} \mathbb{C}[[t-z_i]] \oplus \mathbb{C}\mathsf{K} \right). \tag{3.89}$$

3.3.3. The space of coinvariants. Consider the decomposition

$$\mathring{\mathfrak{g}}_N \oplus \mathbb{C}\mathsf{K} \simeq_{\mathbb{C}} \left(\bigoplus_{i=1}^{N} \mathring{\mathfrak{g}} \otimes \mathbb{C}[[t-z_i]] \oplus \mathbb{C}\mathsf{K} \right) \oplus \mathfrak{c} \tag{3.90}$$

where $\mathfrak{c}$ is a choice of *complement* of $\bigoplus_{i=1}^N \mathring{\mathfrak{g}} \otimes \mathbb{C}[[t-z_i]] \oplus \mathbb{C}\mathsf{K}$. For example, a possible simple choice of complement is the polar subalgebra $\bigoplus_{i=1}^N \mathring{\mathfrak{g}} \otimes (t-z_i)^{-1}\mathbb{C}[(t-z_i)^{-1}]$. A less obvious choice is provided by eq. (3.89).

We define the *space of* $\mathfrak{c}$*-coinvariants* as the quotient

$$\mathbb{M}^k_{\boldsymbol{\lambda}} \big/ \mathfrak{c} := \mathbb{M}^k_{\boldsymbol{\lambda}} \big/ (\mathfrak{c}\mathbb{M}^k_{\boldsymbol{\lambda}}). \tag{3.91}$$

We have the following fundamental result, which will be crucial in the next sections

Lemma 3.3.2. *There is an isomorphism of vector spaces,*

$$\mathbb{M}^k_{\boldsymbol{\lambda}} \big/ \mathfrak{c} \simeq_{\mathbb{C}} M_{\boldsymbol{\lambda}}. \tag{3.92}$$

This follows from the fact that the decomposition (3.90) implies a factorisation of the enveloping algebra $U(\mathring{\mathfrak{g}}_N \oplus \mathbb{C}\mathsf{K}) \simeq_{\mathbb{C}} U(\bigoplus_{i=1}^N \mathring{\mathfrak{g}} \otimes \mathbb{C}[[t-z_i]] \oplus \mathbb{C}\mathsf{K}) \otimes_{\mathbb{C}} U(\mathfrak{c})$. Therefore, as a $\mathfrak{c}$-module,

$$\mathbb{M}^k_{\boldsymbol{\lambda}} \simeq U(\mathfrak{c}) \otimes M^k_{\boldsymbol{\lambda}}, \tag{3.93}$$

from which eq. (3.92) follows, since $M^k_{\boldsymbol{\lambda}} \simeq_{\mathbb{C}} M_{\boldsymbol{\lambda}}$.

3.3.4. The swapping procedure. We can now proceed to describe the *swapping procedure* introduced in [**FFR94**].

First, we introduce one additional site, $z \in \mathbb{CP}^1$, distinct from the other sites of the model. There is a copy of the local Lie algebra $\mathring{\mathfrak{g}} \otimes \mathbb{C}((t-z)) \oplus \mathbb{C}\mathsf{K}$ in the local coordinate $t-z$. To this point, we attach the *vacuum Verma module at level* k, $\mathbb{V}_0^{\mathring{\mathfrak{g}},k}$, defined as follows. One starts by defining the $\mathring{\mathfrak{g}}$-module $\mathbb{C}\mathsf{v}_0$, on which the action of $\mathring{\mathfrak{g}}$ is trivial. It can be made into a $\mathring{\mathfrak{g}} \otimes \mathbb{C}[[t-z]] \oplus \mathbb{C}\mathsf{K}$ module by declaring

$$\mathring{\mathfrak{g}} \otimes \mathbb{C}[[t-z]]\mathsf{v}_0 = 0, \qquad \mathsf{K}\mathsf{v}_0 = k\mathsf{v}_0. \tag{3.94}$$

We finally construct the induced module,

$$\mathbb{V}_0^{\mathring{\mathfrak{g}},k} := U(\mathring{\mathfrak{g}} \otimes \mathbb{C}((t-z)) \oplus \mathbb{C}\mathsf{K}) \otimes_{U(\mathring{\mathfrak{g}} \otimes \mathbb{C}[[t-z]] \oplus \mathbb{C}\mathsf{K})} \mathbb{C}\mathsf{v}_0 \tag{3.95}$$

As vector spaces, we have the identification

$$\mathbb{V}_0^{\mathring{\mathfrak{g}},k} \simeq_{\mathbb{C}} U(\mathring{\mathfrak{g}} \otimes (t-z)^{-1}\mathbb{C}[(t-z)^{-1}])\mathsf{v}_0, \tag{3.96}$$

which means that $\mathbb{V}_0^{\mathring{\mathfrak{g}},k}$ is spanned by vectors

$$A_1[-n_1] \cdots A_\ell[-n_\ell]\mathsf{v}_0, \qquad A_i \in \mathring{\mathfrak{g}}, \quad i = 1, \ldots, \ell, \quad \ell \in \mathbb{Z}_{\geq 1}, \tag{3.97}$$

where we used the standard notation

$$A[n] := A \otimes (t-z)^n. \tag{3.98}$$

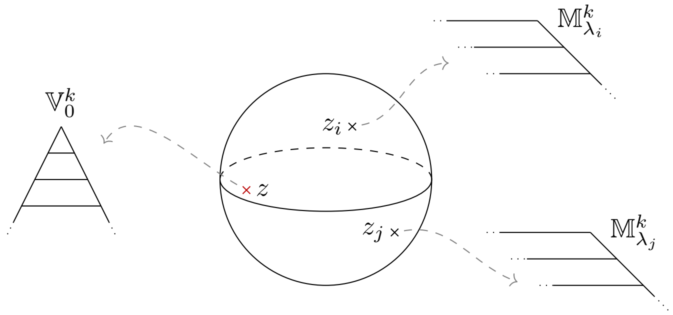


FIGURE 3.2. Graphical representation of the Riemann sphere, with modules $\mathbb{M}^k_{\lambda_i}$ attached to each marked point and a vacuum Verma module to the auxiliary point z.

We can consider the global algebra $\mathring{\mathfrak{g}} \otimes \mathbb{C}^\infty_{\boldsymbol{z},z}(t)$, which is the algebra of rational functions on $\mathbb{CP}^1$, with poles at most at the points $\{z_1, \ldots, z_N\} \cup \{z\}$ and vanish at infinity.

Using the result (3.92), we can construct the space of coinvariants with respect to this global algebra

$$(\mathbb{M}^k_{\boldsymbol{\lambda}} \otimes \mathbb{V}^{\mathring{\mathfrak{g}},k}_0)\Big/(\mathring{\mathfrak{g}} \otimes \mathbb{C}^\infty_{\boldsymbol{z},z}(t)) \simeq_{\mathbb{C}} M_{\boldsymbol{\lambda}} \otimes \mathbb{C}\mathsf{v}_0 \simeq_{\mathbb{C}} M_{\boldsymbol{\lambda}}. \tag{3.99}$$

This fact can be utilised to "swap" the data from one point to another. To clarify what we mean by this, consider the case with just one site at z_1. We attach the vacuum Verma module at z. We can consider the element

$$f = \frac{A}{(t-z)} \in \mathring{\mathfrak{g}} \otimes \mathbb{C}^\infty_{\boldsymbol{z},z}(t), \qquad A \in \mathring{\mathfrak{g}}, \tag{3.100}$$

which is a rational $\mathring{\mathfrak{g}}$-valued function with just a pole at $t = z$, and vanishes at infinity. We can embed this function at z_1 and z to obtain an element of the direct sum of algebras of local functions, as follows

$$\frac{A}{(t-z)} = \frac{A}{(z_1 - z)} \frac{1}{1 - \frac{t-z_1}{z-z_1}} \xhookrightarrow{\iota_{t-z_1}} -\sum_{k\geq 0} \frac{A(t-z_1)^k}{(z-z_1)^{k+1}} \in \mathring{\mathfrak{g}} \otimes \mathbb{C}((t-z_1)) \tag{3.101}$$

$$\frac{A}{(t-z)} \xhookrightarrow{\iota_{t-z}} \frac{A}{(t-z)} \in \mathring{\mathfrak{g}} \otimes \mathbb{C}((t-z)). \tag{3.102}$$

Now consider the state $\mathsf{m} \otimes \mathsf{v} \in \mathbb{M}^k_{\boldsymbol{\lambda}} \otimes \mathbb{V}^{\mathring{\mathfrak{g}},k}_0$. There is a well-defined action of f on this element, given by

$$f.(\mathsf{m} \otimes \mathsf{v}) = \left(-\sum_{k\geq 0} \frac{A(t-z_1)^k}{(z-z_1)^{k+1}} \mathsf{m} \right) \otimes \mathsf{v} + \mathsf{m} \otimes \left(\frac{A}{(t-z)} \mathsf{v} \right). \tag{3.103}$$

Now, by taking the quotient (3.99), the term on the left-hand side is zero by definition and we obtain the following identification of equivalence classes

$$\left[\left(\sum_{k\geq 0} \frac{A(t-z_1)^k}{(z-z_1)^{k+1}} \mathsf{m} \right) \otimes \mathsf{v} \right] = \left[\mathsf{m} \otimes \left(\frac{A}{(t-z)} \mathsf{v} \right) \right]. \tag{3.104}$$

As mentioned above, this can be interpreted as "swapping" the action of the local algebra from the auxiliary point z to the site at z_1, and vice versa.

More generally, consider the element

$$f = \frac{A}{(t-z)^n} = \frac{1}{(n-1)!}\frac{\partial^n}{\partial z^n}\frac{A}{t-z} \in \mathring{\mathfrak{g}} \otimes \mathbb{C}^\infty_{\boldsymbol{z},z}(t). \tag{3.105}$$

By expanding at every site, we have the embedding

$$f \xhookrightarrow{\iota_{t-z_i}} f_i := -\frac{1}{(n-1)!}\frac{\partial^{n-1}}{\partial z^{n-1}}\sum_{k=0}^{\infty}\frac{A(t-z_i)^k}{(z-z_i)^{k+1}} \in \mathring{\mathfrak{g}} \otimes \mathbb{C}((t-z_i)), \qquad i=1,\ldots,N \tag{3.106}$$

$$f \xhookrightarrow{\iota_{t-z}} f_0 := \frac{A}{(t-z)^n} \in \mathring{\mathfrak{g}} \otimes \mathbb{C}((t-z)). \tag{3.107}$$

Consider $\mathsf{m}\otimes\mathsf{v} \in \mathbb{M}^k_{\boldsymbol{\lambda}} \otimes \mathbb{V}^{\mathring{\mathfrak{g}},k}_0$, where $\mathsf{m} = \mathsf{m}_1 \otimes \cdots \otimes \mathsf{m}_N$. Since by definition of coinvariant space we have $[f.(\mathsf{m}\otimes\mathsf{v})] = 0$, where we mean $f \hookrightarrow (f_1,\ldots,f_N,f_0)$, we obtain the most general swapping formula:

$$\left[\left(\sum_{i=1}^{N}\frac{1}{(n-1)!}\frac{\partial^{n-1}}{\partial z^{n-1}}\sum_{k=0}^{\infty}\frac{A[k]}{(z-z_i)^{k+1}}\mathsf{m}\right)\otimes\mathsf{v}\right] = [\mathsf{m}\otimes A[-n]\mathsf{v}]\,, \tag{3.108}$$

where as in eq. (3.98), we used the notation $A[k] = A\otimes(t-z_i)^k$ when referring to $\mathbb{M}^k_{\boldsymbol{\lambda}}$ and $A[k] = A\otimes(t-z)^k$ when referring to $\mathbb{V}^{\mathring{\mathfrak{g}},k}_0$.

In particular, by regarding m as an element in $M_{\boldsymbol{\lambda}}$, only the zero loop mode gives contribution, cfr. relation (3.77), *i.e.*

$$\left[\sum_{i=1}^{N}\left(\frac{1}{(n-1)!}\frac{\partial^{n-1}}{\partial z^{n-1}}\frac{A}{z-z_i}\right)\mathsf{m}\otimes\mathsf{v}\right] = [\mathsf{m}\otimes A[-n]\mathsf{v}]\,. \tag{3.109}$$

Recall that a generic vector in $X \in \mathbb{V}^{\mathring{\mathfrak{g}},k}_0$ is expressed as in eq. (3.97). This is precisely the form on the right-hand side of the equation above. This means that by applying the swapping identity iteratively, one can always express $[\mathsf{w}\otimes X]$ as $[\mathsf{w}'\otimes\mathsf{v}_0]$.

This procedure defines an endomorphism for any $X \in \mathbb{V}^{\mathring{\mathfrak{g}},k}_0$

$$X(z) : M_{\boldsymbol{\lambda}} \longrightarrow M_{\boldsymbol{\lambda}}, \tag{3.110}$$

defined as follows

$$M_{\boldsymbol{\lambda}} \hookrightarrow \mathbb{M}^k_{\boldsymbol{\lambda}} \xrightarrow{\otimes X} \mathbb{M}^k_{\lambda_1} \otimes \mathbb{V}^{\mathring{\mathfrak{g}},k}_0 \twoheadrightarrow (\mathbb{M}^k_{\boldsymbol{\lambda}} \otimes \mathbb{V}^{\mathring{\mathfrak{g}},k}_0)\Big/(\mathring{\mathfrak{g}}\otimes\mathbb{C}^\infty_{\boldsymbol{z},z}(t)) \simeq_{\mathbb{C}} M_{\boldsymbol{\lambda}}. \tag{3.111}$$

More explicitly, we first regard the vector in $M_{\boldsymbol{\lambda}}$ as a vector in $\mathbb{M}^k_{\boldsymbol{\lambda}}$. Then, we consider the tensor product of this with the element X in the vacuum Verma module. After that, we employ the procedure just explained, to "remove" elements from $\mathbb{V}^{\mathring{\mathfrak{g}},k}_0$ by swapping them onto $\mathbb{M}^k_{\boldsymbol{\lambda}}$. Lastly, one takes the quotient. By definition,

$$[X(z)\mathsf{m}\otimes\mathsf{v}_0] = [\mathsf{m}\otimes X]. \tag{3.112}$$

3.3.5. Higher Hamiltonians. A vector $X \in \mathbb{V}_0^{\mathring{\mathfrak{g}},k}$ is *singular* if

$$A.X = 0, \qquad \text{for all } A \in \mathring{\mathfrak{g}} \otimes \mathbb{C}[t], \tag{3.113}$$

i.e. it behaves like a highest weight vector. The space of singular vectors is denoted with $\mathfrak{Z}(\mathfrak{g})$, where $\mathfrak{g}$ is the affine algebra defined in eq. (3.74). We have the following fundamental result,

Proposition 3.3.1. [**FFR94**] *Let $Z_1, Z_2 \in \mathfrak{Z}(\mathfrak{g})$. Then, for any pair of complex parameters $u, v \in \mathbb{CP}^1$, the corresponding linear operators on $M_{\boldsymbol{\lambda}}$, $Z_1(u)$ and $Z_2(v)$, commute.*

The idea behind the proof of this statement is that we can consider the space $\mathbb{M}_{\boldsymbol{\lambda}}^k \otimes \mathbb{V}_0^{\mathring{\mathfrak{g}},-h^\vee} \otimes \mathbb{V}_0^{\mathring{\mathfrak{g}},-h^\vee}$, where we attached two copies of the vacuum Verma module at the sites at u and v, respectively. Then, by the considerations made above, one can always swap the content from the vacuum Verma modules onto $\mathbb{M}_{\boldsymbol{\lambda}}^k$. There is only one small but crucial difference that can easily understood with an example. Consider the equivalence class $[\mathsf{m} \otimes Z_1 \otimes A_1[-n_1] \cdots A_\ell[-n_\ell]\mathsf{v}_0]$, where we explicitly wrote $Z_2 = A_1[-n_1] \cdots A_\ell[-n_\ell]\mathsf{v}_0$, for some $A_i[-n_i]$, $i = 1, \dots, \ell$, $n_i \in \mathbb{Z}_{\geq 1}$. We can now swap the leftmost factor $A_1[-n_1] = A_1 \otimes (t - v)^{-n_1}$ onto the other terms, as follows

$$\begin{aligned}
&[\mathsf{m} \otimes Z_1 \otimes A_1[-n_1] \cdots A_\ell[-n_\ell]\mathsf{v}_0] \\
&\quad = \left[\left(\frac{1}{(n-1)!}\frac{\partial^{n-1}}{\partial z^{n-1}}\sum_{k=0}^{\infty}\frac{A_1[k]}{(v-z_i)^{k+1}}\mathsf{m}\right) \otimes Z_1 \otimes A_2[-n_2] \cdots A_\ell[-n_\ell]\mathsf{v}_0\right] \\
&\qquad + \left[\mathsf{m} \otimes \left(\frac{1}{(n-1)!}\frac{\partial^{n-1}}{\partial z^{n-1}}\sum_{k=0}^{\infty}\frac{A_1[k]}{(v-u)^{k+1}}Z_1\right) \otimes A_2[-n_2] \cdots A_\ell[-n_\ell]\mathsf{v}_0\right].
\end{aligned} \tag{3.114}$$

This should clarify why Z_1 and Z_2 are needed to be singular vectors. Indeed, by definition the term in the last line is always identically zero for the property (3.113). Keeping this in mind, one can swap Z_1 first and then Z_2 or vice-versa,

$$[\mathsf{m} \otimes Z_1 \otimes Z_2] = [Z_1(u)\mathsf{m} \otimes \mathsf{v}_1 \otimes Z_2] = [Z_2(v)Z_1(u)\mathsf{m} \otimes \mathsf{v}_0 \otimes \mathsf{v}_0], \tag{3.115}$$

$$[\mathsf{m} \otimes Z_1 \otimes Z_2] = [Z_2(v)\mathsf{m} \otimes Z_1 \otimes \mathsf{v}_0] = [Z_1(u)Z_2(v)\mathsf{m} \otimes \mathsf{v}_0 \otimes \mathsf{v}_0]. \tag{3.116}$$

Since they define the same rational function $(Z_1, Z_2)(u, v)$, the result follows.

The space of singular vectors is known to be extremely rich when $k = -h^\vee$ [**FF92**]. It has the structure of a polynomial algebra in $\partial^n S^{(k+1)}$, where ∂ is a derivative operator of degree 1, $n \geq 0$ and with k running over the *finite* set of *exponents* of the Lie algebra $\mathring{\mathfrak{g}}$, see e.g. [**FFR94**, Proposition 3].

For any $X \in \mathbb{V}_0^{\mathring{\mathfrak{g}},-h^\vee}$, the function $X(u)$ depends rationally on u, with poles at the marked points. Its Laurent coefficients are valued in $U(\mathring{\mathfrak{g}})^{\otimes N}$, which is identified with the algebra of observables of the Gaudin model, cfr. eq. (3.6).

In particular, when the chosen vector is a singular vector, $Z \in \mathfrak{Z}(\mathfrak{g})$, the coefficients of the Laurent expansion form a subalgebra $\mathcal{Z}_{\boldsymbol{z}}(\mathring{\mathfrak{g}})$, containing the centre $Z(\mathring{\mathfrak{g}})$ as a subalgebra, of the algebra of observables, which is the celebrated *Gaudin/Bethe subalgebra* [**FFR94, MTV06**]. By construction, it is spanned by commuting elements which can be identified with the Hamiltonians of the Gaudin model.

3.3.6. The quadratic Hamiltonian. Consider a finite semi-simple Lie algebra $\mathring{\mathfrak{g}}$, with basis $\{I^a\}_{a=1,\dots,\dim\mathring{\mathfrak{g}}}$ and symmetric invariant bilinear form κ. There is a non-trivial singular vector,

$$\mathsf{S} = \frac{1}{2}\kappa_{ab} I^a[-1] I^b[-1] \mathsf{v}_0 \in \mathbb{V}_0^{\mathring{\mathfrak{g}},-h^\vee}[2], \tag{3.117}$$

which is the Segal-Sugawara vector introduced in eq. (3.68). This singular vector is the one with lowest degree, corresponding to the exponent $k = 1$. It can be defined for *any* Lie algebra $\mathring{\mathfrak{g}}$ and corresponds to the existence of the Gaudin Hamiltonians (3.8), as we will now show.

Introducing a state $\mathsf{m} = \mathsf{m}_1 \otimes \cdots \otimes \mathsf{m}_N \in M_{\boldsymbol{\lambda}} \hookrightarrow \mathbb{M}^k_{\boldsymbol{\lambda}}$, we can use the swapping procedure (3.109), as follows

$$\begin{aligned}\left[\mathsf{m} \otimes \left(\frac{1}{2}\kappa_{ab} I^a[-1] I^b[-1] \mathsf{v}_0\right)\right] &= \left[\left(\sum_{i=1}^N \frac{I^{a,(i)}}{z - z_i}\mathsf{m}\right) \otimes \left(\frac{1}{2}\kappa_{ab} I^b[-1]\mathsf{v}_0\right)\right] \\ &= \left[\left(\sum_{j=1}^N \sum_{i=1}^N \frac{1}{2}\kappa_{ab} \frac{I^{b,(j)} I^{a,(i)}}{(z - z_j)(z - z_i)}\mathsf{m}\right) \otimes \mathsf{v}_0\right]. \end{aligned} \tag{3.118}$$

We clearly see that the swapping procedure defines a map

$$S^{(2)}(z) = \sum_{j=1}^N \sum_{i=1}^N \frac{1}{2}\kappa_{ab} \frac{I^{b,(j)} I^{a,(i)}}{(z - z_j)(z - z_i)}, \tag{3.119}$$

which is precisely the expression of the quadratic Hamiltonian in eq. (3.9). By partial fraction decomposition, one can write it as in eq. (3.11). As argued above, the coefficients of this expansion are precisely commuting operators in the Gaudin/Bethe subalgebra: one is trivial, being the quadratic Casimir and the other are the Gaudin Hamiltonians (3.8).

3.4. Wakimoto modules and Bethe ansatz

In this section, we are going to introduce Wakimoto modules, as certain bosonic free field realisations of the Lie algebra $\mathring{\mathfrak{g}}$. This will allow us to construct the eigenvectors and eigenvalues of the Gaudin Hamiltonians and the Bethe equations will arise as certain consistency conditions from this procedure. To do this we will use the language of coinvariants introduced in the previous section. This section is mainly a summary of the main results from the second part of [**FFR94**].

3.4.1. Feigin-Frenkel-Wakimoto homomorphism. For any finite-dimensional simple Lie algebra $\mathring{\mathfrak{g}} = \mathring{\mathfrak{n}}_- \oplus \mathring{\mathfrak{h}} \oplus \mathring{\mathfrak{n}}_+$ there is a Lie algebra homomorphism

$$\rho : \mathring{\mathfrak{g}} \to \operatorname{Der} \mathcal{O}(\mathring{\mathfrak{n}}_+) \tag{3.120}$$

realising the Lie algebra as differential operators on the algebra of polynomial functions $\mathcal{O}(\mathring{\mathfrak{n}}_+)$ on the unipotent group $U \simeq N_+ \simeq \exp(\mathring{\mathfrak{n}}_+) \simeq \mathring{\mathfrak{n}}_+$. This follows from considering the right action of the group G, whose Lie algebra is $\mathring{\mathfrak{g}}$, on the open subset $U \subset B_- \backslash G$, called the *big cell*, which is isomorphic to N_+, which itself is isomorphic to $\mathring{\mathfrak{n}}_+$ via the exponential map. This gives rise to a left action of G on the space of functions on $B_- \backslash G$, which gives rise (infinitesimally) to a realisation of $\mathring{\mathfrak{g}}$ as vector fields on U. Choosing coordinates X^α on U, labelled by the positive roots, $\alpha \in \mathring{\Delta}_+$, then $\mathcal{O}(\mathring{\mathfrak{n}}_+) = \mathbb{C}[X^\alpha]_{\alpha \in \mathring{\Delta}_+}$. Explicitly,

$$E_\alpha \mapsto \sum_{\beta \in \mathring{\Delta}_+} P^\beta_\alpha(X) D_\beta, \quad H_\alpha \mapsto - \sum_{\beta \in \mathring{\Delta}_+} \beta(H_\alpha) X^\beta D_\beta, \quad F_\alpha \mapsto \sum_{\beta \in \mathring{\Delta}_+} Q^\beta_\alpha(X) D_\beta, \tag{3.121}$$

for $P, Q \in \mathbb{C}[X^\alpha]_{\alpha \in \mathring{\Delta}_+}$ such that $\deg P^\beta_\alpha = \beta - \alpha$ and $\deg Q^\beta_\alpha = \beta + \alpha$. For example, in $\mathfrak{sl}_2$ one finds the well-known realisation

$$E \mapsto D \qquad H \mapsto -2XD \qquad F \mapsto -X^2 D, \tag{3.122}$$

where E, F and H are the Chevalley-Serre generators and X, D are the generator of the Weyl algebra $\operatorname{Weyl}(\mathfrak{sl}_2)$ with commutation relations $[D, X] = 1$.

The main fact underpinning the Wakimoto construction is that this homomorphism can be promoted to a homomorphism of vertex algebras, as follows. First, consider the affine algebra

$$\mathfrak{g} \simeq_{\mathbb{C}} \mathring{\mathfrak{g}}[t, t^{-1}] \oplus \mathbb{C}\mathsf{k} \oplus \mathbb{C}\mathsf{d}, \tag{3.123}$$

where k is the central element and d is the derivation in the homogeneous gradation, *i.e.* $\mathsf{d} = t\partial_t$. It has $\mathfrak{g}_+ := \mathring{\mathfrak{g}} \otimes \mathbb{C}[t] \oplus \mathbb{C}\mathsf{k}$ as a subalgebra. We define $\mathbb{C} \left|0\right\rangle^k$ the one-dimensional representation of $\mathfrak{g}_+$ in which $\mathbb{C}[t]$ acts trivially and the central element acts as multiplication by $k \in \mathbb{C}$, called the *level* of the representation. The vacuum Verma module at level k is the induced module

$$\mathbb{V}^{\mathring{\mathfrak{g}},k}_0 = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_+)} \mathbb{C} \left|0\right\rangle^k .$$

This space has the structure of a vertex algebra (cfr. [**FBZ04**, theorem 2.4.5]).

On the other hand, one can define the Fock module for the $\beta\gamma$-system on $\mathring{\mathfrak{n}}_+$, $\mathsf{M}(\mathring{\mathfrak{n}}_+)$, as the induced module generated by the vector $\left|0\right\rangle$ for the Heisenberg algebra $\mathsf{H}(\mathring{\mathfrak{g}})$ generated by $\beta_\alpha[M]$, $\gamma^\alpha[M]$ and $\mathbf{1}$, with $\alpha \in \mathring{\Delta}_+$, $M \in \mathbb{Z}$, with the following relations

$$[\beta_\alpha[M], \beta_\beta[N]] = 0 \qquad [\gamma^\alpha[M], \gamma^\beta[N]] = 0 \tag{3.124}$$

$$[\beta_\alpha[M], \gamma^\beta[N]] = \delta_{M+N,0} \delta^\beta_\alpha \mathbf{1}. \tag{3.125}$$

Here we are using the notation (3.98) for the vertex algebra modes (this will be helpful in the next sections to distinguish it from the additional loop parameter). Recalling the natural grading in the vacuum Verma module, cfr. eq. (3.62), we can identify $\mathfrak{g} \simeq \mathbb{V}^{\mathring{\mathfrak{g}},k}_0[1]$,

$\mathcal{O}(\mathring{\mathfrak{n}}_+) \simeq \mathsf{M}(\mathring{\mathfrak{n}}_+)[0]$ and $\operatorname{Der}\mathcal{O}(\mathring{\mathfrak{n}}_+) \oplus \Omega_{\mathcal{O}(\mathring{\mathfrak{n}}_+)} \simeq \mathsf{M}(\mathring{\mathfrak{n}}_+)[1]$. One can construct the following map of vector spaces

$$\mathbb{V}_0^{\mathring{\mathfrak{g}},k}[\leq 1] \to \mathsf{M}(\mathring{\mathfrak{n}}_+)[\leq 1], \tag{3.126}$$

identifying the corresponding vacuum vectors and mapping

$$A[-1]\,|0\rangle \mapsto \sum_{\alpha\in\mathring{\Delta}_+} P_A^\alpha(\gamma[0])\beta_\alpha[-1]\,|0\rangle\,. \tag{3.127}$$

Crucially, this map is not a map of vertex algebras, as it does not preserve non-negative products.

This problem can be solved by adding extra terms, as shown in [**FF90, Fre07**]. Formally, there exists a linear map

$$\phi : \mathring{\mathfrak{g}} \to \Omega_{\mathcal{O}(\mathring{\mathfrak{n}}_+)}, \tag{3.128}$$

where $\Omega_{\mathcal{O}(\mathring{\mathfrak{n}}_+)}$ is the space of 1-forms, such that

$$\rho + \phi : \mathring{\mathfrak{g}} \to \operatorname{Der}\mathcal{O}(\mathring{\mathfrak{n}}_+) \oplus \Omega_{\mathcal{O}(\mathring{\mathfrak{n}}_+)} \tag{3.129}$$

can be lifted to a map of vertex algebras,

$$\theta : \mathbb{V}_0^{\mathring{\mathfrak{g}},-h^\vee} \to \mathsf{M}(\mathring{\mathfrak{n}}_+), \tag{3.130}$$

where, crucially, the level has to be set to the critical level $-h^\vee$, $h^\vee$ being the dual Coxeter number of $\mathring{\mathfrak{g}}$. In the case of $\mathfrak{sl}_2$, eq. (3.122) is lifted to the vertex algebra map

$$\begin{gathered} E[-1]\,|0\rangle \mapsto \beta[-1]\,|0\rangle \qquad H[-1]\,|0\rangle \mapsto -2\gamma[0]\beta[-1]\,|0\rangle \\ F[-1]\,|0\rangle \mapsto -\gamma[0]\gamma[0]\beta[-1]\,|0\rangle - 2\gamma[-1]\,|0\rangle\,. \end{gathered} \tag{3.131}$$

Analogously, one can also repeat the same construction to define a *right* action of G on the big cell. It will give rise to a map from $\mathring{\mathfrak{n}}_+$ to the space of derivation, defined as

$$E_\alpha \mapsto G_\alpha := \sum_{\beta\in\mathring{\Delta}_+} R_\alpha^\beta(X) D_\beta, \tag{3.132}$$

for certain polynomials $R \in \mathbb{C}[X^\alpha]_{\alpha\in\mathring{\Delta}}$ with $\deg R_\alpha^\beta = \beta - \alpha$. Similar expressions can be obtained for the other generators of $\mathring{\mathfrak{g}}$, however for our discussions below we just need a realisation of $\mathring{\mathfrak{n}}_+$. This action commutes with the left action introduced above. One can define the vacuum Verma module for this algebra in the same way as above.

3.4.2. Wakimoto modules. At this point, one can consider the loop algebra $L\mathring{\mathfrak{h}} = \mathring{\mathfrak{h}} \otimes \mathbb{C}((t))$ of the Cartan subalgebra $\mathring{\mathfrak{h}}$ of $\mathring{\mathfrak{g}}$. A basis for this algebra is $b_i[n] := b_i \otimes t^n$, where $\{b_i\}_{i=1,\dots,\dim\mathring{\mathfrak{h}}}$ is a basis for $\mathring{\mathfrak{h}}$. One can define the induced Fock module

$$\pi_0 = U(L\mathring{\mathfrak{h}}) \otimes_{U(\mathring{\mathfrak{h}}\otimes\mathbb{C}[[t]])} \mathbb{C}\,|0\rangle \simeq_{\mathbb{C}} U(\mathring{\mathfrak{h}} \otimes t^{-1}\mathbb{C}[t^{-1}])\,|0\rangle\,, \tag{3.133}$$

in which $b_i[k]\,|0\rangle = 0$ for any $k \geq 0$. As described in section 3.2.2, the module π_0 has the structure of a commutative vertex algebra, called the Heisenberg vertex algebra, whose fields are

$$b_i(x) = Y(b_i[-1]\,|0\rangle\,, x) = \sum_{k<0} b_i[k]x^{-k-1}. \tag{3.134}$$

By slightly modifying the construction of the map illustrated above, one obtains a new homomorphism of vertex algebras,

$$\theta_W : \mathbb{V}_0^{\mathfrak{g},-h^\vee} \to \mathsf{M}(\mathfrak{n}_+) \otimes \pi_0 =: \mathbb{W}_0, \tag{3.135}$$

which makes $\mathbb{W}_0$ into a smooth $\mathfrak{g}$-module. This map, which underlies the famous *Wakimoto construction* [**Wak86**], is a realisation of the Lie algebra $\mathfrak{g}$ in terms of the free fields $\beta(x), \gamma(x)$ and an additional boson $b(x)$. For example, the map (3.131) becomes

$$\begin{gathered} E[-1]\,|0\rangle \mapsto \beta[-1]\,|0\rangle \qquad H[-1]\,|0\rangle \mapsto -2\gamma[0]\beta[-1]\,|0\rangle + b[-1]\,|0\rangle \\ F[-1]\,|0\rangle \mapsto -\gamma[0]\gamma[0]\beta[-1]\,|0\rangle - 2\gamma[-1]\,|0\rangle - \gamma[0]b[-1]\,|0\rangle\,. \end{gathered} \tag{3.136}$$

There is another class of similar modules. Consider an $\mathring{\mathfrak{h}}^*$-valued Laurent series $\chi(t) \in \mathring{\mathfrak{h}}^* \otimes \mathbb{C}((t))$. Define $\mathbb{C}\,|0\rangle_\chi$ the one-dimensional $\mathring{\mathfrak{h}} \otimes \mathbb{C}((t))$ module, where

$$f\,|0\rangle_\chi = \frac{1}{2\pi i} \oint_{t=0} \chi(f)\,|0\rangle_\chi\,, \tag{3.137}$$

for any $\mathring{\mathfrak{h}}$-valued Laurent series f, where $\chi(f)$ denotes the pairing between $\mathring{\mathfrak{h}}$ and its dual. The *Wakimoto module* of highest weight χ is

$$W_\chi := \mathsf{M}(\mathfrak{n}_+) \otimes \mathbb{C}\,|0\rangle_\chi\,, \tag{3.138}$$

which is also a smooth $\mathfrak{g}$-module by the homomorphism (3.135). We denote the vacuum state as $\mathsf{w}_0 := |0\rangle \otimes |0\rangle_\chi$. A crucial feature that will be important later is that the zero-th graded component $W_\chi[0] \simeq_{\mathbb{C}} \mathbb{C}[\gamma^\alpha[0]]_{\alpha\in\mathring{\Delta}}\mathsf{w}_0$ is stable under the action of the Lie algebra $\mathring{\mathfrak{g}} \subset \mathfrak{g}$ and as a $\mathring{\mathfrak{g}}$-module it is isomorphic to the *contragredient Verma module*[2] $M^*_{\chi_0}$, with highest weight $\chi_0 := \mathrm{res}_{t=0}\,\chi(t)$.

Note that the main difference between $\mathbb{W}_0$ and W_χ is that the former is induced in both $H(\mathring{\mathfrak{g}})$ and $\mathring{\mathfrak{h}} \otimes \mathbb{C}((t))$, while the latter one only in the first tensor factor.

Before discussing how Bethe ansatz can be performed using the coinvariant language, we need the following

[2]The contragredient Verma module M^*_λ, with $\lambda \in \mathring{\mathfrak{h}}^*$ is obtained as the restricted dual space of the Verma module M_λ. It has the structure of a $\mathring{\mathfrak{g}}$ module, defined via Cartan anti-involution, *i.e.* for any $\mathsf{v} \in M_\lambda$ and $\omega \in M^*_\lambda$ one has $(X\omega)(\mathsf{v}) = \omega(\iota(X).\mathsf{v})$, for any $X \in \mathring{\mathfrak{g}}$, where ι sends E to F and vice versa, and leave H invariant.

LEMMA 3.4.1 ([**FFR94**]). *Let α_k be a simple root. Consider the Wakimoto module W_χ, whose highest weight is given by the following special form*

$$\chi(t) = -\frac{\alpha_k}{t} + \sum_{n\geq 0} \chi^{(n)} t^n, \qquad \chi^{(n)} \in \mathring{\mathfrak{h}}^*. \tag{3.139}$$

The state $G_{\alpha_k}[-1]\mathsf{w}_0 \in W_\chi$ is a singular vector, where G_{α_k} is defined in eq. (3.132) if and only if $(\alpha_k, \chi^{(0)}) = 0$.

3.4.3. Local and global actions, revisited. Consider the Riemann sphere $\mathbb{CP}^1$, with pairwise distinct marked points at $\boldsymbol{z} = \{z_1, \ldots, z_N\}$, $N \in \mathbb{Z}_{>0}$ representing the sites of the Gaudin model and additional auxiliary marked points at $\boldsymbol{w} = \{w_1, \ldots, w_M\}$, $M \in \mathbb{Z}_{\geq 0}$, the Bethe roots. Collectively, we introduce the tuple $\boldsymbol{x} := \{x_1, \ldots, x_p\} = \{z_i, \ldots, z_N, w_1, \ldots, w_M\}$.

In the same spirit as the previous section, denoted by t the global coordinate on the Riemann sphere, for each point we introduce the *local* Heisenberg algebra $\mathsf{H}_{x_i}(\mathring{\mathfrak{g}})$, which is isomorphic to a copy of $(\mathring{\mathfrak{n}}_+ \oplus \mathring{\mathfrak{n}}_+^*) \otimes \mathbb{C}((t-x_i))$, centrally extended by the one-dimensional centre $\mathbb{C}\mathbf{1}$, where we identify

$$\beta_\alpha[k] = E_\alpha \otimes (t-x_i)^k, \qquad \gamma^\alpha[k] = E^\alpha \otimes (t-x_i)^{k-1}, \tag{3.140}$$

where $\{E_\alpha\}_{\alpha\in\mathring{\Delta}}$ is a basis for $\mathring{\mathfrak{n}}_+$ and $\{E^\alpha\}_{\alpha\in\mathring{\Delta}}$ its dual [**FF91**]. We denote $\mathsf{H}_p(\mathring{\mathfrak{g}}) = \bigoplus_{x_i\in\boldsymbol{x}} \mathsf{H}_{x_i}(\mathfrak{g}) \oplus \mathbb{C}\mathbf{1}$, where all central extensions are identified.

We can also introduce the Lie algebra of *global* functions, $H_{\boldsymbol{x}}(t) = (\mathring{\mathfrak{n}}_+ \otimes \mathbb{C}^\infty_{\boldsymbol{x}}(t)) \oplus (\mathring{\mathfrak{n}}_+^* \otimes \mathbb{C}^\infty_{\boldsymbol{x}}(t))$, where $\mathbb{C}^\infty_{\boldsymbol{x}}(t)$ is the algebra of global rational function in t vanishing at infinity introduced in section 3.3.2. As above, there is an embedding of global into local since also in this case the cocycle is trivial, defined by Laurent-expanding at each point

$$\iota : \mathsf{H}_{\boldsymbol{x}}(t) \longhookrightarrow \mathsf{H}_p(\mathring{\mathfrak{g}}). \tag{3.141}$$

Similarly, one can define the local algebra

$$\mathring{\mathfrak{h}}_p := \bigoplus_{i=1}^{p} \mathring{\mathfrak{h}} \otimes \mathbb{C}((t-x_i)), \tag{3.142}$$

and the global algebra

$$\mathring{\mathfrak{h}}_{\boldsymbol{x}}(t) = \mathring{\mathfrak{h}} \otimes \mathbb{C}^\infty_{\boldsymbol{x}}(t). \tag{3.143}$$

Again, there is the embedding of global into local

$$\iota : \mathring{\mathfrak{h}}_{\boldsymbol{x}}(t) \longhookrightarrow \mathring{\mathfrak{h}}_p. \tag{3.144}$$

To the points $\boldsymbol{x}$ we can assign the Wakimoto modules W_{χ_i}, $i = 1, \ldots, p$. There is a well defined action of the local algebras $\mathsf{H}_p(\mathring{\mathfrak{g}}) \oplus \mathring{\mathfrak{h}}_p$ on the tensor product $\bigotimes_{i=1}^p W_{\chi_i}$, as well as an action of the global algebra $\mathfrak{H}_{\boldsymbol{x}}(t) := \mathsf{H}_{\boldsymbol{x}}(t) \oplus \mathring{\mathfrak{h}}_{\boldsymbol{x}}(t)$ obtained by the embeddings above.

At this point, we can construct the space of coinvariants with respect to the global action $\mathfrak{H}_{\boldsymbol{x}}(t)$. One finds

PROPOSITION 3.4.1 ([**FFR94**]). *The space of coinvariants* $\bigotimes_{i=1}^{p} W_{\chi_i} \big/ \mathfrak{H}_{\boldsymbol{x}}(t)$ *is one dimensional if and only if at each point* $x_i \in \boldsymbol{x}$*, the highest weight* χ_i *are the expansion of the same global function* $\chi(t) \in \mathring{\mathfrak{h}}^* \otimes \mathbb{C}_{\boldsymbol{x}}^{\infty}(t)$*. If not, the space of coinvariants is zero-dimensional.*

3.4.4. The Bethe equations. Consider the tuple $\boldsymbol{c} = \{c(1), \ldots, c(M)\}$ of *colours* of the Bethe roots, where $c(i)$ is a node of the Dynkin diagram $\mathring{I}$ of the Lie algebra $\mathring{\mathfrak{g}}$. Consider the special global function

$$\lambda(t) = \sum_{i=1}^{N} \frac{\lambda_i}{t - z_i} - \sum_{j=1}^{M} \frac{\alpha_{c(j)}}{t - w_j}, \qquad \lambda_i \in \mathring{\mathfrak{h}}^*. \tag{3.145}$$

We denote the expansion of this function at each point as

$$\iota(\lambda(t)) := (\chi_1, \ldots, \chi_N, \mu_1, \ldots, \mu_M). \tag{3.146}$$

In particular, if we expand $\lambda(z)$ around $t - w_k$, $k = 1, \ldots, M$, we find that the explicit expression of μ_k is given by

$$\mu_k = \iota_{t-w_k} \lambda(t) = -\frac{\alpha_{c(k)}}{t - w_k} + \left(\sum_{i=1}^{N} \frac{\lambda_i}{w_k - z_i} - \sum_{\substack{j=1\\ j\neq k}}^{M} \frac{\alpha_{c(j)}}{w_k - w_j} \right) + \mathcal{O}(t - w_k). \tag{3.147}$$

Now consider the tensor product of Wakimoto modules $W_{\boldsymbol{\lambda}} := \bigotimes_{i=1}^{N} W_{\chi_i} \bigotimes_{j=1}^{M} W_{\mu_j}$, attached to the points x_i, $i = 1, \ldots, p$, where the highest states are given by the expressions in eq. (3.146) and denote its vacuum state by $\mathsf{w}_0^{\otimes p} := \bigotimes_{i=1}^{p} \mathsf{w}_0^{(i)}$.

For each of the auxiliary marked points w_j, $j = 1, \ldots, M$, we can consider to have the states $G_{c(j)}[-1]\mathsf{w}_0^{(j)} \in W_{\mu_j}$, for some $c(j) \in \mathring{I}$, where $G_{c(j)}$ is defined in eq. (3.132). By Lemma 3.4.1, we can conclude that this vector is singular if and only if

$$\sum_{i=1}^{N} \frac{(\alpha_{c(j)}, \lambda_i)}{w_k - z_i} - \sum_{\substack{j=1\\ j\neq k}}^{M} \frac{(\alpha_{c(j)}, \alpha_{c(j)})}{w_k - w_j} = 0, \qquad j = 1, \ldots, M. \tag{3.148}$$

This is precisely the generalisation of the Bethe equations in eq. (3.24) for arbitrary $\mathring{\mathfrak{g}}$. In the next section, we will show how this condition naturally arises in the construction of eigenvectors and eigenvalues.

3.4.5. Schechtman-Varchenko vector. With the special choice of highest weight vectors (3.146), by proposition Proposition 3.4.1, we can conclude that the space of coinvariants $W_{\boldsymbol{\lambda}} \big/ \mathfrak{H}_{\boldsymbol{x}}(t)$ is one dimensional.

This implies that there exists a unique[3] $\mathfrak{H}_{\boldsymbol{x}}(t)$-invariant linear functional

$$\tau = W_{\boldsymbol{\lambda}} \longrightarrow \mathbb{C}. \tag{3.149}$$

[3]It is unique up to normalisation, which can be fixed by requiring that $\tau(\mathsf{w}_0^{\otimes p}) = 1$.

By "saturating" the last M slots of this map by inserting the states $G_{c(j)}[-1]\mathsf{w}_0^{(j)} \in W_{\mu_j}$ $j = 1, \dots, M$ we can define a map

$$\tau(\cdot, G_{c(1)}[-1]\mathsf{w}_0^{(1)}, \dots, G_{c(M)}[-1]\mathsf{w}_0^{(M)}) : \bigotimes_{i=1}^{N} W_{\chi_i} \longrightarrow \mathbb{C}. \tag{3.150}$$

As mentioned above, by restricting this functional to the zero-depth component, we have the identification $W_{\chi_i}[0] \simeq_{\mathring{\mathfrak{g}}} M^*_{\lambda_i}$, where $\lambda_i = \operatorname{res}_{t=z_i} \chi_i$,

$$\psi : \bigotimes_{i=1}^{N} M^*_{\lambda_i} \longrightarrow \mathbb{C}. \tag{3.151}$$

This map takes a product of dual vectors in the contragredient Verma modules and maps it to the field $\mathbb{C}$, hence it naturally defines a vector in $M_{\boldsymbol{\lambda}}$. This object is the Schechtman-Varchenko vector from [**SV91**]. We will now show that it is an eigenvector for the Gaudin Hamiltonians.

3.4.6. Eigenvalues and eigenvectors. We introduce an additional site at z, and to it we attach the $\mathsf{H}(\mathring{\mathfrak{g}}) \oplus \mathring{\mathfrak{h}} \otimes \mathbb{C}((t-z))$-module $\mathbb{W}_0$ with a local coordinate $t - z$. Recall that $\mathbb{W}_0 = \mathsf{M}(\mathring{\mathfrak{n}}_+) \otimes \pi_0$, where $\pi_0 \simeq_{\mathbb{C}} U(b_i[-n]) \left|0\right\rangle$, $n \in \mathbb{Z}_{>0}$ is a commutative vertex algebra with translation map $T \in \operatorname{End} \pi_0$ defined by $T b_i[-n] = n b_i[-n-1]$.

Let $\mathbb{C}^{\infty}_{\boldsymbol{x},z}(t)$ be the algebra of global functions which vanish at infinity and have poles at most at the marked points and z.

One has the homomorphism of differential algebras

$$r : (\pi_0, T) \longrightarrow (\mathbb{C}^{\infty}_{\boldsymbol{x},z}(t), \partial_z), \tag{3.152}$$

defined as follows

$$r(b_k[-1]\left|0\right\rangle) = \sum_{i=1}^{N} \frac{(\lambda_i, \alpha_k^\vee)}{z - z_i} - \sum_{j=1}^{M} \frac{(\alpha_{c(j)}, \alpha_k^\vee)}{z - w_j}. \tag{3.153}$$

Recalling the definition (3.145), we have

$$r(b_{k_1}[-n_1] \cdots b_{k_p}[-n_p]\left|0\right\rangle) = \prod_{\ell=1}^{p} \frac{1}{(n_\ell - 1)!} \frac{\partial^{n_\ell - 1}}{\partial z^{n_\ell - 1}} \lambda(z)(H_{k_\ell}). \tag{3.154}$$

At this point, we can consider a singular vector $Z \in \mathfrak{Z}(\mathfrak{g}) \subset \mathbb{V}_0^{\mathring{\mathfrak{g}}, -h^\vee}$, which from the previous section it has been identified with the space of the higher Gaudin Hamiltonians. Thanks to the homomorphism (3.135) we can map it to $\theta_W(Z) \in \mathbb{W}_0$. Constructing the space of coinvariants considering the extra site allows one to define a linear functional in a similar manner as in eq. (3.149), and one has

$$\tau(\omega, G_{c(1)}[-1]\mathsf{w}_0^{(1)}, \dots, G_{c(M)}[-1]\mathsf{w}_0^{(M)}, \theta_W(Z)). \tag{3.155}$$

for some $\omega \in M^*_{\boldsymbol{\lambda}}$. There are two ways to evaluate this quantity.

First, as we did in the previous section, we can "swap" from z to the other points. The result of the swapping onto the auxiliary points $G_{c(j)}[-1]\mathsf{w}_0^{(j)}$, $j = 1, \dots, M$, is trivial only if the vectors $G_{c(j)}[-1]\mathsf{w}_0^{(j)}$ are singular. Thanks to Lemma 3.4.1, we know that this is the case when the Bethe ansatz equations are satisfied, as explained in section 3.4.4. Therefore, we only get a contribution from swapping onto the sites of the model. As a result, we have

$$\begin{aligned}\tau(\omega, G_{c(1)}[-1]\mathsf{w}_0^{(1)}, \ldots, G_{c(M)}[-1]\mathsf{w}_0^{(M)}, \theta_W(Z))\\ &= \tau(Z(z)\omega, G_{c(1)}[-1]\mathsf{w}_0^{(1)}, \ldots, G_{c(M)}[-1]\mathsf{w}_0^{(M)}, \mathsf{w}_0)\\ &= \tau(Z(z)\omega, G_{c(1)}[-1]\mathsf{w}_0^{(1)}, \ldots, G_{c(M)}[-1]\mathsf{w}_0^{(M)})\\ &= \psi(Z(z)\omega) = (\iota(Z(u))\psi)(\omega), \end{aligned} \tag{3.156}$$

where ι is the Cartan involution (see footnote 2) and we introduced the rational function $Z(z)$ as in eq. (3.110).

At the same time, recall that a state in π_0 has the following from $b_{k_1}[-n_1]\cdots b_{k_p}[-n_p]\,|0\rangle$, $k_i \in \mathring{I}$, $n_i \in \mathbb{Z}_{>0}$, $i = 1, \ldots, p$. Therefore using eq. (3.152), one finds that for a state $b_k[-n]\mathsf{w} \in \mathbb{W}_0$

$$\begin{aligned}\tau(\omega, G_{c(1)}[-1]\mathsf{w}_0^{(1)}, \ldots, G_{c(M)}[-1]\mathsf{w}_0^{(M)}, b_k[-n]\mathsf{w})\\ &= r(b_k[-n]\,|0\rangle)\tau(\omega, G_{c(1)}[-1]\mathsf{w}_0^{(1)}, \ldots, G_{c(M)}[-1]\mathsf{w}_0^{(M)}, \mathsf{w}).\end{aligned} \tag{3.157}$$

Remarkably, it is known from [**FFR94**] that $\theta_W(\mathfrak{Z}(\mathfrak{g})) \subset \pi_0$. This implies that the image $\theta_W(Z)$ can be expressed as a polynomial in $b_k[-n]$, and applying the formula (3.157) iteratively one finds

$$\begin{aligned}\tau(\omega, G_{c(1)}[-1]\mathsf{w}_0^{(1)}, \ldots, G_{c(M)}[-1]\mathsf{w}_0^{(M)}, \theta_W(Z))\\ &= r(\theta_W(Z))\tau(\omega, G_{c(1)}[-1]\mathsf{w}_0^{(1)}, \ldots, G_{c(M)}[-1]\mathsf{w}_0^{(M)})\\ &= r(\theta_W(Z))\psi(\omega).\end{aligned} \tag{3.158}$$

Finally, these considerations lead to the following

THEOREM 3.4.1 ([**FFR94**]). *If the Bethe ansatz equations (3.148) are satisfied, the Schechtman-Varchenko (3.151) is an eigenvector of the operators $\iota(Z(u))$ for any $Z \in \mathfrak{Z}(\mathfrak{g})$ with eigenvalue $r(\theta_W(Z))$,*

$$\iota(Z(u))\psi = r(\theta_W(Z))\psi. \tag{3.159}$$

CHAPTER 4

Gaudin models of affine type

In this chapter, we start by recalling the definition of Gaudin models of affine type, describing some of their properties. We will continue by outlining the conjecture proposed by [**FF07, LVY18**] on the form that the higher Hamiltonians should have. The main part of the chapter is then occupied by the construction of the first non-trivial Hamiltonian for the $\widehat{\mathfrak{sl}}_2$-Gaudin model. We conclude by computing all other higher Hamiltonians up to next-to-leading order for this model.

The content of this chapter is mainly based on the paper published in collaboration with Charles Young [**FY23**].

4.1. Introduction

Given a finite Lie algebra $\mathring{\mathfrak{g}}$, one defines the loop algebra as the algebra of Laurent series in a formal variable t with coefficient in $\mathring{\mathfrak{g}}$,

$$L(\mathring{\mathfrak{g}}) = \mathring{\mathfrak{g}} \otimes \mathbb{C}((t)). \tag{4.1}$$

One considers the extension

$$0 \longrightarrow \mathbb{C}\mathsf{k} \longrightarrow \mathfrak{g} \longrightarrow L(\mathring{\mathfrak{g}}) \longrightarrow 0, \tag{4.2}$$

where the one-dimensional element k is central, *i.e.* $[\mathsf{k}, \cdot] = 0$.

The affine Kac-Moody algebra is defined by adjoining to this algebra a one-dimensional derivation,

$$\mathfrak{g} = \mathring{\mathfrak{g}} \otimes \mathbb{C}((t)) \oplus \mathbb{C}\mathsf{k} \oplus \mathbb{C}\mathsf{d}, \tag{4.3}$$

which obeys $[\mathsf{d}, \mathsf{k}] = 0$ and $\mathsf{d} = \mathrm{id} \otimes t\partial_t$.

Consider a set of points $\{z_1, \ldots, z_N\}$, $N \in \mathbb{Z}_{\geq 1}$ as the sites of the model. The algebra of observables is now given by

$$\mathrm{Obs}(\mathfrak{g}) := \widetilde{U}(\mathfrak{g})^{\otimes N}, \tag{4.4}$$

where $\widetilde{U}(\mathfrak{g})$ is a completion of the completed N-fold tensor product of the universal enveloping algebra $U(\mathfrak{g})$. The Hilbert space of the model is still defined as the tensor product of Verma modules with highest weights $\{\lambda_1, \ldots, \lambda_N\}$. However, in order to obtain a well-defined action of the algebra of observables on it, we need to require that its factors are *smooth* $\mathfrak{g}$-modules, *i.e.*

$$\mathring{\mathfrak{g}} \otimes t^n \mathbb{C}[[t]]\mathsf{m} = 0, \tag{4.5}$$

for any $\mathsf{m} \in M_\lambda$ and sufficiently large $n > 0$.

The models are defined by their Gaudin Hamiltonians

$$\mathcal{H}_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{\mathsf{k}^{(i)}\mathsf{d}^{(j)} + \mathsf{d}^{(i)}\mathsf{k}^{(j)} + \kappa_{ab} \sum_{n\in\mathbb{Z}} I^{a(i)}_{-n} I^{b(j)}_{n}}{z_i - z_j}, \qquad i = 1, \dots, N, \tag{4.6}$$

where we are using the notation $I^a_n = I^a \otimes t^n$, $\{I^a\}_{a\in\dim\mathring{\mathfrak{g}}}$ being a basis for $\mathring{\mathfrak{g}}$.

The analogue of the Bethe ansatz construction described in section 3.1.2 can be performed also in this setting (for more details, cfr [**Lac18**]). To do that one can introduce a set of auxiliary parameters $w_1, \dots, w_M$, distinct from the marked points, their colours $c(1), \dots, c(M)$ whose values $k = 0, \dots, \operatorname{rank}\mathring{\mathfrak{g}}$ are the vertices of the Dynkin diagram, and the Schechtman-Varchenko vector introduced in section 3.4.5. This vector turns out to be an eigenvector for the Gaudin Hamiltonians if the same Bethe ansatz equations from section 3.4.4 are satisfied. Up to this point, modulo minor technical changes, there are not many differences from the finite-type case. However, as we will describe in the next sections, things do become more intricate in the affine one, when one considers the construction and diagonalisation of higher Hamiltonians.

4.1.1. Higher Hamiltonians as hypergeometric integrals. The natural question one might ask is if also in the affine case it is possible to define higher Hamiltonians and, if yes, how to characterise and diagonalise them. It turns out that both questions are still unanswered to this day, as it is still not known how to generalise the Feigin-Frenkel-Reshetikhin construction to this setting.

Nevertheless, some conjectures were put forward in the seminal work [**FF07**]. In order to understand where they come from, we need to go back to the finite case. In the previous chapter we have seen how the Bethe ansatz construction can be obtained by considering restrictions of a certain functional on the tensor product of Wakimoto modules. There is an even stronger result, which states that the spectrum of the Gaudin Hamiltonians can be identified with the algebra of functions on the space of monodromy-free ${}^L\mathring{\mathfrak{g}}$-opers on the Riemann sphere [**Fre05a, MTV09, Ryb16**].

The idea proposed in [**FF07**] is to assume that the spectrum of higher Hamiltonians in the affine case can be again described by suitable functions on a space of *affine* opers. Some further conjectures of how this might work, at least for the local Hamiltonians, were made in [**LVY18**], where it was conjectured that the eigenvalues of higher local Hamiltonians of the affine Gaudin models, as well as the Hamiltonians themselves, are given by hypergeometric-type integrals on the spectral plane, namely

$$\widehat{Q}^{\gamma}_n = \int_\gamma \mathcal{P}(z)^{-n/2} \varsigma_n(z)_{[0]} \mathrm{d}z, \tag{4.7}$$

where n lives in a (multi)set of indices given by the *exponents* of the affine algebra $\mathfrak{g}$, $\mathcal{P}$ is a certain multi-valued function defined by the data of the levels k_i of the modules attached to the marked points z_i, γ is a Pochhammer contour in the spectral plane around any two of these points (see e.g. fig. 4.1) and $\varsigma_n(z)_{[0]}$ can be thought as the Hamiltonian density.

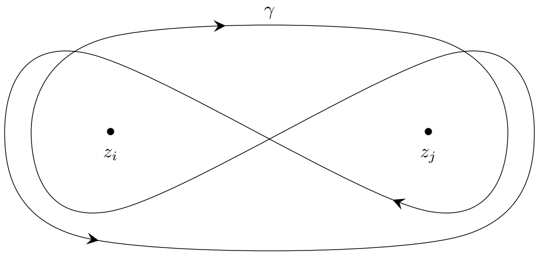


FIGURE 4.1. An example of Pochhammer contour γ around any two marked points.

The key step in computing the higher Hamiltonians is to characterize these Hamiltonian densities, which are obtained by defining a suitable state $\varsigma_n(z) \in \mathcal{V}$ for each given exponent n. In order to do that, it is possible to exploit the general properties consistent Hamiltonians must obey: they have to commute with the generators $\{I_n^\alpha\}_{n\in\mathbb{Z}}^{\alpha=1,\dots,\dim\mathring{\mathfrak{g}}}$ of the algebra $\mathfrak{g}$ defining the model as well as amongst themselves, where as usual $\mathring{\mathfrak{g}}$ denotes the underlying finite algebra. As we will see in section 4.4.7, there requirements can be recasted in the more convenient vertex algebra language, as follows

$$\begin{aligned} \Delta I^\alpha_{n\geq 0}\varsigma_m(z) &= 0 \quad \text{mod twisted derivatives,} \\ \varsigma_n(z)_{(0)}\varsigma_m(z) &= 0 \quad \text{mod twisted derivatives and translates,} \end{aligned} \tag{4.8}$$

where $\Delta I^\alpha_{n\geq 0}$ represents the diagonal action of the positive modes of the generators of the algebra $\mathfrak{g}$ from section 3.1.1 and $\varsigma_n(z)_{(0)}$ is a vertex algebra zero-th product, as defined in section 3.2.1.

As we will see in section 4.4.4, working up to twisted derivatives in the context of hypergeometric integrals has a similar meaning to working up to total derivatives are in the context of standard integration. Moreover, we work modulo translates since the zero-th mode of a translate is by definition always zero (cfr. *iii'*).

The general expectation is that there exists a state $\varsigma_m(z)$, for every exponent m of the affine algebra $\mathfrak{g}$, and that it takes the following form

$$\varsigma_m(z) = t_{i_1,\dots,i_{m+1}} I^{i_1}_{-1}(z)\cdots I^{i_{m+1}}_{-1}(z)\,|0\rangle + \text{quantum corrections,} \tag{4.9}$$

where $I(z)$ is as in eq. (3.10) and t is a certain symmetric invariant tensor of $\mathring{\mathfrak{g}}$. This particular structure is justified by the semi-classical counterpart of these models, which have been thoroughly studied [**EHMM99, Eva01, LMV17**]. In the very simplest cases, including the cubic Hamiltonian in type $\widehat{\mathfrak{sl}}_{M\geq 3}$, there are no quantum corrections needed [**LVY20**].

Already in this case of the exponent $n = 3$, *i.e.* of quartic Hamiltonians, the direct computations needed are very lengthy. This is especially true of the computations needed to show the mutual commutativity of the Hamilonians. For higher exponents $n \geq 5$, direct calculations become computationally difficult even with the aid of computer algebra. In the last part of this chapter we will compute explicitly the next-to-leading order quantum corrections of the leading term.

4.2. Vacuum Verma modules for $\widehat{\mathfrak{sl}}_2$

In this section, we repeat some of the definitions from the previous chapter adapted to the particular case of $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$.

4.2.1. Loop realization of $\widehat{\mathfrak{sl}}_2$. We define the loop algebra $L(\mathfrak{sl}_2) = \mathfrak{sl}_2 \otimes \mathbb{C}((t))$ as the algebra of Laurent series in a formal variable t with coefficient in the finite-dimensional Lie algebra $\mathfrak{sl}_2$. The Lie brackets on this algebra are given by

$$[a \otimes f(t), b \otimes g(t)] = [a, b]_{\mathfrak{sl}_2} \otimes f(t)g(t), \tag{4.10}$$

where $f(t)$ and $g(t)$ are arbitrary Laurent series.

Let $\kappa : \mathfrak{sl}_2 \times \mathfrak{sl}_2 \to \mathbb{C}$ denote the canonically normalized symmetric invariant bilinear form on $\mathfrak{sl}_2$. The extension by a one-dimensional central element $\mathbb{C}\mathsf{k}$, gives rise to the affine Lie algebra $\widehat{\mathfrak{sl}}_2$, whose commutation relations are

$$[a \otimes f(t), b \otimes g(t)] = [a, b]_{\mathfrak{sl}_2} \otimes f(t)g(t) - (\operatorname{res} t f \mathrm{d}g)\kappa(a, b)\mathsf{k}, \tag{4.11}$$

$$[\mathsf{k}, \cdot] = 0. \tag{4.12}$$

We shall use the notation

$$a_n := a \otimes t^n, \quad \text{for } a \in \mathfrak{sl}_2 \text{ and } n \in \mathbb{Z}, \tag{4.13}$$

so that the commutation relations can be equivalently written as

$$[a_m, b_n] = [a, b]_{n+m} - n\delta_{n+m,0}\kappa(a, b)\mathsf{k}. \tag{4.14}$$

We can add to this algebra a one-dimensional derivation d, such that $[\mathsf{d}, \mathsf{k}] = 0$ and $[\mathsf{d}, a \otimes f(t)] = a \otimes t\partial_t f(t)$, for all $a \in \mathfrak{sl}_2$ and $f(t) \in \mathbb{C}((t))$. It is possible to show that this algebra is isomorphic to the Kac-Moody algebra over $\mathbb{C}$ of type $\mathsf{A}_1^{(1)}$, see e.g. [**Kac90**, ch. 7].

4.2.2. $\widehat{\mathfrak{sl}}_2$ as a Kac-Moody algebra. The Cartan matrix for the Kac-Moody algebra of type $\mathsf{A}_1^{(1)}$ is defined as $A = (a_{i,j})_{i,j=0}^1 = (2\delta_{i,j} - \delta_{i+1,j} - \delta_{i-1,j})_{i,j=0}^1$,

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{4.15}$$

The Cartan decomposition is given by $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. The Chevalley-Serre generators are $\{e_i\}_{i=0}^1 \subset \mathfrak{n}_+$, $\{f_i\}_{i=0}^1 \subset \mathfrak{n}_-$ while $\{\check{\alpha}_i\}_{i=0}^1 \subset \mathfrak{h}$ and $\{\alpha_i\}_{i=0}^1 \subset \mathfrak{h}^*$ are respectively a basis for the Cartan subalgebra of simple coroots of $\mathfrak{g}$ and a basis for the dual Cartan subalgebra of simple roots of $\mathfrak{g}$. The latter are related via the canonical pairing between the Cartan algebra and its dual, $\langle\cdot,\cdot\rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$

$$\langle \alpha_i, \check{\alpha}_j \rangle = a_{i,j}. \tag{4.16}$$

The fundamental commutation relations in $\mathfrak{g}$ are

$$\begin{aligned} [x, e_i] &= \langle \alpha_i, x\rangle e_i, \qquad [x, f_i] = -\langle \alpha_i, x\rangle f_i, \\ [x, x'] &= 0, \qquad [e_i, f_j] = \check{\alpha}_i \delta_{ij}, \end{aligned} \tag{4.17}$$

where $x, x' \in \mathfrak{h}$ and $i, j = 0, 1$, together with the Serre relations

$$(\mathrm{ad} e_i)^{1-a_{ij}} e_j = 0, \qquad (\mathrm{ad} f_i)^{1-a_{ij}} f_j = 0. \tag{4.18}$$

The Kac-Moody algebra $\mathfrak{g}$ has a central element $\mathsf{k} = \sum_{i=0}^{1} \check{\alpha}_i$, which spans a one-dimensional centre. A basis for the Cartan subalgebra $\mathfrak{h}$ is given by the coroots $\{\check{\alpha}_i\}_{i=0}^{1}$ together with the derivation element d, which by definition satisfies

$$\langle \alpha_i, \mathsf{d} \rangle = \delta_{i,0}. \tag{4.19}$$

If we remove the zero-th row and column from A, we obtain the Cartan matrix corresponding to the finite dimensional Lie algebra $\mathfrak{sl}_2$, $A = 2$. This subalgebra of $\mathfrak{g}$ is generated by $e_1 \in \mathfrak{n}_+$, $f_1 \in \mathfrak{n}_-$ and $\check{\alpha}_1 \in \mathfrak{h}$.

4.2.3. Local completion and vacuum Verma modules. For any $k \in \mathbb{C}$, let us define $U_k(\widehat{\mathfrak{sl}}_2)$ as the quotient of the universal enveloping algebra $U(\widehat{\mathfrak{sl}}_2)$ of $\widehat{\mathfrak{sl}}_2$ by the two-sided ideal generated by $\mathsf{k} - k$. For each $n \in \mathbb{Z}_{\geq 0}$, let us introduce the left ideal $J_n = U_k(\widehat{\mathfrak{sl}}_2) \cdot (\mathfrak{sl}_2 \otimes t^n \mathbb{C}[t])$. The inverse limit

$$\widetilde{U_k}(\widehat{\mathfrak{sl}}_2) = \varprojlim_n U_k(\widehat{\mathfrak{sl}}_2) \Big/ J_n \tag{4.20}$$

is a complete topological algebra, called the local completion of $U_k(\widehat{\mathfrak{sl}}_2)$ at level k. With this definition, the elements of $\widetilde{U_k}(\widehat{\mathfrak{sl}}_2)$ are possibly infinite sums of the type $\sum_{m\geq 0} a_m$ of elements $a_m \in U_k(\widehat{\mathfrak{sl}}_2)$ which do truncate to finite sums when working modulo any J_n.

A module $\mathcal{M}$ over $\widehat{\mathfrak{sl}}_2$ is said to be smooth if, for all $a \in \mathfrak{sl}_2$ and all $v \in \mathcal{M}$, $a_n v = 0$ for sufficiently large n. A module $\mathcal{M}$ has level k if $\mathsf{k} - k$ acts as zero on $\mathcal{M}$. Any smooth module of level k over $\widehat{\mathfrak{sl}}_2$ is also a module over the completion $\widetilde{U_k}(\widehat{\mathfrak{sl}}_2)$.

We can identify the subalgebra of positive modes $\mathfrak{sl}_2 \otimes \mathbb{C}[[t]] \oplus \mathbb{C}\mathsf{k} \subset \widehat{\mathfrak{sl}}_2$ and introduce the one-dimensional representation $\mathbb{C}\,|0\rangle^k$ defined by

$$(\mathsf{k} - k)\,|0\rangle^k = 0, \qquad a_n\,|0\rangle^k = 0 \quad \text{for all } n \geq 0,\ a \in \mathfrak{sl}_2. \tag{4.21}$$

We define $\mathbb{V}_0^{\mathfrak{sl}_2,k}$, the vacuum Verma module at level k, as the induced smooth $\widehat{\mathfrak{sl}}_2$-module

$$\mathbb{V}_0^{\mathfrak{sl}_2,k} = U(\widehat{\mathfrak{sl}}_2) \otimes_{U(\mathfrak{sl}_2 \otimes \mathbb{C}[[t]] \oplus \mathbb{C}\mathsf{k})} \mathbb{C}\,|0\rangle^k \tag{4.22}$$

This vector space is spanned by monomials of the form $a_p \cdots b_q\,|0\rangle^k$, with $a, \ldots, b \in \mathfrak{sl}_2$ and strictly negative mode numbers $p, \ldots, q \in \mathbb{Z}_{<0}$. We call these vectors *states*.

Let us denote by $[T,\cdot]$ the derivation on $U_k(\widehat{\mathfrak{sl}}_2)$ defined by $[T,a_n] = -na_{n-1}$ and $[T,1]=0$. By setting $T(X\,|0\rangle^k) = [T,X]\,|0\rangle^k$ for any $X \in U_k(\widehat{\mathfrak{sl}}_2)$, one can interpret T as a translation operator $T : \mathbb{V}_0^{\mathfrak{sl}_2,k} \to \mathbb{V}_0^{\mathfrak{sl}_2,k}$. This space has the structure of a vertex algebra, as described in section 3.2.3.

4.3. Construction of higher Hamiltonians

4.3.1. The algebra of observables. Let us introduce a set of complex numbers $\boldsymbol{k} = \{k_i\}_{i=1}^N$, where $N \in \mathbb{Z}_{>0}$ and $k_i \neq -2$ for all $i = 1,\dots,N$. Consider the following tensor product of vacuum Verma modules

$$\mathbb{V}_0^{\mathfrak{sl}_2,\boldsymbol{k}} = \mathbb{V}_0^{\mathfrak{sl}_2,k_1} \otimes \cdots \otimes \mathbb{V}_0^{\mathfrak{sl}_2,k_N}. \tag{4.23}$$

This space can be interpreted as a module over the direct sum of N copies of $\widehat{\mathfrak{sl}}_2$. Let us denote by $A^{(i)} \in \widehat{\mathfrak{sl}}_2^{\oplus N}$ the copy of $A \in \widehat{\mathfrak{sl}}_2$ in the i^{th} direct summand.

Let us denote by $\mathbb{C}\,|0\rangle^{\boldsymbol{k}}$ the one-dimensional vacuum representation of the "positive modes" Lie subalgebra $(\mathfrak{sl}_2 \otimes \mathbb{C}[[t]] \oplus \mathbb{C}\mathsf{k})^{\oplus N} \subset \widehat{\mathfrak{sl}}_2^{\oplus N}$, defined by $(\mathsf{k}^{(i)} - k_i)\,|0\rangle^{\boldsymbol{k}} = 0$ and $a_n^{(i)}\,|0\rangle^{\boldsymbol{k}} = 0$ for all $n \geq 0$, $a \in \mathfrak{sl}_2$ and $i = 1,\dots,N$. Therefore $\mathbb{V}_0^{\mathfrak{sl}_2,\boldsymbol{k}}$ is the induced $\widehat{\mathfrak{sl}}_2^{\oplus N}$-module, namely

$$\mathbb{V}_0^{\mathfrak{sl}_2,\boldsymbol{k}} = U(\widehat{\mathfrak{sl}}_2^{\oplus N}) \otimes_{U(\mathfrak{sl}_2\otimes\mathbb{C}[[t]]\oplus\mathbb{C}\mathsf{k})^{\oplus N}} \mathbb{C}\,|0\rangle^{\boldsymbol{k}}. \tag{4.24}$$

Repeating similar arguments to those of the previous sections, we can define $U_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_2^{\oplus N})$ as the quotient of $U(\widehat{\mathfrak{sl}}_2^{\oplus N})$ by the two-sided ideal generated by $\mathsf{k}^{(i)} - k_i$ for all $i = 1,\dots,N$. We have the isomorphism

$$U_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_2^{\oplus N}) \cong U_{k_1}(\widehat{\mathfrak{sl}}_2) \otimes U_{k_2}(\widehat{\mathfrak{sl}}_2) \otimes \cdots \otimes U_{k_N}(\widehat{\mathfrak{sl}}_2). \tag{4.25}$$

Again, we can introduce the left ideals $J_n^N \in U_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_2^{\oplus N})$ generated by $a_r^{(i)}$ for all $r \geq n$, $a \in \mathfrak{sl}_2$ and $i = 1,\dots,N$. Let $\widetilde{U}_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_2^{\oplus N}) = \varprojlim U_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_2^{\oplus N})/J_n^N$ be the inverse limit. This space is a complete topological algebra and

$$\widetilde{U}_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_2^{\oplus N}) \cong \widetilde{U}_{k_1}(\widehat{\mathfrak{sl}}_2)\hat{\otimes}\cdots\hat{\otimes}\widetilde{U}_{k_N}(\widehat{\mathfrak{sl}}_2), \tag{4.26}$$

where $\hat{\otimes}$ denotes the completed tensor product. This space $\widetilde{U}_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_2^{\oplus N})$ is called the algebra of observables of the Gaudin model.

The tensor product $\mathbb{V}_0^{\mathfrak{sl}_2,\boldsymbol{k}}$ is again a vertex algebra. The state-field map $Y(\cdot,x) : \mathbb{V}_0^{\mathfrak{sl}_2,\boldsymbol{k}} \to \mathrm{Hom}(\mathbb{V}_0^{\mathfrak{sl}_2,\boldsymbol{k}}, \mathbb{V}_0^{\mathfrak{sl}_2,\boldsymbol{k}}((x)))$ is defined as in eqs. (3.63)–(3.65) but decorated with the extra index $^{(i)}$.

Recall from section 3.1.1 the map $\Delta : \widehat{\mathfrak{sl}}_2 \hookrightarrow \widehat{\mathfrak{sl}}_2^{\oplus N}$, which is the diagonal embedding of $\widehat{\mathfrak{sl}}_2$ into $\widehat{\mathfrak{sl}}_2^{\oplus N}$, defined as

$$\Delta x = \sum_{i=1}^N x^{(i)}, \qquad \text{for all } x \in \widehat{\mathfrak{sl}}_2. \tag{4.27}$$

It extends to an embedding of the enveloping algebras, $\Delta : U(\widehat{\mathfrak{sl}}_2) \hookrightarrow U(\widehat{\mathfrak{sl}}_2^{\oplus N}) \cong U(\widehat{\mathfrak{sl}}_2)^{\otimes N}$. It is easy to check that

$$[\Delta X_m, \Delta Y_n] = \Delta[X,Y]_{n+m} - n\delta_{n+m,0}\kappa(X,Y)\sum_{i=1}^{N} \mathsf{k}^{(i)}, \tag{4.28}$$

where κ is the non degenerate symmetric invariant bilinear form.

Therefore Δ descends to an embedding of the quotients $\Delta : U_{|\boldsymbol{k}|}(\widehat{\mathfrak{sl}}_2) \hookrightarrow U_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_2^{\oplus N})$, where $|\boldsymbol{k}| = \sum_{i=1}^N k_i$, and hence of their completions

$$\Delta : \widetilde{U}_{|\boldsymbol{k}|}(\widehat{\mathfrak{sl}}_2) \hookrightarrow \widetilde{U}_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_2^{\oplus N}). \tag{4.29}$$

4.3.2. $\mathfrak{sl}_2$-invariant tensors. Let $\{I^a\}_{a=1}^3$ be a basis of $\mathfrak{sl}_2$, and let $\{I_a\}_{a=1}^3$ be its dual basis with respect to the non-degenerate symmetric invariant bilinear form. Let $f^{ab}{}_c$ denote the structure constants, so that

$$[I^a, I^b] = f^{ab}{}_c I^c. \tag{4.30}$$

Here and in what follows we employ the summation convention on Lie algebra indices. Thanks to the non-degeneracy of the bilinear form, we may suppose our basis is chosen in such a way that

$$\kappa(I^a, I^b) = \delta^{ab}. \tag{4.31}$$

By doing this, we no longer have to distinguish between upper and lower indices. The structure constants are then

$$f^{ab}{}_c = f^{abc} = i\sqrt{2}\epsilon^{abc}, \tag{4.32}$$

where ϵ^{abc} is the usual Levi-Civita symbol. Concretely, in the defining representation we have

$$I^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad I^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad I^3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.33}$$

It is easy to check that eq. (4.31) holds, where $\kappa(a,b) = \mathrm{tr}(ab)$.

Recall that for any finite-dimensional Lie algebra $\mathring{\mathfrak{g}}$, a tensor $t : \mathring{\mathfrak{g}} \times \cdots \times \mathring{\mathfrak{g}} \to \mathbb{C}$ is *invariant* if

$$t([a,x],y,\dots,z) + t(x,[a,y],\dots,z) + \cdots + t(x,y,\dots,[a,z]) = 0, \quad a,x,y,z \in \mathring{\mathfrak{g}}, \tag{4.34}$$

or equivalently, if its components $t^{a_1\dots a_n} := t(I^{a_1},\dots,I^{a_n})$ satisfy

$$f^{ca_1}{}_b t^{ba_2\dots a_n} + f^{ca_2}{}_b t^{a_1 b\dots a_n} + \cdots + f^{ca_n}{}_b t^{a_1a_2\dots b} = 0, \tag{4.35}$$

where the indices take values from 1 to $\dim\mathring{\mathfrak{g}}$. In the case of $\mathfrak{sl}_2$, the ring of invariant tensors is generated by δ^{ab} and f^{abc}. We shall need the following *syzygy relations* between them:

$$\begin{gathered} f^{abc}f^{cde} = 2(\delta^{ae}\delta^{bd} - \delta^{ad}\delta^{be}), \qquad f^{abc}f^{abd} = -4\delta^{cd}, \\ f^{abc}\delta^{de} - f^{bcd}\delta^{ae} + f^{cda}\delta^{be} - f^{dab}\delta^{ce} = 0. \end{gathered} \tag{4.36}$$

Note in particular the last of these, which will play a crucial role in the explicit calculations of the next sections. It can also be generalized to higher rank tensors (see e.g. §369 F from [NS93]).

4.4. Quartic Hamiltonian

4.4.1. Meromorphic states. Let us introduce a set $\{z_1, \dots, z_N\}$ of $N \in \mathbb{Z}_{>0}$ points $z_i \in \mathbb{C}$ in the complex plane, chosen to be pairwise distinct, $z_i \neq z_j$ whenever $i \neq j$. For any element $A \in \widehat{\mathfrak{sl}}_2$ we introduce the $\widehat{\mathfrak{sl}}_2^{\oplus N}$-valued meromorphic functions

$$A(z) := \sum_{i=1}^{N} \frac{A^{(i)}}{z - z_i}. \tag{4.37}$$

We are allowed to take derivatives of such functions, which will be denoted by $A'(z)$ or, in general, for each $p \geq 0$,

$$A^{[p]}(z) := \left(\frac{d}{dz}\right)^p A(z) = \sum_{i=1}^{N} (-1)^p p! \frac{A^{(i)}}{(z - z_i)^{p+1}}. \tag{4.38}$$

Considering two of these functions with different spectral parameters, we obtain the following commutation relations

$$\begin{aligned} [A^{[p]}(z), B^{[q]}(w)] =& (-1)^{p+1}(p+q)! \frac{[A,B](z) - [A,B](w)}{(z-w)^{p+q+1}} \\ &+ \sum_{k=1}^{p} (-1)^{p+1-k} \binom{p}{k} (p+q-k)! \frac{[A,B]^{[k]}(z)}{(z-w)^{p+q+1-k}} \\ &- \sum_{k=1}^{q} (-1)^{p+1} \binom{q}{k} (p+q-k)! \frac{[A,B]^{[k]}(w)}{(z-w)^{p+q+1-k}}. \end{aligned} \tag{4.39}$$

By taking the limit $w \to z$, we get the commutation relations for the same spectral parameter, namely

$$[A^{[p]}(z), B^{[q]}(z)] = -\frac{p!q!}{(p+q+1)!}[A,B]^{[p+q+1]}(z). \tag{4.40}$$

We see that these $A^{[p]}(z)$, for $A \in \widehat{\mathfrak{sl}}_2$ and $p \geq 0$, span a Lie algebra of $\widehat{\mathfrak{sl}}_2^{\oplus N}$-valued meromorphic functions of z with poles at the marked points.

It is helpful to be able to treat this as an abstract Lie algebra. Thus, let $\mathfrak{L}$ denote the Lie algebra over $\mathbb{C}$ with basis consisting of $I_n^{a[p]}(z)$ and $\mathsf{k}^{[p]}(z)$, for $n \in \mathbb{Z}$, $p \in \mathbb{Z}_{\geq 0}$ and $a \in \{1,2,3\}$ with the non-vanishing Lie brackets given by

$$[I_m^{a[p]}(z), I_n^{b[q]}(z)] = -\frac{p!q!}{(p+q+1)!}(f^{ab}_c I^{c[p+q+1]}_{m+n}(z) - n\delta^{ab}\delta_{m+n,0}\mathsf{k}^{[p+q+1]}(z)). \tag{4.41}$$

Let $\mathfrak{L}_+$ denote the subalgebra generated by $I_n^{a[p]}(z)$ for $n \geq 0$, $p \in \mathbb{Z}_{\geq 0}$ and $a \in \{1,2,3\}$, and let

$$\mathcal{V} := U(\mathfrak{L}) \otimes_{U(\mathfrak{L}_+)} \mathbb{C}\,|0\rangle \tag{4.42}$$

denote the module over $\mathfrak{L}$ induced from the trivial one-dimensional module $\mathbb{C}\,|0\rangle$ over $\mathfrak{L}_+$. We call $\mathcal{V}$ the *space of meromorphic states*. It is again a vertex algebra, with the same state-field map as above. For each $z \in \mathbb{C} \setminus \{z_1, \dots, z_N\}$, one has the homomorphism of Lie algebras $\mathfrak{L} \to \widehat{\mathfrak{sl}}_2^{\oplus N}$ given by evaluating at z. It gives rise to a map $\mathcal{V} \to \mathbb{V}_0^{\mathfrak{sl}_2, \boldsymbol{k}}$ of vertex algebras.

There is a bi-gradation of $\mathfrak{L}$ in which $X_{-n}^{[p]}(z)$ (for any $X \in \mathfrak{sl}_2$) has weight $(n, p+1)$ and $\mathsf{k}^{[p]}(z)$ has weight $(0, p+1)$. This yields a bi-gradation of $\mathcal{V}$

$$\mathcal{V} = \bigoplus_{n \geq 0, p \geq 0} \mathcal{V}_{n,p}. \tag{4.43}$$

For each n, let $\mathcal{V}_n := \mathcal{V}_{n,n}$ denote the subspace of grade (n,n). We call elements of $\mathcal{V}_n$ *homogeneous meromorphic states of degree n.*

4.4.2. Diagonal action of the zero modes of $\widehat{\mathfrak{sl}}_2$. There is an evident diagonal action of the Lie algebra $\widehat{\mathfrak{sl}}_2$ on the $\mathfrak{L}$-module $\mathcal{V}$, defined in the same way as the action on $\mathbb{V}_0^{\mathfrak{sl}_2, \boldsymbol{k}}$ in eq. (4.27). In particular, for any $X \in \mathfrak{sl}_2$, the zero modes stabilize each subspace $\mathcal{V}_{n,p}$, namely

$$\Delta X_0 : \mathcal{V}_{n,p} \to \mathcal{V}_{n,p}. \tag{4.44}$$

An important fact is that every state in $\mathcal{V}_n$ properly contracted with an $\mathfrak{sl}_2$-invariant tensor vanishes under the diagonal action of the zero modes. This follows directly from the defining property of invariant tensors in eq. (4.35). Let us denote with $\mathcal{V}_n^{\mathfrak{sl}_2}$ the invariant subspace. We can characterize this space for small n:

- for $n = 0$, $\mathcal{V}_0^{\mathfrak{sl}_2} = \mathcal{V}_0 = \mathbb{C}\,|0\rangle$.
- for $n = 1$, $\mathcal{V}_1^{\mathfrak{sl}_2} = \{0\}$. Indeed, elements of $\mathcal{V}_1$ are of the form $t_a I^a_{-1}(z)\,|0\rangle$. Such an element is in $\mathcal{V}_1^{\mathfrak{sl}_2}$ if and only if t_a are the components of an $\mathfrak{sl}_2$-invariant tensor of rank 1. But there are no nonzero such tensors.
- for $n = 2$, $\mathcal{V}_2^{\mathfrak{sl}_2}$ has dimension 1 and it is spanned by the state

$$\varsigma_1(z) = \delta_{ab} I^a_{-1}(z) I^b_{-1}(z)\,|0\rangle\,. \tag{4.45}$$

- for $n = 3$, $\mathcal{V}_3^{\mathfrak{sl}_2}$ has dimension 2 and it is spanned by the states

$$\begin{aligned} f^{abc} I^a_{-1}(z) I^b_{-1}(z) I^c_{-1}(z)\,|0\rangle =& f^{abc}\frac{1}{2}(I^a_{-1}(z)I^b_{-1}(z) - I^b_{-1}(z)I^a_{-1}(z))I^c_{-1}(z)\,|0\rangle \\ =& -\frac{1}{2} f^{abc} f^{abd} I^{d\prime}_{-2}(z) I^c_{-1}(z)\,|0\rangle \\ =& 2 I^{c\prime}_{-2}(z) I^c_{-1}(z)\,|0\rangle\,, \end{aligned} \tag{4.46}$$

and

$$I^c_{-2}(z) I^c_{-1}(z) \mathsf{k}(z)\,|0\rangle\,. \tag{4.47}$$

- for $n = 4$, $\mathcal{V}_4^{\mathfrak{sl}_2}$ has dimension 14. Below, we will make use of the following explicit choice of basis:

$$
\begin{gathered}
\mathsf{v}_1 := \delta^{(ab}\delta^{cd)} I^a_{-1}(z) I^b_{-1}(z) I^c_{-1}(z) I^d_{-1}(z), \qquad \mathsf{v}_2 := f^{abc} I^a_{-2}(z) I^{b\prime}_{-1}(z) I^c_{-1}(z), \\
\mathsf{v}_3 := I^{a\prime\prime}_{-3}(z) I^a_{-1}(z), \qquad \mathsf{v}_4 := I^a_{-3}(z) I^{a\prime\prime}_{-1}(z), \qquad \mathsf{v}_5 := I^{a\prime\prime}_{-2}(z) I^a_{-2}(z), \\
\mathsf{v}_6 := I^{a\prime}_{-3}(z) I^{a\prime}_{-1}(z), \qquad \mathsf{v}_7 := I^{a\prime}_{-2}(z) I^{a\prime}_{-2}(z), \qquad \mathsf{v}_8 := I^a_{-3}(z) I^a_{-1}(z) \mathsf{k}'(z), \\
\mathsf{v}_9 := I^a_{-2}(z) I^a_{-2}(z) \mathsf{k}'(z), \qquad \mathsf{v}_{10} := I^{a\prime}_{-3}(z) I^a_{-1}(z) \mathsf{k}(z), \\
\mathsf{v}_{11} := I^a_{-3}(z) I^{a\prime}_{-1}(z) \mathsf{k}(z), \qquad \mathsf{v}_{12} := I^{a\prime}_{-2}(z) I^a_{-2}(z) \mathsf{k}(z), \\
\mathsf{v}_{13} := I^a_{-3}(z) I^a_{-1}(z) \mathsf{k}(z)^2, \qquad \mathsf{v}_{14} := I^a_{-2}(z) I^a_{-2}(z) \mathsf{k}(z)^2.
\end{gathered}
\tag{4.48}
$$

Note that to write these terms we have to choose an ordering prescription. Here we sort level first in ascending order from left to right and after that, for a given level, we sort derivatives in descending order from left to right. For example $f^{abc} I^a_{-2} I^{b\prime\prime}_{-2} I^{c\prime}_{-3} = f^{abc} I^{c\prime}_{-3} I^{b\prime\prime}_{-2} I^a_{-2} +$ terms obtained from commutations.

4.4.3. Top terms. We can see from the above construction that in the case $n = 2$ and $n = 4$, there is a particular state, that we will call *top term*, which is the state in $\mathcal{V}_n^{\mathfrak{sl}_2}$ that contains exactly n generators:

$$
\delta_{ab} I^a_{-1}(z) I^b_{-1}(z) |0\rangle , \qquad \delta_{(ab}\delta_{cd)} I^a_{-1}(z) I^b_{-1}(z) I^c_{-1}(z) I^d_{-1}(z) |0\rangle . \tag{4.49}
$$

We do not have such state for $n = 3$, because we can always use the commutation relations to reduce the number of generators, as shown in eq. (4.46). This pattern continues. Indeed, notice that the universal enveloping algebra $U(\mathfrak{L})$ has an increasing filtration

$$
F_0 U(\mathfrak{L}) \subseteq F_1 U(\mathfrak{L}) \subseteq \cdots \subseteq U(\mathfrak{L}), \tag{4.50}
$$

in which the generators $I^{a[p]}_n(z)$ count as $+1$ and the generators $\mathsf{k}^{[p]}(z)$ count as 0, cf. the commutation relations of $\mathfrak{L}$ in eq. (4.41). For example $I^a_{-1}(z) I^{a\prime}_{-2}(z) \in F_2$, and $I^a_{-1}(z) I^{a\prime}_{-2}(z) \mathsf{k}(z) \in F_2$ as well. It gives rise to a corresponding filtration, $F_0\mathcal{V} \subseteq F_1\mathcal{V} \subseteq \cdots \subseteq \mathcal{V}$, on the space $\mathcal{V}$ of meromorphic states.

Observe that if $\mathsf{v} \in \mathcal{V}_N$ then $\mathsf{v} \in F_N \mathcal{V}_N$. We see that

$$
\begin{aligned}
\mathsf{v} &\equiv t_{i_1 \dots i_N} I^{i_1}_{-1}(z) \dots I^{i_N}_{-1}(z) |0\rangle \mod F_{N-1}\mathcal{V}_N, \\
&\equiv t_{(i_1 \dots i_N)} I^{i_1}_{-1}(z) \dots I^{i_N}_{-1}(z) |0\rangle \mod F_{N-1}\mathcal{V}_N,
\end{aligned}
\tag{4.51}
$$

for some $\mathfrak{sl}_2$ tensor $t_{i_1,\dots,i_N}$, where the brackets around the indices denote the operation of symmetrization,

$$
t_{(i_1,\dots,i_n)} = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} t_{\sigma(i_1)\dots\sigma(i_n)}, \tag{4.52}
$$

(and we may symmetrize without loss of generality because the non-symmetric pieces fall into F_{N-1}, as for example in eq. (4.46)). Let us call $t_{(i_1 \dots i_N)} I^{i_1}_{-1}(z) \dots I^{i_N}_{-1}(z) |0\rangle$ the *top term* of the state $\mathsf{v} \in \mathcal{V}_N$.

If this state $\mathsf{v} \in \mathcal{V}$ is $\mathfrak{sl}_2$-invariant, $\mathsf{v} \in \mathcal{V}^{\mathfrak{sl}_2}$, then $t_{(i_1 \dots i_N)}$ is a symmetric invariant tensor. Nonzero such tensors exist only in even degrees, and up to rescaling they are, explicitly,

$$
\begin{aligned}
t_{i_1,i_2} &= \delta_{i_1 i_2} \\
t_{i_1,i_2,i_3,i_4} &= \delta_{(i_1 i_2}\delta_{i_3 i_4)} \\
t_{i_1,i_2,i_3,i_4,i_5,i_6} &= \delta_{(i_1 i_2}\delta_{i_3 i_4}\delta_{i_5 i_6)} \\
&\dots
\end{aligned}
$$

In what follows, our interest is in meromorphic states $\mathsf{v} \in \mathcal{V}^{\mathfrak{sl}_2}$ that have nonzero top term (in other words states whose principal symbol has maximal degree) and that are $\mathfrak{sl}_2$-invariant.

4.4.4. Singular vectors up to twisted derivative. Let us define the twisted derivative operator of degree $j \in \mathbb{Z}$ with respect to the spectral parameter z,

$$D_z^{(j)} = \left(\partial_z - \frac{j}{2}\mathsf{k}(z)\right). \tag{4.53}$$

Note that this operator sends $\mathcal{V}_{n,p} \to \mathcal{V}_{n,p+1}$ in the bigradation we introduced above.

We will say that a vector $\mathsf{v} \in \mathcal{V}_n^{\mathfrak{sl}_2}$ is *singular up to twisted derivatives* if for all $x \in \mathfrak{sl}_2$ we have

$$\Delta x_m \mathsf{v} = 0 \qquad \text{mod } D_z^{(n-1)}\mathcal{V}_{n-m,n-1}. \tag{4.54}$$

for *all* non-negative modes x_m, $m \geq 0$. This defines a subspace

$$\mathcal{V}_n^{\text{sing}} \subset \mathcal{V}_n^{\mathfrak{sl}_2} \tag{4.55}$$

of vectors singular up to twisted derivatives.

PROPOSITION 4.4.1. *The space of singular vectors $\mathcal{V}_2^{sing}$ is spanned by the quadratic state $\varsigma_1(z)$ defined in (4.45).*

PROOF. We need to show that

$$\Delta I_k^r \varsigma_1(z) = 0 \qquad \text{mod } D_z^{(1)}\mathsf{G}_k^r(z), \tag{4.56}$$

for some meromorphic states $\mathsf{G}_k^r \in \mathcal{V}_{2-k,1}$, for all $k \geq 0$ and $r = 1,2,3$. For $k = 0$ there is nothing to check since $\Delta I_0^r \varsigma_1(z) = 0$ identically, by the definition of $\mathcal{V}_2^{\mathfrak{sl}_2}$. It is enough to check the action of the first modes I_1^r, since any higher modes can be expressed in terms of their brackets, *i.e.* $I_2^r = -\frac{1}{4}f^{rbc}[I_1^b, I_1^c]$ etc. From direct calculations we get that

$$\Delta I_1^r \varsigma_1(z) = D_z^{(1)}\mathsf{G}_1^r(z), \tag{4.57}$$

where

$$\mathsf{G}_1^r(z) = -4I_{-1}^r(z)\,|0\rangle\,. \tag{4.58}$$

□

More non-trivially, for $n = 4$ we have the following result.

PROPOSITION 4.4.2. *The space of singular vectors $\mathcal{V}_4^{sing}$ is of dimension 7. A choice of basis is given by the state*

$$\begin{aligned} \varsigma_3(z) = \Big[&\delta_{(ab}\delta_{cd)} I^a_{-1}(z) I^b_{-1}(z) I^c_{-1}(z) I^d_{-1}(z) + \frac{20}{3} f_{abc} I^a_{-2}(z) I^{b\prime}_{-1}(z) I^c_{-1}(z) \\ &+ \frac{40}{9} I^a_{-3}(z) I^{a\prime\prime}_{-1}(z) - \frac{20}{3} I^{a\prime\prime}_{-2}(z) I^a_{-2}(z) + \frac{40}{9} I^{a\prime}_{-3}(z) I^{a\prime}_{-1}(z) \\ &- \frac{10}{3} I^{a\prime}_{-2}(z) I^{a\prime}_{-2}(z) - \frac{20}{3} I^a_{-3}(z) I^a_{-1}(z) \mathsf{k}'(z)\Big] |0\rangle , \end{aligned} \tag{4.59}$$

together with the double translate state

$$T^2\Big(I^{a\prime\prime}_{-1}(z) I^a_{-1}(z) |0\rangle - I^{a\prime}_{-1}(z) I^{a\prime}_{-1}(z) |0\rangle - \frac{3}{4} I^a_{-1}(z) I^a_{-1}(z) \mathsf{k}'(z) |0\rangle\Big) \tag{4.60}$$

and the following twisted derivative states

$$\begin{gathered} D^{(3)}_z\Big(I^a_{-3}(z) I^{a\prime}_{-1}(z) |0\rangle\Big), \quad D^{(3)}_z\Big(I^{a\prime}_{-3}(z) I^a_{-1}(z) |0\rangle\Big), \quad D^{(3)}_z\Big(I^{a\prime}_{-2}(z) I^a_{-2}(z) |0\rangle\Big), \\ D^{(3)}_z\Big(I^a_{-3}(z) I^a_{-1}(z) \mathsf{k}(z) |0\rangle\Big), \quad D^{(3)}_z\Big(I^a_{-2}(z) I^a_{-2}(z) \mathsf{k}(z) |0\rangle\Big). \end{gathered} \tag{4.61}$$

PROOF. Let $s(z) \in \mathcal{V}_4^{\mathfrak{sl}_2}$. We may write it in our basis (4.48),

$$s(z) = \sum_{i=1}^{14} \xi_i \mathsf{v}_i(z) \tag{4.62}$$

for some coefficients $\xi_i \in \mathbb{C}$ with $i = 1, \ldots, 14$, and ask what conditions the requirement of being singular up to twisted derivatives, (4.54), places on these coefficients. It is enough to demand that

$$\Delta I^r_k s(z) = 0 \mod D^{(3)}_z \mathsf{G}^r_k(z) \tag{4.63}$$

for some meromorphic states $\mathsf{G}^r_k(z) \in \mathcal{V}_{4-k,3}$, for all $k \geq 0$ and $r = 1, 2, 3$. For zero modes there is nothing to check since $\Delta I^a_0 s(z) = 0$ exactly, by definition of $\mathcal{V}_4^{\mathfrak{sl}_2}$. It is then enough to check the action of first modes, I^r_1, since any higher modes can be expressed in terms of their brackets, $I^r_2 = -\frac{1}{4} f^{rbc}[I^b_1, I^c_1]$ etc. So we are to check under what conditions

$$\Delta I^r_1 s(z) = D^{(3)}_z \mathsf{G}^r_1(z) \tag{4.64}$$

for some $\mathsf{G}^a_1(z) \in \mathcal{V}_{3,3}$. By direct calculation, one finds that solutions exist precisely if the coefficient ξ_i obey the relations

$$\begin{gathered} \xi_2 = \frac{20}{3}\xi_1, \qquad \xi_3 = \frac{20}{3}\xi_1 - \xi_4 + 2\xi_5 + \xi_6 - 2\xi_7, \qquad \xi_9 = -\frac{5}{4}\xi_1 - \frac{3}{8}\xi_5 + \frac{3}{8}\xi_7 - \frac{2}{3}\xi_{14}, \\ \xi_{10} = \frac{5}{3}\xi_1 + \frac{3}{2}\xi_4 - \frac{3}{2}\xi_5 - \frac{3}{2}\xi_6 + \frac{3}{2}\xi_7 + \xi_8, \qquad \xi_{11} = \frac{55}{3}\xi_1 - \frac{3}{2}\xi_4 + \frac{3}{2}\xi_5 - \frac{3}{2}\xi_7 + \xi_8, \\ \xi_{12} = -\frac{15}{2}\xi_1 - \frac{3}{4}\xi_5 - \frac{3}{4}\xi_7 - \frac{4}{3}\xi_{14}, \qquad \xi_{13} = -\frac{55}{4}\xi_1 - \frac{9}{8}\xi_5 + \frac{9}{8}\xi_7 - \frac{3}{2}\xi_8. \end{gathered} \tag{4.65}$$

When they do obey these relations, the required functions $\mathsf{G}_1^r(z)$ are given by

$$\begin{aligned}\mathsf{G}_1^r(z) = \Big[&\rho_1 I^{(a}_{-1}(z) I^r_{-1}(z) I^{a)}_{-1}(z) \\ &+ f^{rab}\left(\rho_2 I^a_{-2}(z) I^{b\prime}_{-1}(z) + \rho_3 I^{a\prime}_{-2}(z) I^b_{-1}(z) + \rho_4 I^a_{-2}(z) I^b_{-1}(z)\mathsf{k}(z)\right) \\ &+ \rho_5 I^r_{-3}(z)\mathsf{k}(z)^2 + \rho_6 I^r_{-3}(z)\mathsf{k}'(z) + \rho_7 I^{r\prime}_{-3}(z)\mathsf{k}(z) + \rho_8 I^{r\prime\prime}_{-3}(z)\Big] |0\rangle\,,\end{aligned} \tag{4.66}$$

where

$$\rho_1 = -\frac{8}{3}\xi_1, \qquad \rho_2 = \frac{20}{3}\xi_1 - \xi_4 + \xi_5, \qquad \rho_3 = \xi_4 - \xi_5 - \xi_6 + 2\xi_7,$$
$$\rho_4 = -\frac{55}{6}\xi_1 - \frac{3}{4}\xi_5 + \frac{3}{4}\xi_7 - \xi_8 - \frac{4}{3}\xi_{14} \qquad \rho_5 = \frac{55}{6}\xi_1 + \frac{3}{4}\xi_5 - \frac{3}{4}\xi_7 + \xi_8, \qquad \rho_6 = \xi_4,$$
$$\rho_7 = 5\xi_1 - \xi_4 + \frac{3}{2}\xi_5 + \xi_6 - \frac{3}{2}\xi_7 + \frac{8}{3}\xi_{14}, \qquad \rho_8 = -\frac{100}{9}\xi_1 - \frac{4}{3}\xi_5 - \frac{2}{3}\xi_7.$$

The basis reported in the proposition can be obtained by the one defined by the restrictions (4.65) by a change of basis. □

The proposition above is in agreement with the calculation of the quartic Hamiltonian density $S_4(z)$ (the analogue of our $\varsigma_3(z)$), recently presented in [**KLT24**]. In the present conventions, the latter is given by

$$\begin{aligned}S_4(z) = \Big[&\delta^{(ab}\delta^{cd)} I^a_{-1}(z) I^b_{-1}(z) I^c_{-1}(z) I^d_{-1}(z) + \frac{20}{3} f^{abc} I^a_{-2}(z) I^{b\prime}_{-1}(z) I^c_{-1}(z) \\ &- \frac{40}{9} I^{a\prime\prime}_{-3}(z) I^a_{-1}(z) - \frac{140}{9} I^{a\prime\prime}_{-2}(z) I^a_{-2}(z) + \frac{40}{3} I^{a\prime}_{-3}(z) I^{a\prime}_{-1}(z) \\ &- \frac{10}{3} I^{a\prime}_{-2}(z) I^{a\prime}_{-2}(z) + 5 I^a_{-2}(z) I^a_{-2}(z)\mathsf{k}(z)^2\Big] |0\rangle\end{aligned} \tag{4.67}$$

and it does[1] indeed lie in the space $\mathcal{V}_4^{\text{sing}}$.

4.4.5. Hamiltonian densities. Now, to state the main result of the paper, we need two reintroduce rational functions of two different spectral parameters, z and w, cf. eq. (4.39) and eq. (4.40).

Recall the Lie algebra $\mathfrak{L} \equiv \mathfrak{L}_{(z)}$ over $\mathbb{C}$ from section 4.4.1. Let $\mathfrak{L}_{(z,w)}$ be the Lie algebra with generators $I^{a[p]}_n(z)$, $I^{a[p]}_n(w)$, $\mathsf{k}^{[p]}(z)$ and $\mathsf{k}^{[p]}(w)$ for $a = 1, 2, 3$, $n \in \mathbb{Z}$ and $p \in \mathbb{Z}_{\geq 0}$ and

[1]To match conventions, note that for us

$$\delta_{(ab}\delta_{cd)} = \frac{1}{3}\left(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right) \tag{4.68}$$

and in [**KLT24**] the tensor called τ_3^{abcd} is given by

$$\tau_3^{abcd} = \frac{1}{16}\left(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right). \tag{4.69}$$

We thank Sylvain Lacroix for clarifying discussions on this point.

commutation relations

$$
\begin{aligned}
[I^{a[p]}_m(z), I^{b[q]}_n(w)] =& (-1)^{p+1}(p+q)!\,\frac{f^{ab}_c(I^c_{m+n}(z) - I^c_{m+n}(w)) - n\delta_{m+n,0}\delta^{ab}(\mathsf{k}(z) - \mathsf{k}(w))}{(z-w)^{p+q+1}} \\
&+ \sum_{j=1}^{p}(-1)^{p+1-j}\binom{p}{j}(p+q-j)!\,\frac{f^{ab}_c I^{c[j]}_{m+n}(z) - n\delta_{n+m,0}\delta^{ab}\mathsf{k}^{[j]}(z)}{(z-w)^{p+q+1-k}} \\
&- \sum_{k=1}^{q}(-1)^{p+1}\binom{q}{k}(p+q-k)!\,\frac{f^{ab}_c I^{c[j]}_{m+n}(w) - n\delta_{n+m,0}\delta^{ab}\mathsf{k}^{[j]}(w)}{(z-w)^{p+q+1-k}},
\end{aligned}
\tag{4.70}
$$

together with eq. (4.41) for the generators with parameter z and the analogue with parameter w. This Lie algebra $\mathfrak{L}_{(z,w)}$ and its modules are defined over the ground ring $\mathbb{C}[(z-w)^{-1}]$ of polynomials in $(z-w)^{-1}$. We have the vertex algebra $\mathcal{V}_{(z,w)}$ defined analogously to eq. (4.42) and the two obvious embedding maps of vertex algebras $\mathcal{V} \hookrightarrow \mathcal{V}_{(z,w)}$, which we write as $\mathsf{v} \mapsto \mathsf{v}(z)$ and $\mathsf{v} \mapsto \mathsf{v}(w)$.

Moreover, there is a natural notion of "expanding around $z = w$". Namely, there is a homomorphism $\mathfrak{L}_{(z,w)} \to \mathfrak{L}_{(z)}((w-z))$ of Lie algebras over $\mathbb{C}[(z-w)^{-1}]$ defined by

$$
I^{a[p]}_m(w) = I^{a[p]}_m(z) + I^{a[p+1]}_m(z)(w-z) + \frac{1}{2}I^{a[p+2]}_m(z)(w-z)^2 + \dots \tag{4.71}
$$

which is motivated by considering the Taylor expansion $\iota_{w-z}A(w)$ of the function $A(w)$ from eq. (4.37). This gives rise to a map $\mathcal{V}_{(z,w)} \to \mathcal{V}_{(z)}((w-z))$. We say a state $\mathsf{v} \in \mathcal{V}_{(z,w)}$ is *regular at $z = w$ modulo translates* if there exists $Z \in \mathcal{V}_{(z,w)}$ such that the image of $\mathsf{v} - TZ$ under this map has no singularities in $(z-w)$.

Recall from eqs. (4.45) and (4.59) the definitions of the quadratic state $\varsigma_1 \in \mathcal{V}^{\mathrm{sing}}_2$ and of the vector $\varsigma_3 \in \mathcal{V}^{\mathrm{sing}}_4$, respectively.

THEOREM 4.4.1. *The elements $\varsigma_1 \in \mathcal{V}^{\mathrm{sing}}_2$ and $\varsigma_3 \in \mathcal{V}^{\mathrm{sing}}_4$ obey the relations*

$$
\varsigma_1(z)_{(0)}\varsigma_1(w) = (D^{(1)}_z - D^{(1)}_w)\mathsf{A}_{1,1}(z,w) + T\mathsf{B}_{1,1}(z,w), \tag{4.72a}
$$

$$
\varsigma_1(z)_{(0)}\varsigma_3(w) = (3D^{(1)}_z - D^{(3)}_w)\mathsf{A}_{1,3}(z,w) + T\mathsf{B}_{1,3}(z,w), \tag{4.72b}
$$

$$
\varsigma_3(z)_{(0)}\varsigma_1(w) = (D^{(3)}_z - 3D^{(1)}_w)\mathsf{A}_{3,1}(z,w) + T\mathsf{B}_{3,1}(z,w), \tag{4.72c}
$$

$$
\varsigma_3(z)_{(0)}\varsigma_3(w) = (3D^{(3)}_z - 3D^{(3)}_w)\mathsf{A}_{3,3}(z,w) + T\mathsf{B}_{3,3}(z,w), \tag{4.72d}
$$

where $\mathsf{A}_{i,j}(z,w)$ and $\mathsf{B}_{i,j}(z,w)$ are elements of $\mathcal{V}_{(z,w)}$. Moreover, $\mathsf{A}_{i,j}(z,w)$, $i,j \in \{1,3\}$, are regular at $z = w$ modulo translates.

PROOF. The two statements of the theorem follow from direct calculations. In particular, when $m = n = 1$, we get

$$
\mathsf{A}_{1,1}(z,w) = \frac{8}{z-w}I^a_{-2}(z)I^a_{-1}(w)\,|0\rangle\,, \qquad \mathsf{B}_{1,1}(z,w) = \frac{8}{(z-w)^2}I^a_{-1}(z)I^a_{-1}(w)\,|0\rangle\,. \tag{4.73}
$$

We have computed $\mathsf{A}_{1,3}(z,w)$, $\mathsf{B}_{1,3}(z,w)$, $\mathsf{A}_{3,3}(z,w)$ and $\mathsf{B}_{3,3}(z,w)$ explicitly, with the aid of the computer algebra system FORM [**Ver13, KUVV13**]. The expressions for $\mathsf{A}_{1,3}(z,w)$

and $\mathsf{B}_{1,3}(z,w)$, are given in appendix A. The expressions for $\mathsf{A}_{3,3}(z,w)$ and $\mathsf{B}_{3,3}(z,w)$ are extremely lengthy (more that 500 terms in total), and we do not reproduce them here.

Once the expression of $\varsigma_1(z)_{(0)}\varsigma_3(w)$ is known, *i.e.* the functions $\mathsf{A}_{13}(z,w)$ and $\mathsf{B}_{13}(z,w)$ are found, it can be shown that the theorem is automatically satisfied for the product $\varsigma_3(z)_{(0)}\varsigma_1(w)$. This comes from the property of the nth product between two states a, b of a vertex algebra, namely

$$a_{(n)}b = -\sum_{k=0}^{\infty}\frac{1}{k!}(-1)^{k+n}T^k(b_{(n+k)}a). \tag{4.74}$$

Therefore, by swapping two states in a zeroth product, we obtain $\mathsf{A}_{31}(z,w) = -\mathsf{A}_{13}(w,z)$ and a series of terms which are nothing but translates and therefore can be absorbed in the definition of $\mathsf{B}_{31}(z,w) = -\mathsf{B}_{13}(w,z) + \sum_{k=0}^{\infty}(-1)^k T^k(\varsigma_1(w)_{(k+1)}\varsigma_3(z))$.

To prove the second part of the theorem one can expand according to eq. (4.71) and the result follows from direct calculation. □

Having established this statement for the particular choice of quartic density ς_3, we automatically get the following property for *any* element of $\mathcal{V}_4^{\text{sing}}$. It is a slightly weaker property, because the condition on the twisted derivative terms on the right hand side is less rigid. As we shall see in section 4.4.7 below, it is sufficient for defining consistent Hamiltonians.

Corollary 4.4.1. *For* any *element* $\mathsf{v}_3 \in \mathcal{V}_4^{sing}$*, one has*

$$\varsigma_1(z)_{(0)}\mathsf{v}_3(w) = D_z^{(1)}\mathsf{A}_{1,3}^{I}(z,w) + D_w^{(3)}\mathsf{A}_{1,3}^{II}(z,w) + T\mathsf{B}_{1,3}(z,w), \tag{4.75}$$

$$\mathsf{v}_3(z)_{(0)}\varsigma_1(w) = D_z^{(3)}\mathsf{A}_{3,1}^{I}(z,w) + D_w^{(1)}\mathsf{A}_{3,1}^{II}(z,w) + T\mathsf{B}_{3,1}(z,w), \tag{4.76}$$

$$\mathsf{v}_3(z)_{(0)}\mathsf{v}_3(w) = D_z^{(3)}\mathsf{A}_{3,3}^{I}(z,w) + D_w^{(3)}\mathsf{A}_{3,3}^{II}(z,w) + T\mathsf{B}_{3,3}(z,w). \tag{4.77}$$

where $\mathsf{A}_{i,j}^{I,II}(z,w)$ *and* $\mathsf{B}_{i,j}(z,w)$ *are elements of* $\mathcal{V}_{(z,w)}$*. Moreover,* $\mathsf{A}_{ij}^{I,II}(z,w)$*,* $i,j \in \{1,3\}$*, are regular at* $z = w$ *modulo translates.*

Proof. We already know from Theorem 4.4.1 that there exists an element, $\varsigma_3(z)$, satisfying these relations. But we saw in Proposition 4.4.2 that every element $\mathsf{v}_3(z)$ of $\mathcal{V}_4^{\text{sing}}$ is proportional to $\varsigma_3(z)$ up to the addition of certain translates and twisted derivatives.

It follows from the property (4.74) that if we add to $\varsigma_3(z)$ any translate then the statement of Theorem 4.4.1 still holds, the only difference being a re-definition of the states $\mathsf{B}(z,w)$. And it is evident that, if we add to $\varsigma_3(z)$ any linear combination of the twisted derivatives in eq. (4.61) then the resulting vector $\mathsf{v}_3(z)$ still obeys the weaker relations given above. (One might worry about introducing singularities at $z = w$, but note that for *any* meromorphic states $a(z)$ and $b(z)$, the product $a(z)_{(0)}b(w)$ is regular at $z = w$, as is manifest if we expand $b(w)$ about $w = z$ in the spectral plane before taking the vertex-algebra product: $a(z)_{(0)}b(w) = a(z)_{(0)}\left(b(z) + (w-z)b'(z) + \dots\right) = a(z)_{(0)}b(z) + (w-z)a(z)_{(0)}b'(z) + \dots$. □

4.4.6. Gaudin Hamiltonian. Let us define the following state at non-critical level, *i.e.* $k_i \neq -2$,

$$s_1(z) = \frac{1}{2}\Big(\varsigma_1(z) + 4D_z^{(1)}\omega(z)\Big) \in \mathbb{V}_0^{\mathfrak{sl}_2,\boldsymbol{k}}, \tag{4.78}$$

where $\varsigma_1(z)$ is now the image in $\mathbb{V}_0^{\mathfrak{sl}_2,\boldsymbol{k}}$ of the density defined in (4.45) and where

$$\omega(z) := \sum_{i=1}^{N} \frac{1}{z-z_i}\left(\frac{1}{2(k_i+2)}\kappa_{ab}I_{-1}^{a(i)}I_{-1}^{b(i)}\,|0\rangle^{\boldsymbol{k}}\right), \tag{4.79}$$

the term in the brackets being the normalised Segal-Sugawara vector at site i, as defined in eq. (3.67).

It is possible to show (see [**LVY20**]), that the operator $(s_1(z))_{(0)}$ is the image in $\widetilde{U}_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_2^{\oplus N})$ of

$$\sum_{i=1}^{N}\frac{\mathcal{C}^{(i)}}{2(z-z_i)^2} + \sum_{i=1}^{N}\frac{\mathcal{H}_i}{z-z_i} \in \tilde{U}(\mathfrak{sl}^{\oplus N}), \tag{4.80}$$

where

$$\mathcal{C}^{(i)} := 2(\mathsf{k}^{(i)}+2)\mathsf{d}^{(i)} + I_0^{a(i)}I_0^{a(i)} + 2\sum_{n>0} I_{-n}^{a(i)}I_n^{a(i)} \tag{4.81}$$

is the ith copy of the quadratic Casimir operator of $\mathfrak{g}$ in $\tilde{U}(\mathfrak{g}^{\oplus N})$ and $\mathcal{H}_i$ are the Hamiltonians in (4.6). This is nothing but the generalisation of the operator (3.9) to the affine setting.

THEOREM 4.4.2. *Given the images in $\mathbb{V}_0^{\mathfrak{sl}_2,\boldsymbol{k}}$ of the densities ς_i, $i \in \{1,3\}$, we have*

$$s_1(z)_{(0)}\varsigma_i(w) = -\frac{1}{2}D_w^{(1)}\mathsf{A}_{1,i}(z,w) + T\Big(\frac{1}{2}\mathsf{B}_{1,i}(z,w) + 2D_z^{(1)}\frac{\varsigma_i(w)}{z-w}\Big), \tag{4.82}$$

with $\mathsf{A}_{1,i}(z,w)$ and $\mathsf{B}_{1,i}(z,w)$ being the images in $\mathbb{V}_0^{\mathfrak{sl}_2,\boldsymbol{k}}$ of the meromorphic states in Theorem 4.4.1.

PROOF. The result follows from direct calculations, using the definitions of $\mathsf{A}_{1,1}$, $\mathsf{B}_{1,1}$, $\mathsf{A}_{1,3}$, $\mathsf{B}_{1,3}$ in (4.73) and appendix A, respectively. □

As we will see in the next section, this requirement is sufficient to ensure the commutativity of local Hamiltonians, arising from the densities $\varsigma_1(z)$ and $\varsigma_3(z)$, with the usual quadratic Gaudin Hamiltonians which define the model.

4.4.7. Commuting Hamiltonians. In this section, we will simply recall the ideas presented in [**LVY18**]. Consider two states $X, Y \in \mathbb{V}_0^{\mathfrak{sl}_2,\boldsymbol{k}}$ and their formal zero modes $X_{(0)}, Y_{(0)} \in \widetilde{U}_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_2^{\oplus N})$. We have the following commutator formula, coming from eq. (3.27) by setting $m, k = 0$,

$$[X_{(0)}, Y_{(0)}] = (X_{(0)}Y)_{(0)}. \tag{4.83}$$

This means that if one is able to find a family of operators whose zeroth product vanishes (or that can be expressed as a translation, since $(TZ)_{(0)} = 0$ by definition), then their formal zero modes form a commutative subalgebra of the algebra of observables $\widetilde{U}_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_2^{\oplus N})$.

The meromorphic function which is obtained by acting with $\mathsf{k}(z)$ on the module $\mathbb{V}_0^{\mathfrak{sl}_2,\boldsymbol{k}}$, *i.e.* setting the central elements to numbers,

$$k(z) = \sum_{i=1}^{N} \frac{k_i}{z - z_i}, \tag{4.84}$$

has a special role and it is called the *twist function* of the model. Let us define also

$$\mathcal{P}(z) := \prod_{j=1}^{N} (z - z_j)^{k_j}. \tag{4.85}$$

The function $\mathcal{P}$ is multi-valued. It becomes single-valued on a certain multi-sheeted cover of $\mathbb{C} \setminus \{z_1, \ldots, z_N\}$. Let γ be any closed contour in this cover. For example, γ could be the lift to this cover of a Pochhammer contour in $\mathbb{C} \setminus \{z_1, \ldots, z_N\}$ around any two of the marked points. Then $\mathcal{P}^{n/2}$, for any integer n, is single-valued along γ, and one can compute the following hypergeometric integral

$$\int_\gamma \mathcal{P}(z)^{-n/2} f(z) \mathrm{d}z. \tag{4.86}$$

We have the following result from [LVY18]

Lemma 4.4.1. *For any meromorphic function $f(z)$ which is non-singular along γ*

$$\int_\gamma \mathcal{P}(z)^{-n/2} D_z^{(n)} f(z) dz = \int_\gamma \frac{d}{dz} (\mathcal{P}^{-n/2} f(z)) dz = 0, \tag{4.87}$$

where $D_z^{(n)}$ is the twisted derivative operator from eq. (4.53).

Let us now define the following object in $\widetilde{U}_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_2^{\oplus N})$,

$$Q_n^\gamma = \int_\gamma \mathcal{P}(z)^{-n/2} \varsigma_n(z)_{(0)} \mathrm{d}z, \tag{4.88}$$

for $n = 1, 3$, where $\varsigma_n(z)$ are now the images in $\mathbb{V}_0^{\mathfrak{sl}_2,\boldsymbol{k}}$ of the densities we have defined in the previous section.

Proposition 4.4.3. *The operators $Q_n^\gamma \in \widetilde{U}_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_2^{\oplus N})$ commute amongst themselves, with the generators of $\widehat{\mathfrak{sl}}_2$, and with the quadratic Hamiltonians $\mathcal{H}_i$.*

Proof. We can use eq. (4.83) to compute

$$\begin{aligned} [Q_m^\gamma, Q_n^\eta] &= \int_\gamma \int_\eta \mathcal{P}(z)^{-m/2} \mathcal{P}(w)^{-n/2} [\varsigma_m(z)_{(0)}, \varsigma_n(w)_{(0)}] \mathrm{d}z \mathrm{d}w \\ &= \int_\gamma \int_\eta \mathcal{P}(z)^{-m/2} \mathcal{P}(w)^{-n/2} (\varsigma_m(z)_{(0)} \varsigma_n(w))_{(0)} \mathrm{d}z \mathrm{d}w, \end{aligned} \tag{4.89}$$

where we used the commutator formula eq. (3.27). We know from Theorem 4.4.1 that the zeroth product between those states can be expressed as a sum of twisted derivatives and translations. It is now straightforward to check that the result of the commutator is zero: on one side because $(TX)_{(0)} = 0$ for every state $X \in \mathbb{V}_0^{\mathfrak{sl}_2,\boldsymbol{k}}$, on the other because of the property (4.87).

To show the second statement, consider the generators $\{I^a_n\}_{a=1}^3$ with $n \in \mathbb{Z}$. Recalling that $I^a_n = (I^a_{-1}\,|0\rangle^{\boldsymbol{k}})_{(n)}$ and the commutator formula from (3.27), we get

$$\begin{aligned}
[I^a_n, Q^\gamma_m] &= \int_\gamma \mathcal{P}(z)^{-m/2}[I^a_n, \varsigma_m(z)_{(0)}]\mathrm{d}z \\
&= \int_\gamma \sum_{k\geq 0} \binom{n}{k} \mathcal{P}(z)^{-m/2}(I^a_k \varsigma_m(z))_{(n-k)}\mathrm{d}z.
\end{aligned} \tag{4.90}$$

It is now straightforward to check that the result is zero, by using property (4.54), described in the relevant cases in Propositions 4.4.1 and 4.4.2, and the property (4.87).

Using similar arguments, given the result from Theorem 4.4.2, one can show that the charges also commute with the Gaudin Hamiltonians.

□

4.4.7.1. *Fourier modes.* Even though the operators Q^γ_m we have just defined have all the right characteristics to be well-defined Hamiltonians as pointed out in Proposition 4.4.3, there is one last subtlety about these objects, related to the fact that we want their action on highest weight modules to be diagonalisable. In fact, considering $X^{(i)}_n \in \widetilde{U}(\widehat{\mathfrak{sl}}_2^{\oplus N})$ such that $\deg X^{(i)}_n = -n$ and setting $\deg(|0\rangle^{\boldsymbol{k}}) = 0$, induces a $\mathbb{Z}_{\geq 0}$ gradation on the product of vacuum Verma modules, called the *homogeneous gradation*,

$$\mathbb{V}_0^{\mathfrak{sl}_2,\boldsymbol{k}} = \bigoplus_{n\geq 0} \mathbb{V}_0^{\mathfrak{sl}_2,\boldsymbol{k}}[n]. \tag{4.91}$$

Therefore if $X \in \mathbb{V}_0^{\mathfrak{sl}_2,\boldsymbol{k}}[k]$, *i.e.* it has degree $\deg(X) = k$, from Definition 3.2.1 the degree of its modes is $\deg(X_{(m)}) = k - m - 1$. The objects we have constructed $\varsigma_n(z)$, by definition, have $\deg(\varsigma_n(z)) = n + 1$, therefore $\deg(\varsigma_n(z)_{(0)}) = n$.

This shows that in the homogeneous gradation these operators have non-zero degree: this means that if we consider a module over $U(\widehat{\mathfrak{sl}}_2^{\oplus N})$ which has a trivial subspace of grade n for large n, then the operator $\int_{\gamma_n} \mathcal{P}(z)^{-n/2}\varsigma_n(z)_{(0)}dz$ has a non-zero eigenvalue.

A way to overcome this issue is to consider the notion of Fourier mode $X_{[n]} \in \widetilde{U}(\widehat{\mathfrak{sl}}_2^{\oplus N})$ of the state $X \in \mathbb{V}_0^{\mathfrak{sl}_2,\boldsymbol{k}}$: they have the property that we are looking for, namely $\deg(X_{[n]}) = n$. Additionally, they satisfy a similar relation to (4.83),

$$[X_{[0]}, Y_{[0]}] = (X_{(0)}Y)_{[0]}, \tag{4.92}$$

with $(TX)_{[0]} = 0$. One has $(x^{(i)}_{-1}\,|0\rangle^{\boldsymbol{k}})_{[n]} = x_n$ for $x \in \mathfrak{sl}_2$ and it is possible to show that the following recursive formula holds:

$$(A_{(-n)}B)_{[m]} = ((A \otimes f(t))B)_{[m]} + \sum_{k>0} c_k A_{[-k]}B_{[k+m]} + \sum_{k\leq 0} c_k B_{[k+m]}A_{[-k]}, \tag{4.93}$$

where $f(t)$ is the Taylor series in $t := u - v$ given by

$$f = \frac{1}{(n-1)!}(-\partial_u)^{n-1}\Big(\frac{1}{u-v} - \iota_{u-v}\frac{e^v}{e^u - e^v}\Big). \tag{4.94}$$

and where the coefficients c_k are defined by the requirement that $\sum_{k>0} c_k(\frac{z}{w})^k$ and $-\sum_{k\leq 0} c_k(\frac{z}{w})^k$ are the expansions, for $|z|<|w|$ and $|w|<|z|$ respectively, of the function

$$\frac{1}{(n-1)!}(-w\partial_w)^{n-1}\frac{z}{w-z}. \tag{4.95}$$

The first relevant examples are

$$(A_{(-1)}B)_{[m]} = \frac{1}{2}(A_{(0)}B)_{[m]} - \frac{1}{12}(A_{(1)}B)_{[m]} + \dots \\ + \sum_{k>0} A_{[-k]}B_{[k+m]} + \sum_{k\leq 0} B_{[k+m]}A_{[-k]} \tag{4.96a}$$

$$(A_{(-2)}B)_{[m]} = \frac{1}{12}(A_{(0)}B)_{[m]} - \frac{1}{240}(A_{(2)}B)_{[m]} + \dots \\ + \sum_{k>0} kA_{[-k]}B_{[k+m]} + \sum_{k\leq 0}(-k)B_{[k+m]}A_{[-k]} \tag{4.96b}$$

where $A, B \in \mathbb{V}_0^{\mathfrak{sl}_2,\boldsymbol{k}}$. These formulae are the Fourier-analogue of the normal ordered product formula eq. (3.31), and they allow one to compute by recursion the Fourier modes of a general state $X \in \mathbb{V}_0^{\mathfrak{sl}_2,\boldsymbol{k}}$.

Property (4.92) means that if the vertex algebra zeroth product of X and Y vanishes their Fourier zero-modes generate a commutative subalgebra, in homogeneous degree zero, of $\widetilde{U}(\widehat{\mathfrak{sl}}_2^{\oplus N})$. We let

$$\widehat{Q}_n^\gamma = \int_\gamma \mathcal{P}(z)^{-n/2}\varsigma_n(z)_{[0]}\mathrm{d}z, \tag{4.97}$$

for $n=1,3$. By the same logic as for Proposition 4.4.3, we have the following.

Proposition 4.4.4. *The operators $\widehat{Q}_n^\gamma \in \widetilde{U}_{\boldsymbol{k}}(\widehat{\mathfrak{sl}}_2^{\oplus N})$ have homogeneous degree 0 and they commute amongst themselves, with the generators of $\widehat{\mathfrak{sl}}_2$, and with the quadratic Hamiltonians $\mathcal{H}_i$.*

4.5. Higher local Hamiltonians to sub-leading order

In the previous section, we have shown that it is possible to define quartic local Hamiltonians which commute among themselves and with the quadratic ones, together with the generators of $\mathfrak{sl}_2$. Following the same steps, one could in principle try to construct the Hamiltonians for every exponent of $\widehat{\mathfrak{sl}}_2$. However, the direct calculation (already lengthy in the case of $\varsigma_3(z)_{(0)}\varsigma_3(w)$, as we noted above) becomes computationally very demanding. What we shall do instead is work to next-to-leading order in a certain semiclassical limit, which will at least give a strong consistency check on the existence of the expected Hamiltonian densities.

Thus, let us introduce a formal parameter $\hbar$ and work over $\mathbb{C}[[\hbar]]$. In particular, all vector spaces above are now to be regarded as modules over $\mathbb{C}[[\hbar]]$. We consider the following rescaled generators:

$$I^a \longrightarrow \tilde{I}^a := \hbar I^a,$$

$$\mathsf{k} \longrightarrow \tilde{k} := \mathsf{k}. \tag{4.98}$$

With this re-scaling the commutation relations become

$$[\tilde{I}^{a[p]}_m(z), \tilde{I}^{b[q]}_n(z)] = -\frac{p!q!}{(p+q+1)!}(\hbar f^{ab}_c \tilde{I}^{c[p+q+1]}_{m+n}(z) - \hbar^2 n \delta^{ab}\delta_{m+n,0}\tilde{k}^{[p+q+1]}(z)). \tag{4.99}$$

At this point we can identify the various quantum corrections by their $\hbar$ dependence and work grade by grade. We shall work at next-to-leading order, i.e. the next order beyond the usual semi-classical calculation of Poisson brackets. Thus, we consider the densities of Hamiltonians up to and including the leading quantum corrections at order $\hbar$, and we compute commutators up to and including terms of order $\hbar^2$.

Remark. It is worth remarking that the classical limit, eq. (4.98), that we take is not quite the standard one which recovers the usual classical Gaudin model (cf. [**KLT24**] for a very complete discussion of that limit). From our present perspective this is simply for computational convenience – this limit produces the simplest possible non-trivial check, and had we rescaled the central charges there would be more potential quantum correction terms already at next-to-leading order. But it might be interesting to consider this classical limit in its own right.

Having introduced the formal parameter $\hbar$, there is a gradation on the enveloping algebras in which $\tilde{I}^a$ and $\hbar$ have grade one and $\tilde{k}$ has grade zero. Recall that $\mathcal{V}_n$ denotes the space of homogeneous meromorphic states of degree n, eq. (4.43). Let now $\widetilde{\mathcal{V}}_n \subset \mathcal{V}_n$ denote the subspace consisting of states that are also of grade n in this new gradation (*i.e.* which are sums of terms having exactly n factors of $\tilde{I}^a$ or $\hbar$).

Proposition 4.5.1. *Modulo terms of order $\hbar^2$ there is, up to rescaling, exactly one state $\tilde{\varsigma}_{2n-1} \in \widetilde{\mathcal{V}}^{\mathfrak{sl}_2}_{2n}$, $n \in \mathbb{Z}_{\geq 1}$, such that, for all $x \in \mathfrak{sl}_2$ and $m \in \mathbb{Z}_{\geq 0}$,*

$$\Delta x_m \tilde{\varsigma}_{2n-1} = 0 \qquad \mathrm{mod}\, \hbar^2 D^{(2n-1)}_z \mathcal{V}_{2n-m,2n-1}, \quad \mathrm{mod}\, \hbar^3 \mathcal{V}_{2n-m,2n}. \tag{4.100}$$

Explicitly, modulo terms in $\hbar^2\mathcal{V}_{2n}$,

$$\begin{aligned}\tilde{\varsigma}_{2n-1}(z) &= t_{i_1,\dots,i_{2n}} \tilde{I}^{i_1}_{-1}(z)\tilde{I}^{i_2}_{-1}(z)\cdots\tilde{I}^{i_{2n}}_{-1}(z)\,|0\rangle \\ &\quad + \hbar\frac{n(2n+1)(2n-2)}{(2n-1)} t_{i_1,\dots,i_{2n-4}} f^{abc}\tilde{I}^a_{-2}(z)\tilde{I}^{b\prime}_{-1}(z)\tilde{I}^{(c}_{-1}(z)\tilde{I}^{i_1}_{-1}(z)\cdots\tilde{I}^{i_{2n-4})}_{-1}(z)\,|0\rangle\,.\end{aligned} \tag{4.101}$$

Proof. Given the basis $\{I^r\}^3_{r=1}$ for $\mathfrak{sl}_2$, we need to show that there exist a function $\mathsf{G}_m(z)$ such that

$$\Delta I^r_m \tilde{\varsigma}_{2n-1}(z) = D^{(2n-1)}_z \mathsf{G}^r_m(z) + \mathcal{O}(\hbar^3). \tag{4.102}$$

For $m = 0$, this is always true thanks to the invariance of the tensor (4.35). For the same reasons explained in the previous section, the only relevant check that one needs to make is the one for $m = 1$. From direct calculation, we get

$$\mathsf{G}_1^r(z) = -\frac{4n}{(2n-1)}\hbar^2 t_{i_1,\dots,i_{2n-2}} \tilde{I}_{-1}^{(r}(z)\tilde{I}_{-1}^{i_1}(z)\cdots\tilde{I}_{-1}^{i_{2n-2})}(z)\,|0\rangle\,. \tag{4.103}$$

Note that this result is in accordance with the exact ones obtained for the quadratic ($n = 1$) and the leading order of quartic ($n = 2$) states, cf. eqs. (4.58) and (4.66). □

(Observe that this is consistent with Proposition 4.4.2 because the vectors in eqs. (4.60) and (4.61) all come with factors of $\hbar^2$ in the limit.)

We can now state the following theorem

Theorem 4.5.1. *Let $\tilde{\varsigma}_n(z)$ be as in eq. (4.101) above, for all odd $m, n \in \mathbb{Z}_{\geq 1}$. We have*

$$\tilde{\varsigma}_m(z)_{(0)}\tilde{\varsigma}_n(w) = (nD_z^{(m)} - mD_w^{(n)})\mathsf{A}_{m,n}(z,w) + T\mathsf{B}_{m,n}(z,w) + \mathcal{O}(\hbar^3), \tag{4.104}$$

where $\mathsf{A}_{m,n}(z,w), \mathsf{B}_{m,n}(z,w) \in \widetilde{\mathcal{V}}_{(z,w)}$ are given by

$$\begin{aligned}\mathsf{A}_{m,n}(z,w) =& \zeta_{m,n}\frac{\hbar^2}{z-w}t_{i_1,\dots,i_{m-1}}t_{j_1,\dots,j_{n-1}} \\ &\times T\Big(\tilde{I}_{-1}^{(a}(z)\tilde{I}_{-1}^{i_1}(z)\dots\tilde{I}_{-1}^{i_{m-1})}(z)\Big)\tilde{I}_{-1}^{(a}(w)\tilde{I}_{-1}^{j_1}(w)\dots\tilde{I}_{-1}^{j_{n-1})}(w)\,|0\rangle + \mathcal{O}(\hbar^3),\end{aligned} \tag{4.105}$$

$$\begin{aligned}\mathsf{B}_{m,n}(z,w) =& \zeta_{m,n}\frac{\hbar^2}{(z-w)^2}t_{i_1,\dots,i_{m-1}}t_{j_1,\dots,j_{n-1}} \\ &\times \tilde{I}_{-1}^{(a}(z)\tilde{I}_{-1}^{i_1}(z)\dots\tilde{I}_{-1}^{i_{m-1})}(z)\tilde{I}_{-1}^{(a}(w)\tilde{I}_{-1}^{j_1}(w)\dots\tilde{I}_{-1}^{j_{n-1})}(w)\,|0\rangle + \mathcal{O}(\hbar^3),\end{aligned} \tag{4.106}$$

where

$$\zeta_{m,n} = \frac{2(m+1)(n+1)}{mn}. \tag{4.107}$$

Moreover, $\mathsf{A}_{m,n}(z,w)$ is a regular function for $z = w$ modulo translates and modulo terms proportional to $\hbar^3$.

Proof. The zeroth mode of $\varsigma_m(z) \in \widetilde{\mathcal{V}}_{m+1}$ can be inferred from a purely combinatorial reasoning. Let us start with the top term $\tilde{\varsigma}_m^{\mathrm{TT}}(z)$ of (4.101). We know that computing the zeroth mode, the number of generators in any term we get does not change, but the result will be a state of total depth m and therefore we know there must be at least one generator with a positive mode. We can also use the fact that we are working at leading order in $\hbar$, therefore we could get at least one $\tilde{I}_0$, one $\tilde{I}_1$ or a term with two $\tilde{I}_0$, every other term will be $\mathcal{O}(\hbar^3)$. The only thing to fix is the combinatorial factor describing the number of possible ways to write such terms. The result is

$$\begin{aligned}\tilde{\varsigma}_m^{\mathrm{TT}}(z)_{(0)} = t_{i_1,\dots,i_{m+1}}\Big[&\frac{(m+1)!}{(m-1)!}\tilde{I}_{-2}^{i_1}(z)\tilde{I}_{-1}^{i_2}(z)\dots\tilde{I}_{-1}^{i_m}(z)\tilde{I}_1^{i_{m+1}}(z) \\ &+\frac{(m+1)!}{(m)!}\tilde{I}_{-1}^{i_1}(z)\dots\tilde{I}_{-1}^{i_m}(z)\tilde{I}_0^{i_{m+1}}(z) \\ &+\frac{(m+1)!(m-1)}{2(m-1)!}\tilde{I}_{-2}^{i_1}(z)\tilde{I}_{-1}^{i_2}(z)\dots\tilde{I}_{-1}^{i_{m-1}}(z)\tilde{I}_0^{i_m}(z)\tilde{I}_0^{i_{m+1}}(z)\Big] + \mathcal{O}(\hbar^3).\end{aligned}$$

With similar arguments we can compute the zeroth mode of the correction term $\tilde{\varsigma}^{\mathrm{C}}_m(z)$ of (4.101), the result reads

$$\begin{aligned}
\tilde{\varsigma}^{\mathrm{C}}_m(z)_{(0)} = &-\hbar\xi(m-2)!t_{i_1,\dots,i_{m-3}}f^{abc}\tilde{I}^a_{-2}(z)\tilde{I}^{(b}_{-1}(z)\tilde{I}^{i_1}_{-1}(z)\dots\tilde{I}^{i_{m-3})}_{-1}(z)\tilde{I}^c_0(z)\\
&+\hbar\xi 2(m-2)(m-3)!t_{i_1,\dots,i_{m-3}}f^{abc}\tilde{I}^a_{-2}(z)\tilde{I}^{b\prime}_{-1}(z)\tilde{I}^{(i_1}_{-1}(z)\dots\tilde{I}^{i_{m-3})}_{-1}(z)\tilde{I}^c_0(z)\\
&+\hbar\xi(m-2)(m-3)(m-3)!t_{i_1,\dots,i_{m-5}}f^{abc}\tilde{I}^a_{-2}(z)\tilde{I}^{b\prime}_{-1}(z)\\
&\qquad\times\tilde{I}^{(c}_{-1}(z)\tilde{I}^d_{-1}(z)\tilde{I}^{i_1}_{-1}(z)\dots\tilde{I}^{i_{m-5})}_{-1}\tilde{I}^d_0(z)\\
&-\hbar\xi(m-2)(m-3)(m-3)!t_{i_1,\dots,i_{m-5}}f^{abc}\tilde{I}^d_{-2}(z)\tilde{I}^{a\prime}_{-1}(z)\\
&\qquad\times\tilde{I}^{(b}_{-1}(z)\tilde{I}^d_{-1}(z)\tilde{I}^{i_1}_{-1}(z)\dots\tilde{I}^{i_{m-5})}_{-1}(z)\tilde{I}^c_0(z)+\mathcal{O}(\hbar^3)
\end{aligned}$$

where $\xi = \frac{(m+2)(m+1)(m-1)}{2m}$. At this point, acting with what we have obtained on $\varsigma_n(w)$ and using repeatedly the commutation relations (4.39), we obtain eq. (4.104). $\square$

CHAPTER 5

Wakimoto construction for double loop algebras and ζ-function regularisation

As we have seen in section 3.4, Wakimoto modules play a central role in the description of the Bethe ansatz of Gaudin models of finite type. At the core of this construction, there is the Feigin-Frenkel homomorphism of vertex algebras which allows one to realise an affine Lie algebra in terms of free fields, also known as Wakimoto construction.

In this chapter, we are going to study a possible generalisation of the Feigin-Frenkel homomorphism to the case of double-loop algebras. Following an observation made in [You21], we will construct a vertex algebra depending on some parameter z, with the role of a regulator. This will allow us to introduce a "renormalisation procedure" for the n-th products of this space.

This chapter is entirely based on the recent paper [Fra24].

5.1. The Feigin-Frenkel homomorphism in the affine case

As we have described in section 3.4.1, one can introduce a free field realisation of an affine Lie algebra in terms of a Heisenberg algebra. This was first proposed by Wakimoto [Wak86] in the case of $\widehat{\mathfrak{sl}}_2$, and later generalised to any affine Lie algebra by Feigin and Frenkel in the untwisted case [FF90] and by Szczesny in the twisted one [Szc01].

For any simple Lie algebra $\mathfrak{g}$ of finite type, this realisation can be mathematically formulated as a vertex algebra homomorphism between the vacuum Verma module at critical level over the corresponding untwisted affine algebra $\widehat{\mathfrak{g}}$ and the Fock space for the $\beta\gamma$-system of free fields [FBZ04, Fre07],

$$\theta : \mathbb{V}_0^{\mathfrak{g},-h^\vee} \longrightarrow \mathsf{M}(\mathfrak{n}_+). \tag{5.1}$$

It is natural to ask whether this construction generalises to the case where $\mathfrak{g}$ itself is an untwisted affine algebra. Perhaps surprisingly, as shown in [You21], much of it does, as follows.

The algebra $\mathfrak{g}$ still admits a triangular decomposition, where the various subalgebras are now infinite-dimensional and in general not nilpotent, namely $\mathfrak{g} \cong \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. We denote by $\mathring{\mathfrak{g}} \cong \mathring{\mathfrak{n}}_- \oplus \mathring{\mathfrak{h}} \oplus \mathring{\mathfrak{n}}_+$ the corresponding underlying finite-type Lie algebra, with $\mathring{\Delta}$ its space of roots. A basis of $\mathfrak{g}$ is given by

$$\{\mathsf{k}, \mathsf{d}\} \cup \{J_{a,n}\}_{a\in\mathcal{I};n\in\mathbb{Z}}, \tag{5.2}$$

where $J_{a,n} := J_a \otimes t^n$ and $\mathcal{I} := (\mathring{\Delta} \setminus \{0\}) \cup \{1, \ldots, \operatorname{rank} \mathring{\mathfrak{g}}\}$. Defining the index set $\mathsf{A} := \{(a,0)\}_{a \in \mathring{\Delta}_+} \cup (\mathcal{I} \times \mathbb{Z}_{\geq 1})$, a basis for $\mathfrak{n}_+$ is given by $\{J_{a,n}\}_{(a,n)\in\mathsf{A}}$.

Following [**Kum02**], one can define the pro-nilpotent algebra $\widetilde{\mathfrak{n}}_+ = \prod_{\alpha\in\Delta_+} \mathfrak{g}_\alpha$. More explicitly, this space can be defined as the inverse limit of a system of nilpotent Lie algebras $\mathfrak{n}_+/\mathfrak{n}_{\geq k}$, where

$$\mathfrak{n}_{\geq k} = \bigoplus_{\substack{\alpha\in\Delta_+ \\ \operatorname{ht}(\alpha)\geq k}} \mathfrak{g}_\alpha$$

is the ideal containing elements with degree higher than k. Here ht denotes the grade of a root in the homogeneous gradation, *i.e.* $\operatorname{ht}(\delta n + \alpha) = n$, for $n \in \mathbb{Z}$, where $\delta \in \mathfrak{h}^*$ is the imaginary root, defined as the sum of all roots multiplied by their corresponding Dynkin label. Therefore, elements of this completion are possibly infinite sums of the form $\sum_{\alpha\in\Delta_+} x^\alpha$, with $x^\alpha \in \mathfrak{g}_\alpha$, provided they truncate to finite sum modulo $\mathfrak{n}_{\geq k}$ for any k.

Via the exponential map, one can similarly define the pro-group U as the inverse limit of the groups $\exp(\mathfrak{n}_+/\mathfrak{n}_{\geq k})$, where the multiplicative structure is given by the Baker-Campbell-Hausdorff formula and whose elements are infinite products of the form $\prod_{(a,n)\in\mathsf{A}} \exp(x^{a,n} J_{a,n})$ with $x^{a,n} \in \mathbb{C}$, provided they truncate to finite ones modulo terms $\exp(\mathfrak{n}_+/\mathfrak{n}_{\geq k})$ for any $k \in \mathbb{Z}_{\geq 0}$.

One introduces a set of coordinates $X^{a,n} : U \to \mathbb{C}$ on U, such that for $g = \prod_{(a,n)\in\mathsf{A}} \exp(x^{a,n} J_{a,n})$, $X^{a,n}(g) = x^{a,n}$, which define the $\mathbb{C}$-algebra of polynomial functions on U,

$$\mathcal{O}(\mathfrak{n}_+) = \mathbb{C}[X^{a,n}]_{(a,n)\in\mathsf{A}}. \tag{5.3}$$

In this setting, the Weyl algebra is the free unital $\mathbb{C}$-algebra generated by $X^{a,n}$ and $D_{a,n}$, quotiented by the relations

$$[X^{a,n}, X^{b,m}] = 0 = [D_{a,n}, D_{b,m}], \qquad [D_{a,m}, X^{b,n}] = \delta_a^b \delta_m^n. \tag{5.4}$$

The space of derivations on $\mathcal{O}(\mathfrak{n}_+)$, $\operatorname{Der}\mathcal{O}(\mathfrak{n}_+)$, is a subalgebra of the Weyl algebra with elements of the form $\sum_{(a,n)\in\mathsf{A}} P^{a,n}(X) D_{a,n}$, where $P^{a,n}(X) \in \mathcal{O}(\mathfrak{n}_+)$, where only a finite number of terms is non-zero; the respective completion $\widetilde{\operatorname{Der}}\mathcal{O}(\mathfrak{n}_+)$, is the algebra where this last restriction is lifted.

There is a continuous homomorphism of Lie algebras

$$\begin{aligned} \varrho : \mathfrak{g} &\longrightarrow \widetilde{\operatorname{Der}}\mathcal{O}(\mathfrak{n}_+) \\ A &\longmapsto \sum_{(a,n)\in\mathsf{A}} P_A^{a,n}(X) D_{a,n} \end{aligned} \tag{5.5}$$

The case of $\widehat{\mathfrak{sl}}_2$ has been explicitly worked out in [**You21**]; as a novel example, consider the Cartan-Weyl basis for $\mathfrak{sl}_3$ given by $\{E_{\pm\alpha}\}_{\alpha\in\mathring{\Delta}_+}$ where $\mathring{\Delta}_+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2\}$, together with the Cartan generators $\{H_i\}_{i=1,2}$. An explicit matrix representation is given in terms of 3×3 matrices, by the identification $E_{\alpha_1} \mapsto \mathsf{e}_{12}$, $E_{\alpha_2} \mapsto \mathsf{e}_{23}$, $E_{\alpha_1+\alpha_2} \mapsto \mathsf{e}_{13}$, $E_{-\alpha_1} \mapsto \mathsf{e}_{21}$, $E_{-\alpha_2} \mapsto \mathsf{e}_{32}$, $E_{-\alpha_1-\alpha_2} \mapsto \mathsf{e}_{31}$ and $H_1 \mapsto \mathsf{e}_{11} - \mathsf{e}_{22}$, $H_2 \mapsto \mathsf{e}_{22} - \mathsf{e}_{33}$, where e_{ij} is the matrix with a 1 in position (i,j) and zero elsewhere.

One finds for example

$$\begin{aligned}\varrho(J_{\alpha_1,1}) =& D_{\alpha_1,1} - \sum_{k\geq 2} X^{\alpha_2,k-1} D_{\alpha_1+\alpha_2,k} - \sum_{k\geq 3} X^{-\alpha_1,k-1} D_{1,k} \\ &+ \sum_{k\geq 3} X^{-\alpha_1-\alpha_2,k-1} D_{\alpha_2,k} + 2\sum_{k\geq 3} X^{1,k-1} D_{\alpha_1,k} - \sum_{k\geq 3} X^{2,k-1} D_{\alpha_1\, k} \\ &+ (-X^{-\alpha_2,2} X^{\alpha_2,1} + \dots) D_{\alpha_1,4} + (-X^{-\alpha_1-\alpha_2,2} X^{\alpha_2,1} + \dots) D_{1,4} \\ &+ (-X^{-\alpha_1-\alpha_2,2} X^{\alpha_2,1} + \dots) D_{2,4} + (X^{1,2} X^{\alpha_2,1} + \dots) D_{\alpha_1+\alpha_2,4} + \dots\end{aligned} \tag{5.6}$$

In appendix B we present more examples of this realisation.

5.1.1. The widening gap subalgebra. Let us now briefly comment on the general structure of the terms in eq. (5.6). For each basis element $J_{a,n} \in \mathfrak{g}$ one finds that

$$\varrho(J_{a,n}) = f_{ab}{}^{c} \sum_{k\geq N} X^{b,k-n} D_{c,k} + \sum_{(b,m)\in \mathsf{A}} R^{b,m}_{J_{a,n}}(X) D_{b,m}. \tag{5.7}$$

for some $N \in \mathbb{Z}_{\geq 0}$ depending on a, b, c and n. Here, the first term is a quadratic infinite sum in the generators and will be of central importance in the sections below. Indeed, this kind of sums will be the main source of problems when trying to lift the homomorphism of Lie algebras to one of vertex algebras. The second term is part of a subalgebra of $\widetilde{\mathrm{Der}}\,\mathcal{O}(\mathfrak{n}_+)$, of derivations of $\mathcal{O}(\mathfrak{n}_+)$ with *widening gap*, $\overline{\mathrm{Der}}\,\mathcal{O}(\mathfrak{n}_+)$. We will give a precise definition in section 5.2.4, but roughly speaking these are those possibly infinite sums where the loop degree of each X factor in the polynomial R grows "slower" than the loop degree of the corresponding D, creating a gap between them that eventually widens. For example in the case of $\widehat{\mathfrak{sl}}_3$, the polynomial part of $R^{\alpha_1,7}_{J_{\alpha_1,0}}(X) D_{\alpha_1,7}$ is

$$\begin{aligned}R^{\alpha_1,7}_{J_{\alpha_1,0}}(X) =& X^{-\alpha_2,2} X^{-\alpha_1,2} X^{\alpha_1+\alpha_2,3} + 4X^{1,3}(X^{1,2})^2 \\ &- 4X^{2,2}X^{1,3}X^{1,2} + (X^{2,2})^2 X^{1,3} \\ &- 4(X^{1,1})^2 X^{-\alpha_1,2} X^{\alpha_1,3} - (2X^{1,1})^2 X^{-\alpha_1-\alpha_2,2} X^{\alpha_1+\alpha_2,3} \\ &- 8X^{1,3}X^{1,2}(X^{1,1})^2 + \dots\end{aligned} \tag{5.8}$$

and one sees that each of these terms is a product of monomials with loop degree strictly less than $\lfloor 7/2 \rfloor$; this is part of a pattern, and the gap, $n - n/2 = n/2$ grows unboundedly with n. We define the map

$$\begin{aligned}\nu : \mathfrak{g} &\longrightarrow \widetilde{\mathrm{Der}}\mathcal{O}(\mathfrak{n}_+) \\ J_{a,n} &\longmapsto f_{ab}{}^{c} \sum_{k\geq \max(0,n)} X^{b,k-n} D_{c,k},\end{aligned} \tag{5.9}$$

where $f_{ab}{}^{c}$ are the structure constants of $\mathring{\mathfrak{g}}$.[1]

---

[1]From the definition, if $a \in \mathring{\Delta}_+$, $X^{a,0}$ is part of the coordinate system on U, while if $a \in \mathring{\Delta}_-$ or $a \in \{1,\dots,\mathrm{rank}\,\mathfrak{g}\}$, it is not. In order to keep this fact into account, one should really consider as the lower bound of the sum in eq. (5.9) the expression $\max(\eta(b), n+\eta(a))$, where $\eta : (\mathring{\Delta} \cup \{1,\dots,\mathrm{rank}\,\mathfrak{g}\}) \to \{0,1\}$,

The following theorem makes this observation more precise, showing that apart from the leading monomial in each $P^{a,n}_{J_{a,n}}(X)$, the remaining part always has widening gap:

THEOREM ([**You21**]). *For all* $(a,n) \in \mathcal{I} \times \mathbb{Z}$,

$$\varrho(J_{a,n}) - \nu(J_{a,n}) =: \sum_{(b,m)\in \mathsf{A}} R^{b,m}_{J_{a,n}}(X) D_{b,m} \in \overline{\mathrm{Der}}\, \mathcal{O}(\mathfrak{n}_+), \tag{5.10}$$

where $R \in \mathcal{O}(\mathfrak{n}_+)$.

It is important to stress that the widening gap part may contain *finite* sums and in particular, because of definition eq. (5.9), finite sums of quadratic terms of the form $X^{a,n}D_{a,n}$.

5.1.2. Vertex algebras and splitting map. Following the finite-type construction, the next natural step would be to consider the corresponding vertex algebras and repeat the same construction as above. As before, one can define the vacuum Verma module over the central extension by K of the loop algebra of $\mathfrak{g}$, $\mathfrak{g} \otimes \mathbb{C}[s, s^{-1}]$, which naturally carries the structure of a vertex algebra. We denote it by $\mathbb{V}_0^{\mathfrak{g},K}$, where $\mathbb{V}_0^{\mathfrak{g},K}[1] \simeq \mathfrak{g}$, where we are using the notation from eq. (3.62).

In what follows we use the following convention: the loop mode of the original untwisted affine algebra $\mathfrak{g}$ is always written as a subscript, while the double loop mode, *i.e.* the vertex algebra mode, is in square brackets. We have used a similar notation in eq. (3.98).

Similarly to the finite-type construction, one also defines the Fock module for the $\beta\gamma$-system on $\mathfrak{n}_+$, $\mathsf{M}(\mathfrak{n}_+)$, which has the structure of a vertex algebra. Having in mind the idea of embedding expressions like the one in eq. (5.6), we need to enlarge this space by allowing infinite sums, hence we have to work in a certain completion of this space, $\widetilde{\mathsf{M}}(\mathfrak{n}_+)$.

As before, from (5.5) one can introduce a map of vector spaces

$$\vartheta : \mathbb{V}_0^{\mathfrak{g},K} \to \widetilde{\mathsf{M}}(\mathfrak{n}_+), \tag{5.11}$$

from the vacuum Verma module over the double loop algebra of $\mathfrak{g}$ at level K. As in the finite-type case it turns out that non-negative products are not preserved. Nevertheless, the remarkable result from [**You21**], is that also in the affine setting there exists an analogue of the splitting map (3.128), namely

$$\varphi : \mathfrak{g} \to \Omega_{\mathcal{O}(\mathfrak{n}_+)}, \tag{5.12}$$

where $\Omega_{\mathcal{O}(\mathfrak{n}_+)}$ is the space of 1-forms. It maps

$$J_{a,n} \mapsto \sum_{(b,m)\in \mathsf{A}} Q_{J_{a,n};b,m}(X) dX^{b,m}. \tag{5.13}$$

It has the property that the map

$$\varrho + \varphi : \mathfrak{g} \to \widetilde{\mathrm{Der}}\, \mathcal{O}(\mathfrak{n}_+) \oplus \Omega_{\mathcal{O}(\mathfrak{n}_+)}, \tag{5.14}$$

defined as $\eta(\mathring{\Delta}_+) = 0$, $\eta(\mathring{\Delta}_-) = 1$, $\eta(\{1, \dots, \operatorname{rank} \mathfrak{g}\}) = 1$. In order to keep the notation cleaner, we will implicitly assume this below.

can be lifted to a linear map

$$\vartheta : \mathbb{V}_0^{\mathfrak{g},0} \to \widetilde{\mathsf{M}}(\mathfrak{n}_+), \tag{5.15}$$

from the vacuum Verma module over the *loop* algebra of $\mathfrak{g}$, *i.e.* at zero level, so that the zeroth products of generators are preserved (cfr. [**You21**, Theorem 33]):

$$\vartheta(J_{a,n}[-1]\,|0\rangle)_{(0)}\vartheta(J_{b,m}[-1]\,|0\rangle) = \vartheta(J_{a,n}[-1]\,|0\rangle_{(0)}\,J_{b,m}[-1]\,|0\rangle) \tag{5.16}$$

Unfortunately, the space $\widetilde{\mathsf{M}}(\mathfrak{n}_+)$ is not a vertex algebra. For example, the would-be first products may result in ill-defined divergent sums. Consider the state $a = \sum_{a\in\mathcal{I},k\geq 0}\gamma^{a,k}[0]\beta_{a,k}[-1]\,|0\rangle$; the first product of this element with itself would be

$$\begin{aligned} a_{(1)}a &= \sum_{a\in\mathcal{I},k\geq 0}\gamma^{a,k}[0]\beta_{a,k}[-1]\,|0\rangle_{(1)}\sum_{b\in\mathcal{I},j\geq 0}\gamma^{b,j}[0]\beta_{b,j}[-1]\,|0\rangle \\ &= \sum_{a\in\mathcal{I},k\geq 0}\sum_{b\in\mathcal{I},j\geq 0}\gamma^{a,k}[1]\beta_{a,k}[0]\gamma^{b,j}[0]\beta_{b,j}[-1]\,|0\rangle = \sum_{a\in\mathcal{I}}\sum_{k\geq 0}1\,|0\rangle\,, \end{aligned} \tag{5.17}$$

which clearly diverges. The quadratic infinite sums we saw above also suffer from this problem. In the next sections, we solve this problem by suitably "renormalising" these products.

5.2. The vertex Lie algebra $\widehat{\mathsf{M}}_z[\leq 1]$

In order to define a regularisation procedure to cure the expressions above, we need to define the appropriate space we will work with. To do this, we now proceed by introducing a regulating parameter in the commutation relations of the algebra. The Fock module over this algebra has the structure of a vertex algebra. Finally, we will suitably complete this space to obtain the vertex Lie algebra $\widehat{\mathsf{M}}_z[\leq 1]$, which is our object of interest.

5.2.1. Heisenberg algebra and Fock module. Let $\mathfrak{g}$ be an affine Kac-Moody algebra with triangular decomposition $\mathfrak{g} \simeq \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, whose underlying finite algebra is denoted by $\mathring{\mathfrak{g}}$. Let $\mathsf{A} = \{(a,0)\}_{a\in\mathring{\Delta}_+} \cup (\mathcal{I}\times\mathbb{Z}_{\geq 1})$ be an index set indexing a basis of $\mathfrak{n}_+$, where $\mathring{\Delta}_+$ is the set of positive roots of $\mathring{\mathfrak{g}}$ and $\mathcal{I} = (\mathring{\Delta}\setminus\{0\})\cup\{1,\ldots,\operatorname{rank}\mathring{\mathfrak{g}}\}$.

Let z be a formal parameter. We denote by $\mathbb{C}[z]$, the polynomial ring in z with complex coefficients. Given $M, N \in \mathbb{Z}$ and $(a,m)\in\mathsf{A}$, consider the free unital associative algebra H_z generated by $\beta_{a,m}[M]$, $\gamma^{a,m}[N]$ and $\mathbf{1}$, quotiented by the ideal generated by the commutation relations

$$\begin{gathered} [\beta_{a,m}[M],\beta_{b,n}[N]] = 0, \qquad [\gamma^{a,m}[M],\gamma^{b,n}[N]] = 0, \\ [\beta_{a,m}[M],\gamma^{b,n}[N]] = z^m\delta_{N+M,0}\delta_a^b\delta_m^n\mathbf{1}. \end{gathered} \tag{5.18}$$

This algebra can be seen as a "deformation" of the *Heisenberg algebra* H, that can be recovered by taking the limit $z\to 1$. We introduce the parameter z with the role of *regulator*: its meaning will be clear in the following sections.

In this section, unless otherwise stated, we work over the ring $\mathbb{C}[[z]]$ of formal power series. As we will see, in certain special cases we can work over $\mathbb{C}[z]$ or $\mathbb{C}(z)$.

This algebra represents a system of free fields as it can be decomposed into $\mathsf{H}_z \cong \mathsf{H}_z^+ \otimes \mathsf{H}_z^-$, where

$$\mathsf{H}_z^- \simeq_{\mathbb{C}} \{\gamma^{a,m}[N], \beta_{a,m}[N-1]\}_{(a,m)\in\mathsf{A};N\leq 0}, \tag{5.19}$$

$$\mathsf{H}_z^+ \simeq_{\mathbb{C}} \{\gamma^{a,m}[N], \beta_{a,m}[N-1]\}_{(a,m)\in\mathsf{A};N> 0}, \tag{5.20}$$

are called the *creation* and *annihilation* subalgebras, respectively.

Introducing a vacuum vector $|0\rangle$, we call M_z the induced H_z-module annihilated by H_z^+, on which $\mathbf{1}\,|0\rangle = |0\rangle$.

Denoting by Q the root lattice of $\mathfrak{g}$, there is the $\mathbb{Z}\times Q$ gradation of H_z and consequently of M_z, in which $\beta_{a,n}[N]$ has grade (N,α) and $\gamma^{a,n}[N]$ has grade $(N,-\alpha)$, whenever $J_{a,n}\in\mathfrak{g}_\alpha$ and $|0\rangle$ has degree $(0,0)$. The natural depth gradation of M_z is given by

$$\mathsf{M}_z = \bigoplus_{i=0}^{\infty} \mathsf{M}_z[i] \tag{5.21}$$

and a corresponding filtration $\mathsf{M}_z[\leq K] = \bigoplus_{i=0}^{K} \mathsf{M}_z[i]$ for $K\geq 0$.

More explicitly, the space $\mathsf{M}_z[0]$ is spanned by elements of the form $R(\gamma[0])\,|0\rangle$, while $\mathsf{M}_z[1]$ by finite linear combinations of elements of the form

$$P(\gamma[0])\beta_{a,m}[-1]\,|0\rangle + Q(\gamma[0])\gamma^{a,m}[-1]\,|0\rangle \tag{5.22}$$

where P, Q, R are polynomials in $\mathbb{C}[\gamma^{a,n}[0]]_{(a,n)\in\mathsf{A}}$.

5.2.2. Vertex algebra structure on M_z. The space M_z is called the *Fock module of the* $\beta\gamma$*-system of free fields* and it has the structure of a vertex algebra over $\mathbb{C}[z]$, as defined in section 3.2.1. The state-field-map Y is

$$Y:\mathsf{M}_z \to \operatorname{End}\mathsf{M}_z((x)), \qquad A\mapsto Y(A,x) := \sum_{k\in\mathbb{Z}} A[k]x^{-k-1} \tag{5.23}$$

satisfying the axioms *i*–*iv*, where the kth-mode is the map in $\operatorname{End}(\mathsf{M}_z)$ denoted by

$$\mathsf{M}_z \to \operatorname{End}\mathsf{M}_z, \qquad a\mapsto a[k], \qquad k\in\mathbb{Z}. \tag{5.24}$$

The fields are

$$\beta_{a,n}(x) := \sum_{k\in\mathbb{Z}} \beta_{a,n}[k]x^{-k-1}, \qquad \gamma^{a,n}(x) := \sum_{k\in\mathbb{Z}} \gamma^{a,n}[k]x^{-k}, \tag{5.25}$$

where $\beta_{a,n}[k] := (\beta_{a,n}[-1]\,|0\rangle)_{(k)}$ and $\gamma^{a,n}[k] := (\gamma^{a,n}[0]\,|0\rangle)_{(k-1)}$. Composite fields are obtained by using iteratively eq. (3.29). The translation map is defined as follows

$$T\gamma^{a,n}[N]\,|0\rangle = -(N-1)\gamma^{a,n}[N-1]\,|0\rangle\,, \qquad T\beta_{a,n}[N]\,|0\rangle = -N\beta_{a,n}[N-1]\,|0\rangle\,. \tag{5.26}$$

for $N\in\mathbb{Z}_{\leq 0}$ and $T\,|0\rangle = 0$.

5.2.3. Completion of the Fock module. Having in mind to lift the homomorphism in eq. (5.5) to the vertex algebra case, we also need to be able to work with certain infinite sums.

This is achieved by working in a suitable completion of the space M_z. This means that infinite sums will be allowed provided they truncate to finite ones, modulo terms containing $\beta_{a,m}$ and z^m, for m big enough.

The explicit construction goes as follows. Let us denote by $\mathsf{H}^-_{z,\geq k}$ the two-sided ideal in H^-_z generated by $\{\beta_{a,m}[N] : m \geq k, N \in \mathbb{Z}\}$. Let

$$\mathcal{I}_z[\leq M]_k := \mathsf{M}_z[\leq M] \cap (\mathsf{H}^-_{z,\geq k} |0\rangle), \tag{5.27}$$

for some $M \in \mathbb{Z}_{\geq 0}$. Therefore, $\mathcal{I}_z[\leq M]_k$ is the subspace of $\mathsf{M}_z[\leq M]$ spanned by monomials of depth less or equal to M in the creation operators with some factor $\beta_{a,m}[N]$, with $m \geq k$, $N \in \mathbb{Z}$. One has

$$\mathcal{I}_z[\leq M]_0 \supset \mathcal{I}_z[\leq M]_1 \supset \mathcal{I}_z[\leq M]_2 \supset \dots \tag{5.28}$$

with $\bigcap_{i=0}^{\infty} \mathcal{I}_z[\leq M]_i = \{0\}$. One defines the completed subspaces

$$\widetilde{\mathsf{M}}_z[\leq M] := \varprojlim_k \mathsf{M}_z[\leq M]/\mathcal{I}_z[\leq M]_k. \tag{5.29}$$

To give an element of this inverse limit means to give an element of each space of the inverse system, in a manner compatible with the inclusion maps between them. In this sense one allows infinite sums: every element of the sum is well-defined because each of its truncations is well-defined. Finally, one can consider a completion in the depth direction, from the system of inclusions

$$\widetilde{\mathsf{M}}_z[0] \subset \widetilde{\mathsf{M}}_z[\leq 1] \subset \dots \tag{5.30}$$

by taking the direct limit

$$\widetilde{\mathsf{M}}_z := \varinjlim_M \widetilde{\mathsf{M}}_z[\leq M]. \tag{5.31}$$

Any element of $\widetilde{\mathsf{M}}_z$ is therefore a well-defined element in $\widetilde{\mathsf{M}}_z[\leq M]$ for some M. We have the following result:

Proposition 5.2.1. *The space* $(\widetilde{\mathsf{M}}_z[\leq 1], |0\rangle, T, Y(\bullet, x))$ *is a vertex Lie algebra.*

Proof. The results follows from the fact that each space $\mathsf{M}_z[\leq 1]/\mathcal{I}_z[\leq 1]_k$ has the structure of a vertex Lie algebra over $\mathbb{C}[z]/z^k\mathbb{C}[z]$, with $k \in \mathbb{Z}_{\geq 0}$. Therefore, the inverse limit eq. (5.29) defines a vertex algebra over the inverse limit of rings $\mathbb{C}[z]/z^k\mathbb{C}[z]$, which is the ring of power series $\mathbb{C}[[z]]$. $\square$

For example, for $p(z), q(z) \in \mathbb{C}[[z]]$, one would get

$$(p(z) \sum_{k\geq 1} \gamma^{a,k-1}[0]\beta_{b,k}[-1] |0\rangle)_{(1)} q(z) \sum_{j\geq 0} \gamma^{a,j+1}[0]\beta_{b,j}[-1] |0\rangle$$

$$= p(z)q(z)\sum_{k\geq 1}\sum_{j\geq 1} \gamma^{a,k-1}[1]\beta_{b,k}[0]\gamma^{c,j+1}[0]\beta_{d,j}[-1]\,|0\rangle$$

$$= -\delta^a_d\delta^c_b p(z)q(z)\sum_{k\geq 1} z^{2k-1}\,|0\rangle\,, \tag{5.32}$$

whose k-truncations are all well defined over $\mathbb{C}[z]/z^k\mathbb{C}[z]$, $k \in \mathbb{Z}_{\geq 0}$.

From this simple example we see the fact mentioned above: when $z \to 1$, the vertex algebra structure breaks down since ill-defined quantities start to appear.

5.2.4. States with widening gap. We now restrict the space $\widetilde{\mathsf{M}}_z$, by excluding all those sums that give rise to infinite power series in z, in such a way that the vertex algebra structure is preserved. This follows by extending the idea of widening gap introduced in [**You21**] to this setting.

A family of polynomials $\{P^{a,n}\}_{(a,n)\in\mathsf{A}}$ in $\mathbb{C}[\gamma^{b,m}]_{(b,m)\in\mathsf{A}}$ has widening gap if for all $K \geq 1$, there exists a $\overline{n} \in \mathbb{Z}_{>0}$, such that for all $n \geq \overline{n}$

$$P^{a,n}(\gamma) \in \mathbb{C}[\gamma^{b,m} : m < n - K, b \in \mathcal{I}]. \tag{5.33}$$

Define $\overline{\mathsf{M}}_z \subset \widetilde{\mathsf{M}}_z$, as the space spanned by sums of the form

$$\sum_{(a_i,m_i)_{i=1,\ldots,n}\in\mathsf{A}} P^{a_1,m_1}(\gamma)\cdots P^{a_n,m_n}(\gamma)\beta_{a_1,m_1}[-N_1]\cdots\beta_{a_n,m_n}[-N_n]\,|0\rangle \tag{5.34}$$

where $N_i \in \mathbb{Z}_{>0}$, $i = 1,\ldots,n$ and the polynomials $P^{a_i,m_i}(\gamma)$ have widening gap. By construction, we have that $\mathsf{M}_z \subset \overline{\mathsf{M}}_z$, since any finite sum has obviously a widening gap.

We have the following useful result,

Lemma 5.2.1. *Given a collection of polynomials with widening gap* $\{P^{b,m}(\gamma)\}_{(b,m)\in\mathsf{A}}$,

$$\beta_{a,n}[N]P^{b,m}(\gamma)\,|0\rangle = z^n\frac{\partial P^{b,m}(\gamma)}{\partial\gamma^{a,n}[-N]}\,|0\rangle \tag{5.35}$$

is again a collection of polynomials with widening gap, with $(a,n)\in\mathsf{A}$ *and* $N \in \mathbb{Z}_{\geq 0}$.

The following statement characterises the space of states with widening gap

Lemma 5.2.2. *The space* $(\overline{\mathsf{M}}_z[\leq 1], |0\rangle, T, Y(\bullet, z))$ *is a vertex Lie algebra.*

The proof is essentially the same of [**You21**, Lemma 21], with the addition of the regulator which is this setting does not produce any particular difference. As a matter of fact, when taking the limit $z \to 1$, the vertex Lie algebra structure is not spoiled.

5.2.5. The space $\widehat{\mathsf{M}}_z[\leq 1]$. Recall from section 5.1.1 that the image of the element $J_{a,n} \in \mathfrak{g}$ under the map ϱ from eq. (5.5) contains sums that don't develop a widening gap, namely the image of the map ν from eq. (5.9). To take care of these terms, in this section we introduce a slightly bigger space obtained by adjoining the specific type of infinite sums from eq. (5.9).

Define by $\mathcal{O}_z = \mathbb{C}(z)[\gamma^{a,n}[0]]_{(a,n)\in\mathsf{A}}$ the ring of polynomials in the generators $\gamma^{a,n}[0]$, $a \in \mathcal{I}$ with coefficients in $\mathbb{C}(z)$, the ring of rational functions.

We define the space of *quadratic infinite sums* Q_z as follows

$$\mathsf{Q}_z := \left\{ \sum_{k\geq \max(0,n)} z^{\alpha k}\gamma^{a,k-n}[0]\beta_{b,k}[-1]\,|0\rangle : \alpha \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z} \right\}, \tag{5.36}$$

which clearly are sums that do not develop a widening gap.

Working now over $\mathbb{C}(z)$, we introduce the direct sum

$$\widehat{\mathsf{M}}_z[\leq 1] := \mathsf{Q}_z \oplus \overline{\mathsf{M}}_z[\leq 1] \subset \widetilde{\mathsf{M}}_z[\leq 1]. \tag{5.37}$$

Elements in $\widehat{\mathsf{M}}_z[0]$ are finite sums of terms $R(\gamma)\,|0\rangle$, $R(\gamma) \in \mathcal{O}_z$, while elements in $\widehat{\mathsf{M}}_z[1]$ are of the form

$$\begin{aligned} &\sum_{k\geq \max(0,n)} z^{\alpha k}\gamma^{a,k-n}[0]\beta_{b,k}[-1]\,|0\rangle \\ &\quad + \sum_{(a,n)\in\mathsf{A}} P^{a,n}(\gamma)\beta_{a,n}[-1]\,|0\rangle + \sum_{b,m} Q_{b,m}(\gamma)\gamma^{b,m}[-1]\,|0\rangle\,, \end{aligned} \tag{5.38}$$

where the second possibly infinite sum is over a family of polynomials $P^{a,n}(\gamma) \in \mathcal{O}_z$ with widening gap, while the last sum is finite and $Q(\gamma) \in \mathcal{O}_z$.

5.2.6. Vertex Lie algebra $\widehat{\mathsf{M}}_z[\leq 1]$. Extending the definition of the modes of the states from section 5.2.2 to this space, we see that some products can generate infinite power series in z. In particular, we find that the first products between quadratic infinite sums have coefficients in $\mathbb{C}[[z]]$:

$$\begin{aligned} &\Big(\sum_{k\geq \max(0,n)} z^{\alpha k}\gamma^{a,k-n}[0]\beta_{b,k}[-1]\,|0\rangle\Big)_{(1)} \sum_{j\geq \max(0,m)} z^{\beta j}\gamma^{c,j-m}[0]\beta_{d,j}[-1]\,|0\rangle \\ &= \sum_{k\geq \max(0,n)}\ \sum_{j\geq \max(0,m)} z^{\alpha k}z^{\beta j}\gamma^{a,k-n}[1]\beta_{b,k}[0]\gamma^{c,j-m}[0]\beta_{d,j}[-1]\,|0\rangle \\ &= -\,\delta^a_d\delta^c_b\delta_{m+n,0}z^{-n(\beta+1)} \sum_{k\geq \max(0,m+n)} z^{k(\alpha+\beta+2)}\,|0\rangle \end{aligned} \tag{5.39}$$

$$-\,\delta^a_d\delta^c_b\delta_{m+n,0}z^{-n(\beta+1)} \sum_{k\geq \max(0,n,m+n)}^{\max(0,m+n)-1} z^{k(\alpha+\beta+2)}\,|0\rangle\,. \tag{5.40}$$

In the last step, note that the second sum is quadratic but finite, hence it is a well defined element in $\mathbb{C}[z] \subset \mathbb{C}(z)$. Conversely, the first term is an infinite sum in $\mathbb{C}[[z]]$. Crucially, we can regard it as the expansion of a rational function for $|z| < 1$. By doing this, we can rewrite the first term as

$$\sum_{k\geq \max(0,m+n)} z^{k(\alpha+\beta+2)} = \frac{z^{\max(0,m+n)(\alpha+\beta+2)}}{1 - z^{\alpha+\beta+2}} \in \mathbb{C}(z). \tag{5.41}$$

As a result, we have the following

PROPOSITION 5.2.2. *Regarding infinite sums as the small-z expansions of rational functions, $(\widehat{\mathsf{M}}_z[\leq 1], |0\rangle, T, Y(\bullet, x))$ is a vertex Lie algebra over $\mathbb{C}(z)$.*

PROOF. We first show that the products on $\widehat{\mathsf{M}}_z$ close over $\mathbb{C}[[z]]$.

Since we are considering the first two graded subspaces of $\widehat{\mathsf{M}}_z[0]$ and $\widehat{\mathsf{M}}_z[1]$, we restrict our analysis on the only two non-trivial products on this space: the $_{(0)}$ and $_{(1)}$ product. In more explicit terms, for $f \in \widehat{\mathsf{M}}_z[0]$ and $u, v \in \widehat{\mathsf{M}}_z[1]$, we have the following possible non-trivial combinations

$$f_{(0)}u \in \widehat{\mathsf{M}}_z[0], \qquad u_{(0)}f \in \widehat{\mathsf{M}}_z[0], \qquad u_{(0)}v \in \widehat{\mathsf{M}}_z[1], \qquad u_{(1)}v \in \widehat{\mathsf{M}}_z[0]. \tag{5.42}$$

Moreover, we just need to focus on cross products, *i.e.* products between elements in Q_z and $\overline{\mathsf{M}}_z[\leq 1]$, and products of two infinite quadratic sums in Q_z. The closure of the products between elements in $\overline{\mathsf{M}}_z[\leq 1]$ follows from the fact that it already has the structure of a vertex Lie algebra, as pointed out in Lemma 5.2.2.

Let us start considering the $_{(0)}$ products. It is easy to show that expressions like $(\mathsf{Q}_z)_{(0)}R(\gamma)\,|0\rangle$ and $(\mathsf{Q}_z)_{(0)}R(\gamma)\gamma[-1]\,|0\rangle$ or those with the factors flipped close in $\widehat{\mathsf{M}}_z[\leq 1]$ for any element $R(\gamma) \in \mathcal{O}_z$, as they give rise to finite sums of polynomials of depth 0 or 1 which are well-defined elements in $\widehat{\mathsf{M}}_z[\leq 1]$, respectively.

We need to show the closure for $(\mathsf{Q}_z)_{(0)}\mathsf{Q}_z$ and $(\mathsf{Q}_z)_{(0)}\overline{\mathsf{M}}_z[\leq 1]$. The former reads

$$\begin{aligned}
&\Big(\sum_{k\geq \max(0,n)} z^{\alpha k}\gamma^{a,k-n}[0]\beta_{b,k}[-1]\,|0\rangle\Big)_{(0)} \sum_{j\geq \max(0,m)} z^{\beta j}\gamma^{c,j-m}[0]\beta_{d,j}[-1]\,|0\rangle \\
&= \sum_{k\geq \max(0,n)}\sum_{j\geq \max(0,m)} z^{\alpha k}z^{\beta j}\gamma^{a,k-n}[0]\overbracket{\beta_{b,k}[0]\gamma}^{c,j-m}[0]\beta_{d,j}[-1]\,|0\rangle \\
&+ \sum_{k\geq \max(0,n)}\sum_{j\geq \max(0,m)} z^{\alpha k}z^{\beta j}\overbracket{\gamma^{a,k-n}[1]\beta_{b,k}[-1]\gamma^{c,j-m}[0]\beta}_{d,j}[-1]\,|0\rangle \\
&= + \delta^c_b z^{-m(\alpha+1)} \sum_{k\geq \max(0,m,m+n)} z^{(\alpha+\beta+1)k}\gamma^{a,k-(m+n)}[0]\beta_{d,k}[-1]\,|0\rangle \\
&\quad - \delta^a_d z^{-n(\beta+1)} \sum_{k\geq \max(0,n,m+n)} z^{(\alpha+\beta+1)k}\gamma^{c,k-(m+n)}[0]\beta_{b,k}[-1]\,|0\rangle \\
&= \delta^c_b z^{-m(\alpha+1)} \sum_{k\geq \max(0,m+n)} z^{(\alpha+\beta+1)k}\gamma^{a,k-(m+n)}[0]\beta_{d,k}[-1]\,|0\rangle \\
&\quad - \delta^a_d z^{-n(\beta+1)} \sum_{k\geq \max(0,m+n)} z^{(\alpha+\beta+1)k}\gamma^{c,k-(m+n)}[0]\beta_{b,k}[-1]\,|0\rangle \\
&+ \delta^c_b z^{-m(\alpha+1)} \sum_{k\geq \max(0,m,m+n)}^{\max(0,m+n)-1} z^{(\alpha+\beta+1)k}\gamma^{a,j-(m+n)}[0]\beta_{d,k}[-1]\,|0\rangle \\
&\quad - \delta^a_d z^{-n(\beta+1)} \sum_{k\geq \max(0,n,m+n)}^{\max(0,m+n)-1} z^{(\alpha+\beta+1)k}\gamma^{c,j-(m+n)}[0]\beta_{b,k}[-1]\,|0\rangle
\end{aligned} \tag{5.43}$$

In the last step, we have further decomposed the sums into two terms that manifestly live in Q_z, while the rest are finite sums. Note *en passant* the reason why in the definition

(5.36) we had to include the power of z^α; indeed, setting $\alpha, \beta = 0$, we see that the first two terms in the last expression would not be well-defined.

Consider now the product between one element in Q_z and one infinite sum with widening gap in $\overline{\mathsf{M}}_z[1]$,

$$
\begin{aligned}
&\Big(\sum_{k\geq \max(0,m)} z^{\alpha k}\gamma^{a,k-m}[0]\beta_{b,k}[-1]\,|0\rangle\Big)_{(0)} \sum_{(c,n)\in\mathsf{A}} P^{c,n}\beta_{c,n}[-1]\,|0\rangle\\
&= \sum_{k\geq \max(0,m)} z^{\alpha k}\gamma^{a,k-m}[0]\beta_{b,k}[0] \sum_{(c,n)\in\mathsf{A}} P^{c,n}\beta_{c,n}[-1]\,|0\rangle\\
&\quad + \sum_{k\geq \max(0,m)} z^{\alpha k}\gamma^{a,k-m}[1]\beta_{b,k}[-1] \sum_{(c,n)\in\mathsf{A}} P^{c,n}\beta_{c,n}[-1]\,|0\rangle\\
&= \sum_{k\geq \max(0,m)}\sum_{(c,n)\in\mathsf{A}} z^{k(\alpha+1)}\frac{\partial P^{c,n}}{\partial\gamma^{b,k}}\gamma^{a,k-m}[0]\beta_{c,n}[-1]\,|0\rangle\\
&\quad - \sum_{k\geq \max(0,m)}\sum_{(c,n)\in\mathsf{A}} z^{k(\alpha+1)-m}\delta_{k-m,n}\delta^a_c P^{c,n}\beta_{b,k}[-1]\,|0\rangle
\end{aligned}
\tag{5.44}
$$

The first sum is well defined because, from Lemma 5.2.1, it has widening gap. The second sum is non-zero only if $n > -m$, for all $n, m \in \mathbb{Z}$: this ensures that the combination $k = n+m$ is non-negative, and therefore also the second sum develops widening gap.

Let us consider now the $_{(1)}$ product. We need to show the closure only for products of elements of depth 1, namely $(\mathsf{Q}_z)_{(1)}\mathsf{Q}_z$ and $(\mathsf{Q}_z)_{(1)}\overline{\mathsf{M}}_z[1]$. The first case is

$$
\begin{aligned}
&\Big(\sum_{k\geq \max(0,n)} z^{\alpha k}\gamma^{a,k-n}[0]\beta_{b,k}[-1]\,|0\rangle\Big)_{(1)} \sum_{j\geq \max(0,m)} z^{\beta j}\gamma^{c,j-m}[0]\beta_{d,j}[-1]\,|0\rangle\\
&= \sum_{k\geq \max(0,n)}\sum_{j\geq \max(0,m)} z^{\alpha k}z^{\beta j}\gamma^{a,k-n}[1]\beta_{b,k}[0]\gamma^{c,j-m}[0]\beta_{d,j}[-1]\,|0\rangle\\
&= -\,\delta^a_d\delta^c_b\delta_{m+n,0}z^{-n(\beta+1)} \sum_{k\geq \max(0,n,m+n)} z^{k(\alpha+\beta+2)}\,|0\rangle
\end{aligned}
\tag{5.45}
$$

Consider now the product between one element in Q_z and one infinite sum with widening gap in $\overline{\mathsf{M}}_z[1]$,

$$
\begin{aligned}
&\Big(\sum_{k\geq \max(0,m)} z^{\alpha k}\gamma^{a,k-m}[0]\beta_{b,k}[-1]\,|0\rangle\Big)_{(1)} \sum_{(c,n)\in\mathsf{A}} P^{c,n}\beta_{c,n}[-1]\,|0\rangle\\
&= \sum_{k\geq \max(0,m)}\sum_{(c,n)\in\mathsf{A}} z^{\alpha k}\gamma^{a,k-m}[1]\beta_{b,k}[0] \sum_{(c,n)\in\mathsf{A}} P^{c,n}\beta_{c,n}[-1]\,|0\rangle\\
&= -\sum_{k\geq \max(0,m)}\sum_{(c,n)\in\mathsf{A}} z^{k(\alpha+2)-n}\delta_{k-m,n}\delta^a_c\frac{\partial P^{c,k-m}}{\partial\gamma^{b,k}}\,|0\rangle
\end{aligned}
\tag{5.46}
$$

Since this sum is finite, it represents a well-defined element in $\widehat{\mathsf{M}}_z[0]$. All other combinations of elements give rise to well-defined elements in $\widehat{\mathsf{M}}_z[\leq 1]$.

This shows that the products close on $\widehat{\mathsf{M}}_z[\leq 1]$ over $\mathbb{C}[[z]]$. In particular, since $\widetilde{\mathsf{M}}_z[\leq 1]$ has the structure of a vertex Lie algebra (cfr. Proposition 5.2.1), this implies that $\widehat{\mathsf{M}}_z[\leq 1]$ is a vertex subalgebra of $\widetilde{\mathsf{M}}_z[\leq 1]$ over $\mathbb{C}[[z]]$, as all axioms (*i*)-(*iv*) are satisfied.

These products also close on $\widehat{\mathsf{M}}_z[\leq 1]$ over $\mathbb{C}(z)$, when regarding infinite sums as expansions of rational functions. Indeed, looking at the calculations above, the only difference is in the first product between quadratic states in eq. (5.45), which reads

$$\begin{aligned}&\Big(\sum_{k\geq \max(0,n)} z^{\alpha k}\gamma^{a,k-n}[0]\beta_{b,k}[-1]\,|0\rangle\Big)_{(1)}\sum_{j\geq\max(0,m)} z^{\beta j}\gamma^{c,j-m}[0]\beta_{d,j}[-1]\,|0\rangle\\ &= -\delta^a_d\delta^c_b\delta_{m+n,0}z^{-n(\beta+1)}\left(\frac{z^{\max(0,n)(\alpha+\beta+2)}}{1-z^{\alpha+\beta+2}}+\sum_{k\geq\max(0,n,m+n)}^{\max(0,m+n)-1} z^{k(\alpha+\beta+2)}\right)|0\rangle\,,\end{aligned}\tag{5.47}$$

To finally prove the statement, one has to ensure that the vertex Lie algebra axioms are still satisfied after regarding infinite sums as expansions of rational functions. This is the case for vacuum, translation and skew-symmetry as they are all equality involving single products. More subtle is the case for Borcherds' identities, since nested products appear. However, as pointed out above, resummation is only needed when computing the first products of quadratic states. Writing down explicitly the identities (3.70) for all possible combinations of the products (5.42), we find that the only non-trivial identity which presents a nesting of first products is

$$u_{(1)}(v_{(1)}w) - v_{(1)}(u_{(1)}w) = (u_{(1)}v)_{(1)}w, \tag{5.48}$$

for $u, v, w \in \mathsf{Q}_z$. However, as the first product of such states is proportional to the vacuum and the action of positive modes on the vacuum is zero, this identity is trivially satisfied. □

5.3. Regularising the products

In sections 5.2.3 and 5.2.4, we defined the completion $\widetilde{\mathsf{M}}_z$ of the Fock module of a Heisenberg algebra and we restricted it to the subspace $\overline{\mathsf{M}}_z[\leq 1]$ of sums with widening gap which has the structure of a vertex Lie algebra. Then, in the section above we have introduced the space of main interest for this work, the vertex Lie algebra $\widehat{\mathsf{M}}_z[\leq 1]$ over $\mathbb{C}(z)$, which is spanned by a specific type of quadratic infinite sums and possibly infinite sums with widening gap. We will now proceed to regularise the products.

5.3.1. ζ-function regularisation. For any given expression in $\mathbb{C}(z)$, we introduce the following regularisation procedure

$$\texttt{reg} : \mathbb{C}(z) \to \mathbb{C} \tag{5.49}$$

defined as follows

i) perform the transformation $z \to e^y$;

ii) power expand the resulting term for small values of y;

iii) regard the result as the ratio of Laurent series, which is again a Laurent series;

iv) remove the singular terms and perform the limit $y \to 0$, *i.e.* $z \to 1$ to remove the regulator. This is equivalent to extracting the constant term of the series obtained.

As an example, consider the following

$$\frac{z^3}{1-z^2} \rightsquigarrow \frac{e^{3y}}{1-e^{2y}} \tag{5.50}$$

$$\rightsquigarrow \frac{(1+3y+9y^2/2+\dots)}{1-1-2y-2y^2-\dots} = -\frac{1}{2y}\frac{(1+3y+\dots)}{(1+y+\dots)} \tag{5.51}$$

$$\rightsquigarrow -\frac{1}{2y}(1+3y+\dots)(1-y+\dots) = -\frac{1}{2y} - 1 + \mathcal{O}(y) \rightsquigarrow -1 \tag{5.52}$$

where each arrow corresponds to one of the steps above. Therefore we would write

$$\mathtt{reg}\left[\frac{z^3}{1-z^2}\right] = -1. \tag{5.53}$$

In particular, for any polynomial or rational function which is regular at $z=1$, this procedure is equivalent to the evaluation $z=1$. For this reason, roughly speaking, we can regard this procedure as a "renormalised" version of the limiting procedure $z \to 1$.

5.3.2. Regularisation of the first products. Recall from eq. (5.14) the definition of the Lie algebra map

$$\varrho + \varphi : \mathfrak{g} \to \widetilde{\mathrm{Der}}\,\mathcal{O}(\mathfrak{n}_+) \oplus \Omega_{\mathcal{O}(\mathfrak{n}_+)}. \tag{5.54}$$

We have the embedding into the Fock space

$$\jmath : \widetilde{\mathrm{Der}}\,\mathcal{O}(\mathfrak{n}_+) \oplus \Omega_{\mathcal{O}(\mathfrak{n}_+)} \hookrightarrow \widehat{\mathsf{M}}_z[\leq 1] \tag{5.55}$$

by simply replacing $X^{a,n}$ with $\gamma^{a,n}[0]$, $D_{a,n}$ with $\beta_{a,n}[-1]$ and $dX^{a,n}$ with $\gamma^{a,n}[-1]$.

By identifying $\mathbb{V}_0^{k,0}[1] \simeq \mathfrak{g}$ and then composing the Lie algebra map (5.54) with the embedding eq. (5.55) we obtain a map

$$\vartheta := \jmath \circ (\varrho + \varphi) : \mathbb{V}_0^{\mathfrak{g},0}[\leq 1] \longrightarrow \widehat{\mathsf{M}}_z[\leq 1]. \tag{5.56}$$

More explicitly, any element $J_{a,n}[-1]\,|0\rangle \in \mathbb{V}_0^{\mathfrak{g},0}[1]$ gets mapped to

$$\begin{aligned} &f_{ab}{}^{c} \sum_{k\geq \max(0,n)} \gamma^{b,k-n}[0]\beta_{c,k}[-1]\,|0\rangle \\ &\qquad + \sum_{(b,m)\in \mathsf{A}} R^{b,m}_{J_{a,n}}(\gamma)\beta_{b,m}[-1]\,|0\rangle + \sum_{(b,m)\in\mathsf{A}} Q_{J_{a,n};b,m}(\gamma)\gamma^{b,m}[-1]\,|0\rangle\,. \end{aligned} \tag{5.57}$$

where $R^{b,m}_{J_{a,n}}(\gamma) = \jmath(R^{b,m}_{J_{a,n}}(X))$ from eq. (5.10) and $Q_{J_{a,n};b,m}$ is the image of $\jmath \circ \varphi$, cfr. eq. (5.13), while $|0\rangle \mapsto |0\rangle$.

We have the following

Lemma 5.3.1. *For any two states $J_{a,n}[-1]\,|0\rangle\,, J_{b,m}[-1]\,|0\rangle \in \mathbb{V}_0^{\mathfrak{g},0}[\leq 1]$ one has*

$$\mathtt{reg}[\vartheta(J_{a,n}[-1]\,|0\rangle)_{(0)}\vartheta(J_{b,m}[-1]\,|0\rangle)] = \mathtt{reg}[\vartheta(J_{a,n}[-1]\,|0\rangle_{(0)}\,J_{b,m}[-1]\,|0\rangle)]. \tag{5.58}$$

The proof is essentially the same of [**You21**, Theorem 33]. In particular, the fact that there are no possible double contractions implies that there will never be terms of the form $q(z)\,|0\rangle$, with $q \in \mathbb{C}(z)$. For this reason, the action of the regularisation procedure is simply to compute the limit $z \to 1$, *i.e.* it is equivalent of working in the unregulated setting.

We will now move our attention to first products. As illustrated above, in this case double contractions can appear and the regularisation procedure becomes of central importance. The main result of this chapter is the following

THEOREM 5.3.1. *For any two states* $J_{a,n}[-1]\,|0\rangle\,, J_{b,m}[-1]\,|0\rangle \in \mathbb{V}_0^{\mathfrak{g},0}[\leq 1]$ *one has*

$$\mathtt{reg}[\vartheta(J_{a,n}[-1]\,|0\rangle)_{(1)}\vartheta(J_{b,m}[-1]\,|0\rangle)] = 0. \tag{5.59}$$

PROOF. The proof occupies section 5.4. $\square$

As an example, consider the elements from $\mathbb{V}_0^{\widehat{\mathfrak{sl}}_2,0}$

$$\begin{aligned}
J_{E,2}[-1]\,|0\rangle &\overset{\vartheta}{\longmapsto} \beta_{E,2}[-1]\,|0\rangle - \sum_{k\geq 5}\gamma^{F,k-2}[0]\beta_{H,k}[-1]\,|0\rangle + 2\sum_{k\geq 5}\gamma^{H,k-2}[0]\beta_{E,k}[-1]\,|0\rangle + \dots\\
J_{F,-2}[-1]\,|0\rangle &\overset{\vartheta}{\longmapsto} -14\gamma^{E,2}[-1]\,|0\rangle - \sum_{k\geq 1}\gamma^{E,k+2}[0]\beta_{H,k}[-1]\,|0\rangle + 2\sum_{k\geq 1}\gamma^{H,k+2}[0]\beta_{F,k}[-1]\,|0\rangle + \dots
\end{aligned} \tag{5.60}$$

where we only wrote the quadratic infinite sums and the other terms that could contribute with terms of the form $\mathbb{C}(z)\,|0\rangle$ in the computation of first products; the dots denote all other terms with widening gap. Their first product is

$$\begin{aligned}
&(\beta_{E,2}[-1]\,|0\rangle - \sum_{k\geq 5}\gamma^{F,k-2}[0]\beta_{H,k}[-1]\,|0\rangle + 2\sum_{k\geq 5}\gamma^{H,k-2}[0]\beta_{E,k}[-1]\,|0\rangle + \dots)_{(1)}\\
&\qquad(-14\gamma^{E,2}[-1]\,|0\rangle + \sum_{k\geq 1}\gamma^{E,k+2}[0]\beta_{H,k}[-1]\,|0\rangle - 2\sum_{k\geq 1}\gamma^{H,k+2}[0]\beta_{F,k}[-1]\,|0\rangle + \dots)\\
=&(\beta_{E,2}[1] - \sum_{k\geq 5}\gamma^{F,k-2}[1]\beta_{H,k}[0] + 2\sum_{k\geq 5}\gamma^{H,k-2}[-1]\beta_{E,k}[0] + \dots)\\
&\qquad(-14\gamma^{E,2}[-1]\,|0\rangle + \sum_{k\geq 1}\gamma^{E,k+2}[0]\beta_{H,k}[-1]\,|0\rangle - 2\sum_{k\geq 1}\gamma^{H,k+2}[0]\beta_{F,k}[-1]\,|0\rangle + \dots)\\
=&(-14z^2 - 4\sum_{k\geq 5} z^{2k-2})\,|0\rangle + \dots = (-14z^2 - 4\frac{z^8}{1-z^2})\,|0\rangle + \dots.
\end{aligned} \tag{5.61}$$

In the last line, the dots would correspond to terms which are not of the form $\mathbb{C}(z)\,|0\rangle$. However, the theorem above ensures that they give trivial contribution.

Applying the regularisation procedure to this expression, we obtain

$$\begin{aligned}
&-14z^2 - 4\frac{z^8}{1-z^2} \rightsquigarrow -14e^{2y} - 4\frac{e^{8y}}{1-e^{2y}}\\
&\rightsquigarrow -\frac{1}{2y}\Big(-14\frac{1+2y+\dots}{1+y+\dots} + 14\frac{1+4y+\dots}{1+y+\dots} - 4\frac{1+8y+\dots}{1+y+\dots}\Big)\\
&\quad= \frac{14}{2y}(1+2y+\dots)(1-y+\dots) - \frac{14}{2y}(1+4y+\dots)(1-y+\dots)
\end{aligned}$$

$$-\frac{4}{2y}(1+8y+\dots)(1-y+\dots)\rightsquigarrow 0 \tag{5.62}$$

Remark. It might be tempting to think that one could use this procedure to systematically regularise the products of the vertex Lie algebra $\widehat{\mathsf{M}}_z[\leq 1]$ as follows

$$_{[i]}:=\mathtt{reg}\circ_{(i)}:\widehat{\mathsf{M}}_z[\leq 1]\times\widehat{\mathsf{M}}_z[\leq 1]\to\widehat{\mathsf{M}}[\leq 1], \tag{5.63}$$

and conclude that Lemma 5.3.1 and Theorem 5.3.1 define a homomorphism of vertex Lie algebras. However, the space with these new products has *not* the structure of a vertex Lie algebra, since Borcherds' identities are in general not satisfied.

5.4. Proof of the main theorem

The proof of the theorem makes use of the doubling procedure introduced in [**You21**]. For the sake of completeness, we will first recall the main ideas of that construction, which will be used below.

5.4.1. The doubling trick. Recall from section 5.1.2 that it is not possible to lift the Lie algebra homomorphism at the vertex algebra level because first products are in general not well-defined.

The so-called *doubling trick* was introduced in order to make sense of such products and construct a genuine homomorphism of vertex algebra.

The idea is that the problem can be solved by suitably "glueing together" the algebra $\widetilde{\mathrm{Der}}\,\mathcal{O}(\mathfrak{n}_+)$ with a "negative copy" of itself, $\widetilde{\mathrm{Der}}\,\mathcal{O}(\mathfrak{n}_-)$ acting on the polynomial algebra $\mathcal{O}(\mathfrak{n}_-)=\mathbb{C}[X^{a,n}]_{(a,n)\in\mathsf{A}_-}$, with $\mathsf{A}_-:=(\alpha,0)_{\alpha\in\mathring{\Delta}_-}\cup\mathcal{I}\times\mathbb{Z}_{\leq -1}$. With a construction analogous to the one outlined in the previous subsection, one can define $\mathrm{Der}\,\mathcal{O}(\mathfrak{n}_-)$, its completion $\widetilde{\mathrm{Der}}\,\mathcal{O}(\mathfrak{n}_-)$ and the subalgebra of elements with widening gap $\overline{\mathrm{Der}}\,\mathcal{O}(\mathfrak{n}_-)$. Also, one defines the space $\mathcal{O}:=\mathbb{C}[X^{a,n}]_{(a,n)\in\mathcal{I}\times\mathbb{Z}}$, and accordingly the completion $\widetilde{\mathrm{Der}}\,\mathcal{O}$ and the subalgebra of terms with widening gap $\widetilde{\mathrm{Der}}\,\mathcal{O}$ and $\overline{\mathrm{Der}}\,\mathcal{O}$.

By using the involution map $\tau:\widetilde{\mathrm{Der}}\,\mathcal{O}\to\widetilde{\mathrm{Der}}\,\mathcal{O}$, with the property of exchanging $\widetilde{\mathrm{Der}}\,\mathcal{O}(\mathfrak{n}_+)$ with $\widetilde{\mathrm{Der}}\,\mathcal{O}(\mathfrak{n}_-)$ and vice-versa, one defines

$$\mathsf{\rho}:=\varrho+\tau\circ\varrho\circ\sigma:\mathfrak{g}\to\widetilde{\mathrm{Der}}\,\mathcal{O}(\mathfrak{n}_+)\oplus\widetilde{\mathrm{Der}}\,\mathcal{O}(\mathfrak{n}_-)\hookrightarrow\widetilde{\mathrm{Der}}\,\mathcal{O} \tag{5.64}$$

where $\sigma:\mathfrak{g}\to\mathfrak{g}$ is the Cartan involution, with the property of exchanging $\mathfrak{n}_+$ with $\mathfrak{n}_-$, namely

$$\sigma(J_{E_\alpha,n})=J_{F_\alpha,-n},\qquad \sigma(J_{H_i},n)=-J_{H_i,-n}. \tag{5.65}$$

One can prove a similar result to section 5.1.1, now adapted to the doubled case:

Lemma 5.4.1. *For all $(a,n)\in\mathcal{I}\times\mathbb{Z}$*

$$\mathsf{\rho}(J_{a,n})-f_{ab}{}^{c}\sum_{k\in\mathbb{Z}}X^{b,k-n}D_{c,k}:=\sum_{(b,m)\in\mathcal{I}\times\mathbb{Z}}R^{b,m}_{J_{a,n}}(X)D_{b,m}\in\overline{\mathrm{Der}}\,\mathcal{O} \tag{5.66}$$

where $R\in\mathcal{O}$.

The most important feature is that the infinite quadratic sum now runs over all $k \in \mathbb{Z}$. Let us make an explicit example, in order to understand what this realisation looks like. Consider the element $J_{E,1} \in \widehat{\mathfrak{sl}}_2$. It is realised as

$$\begin{aligned}
\rho(J_{E,1}) =& \underbrace{-\sum_{k\geq 3} X^{F,k-1} D_{H,k} + 2\sum_{k\geq 3} X^{H,k-1} D_{E,k} + \sum_{(b,m)\in \mathsf{A}} {}^{+}R^{b,m}_{E,1}(X) D_{b,m}}_{\varrho(J_{E,1})} \\
& \underbrace{- \sum_{k\leq -1} X^{F,k-1} D_{H,k} + 2\sum_{k\leq -1} X^{H,k-1} D_{E,k} + \sum_{(b,m)\in \mathsf{A}_-} {}^{-}R^{b,m}_{F,-1}(X) D_{b,m}}_{\tau\circ\varrho\circ\sigma(J_{E,1})} \\
=& -\sum_{k\in\mathbb{Z}} X^{F,k-1} D_{H,k} + 2\sum_{k\in\mathbb{Z}} X^{H,k-1} D_{E,k} + \sum_{(b,m)\in \mathsf{A}} {}^{+}R^{b,m}_{E,1}(X) D_{b,m} \\
& + \sum_{(b,m)\in \mathsf{A}_-} {}^{-}R^{b,m}_{F,-1}(X) D_{b,m} + \sum_{k=0}^{2} X^{F,k-1} D_{H,k} - 2\sum_{k=0}^{2} X^{H,k-1} D_{E,k} \\
=& -\sum_{k\in\mathbb{Z}} X^{F,k-1} D_{H,k} + 2\sum_{k\in\mathbb{Z}} X^{H,k-1} D_{E,k} + \sum_{(b,m)\in \mathcal{I}\times\mathbb{Z}} R^{b,m}_{E,1}(X) D_{b,m} \qquad (5.67)
\end{aligned}$$

where $\sum_{(b,m)\in\mathsf{A}_-} {}^{-}R^{b,m}_{F,-1}(X) D_{b,m} = \tau\circ\varrho\circ\sigma(\sum_{(b,m)\in\mathsf{A}} {}^{+}R^{b,m}_{E,1}(X) D_{b,m})$. In the second-to-last step one "fills the gap" between the semi-infinite sums in the positive and negative directions, which is the reason for the appearance of a finite number of *quadratic compensating terms*. The sum $\sum_{(b,m)\in\mathcal{I}\times\mathbb{Z}} R^{b,m}_{E,1}(X) D_{b,m} = \sum_{(b,m)\in\mathsf{A}} {}^{+}R^{b,m}_{E,1}(X) D_{b,m} + \sum_{(b,m)\in\mathsf{A}_-} {}^{-}R^{b,m}_{F,-1}(X) D_{b,m} +$ *finite quadratic compensating terms* in the last line is precisely the r.h.s. of eq. (5.66).

As before, $\widetilde{\mathrm{Der}}\,\mathcal{O}$ can be naturally embedded into the "doubled" Fock space of the $\beta\gamma$-system $\widetilde{\mathsf{M}}_{\mathrm{d}}$ and $\overline{\mathrm{Der}}\,\mathcal{O}$ into $\overline{\mathsf{M}}_{\mathrm{d}} \subset \widetilde{\mathsf{M}}_{\mathrm{d}}$, by the identification $X^{a,n} \mapsto \gamma^{a,n}[0]$ and $D_{a,n} \mapsto \beta_{a,n}[-1]$.

The main advantage of the glueing procedure is that one can now regard the infinite sums of quadratic terms $\sum_{k\in\mathbb{Z}} X^{a,k-n} D_{a,k}$ as new abstract generator $\mathsf{S}^a_{b,n}$ of the loop algebra $\mathfrak{gl}(\mathring{\mathfrak{g}})[t,t^{-1}]$, with commutation relations

$$[\mathsf{S}^a_{b,m}, \mathsf{S}^c_{d,n}] = \delta^c_b \mathsf{S}^a_{d,n+m} - \delta^a_d \mathsf{S}^c_{b,n+m}. \qquad (5.68)$$

Let D be the derivation element for the homogeneous gradation of this algebra, obeying $[\mathsf{D}, \mathsf{S}^a_{b,n}] = n\mathsf{S}^a_{b,n}$. One has the homomorphism $\mathring{\mathfrak{g}}[t,t^{-1}] \to \mathfrak{gl}(\mathring{\mathfrak{g}})[t,t^{-1}]$, given by

$$J_{a,n} \mapsto f_{ab}{}^{c}\, \mathsf{S}^b_{c,n}, \qquad n \in \mathbb{Z}, \qquad (5.69)$$

where we are using the index summation convention for the Lie algebra indices.

This can be extended to the whole affine algebra by declaring

$$\mathsf{k} \mapsto 0, \qquad \mathsf{d} \mapsto \mathsf{D}. \qquad (5.70)$$

One can then introduce the loop algebra $\mathcal{D} = L(\mathfrak{gl}(\mathring{\mathfrak{g}})[t, t^{-1}] \rtimes \mathbb{C}\mathsf{D})$, with generators $\mathsf{S}^a_{b,n}[N]$ and $\mathsf{D}[N]$, where $a, b \in \mathcal{I}$ and $n, N \in \mathbb{Z}$ with the following commutation relations

$$\begin{aligned}[\mathsf{S}^a_{b,n}[N], \mathsf{S}^c_{d,m}[M]] &= \delta^c_b \mathsf{S}^a_{d,n+m}[N+M] - \delta^a_d \mathsf{S}^c_{b,n+m}[N+M] \\ [\mathsf{D}[N], \mathsf{S}^a_{b,n}[M]] &= n\mathsf{S}^a_{b,n}[N+M]\end{aligned} \tag{5.71}$$

Introducing a vacuum vector $|0\rangle$, one defines the vacuum Verma module $\mathbb{V}_0^{\mathfrak{gl}(\mathring{\mathfrak{g}})[t,t^{-1}]\rtimes\mathbb{C}\mathsf{D},0}$ over this loop algebra at level zero. The tensor product of $\mathcal{D} \ltimes \mathsf{H}$ modules,

$$\overline{\mathbf{M}} := \overline{\mathsf{M}}_d \otimes \mathbb{V}_0^{\mathfrak{gl}(\mathring{\mathfrak{g}})[t,t^{-1}]\rtimes\mathbb{C}\mathsf{D},0}, \tag{5.72}$$

has the structure of a vertex algebra.

As in the finite-type case, lifting the homomorphism of Lie algebras ρ to one of vertex algebras from $\mathbb{V}_0^{\mathfrak{g},k}$ to $\overline{\mathbf{M}}$, does not preserve the non-negative products and therefore does not define a homomorphism of vertex algebras. However, one has the doubled analogue of the map eq. (5.12)

$$\phi : \mathfrak{g} \longrightarrow \Omega_{\mathcal{O}}, \tag{5.73}$$

where $\Omega_{\mathcal{O}}$ is the space of one forms $dX^{a,n}$, such that the map $\rho + \phi$ can be lifted to a homomorphism of vertex algebras

$$\theta : \mathbb{V}_0^{\mathfrak{g},0} \longrightarrow \overline{\mathbf{M}}, \tag{5.74}$$

where the level of the vacuum Verma module has to be set to the very particular value $k = 0$.

5.4.2. Undoubling. In order to prove our statement, we need to make contact between the double setting and the undoubled one.

There is the embedding map $p : \overline{\mathbf{M}} \to \widetilde{\mathsf{M}}_{\mathrm{d}}$, mapping the abstract generators to doubly infinite sums, namely

$$\mathsf{S}^a{}_{b,n}[-1]\,|0\rangle \xmapsto{\;p\;} \sum_{k\in\mathbb{Z}} \gamma^{a,k-n}[0]\beta_{b,k}[-1]\,|0\rangle\,, \tag{5.75}$$

$$\mathsf{D}[-1]\,|0\rangle \xmapsto{\;p\;} \sum_{k\in\mathbb{Z}} k\gamma^{a,k}[0]\beta_{a,k}[-1], |0\rangle \tag{5.76}$$

and acting as the identity on the widening gap subspace $\overline{\mathsf{M}}_{\mathrm{d}} \subset \overline{\mathbf{M}}$. We can introduce the projectors onto the positive and negative subspaces

$$\pi_+ : \widetilde{\mathsf{M}}_{\mathrm{d}} \to \widetilde{\mathsf{M}}(\mathfrak{n}_+), \qquad \pi_- : \widetilde{\mathsf{M}}_{\mathrm{d}} \to \widetilde{\mathsf{M}}(\mathfrak{n}_-), \tag{5.77}$$

defined in the obvious way. However, recall that "overlapping terms" with both positive and negative loop modes, like $\gamma^{a,1}[0]\beta_{b,-1}[-1]\,|0\rangle$, are also well defined states in $\widetilde{\mathsf{M}}_{\mathrm{d}}$. Hence, we also define $\pi_0 : \widetilde{\mathsf{M}}_{\mathrm{d}} \to \widetilde{\mathsf{M}}_{\mathrm{d}}$ as follows

$$\pi_0 := \mathrm{id}_{\widetilde{\mathsf{M}}_{\mathrm{d}}} - \pi_+ - \pi_-. \tag{5.78}$$

We define the compositions $p_+ := \pi_+ \circ p : \overline{\mathsf{M}} \to \widetilde{\mathsf{M}}(\mathfrak{n}_+)$, $p_- := \pi_- \circ p : \overline{\mathsf{M}} \to \widetilde{\mathsf{M}}(\mathfrak{n}_-)$ and $p_0 := \pi_0 \circ p : \overline{\mathsf{M}} \to \widetilde{\mathsf{M}}_{\mathrm{d}}$, and therefore we have

$$p = p_+ + p_- + p_0. \tag{5.79}$$

In particular, by the definition of $\overline{\mathsf{M}}$, we have $p_+(\overline{\mathsf{M}}) \subseteq \widehat{\mathsf{M}}_z(\mathfrak{n}_+)$, because the elements in $\mathbb{V}_0^{\mathfrak{gl}(\mathring{\mathfrak{g}})[t,t^{-1}]\rtimes\mathbb{C}\mathsf{D},0}$ are mapped to semi-infinite sums that can always be regarded as elements of Q_z in eq. (5.36) setting $\alpha = 0$, while elements in $\overline{\mathsf{M}}_{\mathrm{d}}$ are mapped to $\overline{\mathsf{M}}_z(\mathfrak{n}_+)$. Recall from section 5.2 that this space is a vertex Lie algebras over $\mathbb{C}(z)$ when regarding infinite sums as the expansion for small z of some rational functions. Similarly, one can repeat similar arguments for the image $p_-(\overline{\mathsf{M}}) \subseteq \widehat{\mathsf{M}}_z(\mathfrak{n}_-)$, finding that it is a vertex Lie algebra over $\mathbb{C}(z^{-1})$ when regarding infinite sums as the *large-z* expansions of rational functions. This allows us to employ the same regularisation procedure as described in section 5.3 to regularise the products on both spaces.

In particular, we have the following result, which relates the values of regularised rational functions in z and z^{-1},

LEMMA 5.4.2. *The regularisation map* $\mathtt{reg}$ *is invariant under the inversion map* $\varpi : z \mapsto z^{-1}$, i.e. $\mathtt{reg}[f(z)] = \mathtt{reg}[\varpi(f(z))]$, *for any* $f \in \mathbb{C}(z)$.

PROOF. We denote by $\{z_1, \dots, z_n\} \subset \mathbb{C}$ the set of poles of $f \in \mathbb{C}(z)$ and by $\{k_1, \dots, k_n\}$ their multiplicities. By partial fraction decomposition, we can write

$$f(z) = \sum_{i=1}^{n} \frac{f_i(z)}{(z - z_i)^{k_i}} + f_0(z) \tag{5.80}$$

where $f_i \in \mathbb{C}[z]$, $i = 0, \dots, n$. Recall that by definition, the regularisation procedure is essentially the evaluation $z \to 1$, whenever this does not produce ill-defined quantities. Hence, if $z_i \neq 1$ one can explicitly evaluate the limit $z \to 1$. In this case, since $z = 1$ is a fixed point for the map ϖ, the result will not change under inversion.

Consider now the case when $z_i = 1$ is one of the poles. We have

$$\frac{1}{(z-1)^{k_i}}, \tag{5.81}$$

where, without loss of generality, we set $f_i(z) = 1$. Recall the expansion

$$\frac{1}{1 - e^y} = -\sum_{k \geq 0} \frac{B_k}{k!} y^{k-1}, \tag{5.82}$$

where B_k are the k-th Bernoulli numbers [Lep99]. By performing the steps *i)* - *iii)* on this expression, we obtain

$$\frac{1}{(z-1)^{k_i}} \rightsquigarrow \frac{(-1)^{k_i}}{(1 - e^y)^{k_i}} = \left[\sum_{j \geq 0} \frac{B_j}{j!} y^{j-1}\right]^{k_i} := y^{-k_i} \left[\sum_{j \geq 0} c_j y^j\right]^{k_i}. \tag{5.83}$$

The constant term can be obtained by extracting the coefficient of the k_i-th power in y of the series in brackets, which will have the following form

$$\sum_{\{j_1,\dots,j_{k_i}\}} c_{j_1}\cdots c_{j_{k_i}} y^{j_1+\cdots+j_{k_i}}, \tag{5.84}$$

where the sum is over all the tuples of reals $\{j_1,\dots,j_{k_i}\}$, with the constraint $\sum_{i=1}^{k_i} j_i = k_i$. For example, if $k_i = 1$, the only tuple one can choose is $\{j_1\} = \{1\}$ and the constant term is just c_1. If $k_i = 2$ the tuples are $\{j_1, j_2\} = \{1,1\},\{0,2\},\{2,0\}$; plugging the values in eq. (5.84) we obtain $c_1^2 + 2c_0c_2$ as constant term.

Repeating analogous steps for the same expression where we first send $z \mapsto z^{-1}$, we get

$$\frac{1}{(z^{-1}-1)^{k_i}} \rightsquigarrow \frac{(-1)^{k_i}}{(1-e^{-y})^{k_i}} = \left[\sum_{j\geq 0}\frac{(-1)^{j-1}B_j}{j!}y^{j-1}\right]^{k_i} := y^{-k_i}\left[\sum_{j\geq 0}(-1)^{j-1}c_j y^j\right]^{k_i}. \tag{5.85}$$

and using the same notations as above the constant term will have the form

$$\sum_{\{j_1,\dots,j_{k_i}\}} (-1)^{j_1+\cdots+j_{k_i}-k_i} c_{j_1}\cdots c_{j_{k_i}} y^{j_1+\cdots+j_{k_i}}. \tag{5.86}$$

Since we have the constraint $\sum_{i=1}^{k_i} j_i = k_i$, the sign disappears. Therefore the constant terms from eq. (5.83) and eq. (5.85) agree. This concludes the proof. $\square$

We can now compute first product in the doubled setting, without abstract generators, using the regulation procedure, as defined in the following lemma

LEMMA 5.4.3. *For any $x, y \in \overline{\mathsf{M}}[\leq 1]$, we have*

$$p(x_{(1)}y) := \mathtt{reg}[p_+(x)_{(1)}p_+(y)] + \mathtt{reg}[p_-(x)_{(1)}p_-(y)] + \lim_{z\to 1}[p_0(x)_{(1)}p_0(y)] \tag{5.87}$$

where $\mathtt{reg}$ *is the regularisation procedure introduced in eq. (5.49).*

PROOF. We need to check all different possible combinations of products, namely when both x and y are in $\mathbb{V}_0^{\mathfrak{gl}(\mathring{\mathfrak{g}})[t,t^{-1}]\rtimes\mathbb{C}\mathsf{D},0}$, when they are both in $\overline{\mathsf{M}}_{\mathrm{d}}[\leq 1]$ and the mixed case.

Consider $\overline{\mathsf{M}}_{\mathrm{d}}[\leq 1]_{(1)}\overline{\mathsf{M}}_{\mathrm{d}}[\leq 1]$. In this case, the embedding map p acts on this space as the identity, by definition. The vertex algebra products are all well-defined, since $\overline{\mathsf{M}}_{\mathrm{d}}$ is a vertex subalgebra over $\mathbb{C}$, hence the regularisation procedure is simply the evaluation at $z \to 1$. Therefore, the right-hand side of eq. (5.87) is just the decomposition of such products relatively to the maps (5.78).

For the mixed case, note that $\overline{\mathsf{M}}_{\mathrm{d}}[\leq 1]$ is a vertex algebra ideal in $\overline{\mathsf{M}}[\leq 1]$, and therefore $\mathbb{V}_0^{\mathfrak{gl}(\mathring{\mathfrak{g}})[t,t^{-1}]\rtimes\mathbb{C}\mathsf{D},0}{}_{(1)}\overline{\mathsf{M}}_{\mathrm{d}}[\leq 1] \subset \overline{\mathsf{M}}_{\mathrm{d}}[\leq 1]$. Moreover, we have

$$\begin{aligned} &p(\mathsf{S}^a_{b,n}[-1]\,|0\rangle)_{(1)}p\Big(\sum_{(c,m)\in\mathcal{I}\times\mathbb{Z}} P^{c,m}(\gamma)\beta_{c,m}[-1]\,|0\rangle\Big) \\ &\quad= \sum_{k\in\mathbb{Z}}\gamma^{a,k-n}[0]\beta_{b,k}[-1]\,|0\rangle_{(1)}\sum_{(c,m)\in\mathcal{I}\times\mathbb{Z}} P^{c,m}(\gamma)\beta_{c,m}[-1]\,|0\rangle \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k\in\mathbb{Z}} \gamma^{a,k-n}[1]\beta_{b,k}[0] \sum_{(c,m)\in\mathcal{I}\times\mathbb{Z}} P^{c,m}(\gamma)\beta_{c,m}[-1]\left|0\right\rangle \\
&= -\sum_{k\in\mathbb{Z}}\sum_{(c,m)\in\mathcal{I}\times\mathbb{Z}} z^{2k-n}\frac{\partial P^{c,m}(\gamma)}{\partial\gamma^{b,k}[0]}\delta^a_c\delta_{k-n,m}\left|0\right\rangle = -\sum_{(c,m)\in\mathcal{I}\times\mathbb{Z}}\delta^a_c z^{2m+n}\frac{\partial P^{c,m}(\gamma)}{\partial\gamma^{b,n+m}[0]}\left|0\right\rangle ,
\end{aligned}
\tag{5.88}
$$

which is precisely the same result one would obtain without embedding the two terms, considering the additional regulated commutation relations

$$
\begin{aligned}
[S^a_{b,n}[N],\gamma^{c,m}[M]] &= z^m\delta^c_b\gamma^{a,m-n}[N+M], \\
[S^a_{b,n}[N],\beta_{c,m}[M]] &= -z^m\delta^a_c\beta_{b,m+n}[N+M].
\end{aligned}
\tag{5.89}
$$

Since the final expression in eq. (5.88) only involves a finite number of non-zero terms, the regularisation procedure is just the evaluation $z\to 1$. As before, the right-hand side of eq. (5.87) just follows from the decomposition eq. (5.79).

The only non-trivial check is when considering two elements in $\mathbb{V}_0^{\mathfrak{gl}(\mathring{\mathfrak{g}})[t,t^{-1}]\rtimes\mathbb{C}\mathsf{D},0}$. In this case, the left-hand side of eq. (5.87) is identically zero by definition. First, we decompose each infinite sum, obtaining

$$
\begin{aligned}
p(\mathsf{S}^a_{b,n}[-1]\left|0\right\rangle) =&\sum_{k\in\mathbb{Z}}\gamma^{a,k-n}[0]\beta_{b,k}[-1]\left|0\right\rangle = \sum_{k\geq\max(0,n)}\gamma^{a,k-n}[0]\beta_{b,k}[-1]\left|0\right\rangle \\
&+\sum_{k\leq-\max(0,n)}\gamma^{a,k-n}[0]\beta_{b,k}[-1]\left|0\right\rangle+\sum_{k=-\max(0,n)+1}^{\max(0,n)+1}\gamma^{a,k-n}[0]\beta_{b,k}[-1]\left|0\right\rangle \\
=&p_+(\mathsf{S}^a_{b,n}[-1]\left|0\right\rangle)+p_-(\mathsf{S}^a_{b,n}[-1]\left|0\right\rangle)+p_0(\mathsf{S}^a_{b,n}[-1]\left|0\right\rangle).
\end{aligned}
\tag{5.90}
$$

where crucially the last sum is always finite. In the case $n=0$, one should instead consider $(\sum_{k>0}+\sum_{k<0}+\delta_{k,0})\gamma^{a,k}[0]\beta_{b,k}[-1]\left|0\right\rangle$. This however does not alter the proof.

Using this identity, we can compute the regularised first products of two such states

$$
\begin{aligned}
\mathtt{reg}[p_+(\mathsf{S}^a_{b,n}[-1]\left|0\right\rangle)_{(1)}p_+(\mathsf{S}^c_{d,m}[-1]\left|0\right\rangle)]+\mathtt{reg}[p_-(\mathsf{S}^a_{b,n}[-1]\left|0\right\rangle)_{(1)}p_-(\mathsf{S}^c_{d,m}[-1]\left|0\right\rangle)] \\
+\mathtt{reg}[p_0(\mathsf{S}^a_{b,n}[-1]\left|0\right\rangle)_{(1)}p_0(\mathsf{S}^c_{d,m}[-1]\left|0\right\rangle)].
\end{aligned}
\tag{5.91}
$$

where no additional cross-terms appear, since they would only give trivial contractions. Since the image $p_0(\mathsf{S}^a_{b,n}[-1]\left|0\right\rangle)$ is always a finite sum, the last term in the previous expression has only a finite number of non-zero contractions. For this reason, strictly speaking, the full regularisation procedure is not needed, as it just corresponds to the limit $z\longrightarrow 1$.

Writing out explicitly the images of the various projectors and computing the products, we obtain

$$
-\delta_{m+n,0}\delta^a_d\delta^c_b\left(\mathtt{reg}\Big[\sum_{k\geq\max(0,n)}z^{2k-n}\Big]+\mathtt{reg}\Big[\sum_{k\leq-\max(0,n)}z^{2k-n}\Big]+\lim_{z\to1}\sum_{k=-\max(0,n)+1}^{\max(0,n)-1}z^{2k-n}\right)
\tag{5.92}
$$

Regarding the infinite sums as expansions of rational functions, we obtain

$$-\delta_{m+n,0}\delta^a_d\delta^c_b\left(\mathtt{reg}\left[\iota_{z=0}\frac{z^{2\max(0,n)-n}}{1-z^2}\right]+\mathtt{reg}\left[\iota_{z=\infty}\frac{z^{-2\max(0,n)-n}}{1-z^{-2}}\right]+\lim_{z\to 1}\sum_{k=-\max(0,n)+1}^{\max(0,n)-1}z^{2k-n}\right). \tag{5.93}$$

Finally, by explicitly performing the regularisation procedure, one finds

$$\begin{aligned}&\mathtt{reg}[p(\mathsf{S}^a_{b,n}[-1]\,|0\rangle)_{(1)}p(\mathsf{S}^c_{d,n}[-1]\,|0\rangle)]\\ &\qquad=-\delta_{m+n,0}\delta^a_d\delta^c_b\Big(1-2\max(0,n)+2\max(0,n)-1\Big)=0.\end{aligned} \tag{5.94}$$

□

As a remark, note that the expression in eq. (5.93) is zero on the nose, even *without* employing the full regularisation procedure outlined in section 5.3. However, as we will see in the proof of the main theorem below, the remarkable cancellations only happen when we perform the other steps of the procedure.

5.4.3. Proof of the main theorem. We now have all the necessary tools to prove the main theorem. Consider the states $J_{a,n}[-1]\,|0\rangle\,, J_{b,m}[-1]\,|0\rangle\in\mathbb{V}^{\mathfrak{g},0}_0$. They can be mapped into $\overline{\mathbf{M}}$, using the vertex algebra map θ in eq. (5.74). Since it is a homomorphism of vertex algebras, we have

$$\theta(J_{a,n}[-1]\,|0\rangle)_{(1)}\theta(J_{b,m}[-1]\,|0\rangle)=\theta(J_{a,n}[-1]\,|0\rangle_{(1)}\,J_{b,m}[-1]\,|0\rangle)=0. \tag{5.95}$$

Here the right-hand side is zero because it is defined from the vacuum Verma module at zero level, cfr. eq. (5.74). Acting on both sides with the map p from eq. (5.79) we get

$$p\left(\theta(J_{a,n}[-1]\,|0\rangle)_{(1)}\theta(J_{b,m}[-1]\,|0\rangle)\right)=0. \tag{5.96}$$

Recall from the doubling construction summarised in section 5.4.1 that the image of θ is obtained by glueing together a positive and a negative copy of the Lie algebra homomorphism (5.5). For this reason, there are no overlapping terms, *i.e.*

$$p_0\circ\theta(x)=0\qquad\text{for all } x\in\mathbb{V}^{\mathfrak{g},0}_0. \tag{5.97}$$

Moreover, for all $x\in\mathbb{V}^{\mathfrak{g},0}_0$ we can identify

$$p_+\circ\theta(x)=\vartheta(x),\qquad p_-\circ\theta(x)=\tau\circ\vartheta\circ\sigma(x) \tag{5.98}$$

These facts can be understood by looking at the first two lines of the example in (5.67). Keeping this in mind, using the result of Lemma 5.4.3 on eq. (5.96), we find

$$\mathtt{reg}[\vartheta(x)_{(1)}\vartheta(y)]+\mathtt{reg}[\tau\circ\vartheta\circ\sigma(x)_{(1)}\tau\circ\vartheta\circ\sigma(y)]=0. \tag{5.99}$$

We will now proceed to show that these terms are in fact equal and therefore independently zero.

The first term will produce either a term proportional to the vacuum, when two quadratic semi-infinite sums are contracted together or when $(\beta_{a,n}[-1]\,|0\rangle)_{(1)} = \beta_{a,n}[1]$ is contracted with a single $\gamma^{a,n}[-1]\,|0\rangle$, or terms of the form $R(\gamma)\,|0\rangle$, with $R \in \mathbb{C}[\gamma^{a,n}]_{(a,n)\in \mathsf{A}_+}$ of degree > 0. Similarly, the second term, being the "negative copy" coming from the glueing procedure, will produce either terms proportional to the vacuum or terms of the form $Q(\gamma)\,|0\rangle$, with $Q \in \mathbb{C}[\gamma^{a,n}]_{(a,n)\in \mathsf{A}_-}$ of degree > 0.

It follows that the two contributions have to be independently zero, exception made for terms of the form $\mathbb{C}(z)\,|0\rangle$.

By direct calculation one finds that this product is proportional to the Killing form, and therefore it is non-trivial only in two cases: $(a,n) = (E_\alpha, n)$, $(b,m) = (E_{-\alpha}, -n)$ for some $\alpha_+ \in \mathring{\Delta}$, $n \in \mathbb{Z}$, and $(a,n) = (H_i, n)$, $(b,m) = (H_i, -n)$, where $H_i \in \mathring{\mathfrak{h}}$ $i = 1, \ldots, \operatorname{rank} \mathring{\mathfrak{g}}$, is an orthogonal basis for the Cartan subalgebra.

In the first case, we can act with the involution σ explicitly and use the symmetry property of the $_{(1)}$ product to get

$$
\begin{aligned}
&\mathtt{reg}[\vartheta(J_{E_\alpha,n}[-1]\,|0\rangle)_{(1)}\vartheta(J_{E_{-\alpha},-n}[-1]\,|0\rangle)] \\
&\qquad\qquad + \mathtt{reg}[\tau\circ\vartheta\circ\sigma(J_{E_\alpha,n}[-1]\,|0\rangle)_{(1)}\tau\circ\vartheta\circ\sigma(J_{E_{-\alpha},-n}[-1]\,|0\rangle)] \\
&= \mathtt{reg}[\vartheta(J_{E_\alpha,n}[-1]\,|0\rangle)_{(1)}\vartheta(J_{E_{-\alpha},-n}[-1]\,|0\rangle)] \\
&\qquad\qquad + \mathtt{reg}[\tau\circ\vartheta(J_{E_{-\alpha},-n}[-1]\,|0\rangle)_{(1)}\tau\circ\vartheta(J_{E_\alpha,n}[-1]\,|0\rangle)] \\
&= \mathtt{reg}[\vartheta(J_{E_\alpha,n}[-1]\,|0\rangle)_{(1)}\vartheta(J_{E_{-\alpha},-n}[-1]\,|0\rangle)] \\
&\qquad\qquad + \mathtt{reg}[\tau\circ\vartheta(J_{E_\alpha,n}[-1]\,|0\rangle)_{(1)}\tau\circ\vartheta(J_{E_{-\alpha},-n}[-1]\,|0\rangle)],
\end{aligned}
\tag{5.100}
$$

where in the last step we have used the symmetry of the first product on the second term (cfr. eq. (3.34)).

In the second case, we similarly obtain

$$
\begin{aligned}
&\mathtt{reg}[\vartheta(J_{H_i,n}[-1]\,|0\rangle)_{(1)}\vartheta(J_{H_i,-n}[-1]\,|0\rangle)] \\
&\qquad\qquad + \mathtt{reg}[\tau\circ\vartheta\circ\sigma(J_{H_i,n}[-1]\,|0\rangle)_{(1)}\tau\circ\vartheta\circ\sigma(J_{H_i,-n}[-1]\,|0\rangle)] \\
&= \mathtt{reg}[\vartheta(J_{H_i,n}[-1]\,|0\rangle)_{(1)}\vartheta(J_{H_i,-n}[-1]\,|0\rangle)] \\
&\qquad\qquad + \mathtt{reg}[\tau\circ\vartheta(J_{H_i,-n}[-1]\,|0\rangle)_{(1)}\tau\circ\vartheta(J_{H_i,n}[-1]\,|0\rangle)] \\
&= \mathtt{reg}[\vartheta(J_{H_i,n}[-1]\,|0\rangle)_{(1)}\vartheta(J_{H_i,-n}[-1]\,|0\rangle)] \\
&\qquad\qquad + \mathtt{reg}[\tau\circ\vartheta(J_{H_i,n}[-1]\,|0\rangle)_{(1)}\tau\circ\vartheta(J_{H_i,-n}[-1]\,|0\rangle)].
\end{aligned}
\tag{5.101}
$$

where in the last term we have used the symmetry of the $_{(1)}$ product.

By definition, the effect of the map τ is to "mirror" the elements in $\widehat{\mathsf{M}}_z[\leq 1]$, to get a realisation of the algebra as derivations on $\mathcal{O}(\mathfrak{n}_-)$. This means that whenever we have a non-trivial contraction between $a, b \in \widehat{\mathsf{M}}_z[\leq 1]$, we also have it between $\tau(a)$ and $\tau(b)$, with the only difference that z is replaced with z^{-1}, *i.e.*

$$
\tau(a)_{(1)}\tau(b) = \varpi(a_{(1)}b) \tag{5.102}
$$

where ϖ is the inversion map $z \mapsto z^{-1}$.

By Lemma 5.4.2, we conclude that after regularisation, the two terms are equal and therefore they must be both independently zero.

Conclusions and outlooks

This thesis examines different aspects within the broad field of quantum integrable field theories. Some of the results led to the explicit definition of new examples of integrable models, while others addressed more abstract questions related to the algebraic formulation of integrable systems. This opens up a series of interesting directions for further investigation, which we will now summarise.

Theories with Hagedorn-like singularity. In chapter 2, we defined a new family of minimal integrable quantum field theories by explicitly constructing their S-matrices in the repulsive regime. In order to achieve this, we imposed all the relevant properties, namely the Yang-Baxter equation, unitarity and crossing symmetry. In accordance with the quantum-inverse scattering program, the resulting scattering matrices can be regarded as exact S-matrices of a factorisable scattering theory. The TBA equations for these models were derived, and thanks to the remarkable relation between the integral kernels in Fourier space, they were recast in a universal form. Despite their apparent simplicity, numerical solutions for spin $s \geq 3/2$ revealed the emergence of singular behaviours, indicative of a second-order phase transition. This result is unexpected, given that the S-matrices constructed using this method typically describe UV-complete quantum field theories. This is exemplified by the sine-Gordon and the sausage models, with spin $s = 1/2$ and $s = 1$, respectively.

Dynkin structure. An intriguing and suggestive fact is that the graphs encoding the structure of TBA equations for these singular cases are of non-Dynkin type, in contrast to those for the sine-Gordon and the sausage models, which are of type D and $\widehat{D}$, respectively. This fact requires further investigation, as it was previously observed that there is a profound relationship between the TBA equations and Dynkin diagrams [**Zam91b, RTV93**]. Furthermore, it is known that the TBA equations can be expressed as the so-called Y-system, whose periodicity properties are directly related with the Dynkin structure of the equations [**KNS11**]. These periodicity relations have been understood in the context of cluster algebras [**Nak10**] and for this reason it would be interesting to explore the non-UV-completeness of the corresponding theories with this language.

It would be interesting to determine whether it is possible to construct minimal theories with similar behaviours. One potential approach to achieving this goal is to define analogous higher-spin theories based on different quantum group symmetries. Another interesting direction is to consider supersymmetric theories. Indeed, the S-matrix for the supersymmetric sine-Gordon model has been constructed in [**Ahn91**] and its TBA is known to be described by a D-type Dynkin diagram, with an additional "fermionic" node attached to the one associated with the massive particle. In light of the results we obtained, one might conjecture that the supersymmetric version of the sausage model develops a singular point, as its TBA graph would be of non-Dynkin type.

Attractive regime. To get a complete description of these theories, one has to study their attractive regime, $\gamma > 1/2s$. It is known that in this case the S-matrix develops poles, which can be interpreted as bound states of solitonic solutions. The first natural question is whether the bootstrap can be "closed", in the spirit of the procedure outlined in section 1.2.6. This is a notoriously challenging task, to the extent that it has not been completed even for the sausage model. Nevertheless, it would be intriguing to investigate the possibility of whether the presence of bound states in the theory could have the effect of "moving the singularity", and eventually remove it.

Affine Gaudin models. In chapters 3 and 4 we introduced the language of Gaudin models, with a particular emphasis on those of affine type. As pointed out in the introduction to this thesis, the reason why they are worthy of study is that they are expected to provide a general framework to describe a vast class of integrable quantum field theories, in analogy with their classical counterparts. In particular, in chapter 4 we have constructed the first example of non-trivial higher Hamiltonian for the affine $\mathfrak{sl}_2$ Gaudin model. Furthermore, we provided the explicit expression up to the next-to-leading order for all higher charges. It is evident from these calculations that the computational power needed for this construction is too high. Consequently, a significant challenge remains in characterising the Gaudin/Bethe subalgebra of commuting charges within this framework.

Higher structures. The most promising prospect would be to undertake a construction *á la* Feigin-Frenkel-Reshetikhin, *i.e.* by identifying the higher Hamiltonians with certain singular vectors in an appropriate space. To do this, the naive expectation is that one would need a "higher analogue" of vertex algebras, depending on two parameters. Unfortunately, to the present day these structures are still not known, although some very recent developments appeared in the context of higher current algebras and factorisation algebras [**CG17**, **FHK19**, **AKY24**].

The meaning of the regularisation procedure. The general expectation is that the "higher analogue" of the Feigin-Frenkel homomorphism of vertex algebra will play a central role in the characterisation of affine Gaudin models. Should this be the case, it would be interesting to understand what role is played by the construction presented in chapter 5 or, in other words, to determine the meaning of the regularisation procedure which, as we saw, one is somehow forced to introduce to make sense of the higher products.

Finally, recall from section 5.3.2 that defining a regularised version of the Feigin-Frenkel map in the affine setting does not give rise to a homomorphism if one regularises systematically every product. However, it would be interesting to show if the map works if one computes all the products and regularises only *at the very end.* This would suggest that the homomorphism works “up to regularisation”, which leads us to wonder if this construction has a more natural interpretation in the language of homotopy theory.

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APPENDIX A

Full expression for $\mathsf{A}_{1,3}(z,w)$ and $\mathsf{B}_{1,3}(z,w)$

The explicit expressions for $\mathsf{A}_{1,3}(z,w)$ and $\mathsf{B}_{1,3}(z,w)$ in $\mathcal{V}_{(z,w)}$ obtained by direct calculations are

$$
\begin{aligned}
\mathsf{A}_{1,3}(z,w) = \Big[& \frac{8}{3}\frac{1}{z-w} I^a_{-2}(z) I^{(a}_{-1}(w) I^b_{-1}(w) I^{b)}_{-1}(w) \\
& + f^{abc}\Big(\frac{80}{9}\frac{1}{z-w} I^a_{-2}(z) I^b_{-2}(w) I^{c\prime}_{-1}(w) - \frac{80}{9}\frac{1}{(z-w)^2} I^a_{-2}(z) I^b_{-2}(w) I^c_{-1}(w) \\
& \qquad + \frac{160}{9}\frac{1}{z-w} I^a_{-3}(z) I^{b\prime}_{-1}(w) I^c_{-1}(w) \Big) \\
& - \frac{80}{9}\frac{1}{z-w} I^a_{-2}(z) I^a_{-3}(w)\mathsf{k}'(w) - \frac{80}{3}\frac{1}{z-w} I^a_{-4}(z) I^a_{-1}(w)\mathsf{k}'(w) \\
& + \frac{320}{27}\frac{1}{(z-w)^3} I^a_{-2}(z) I^a_{-3}(w) + \frac{160}{9}\frac{1}{z-w} I^a_{-4}(z) I^{a\prime\prime}_{-1}(w) \\
& + \frac{320}{9}\frac{1}{(z-w)^3} I^a_{-4}(z) I^a_{-1}(w) + \frac{160}{27}\frac{1}{(z-w)^2} I^a_{-2}(z) I^{a\prime}_{-3}(w) \\
& - \frac{160}{9}\frac{1}{z-w} I^a_{-3}(z) I^{a\prime\prime}_{-2}(w) - \frac{160}{9}\frac{1}{(z-w)^2} I^a_{-3}(z) I^{a\prime}_{-2}(w) \\
& - \frac{160}{9}\frac{1}{(z-w)^2} I^a_{-4}(z) I^{a\prime}_{-1}(w) \Big] |0\rangle\,.
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{B}_{1,3}(z,w) = \Big[& \frac{8}{3}\frac{1}{(z-w)^2} I^a_{-1}(z) I^{(a}_{-1}(w) I^b_{-1}(w) I^{b)}_{-1}(w) \\
& + f^{abc}\Big(- \frac{160}{9}\frac{1}{(z-w)^3} I^a_{-1}(z) I^b_{-2}(w) I^c_{-1}(w) - \frac{80}{9}\frac{1}{(z-w)^2} I^a_{-1}(z) I^{b\prime}_{-1}(w) I^c_{-2}(w) \\
& \qquad - \frac{160}{9}\frac{1}{(z-w)^2} I^a_{-2}(z) I^{b\prime}_{-1}(w) I^c_{-1}(w) \Big) \\
& - \frac{160}{9}\frac{1}{(z-w)^2} I^a_{-1}(z) I^a_{-3}(w)\mathsf{k}'(w) + \frac{1120}{27}\frac{1}{(z-w)^2} I^a_{-1}(z) I^{a\prime\prime}_{-3}(w) \\
& + \frac{640}{27}\frac{1}{(z-w)^3} I^a_{-1}(z) I^{a\prime}_{-3}(w) - \frac{320}{9}\frac{1}{(z-w)^2} I^a_{-2}(z) I^{a\prime}_{-2}(w) \\
& + \frac{320}{9}\frac{1}{(z-w)^4} I^a_{-1}(z) I^a_{-3}(w) - \frac{640}{9}\frac{1}{(z-w)^3} I^a_{-3}(z) I^{a\prime}_{-1}(w) \\
& + \frac{320}{3}\frac{1}{(z-w)^4} I^a_{-3}(z) I^a_{-1}(w) \Big] |0\rangle\,.
\end{aligned}
$$

APPENDIX B

Realisation of $\widehat{\mathfrak{sl}}_3$ as differential operators

Explicit expression for the images of $J_{\alpha_1,n}$, with $n = -1, 0, 1$, through the Lie algebra homomorphism (5.5), truncated up to loop order 3,4 or 5 (depending on the length of the expressions).

$$\begin{aligned}
\varrho(J_{\alpha_1,1}) = D_{\alpha_1,1} &- \sum_{k\geq 2} X^{\alpha_2,k-1} D_{\alpha_1+\alpha_2,k} - \sum_{k\geq 3} X^{-\alpha_1,k-1} D_{1,k} + \sum_{k\geq 3} X^{-\alpha_1-\alpha_2,k-1} D_{-\alpha_2,k} \\
&+ 2\sum_{k\geq 3} X^{1,k-1} D_{\alpha_1,k} - \sum_{k\geq 3} X^{2,k-1} D_{\alpha_1,k} \\
&+ (-X^{-\alpha_2,2} X^{\alpha_2,1} + \dots) D_{\alpha_1,4} + (-X^{-\alpha_1-\alpha_2,2} X^{\alpha_2,1} + \dots) D_{1,4} \\
&+ (-X^{-\alpha_1-\alpha_2,2} X^{\alpha_2,1} + \dots) D_{2,4} + (X^{1,2} X^{\alpha_2,1} + X^{2,2} X^{\alpha_2,1} + \dots) D_{\alpha_1+\alpha_2,4} \\
&+ (-2X^{-\alpha_1,2} X^{\alpha_1,2} - X^{-\alpha_1-\alpha_2,2} X^{\alpha_1+\alpha_2,2} \\
&\qquad - 2X^{1,2} X^{1,2} + 2X^{2,2} X^{1,2} - \frac{1}{2} X^{2,2} X^{2,2} + \dots) D_{\alpha_1,5} \\
&+ (X^{-\alpha_1,2} X^{\alpha_2,2} + \dots) D_{\alpha_2,5} \\
&+ (-X^{-\alpha_1,2} X^{\alpha_1+\alpha_2,2} + 2X^{1,2} X^{\alpha_2,2} - X^{2,2} X^{\alpha_2,2} + \dots) D_{\alpha_1+\alpha_2,5} \\
&+ (X^{-\alpha_1,2} X^{-\alpha_1,2} + \dots) D_{-\alpha_1,5} \\
&+ (-X^{-\alpha_2,2} X^{-\alpha_1,2} + -X^{1,2} X^{-\alpha_1-\alpha_2,2} + 2X^{2,2} X^{-\alpha_1-\alpha_2,2} + \dots) D_{-\alpha_2,5} \\
&- (X^{-\alpha_1-\alpha_2,2} X^{\alpha_2,2} + \dots) D_{2,5} + (X^{-\alpha_1-\alpha_2,2} X^{-\alpha_1,2} + \dots) D_{-\alpha_1-\alpha_2,5} + \dots
\end{aligned} \tag{B.1}$$

$$\begin{aligned}
\varrho(J_{\alpha_1,0}) = D_{\alpha_1,0} &- \sum_{k\geq 0} X^{\alpha_2,k} D_{\alpha_1+\alpha_2,k} - \sum_{k\geq 1} X^{-\alpha_1,k} D_{1,k} + \sum_{k\geq 1} X^{-\alpha_1-\alpha_2,k} D_{-\alpha_2,k} \\
&+ 2\sum_{k\geq 1} X^{1,k} D_{\alpha_1,k} - \sum_{k\geq 1} X^{2,k} D_{\alpha_1,k} \\
&+ (-2X^{1,1} X^{1,1} + 2X^{2,1} X^{1,1} - \frac{1}{2} X^{2,1} X^{2,1} D_{\alpha_1,2} + \dots) D_{\alpha_1,2} + (X^{-\alpha_1,1} X^{-\alpha_1,1} + \dots) D_{-\alpha_1,2} \\
&+ (-X^{-\alpha_2,1} X^{-\alpha_1,1} + \dots) D_{-\alpha_2,2} + (X^{-\alpha_1-\alpha_2,1} X^{-\alpha_1,1} + \dots) D_{-\alpha_1-\alpha_2,2} \\
&+ (X^{-\alpha_2,1} X^{-\alpha_1,1} X^{\alpha_1+\alpha_2,1} + \frac{4}{3} X^{1,1} X^{1,1} X^{1,1} - 2X^{2,1} X^{1,1} X^{1,1} \\
&\qquad + X^{2,1} X^{2,1} X^{1,1} - \frac{1}{6} X^{2,1} X^{2,1} X^{2,1} D_{\alpha_1,3} + \dots) D_{\alpha_1,3} \\
&+ (X^{-\alpha_1,1} X^{-\alpha_1,1} X^{\alpha_1+\alpha_2,1} + \dots) D_{\alpha_2,3}
\end{aligned}$$

$$
\begin{aligned}
&+(-2X^{1,1}X^{1,1}X^{\alpha_2,1}+2X^{2,1}X^{1,1}X^{\alpha_2,1}-\frac{1}{2}X^{2,1}X^{2,1}X^{\alpha_2,1}+\dots)D_{\alpha_1+\alpha_2,3}\\
&+(-X^{-\alpha_1-\alpha_2,1}X^{-\alpha_1,1}X^{\alpha_2,1}+2X^{1,1}X^{-\alpha_1,1}X^{-\alpha_1,1}-X^{2,1}X^{-\alpha_1,1}X^{-\alpha_1,1}+\dots)D_{-\alpha_1,3}\\
&+(X^{-\alpha_1-\alpha_2,1}X^{-\alpha_1,1}X^{\alpha_1,1}+X^{1,1}X^{-\alpha_2,1}X^{-\alpha_1,1}-2X^{2,1}X^{-\alpha_2,1}X^{-\alpha_1,1}+\dots)D_{-\alpha_2,3}\\
&+(-X^{-\alpha_2,1}X^{-\alpha_1,1}X^{-\alpha_1,1}+X^{1,1}X^{-\alpha_1-\alpha_2,1}X^{-\alpha_1,1}+X^{2,1}X^{-\alpha_1-\alpha_2,1}X^{-\alpha_1,1}+\dots)D_{-\alpha_1-\alpha_2,3}\\
&+(-X^{-\alpha_1,1}X^{-\alpha_1,1}X^{\alpha_1,1}-X^{-\alpha_1-\alpha_2,1}X^{-\alpha_1,1}X^{\alpha_1+\alpha_2,1}\dots)D_{1,3}\\
&+(X^{-\alpha_2,1}X^{-\alpha_1,1}X^{\alpha_2,1}-X^{-\alpha_1-\alpha_2,1}X^{-\alpha_1,1}X^{\alpha_1+\alpha_2,1}+\dots)D_{2,3}+\dots
\end{aligned}
\tag{B.2}
$$

$$
\begin{aligned}
\varrho(J_{\alpha_1,-1}) = &-\sum_{k\geq 0}X^{\alpha_2,k+1}D_{\alpha_1+\alpha_2,k}-\sum_{k\geq 1}X^{-\alpha_1,k+1}D_{1,k}+\sum_{k\geq 1}X^{-\alpha_1-\alpha_2,k+1}D_{-\alpha_2,k}\\
&+2\sum_{k\geq 0}X^{1,k+1}D_{\alpha_1,k}-\sum_{k\geq 0}X^{2,k+1}D_{\alpha_1,k}\\
&+(2X^{-\alpha_1,1}X^{\alpha_1,0}+X^{-\alpha_1-\alpha_2,1}X^{\alpha_2,0}X^{\alpha_1,0}+X^{-\alpha_1-\alpha_2,1}X^{\alpha_1+\alpha_2,0}+\dots)D_{\alpha_1,0}\\
&+(-X^{-\alpha_1,1}X^{\alpha_2,0}-X^{-\alpha_1-\alpha_2,1}X^{\alpha_2,0}X^{\alpha_2,0}+\dots)D_{\alpha_2,0}\\
&+(X^{-\alpha_1,1}X^{\alpha_1+\alpha_2,0}-X^{-\alpha_1-\alpha_2,1}X^{\alpha_1+\alpha_2,0}X^{\alpha_2,0}D_{\alpha_1+\alpha_2,0}-2X^{1,1}X^{\alpha_2,0}+X^{2,1}X^{\alpha_2,0}+\dots)D_{\alpha_1+\alpha_2,0}\\
&+(X^{-\alpha_2,1}X^{\alpha_2,1}+2X^{1,1}X^{1,1}-2X^{2,1}X^{1,1}+\frac{1}{2}X^{2,1}X^{2,1}D_{\alpha_1,1}+\dots)D_{\alpha_1,1}\\
&+(-X^{-\alpha_1,1}X^{\alpha_2,1}+\dots)D_{\alpha_2,1}+(-X^{1,1}X^{\alpha_2,1}X^{2,1}X^{\alpha_2,1}+\dots)D_{\alpha_1+\alpha_2,1}\\
&+(-X^{-\alpha_1,1}X^{-\alpha_1,1}+\dots)D_{-\alpha_1,1}+(2X^{1,1}X^{-\alpha_1-\alpha_2,1}-X^{2,1}X^{-\alpha_1-\alpha_2,1}+\dots)D_{-\alpha_2,1}\\
&+(-X^{-\alpha_1-\alpha_2,1}X^{-\alpha_1,1}+\dots)D_{-\alpha_1-\alpha_2,1}+(X^{-\alpha_1-\alpha_2,1}X^{\alpha_2,1}-2X^{1,1}X^{-\alpha_1,1}+X^{2,1}X^{-\alpha_1,1}+\dots)D_{1,1}\\
&+(X^{-\alpha_1-\alpha_2,1}X^{\alpha_2,1}+\dots)D_{2,1}\\
&+(2X^{-\alpha_1,2}X^{\alpha_1,1}+X^{-\alpha_1-\alpha_2,2}X^{\alpha_1+\alpha_2,1}-2X^{1,1}X^{-\alpha_2,1}X^{\alpha_2,1}-\frac{8}{3}X^{1,1}X^{1,1}X^{1,1}\\
&\qquad+X^{2,1}X^{-\alpha_2,1}X^{\alpha_2,1}+4X^{2,1}X^{1,1}X^{1,1}-2X^{2,1}X^{2,1}X^{1,1}+\frac{1}{3}X^{2,1}X^{2,1}X^{2,1}D_{\alpha_1,2}+\dots)D_{\alpha_1,2}\\
&+(-X^{-\alpha_1,2}X^{\alpha_2,1}-X^{1,1}X^{-\alpha_1,1}X^{\alpha_2,1}+2X^{2,1}X^{-\alpha_1,1}X^{\alpha_2,1}+\dots)D_{\alpha_2,2}\\
&+(X^{-\alpha_1,1}X^{\alpha_2,1}X^{\alpha_1,1}+X^{-\alpha_1,2}X^{\alpha_1+\alpha_2,1}+\frac{1}{2}X^{1,1}X^{1,1}X^{\alpha_2,1}\\
&\qquad-2X^{1,2}X^{\alpha_2,1}+X^{2,1}X^{1,1}X^{\alpha_2,1}+\frac{1}{2}X^{2,1}X^{2,1}X^{\alpha_2,1}+X^{2,2}X^{\alpha_2,1}D_{\alpha_1+\alpha_2,2}+\dots)D_{\alpha_1+\alpha_2,2}\\
&+(-X^{-\alpha_1-\alpha_2,1}X^{-\alpha_1,1}X^{\alpha_2,1}+2X^{1,1}X^{-\alpha_1,1}X^{-\alpha_1,1}-X^{2,1}X^{-\alpha_1,1}X^{-\alpha_1,1}+\dots)D_{-\alpha_1,2}\\
&+(-X^{-\alpha_1-\alpha_2,1}X^{-\alpha_2,1}X^{\alpha_2,1}-2X^{1,1}X^{-\alpha_2,1}X^{-\alpha_1,1}+X^{1,1}X^{-\alpha_1-\alpha_2,2}\\
&\qquad+X^{2,1}X^{-\alpha_2,1}X^{-\alpha_1,1}-2X^{2,1}X^{-\alpha_1-\alpha_2,2}D_{-\alpha_2,2}+\dots)D_{-\alpha_2,2}\\
&+(-X^{-\alpha_1-\alpha_2,1}X^{-\alpha_1-\alpha_2,1}X^{\alpha_2,1}+2X^{1,1}X^{-\alpha_1-\alpha_2,1}X^{-\alpha_1,1}\\
&\qquad-X^{2,1}X^{-\alpha_1-\alpha_2,1}X^{-\alpha_1,1}D_{-\alpha_1-\alpha_2,2}+\dots)D_{-\alpha_1-\alpha_2,2}\\
&+(-X^{-\alpha_2,1}X^{-\alpha_1,1}X^{\alpha_2,1}+X^{-\alpha_1-\alpha_2,2}X^{\alpha_2,1}+\dots)D_{2,2}
\end{aligned}
$$

$$
\begin{aligned}
&+ (X^{-\alpha_2,1}X^{-\alpha_1,1}X^{\alpha_2,1}X^{\alpha_1,1} + X^{-\alpha_2,2}X^{\alpha_2,2} + X^{-\alpha_1-\alpha_2,1}X^{-\alpha_2,1}X^{\alpha_1+\alpha_2,1}X^{\alpha_2,1}\\
&\quad + 2X^{1,1}X^{-\alpha_2,1}X^{-\alpha_1,1}X^{\alpha_1+\alpha_2,1} - X^{1,1}X^{-\alpha_1-\alpha_2,2}X^{\alpha_1+\alpha_2,1} + 2X^{1,1}X^{1,1}X^{-\alpha_2,1}X^{\alpha_2,1}\\
&\quad + 2X^{1,1}X^{1,1}X^{1,1}X^{1,1} + 2X^{1,2}X^{1,2} - X^{2,1}X^{-\alpha_2,1}X^{-\alpha_1,1}X^{\alpha_1+\alpha_2,1} + 2X^{2,1}X^{-\alpha_1-\alpha_2,2}X^{\alpha_1+\alpha_2,1}\\
&\quad - 2X^{2,1}X^{1,1}X^{-\alpha_2,1}X^{\alpha_2,1} - 4X^{2,1}X^{1,1}X^{1,1}X^{1,1} + \frac{1}{2}X^{2,1}X^{2,1}X^{-\alpha_2,1}X^{\alpha_2,1}\\
&\quad + 3X^{2,1}X^{2,1}X^{1,1}X^{1,1} - X^{2,1}X^{2,1}X^{2,1}X^{1,1} + \frac{1}{8}X^{2,1}X^{2,1}X^{2,1}X^{2,1}\\
&\quad - 2X^{2,2}X^{1,2} + \frac{1}{2}X^{2,2}X^{2,2} + \dots)D_{\alpha_1,3}\\
&+ (-X^{-\alpha_1,2}X^{\alpha_2,2} - 2X^{-\alpha_2,1}X^{-\alpha_1,1}X^{\alpha_2,1}X^{\alpha_2,1} - X^{-\alpha_1-\alpha_2,1}X^{-\alpha_1,1}X^{\alpha_1+\alpha_2,1}X^{\alpha_2,1}\\
&\quad + X^{-\alpha_1-\alpha_2,2}X^{\alpha_2,1}X^{\alpha_2,1} + 2X^{1,1}X^{-\alpha_1,1}X^{-\alpha_1,1}X^{\alpha_1+\alpha_2,1} - \frac{1}{2}X^{1,1}X^{1,1}X^{-\alpha_1,1}X^{\alpha_2,1}\\
&\quad - X^{2,1}X^{-\alpha_1,1}X^{-\alpha_1,1}X^{\alpha_1+\alpha_2,1} + 2X^{2,1}X^{1,1}X^{-\alpha_1,1}X^{\alpha_2,1} - 2X^{2,1}X^{2,1}X^{-\alpha_1,1}X^{\alpha_2,1} + \dots)D_{\alpha_2,3}\\
&+ (2X^{-\alpha_1,2}X^{\alpha_2,1}X^{\alpha_1,1} - X^{-\alpha_2,1}X^{-\alpha_1,1}X^{\alpha_1+\alpha_2,1}X^{\alpha_2,1} + X^{-\alpha_1-\alpha_2,2}X^{\alpha_1+\alpha_2,1}X^{\alpha_2,1}\\
&\quad + X^{1,1}X^{-\alpha_1,1}X^{\alpha_2,1}X^{\alpha_1,1} - 2X^{1,1}X^{-\alpha_2,1}X^{\alpha_2,1}X^{\alpha_2,1} - \frac{17}{6}X^{1,1}X^{1,1}X^{1,1}X^{\alpha_2,1}\\
&\quad - X^{1,2}X^{\alpha_2,2} - 2X^{2,1}X^{-\alpha_1,1}X^{\alpha_2,1}X^{\alpha_1,1} + X^{2,1}X^{-\alpha_2,1}X^{\alpha_2,1}X^{\alpha_2,1} + \frac{7}{2}X^{2,1}X^{1,1}X^{1,1}X^{\alpha_2,1}\\
&\quad - \frac{5}{2}X^{2,1}X^{2,1}X^{1,1}X^{\alpha_2,1} + \frac{1}{6}X^{2,1}X^{2,1}X^{2,1}X^{\alpha_2,1} - X^{2,2}X^{\alpha_2,2} + \dots)D_{\alpha_1+\alpha_2,3}\\
&+ (-X^{-\alpha_1,2}X^{-\alpha_1,2} + X^{-\alpha_1-\alpha_2,1}X^{-\alpha_1-\alpha_2,1}X^{\alpha_2,1}X^{\alpha_2,1} - 4X^{1,1}X^{-\alpha_1-\alpha_2,1}X^{-\alpha_1,1}X^{\alpha_2,1}\\
&\quad + 4X^{1,1}X^{1,1}X^{-\alpha_1,1}X^{-\alpha_1,1} + 2X^{2,1}X^{-\alpha_1-\alpha_2,1}X^{-\alpha_1,1}X^{\alpha_2,1} - 4X^{2,1}X^{1,1}X^{-\alpha_1,1}X^{-\alpha_1,1}\\
&\quad + X^{2,1}X^{2,1}X^{-\alpha_1,1}X^{-\alpha_1,1} + \dots)D_{-\alpha_1,3}\\
&+ (X^{-\alpha_2,1}X^{-\alpha_2,1}X^{-\alpha_1,1}X^{\alpha_2,1} - X^{-\alpha_1-\alpha_2,1}X^{-\alpha_1-\alpha_2,1}X^{\alpha_2,1}X^{\alpha_1,1}\\
&\quad + 2X^{1,1}X^{-\alpha_1-\alpha_2,1}X^{-\alpha_1,1}X^{\alpha_1,1} + X^{1,1}X^{-\alpha_1-\alpha_2,1}X^{-\alpha_2,1}X^{\alpha_2,1} + 2X^{1,1}X^{1,1}X^{-\alpha_2,1}X^{-\alpha_1,1}\\
&\quad - \frac{1}{2}X^{1,1}X^{1,1}X^{-\alpha_1-\alpha_2,2} + 2X^{1,2}X^{-\alpha_1-\alpha_2,2} - X^{2,1}X^{-\alpha_1-\alpha_2,1}X^{-\alpha_1,1}X^{\alpha_1,1}\\
&\quad - 2X^{2,1}X^{-\alpha_1-\alpha_2,1}X^{-\alpha_2,1}X^{\alpha_2,1} - 5X^{2,1}X^{1,1}X^{-\alpha_2,1}X^{-\alpha_1,1} + 2X^{2,1}X^{1,1}X^{-\alpha_1-\alpha_2,2} + \dots)D_{-\alpha_2,3}\\
&+ (X^{-\alpha_1-\alpha_2,1}X^{-\alpha_2,1}X^{-\alpha_1,1}X^{\alpha_2,1} - X^{-\alpha_1-\alpha_2,2}X^{-\alpha_1,2} - 2X^{1,1}X^{-\alpha_2,1}X^{-\alpha_1,1}X^{-\alpha_1,1}\\
&\quad - X^{1,1}X^{-\alpha_1-\alpha_2,1}X^{-\alpha_1-\alpha_2,1}X^{\alpha_2,1} + 2X^{1,1}X^{1,1}X^{-\alpha_1-\alpha_2,1}X^{-\alpha_1,1} + X^{2,1}X^{-\alpha_2,1}X^{-\alpha_1,1}X^{-\alpha_1,1}\\
&\quad - X^{2,1}X^{-\alpha_1-\alpha_2,1}X^{-\alpha_1-\alpha_2,1}X^{\alpha_2,1} + X^{2,1}X^{1,1}X^{-\alpha_1-\alpha_2,1}X^{-\alpha_1,1}\\
&\quad - X^{2,1}X^{2,1}X^{-\alpha_1-\alpha_2,1}X^{-\alpha_1,1} + \dots)D_{-\alpha_1-\alpha_2,3}\\
&+ (X^{-\alpha_1-\alpha_2,1}X^{-\alpha_1,1}X^{\alpha_2,1}X^{\alpha_1,1} + X^{-\alpha_1-\alpha_2,1}X^{-\alpha_1-\alpha_2,1}X^{\alpha_1+\alpha_2,1}X^{\alpha_2,1} + X^{-\alpha_1-\alpha_2,2}X^{\alpha_2,2}\\
&\quad - 2X^{1,1}X^{-\alpha_1,1}X^{-\alpha_1,1}X^{\alpha_1,1} - 2X^{1,1}X^{-\alpha_1-\alpha_2,1}X^{-\alpha_1,1}X^{\alpha_1+\alpha_2,1} - 2X^{1,2}X^{-\alpha_1,2}\\
&\quad + X^{2,1}X^{-\alpha_1,1}X^{-\alpha_1,1}X^{\alpha_1,1} + X^{2,1}X^{-\alpha_1-\alpha_2,1}X^{-\alpha_1,1}X^{\alpha_1+\alpha_2,1} + X^{2,2}X^{-\alpha_1,2} + \dots)D_{1,3}\\
&+ (X^{-\alpha_1-\alpha_2,1}X^{-\alpha_2,1}X^{\alpha_2,1}X^{\alpha_2,1} + X^{-\alpha_1-\alpha_2,1}X^{-\alpha_1-\alpha_2,1}X^{\alpha_1+\alpha_2,1}X^{\alpha_2,1}\\
&\quad + X^{-\alpha_1-\alpha_2,2}X^{\alpha_2,2} + 2X^{1,1}X^{-\alpha_2,1}X^{-\alpha_1,1}X^{\alpha_2,1} - 2X^{1,1}X^{-\alpha_1-\alpha_2,1}X^{-\alpha_1,1}X^{\alpha_1+\alpha_2,1}
\end{aligned}
$$

$$
\begin{aligned}
&- X^{1,1}X^{-\alpha_1-\alpha_2,2}X^{\alpha_2,1} - X^{2,1}X^{-\alpha_2,1}X^{-\alpha_1,1}X^{\alpha_2,1} + X^{2,1}X^{-\alpha_1-\alpha_2,1}X^{-\alpha_1,1}X^{\alpha_1+\alpha_2,1}\\
&+ 2X^{2,1}X^{-\alpha_1-\alpha_2,2}X^{\alpha_2,1} + 2X^{2,1}X^{2,1}X^{-\alpha_2,1}X^{-\alpha_1,1}\\
&- 2X^{2,1}X^{2,1}X^{-\alpha_1-\alpha_2,2} - X^{2,2}X^{-\alpha_1-\alpha_2,2} + \dots)D_{2,3} + \dots \qquad \text{(B.3)}
\end{aligned}
$$

For the sake of completeness, we also report an example of the image of a generator from $\mathfrak{n}_-$, truncated at order 4.

$$
\begin{aligned}
\varrho(J_{-\alpha_1,1}) &= D_{-\alpha_1,1} + \sum_{k\geq 2} X^{\alpha_1,k-1}D_{1,k} - \sum_{k\geq 3} X^{\alpha_1+\alpha_2,k-1}D_{\alpha_2,k}\\
&+ \sum_{k\geq 2} X^{-\alpha_2,k-1}D_{-\alpha_1-\alpha_2,k} - 2\sum_{k\geq 2} X^{1,k-1}D_{-\alpha_1,k} + \sum_{k\geq 2} X^{2,k-1}D_{-\alpha_1,k}\\
&+ (X^{\alpha_1,1}X^{\alpha_1,1} + \dots)D_{\alpha_1,3} + (-X^{\alpha_2,1}X^{\alpha_1,1} - 2X^{1,1}X^{\alpha_1+\alpha_2,1} + X^{2,1}X^{\alpha_1+\alpha_2,1} + \dots)D_{\alpha_2,3}\\
&+ (-X^{-\alpha_2,1}X^{\alpha_2,1} - 2X^{1,1}X^{1,1} + 2X^{2,1}X^{1,1} - \frac{1}{2}X^{2,1}X^{2,1} + \dots)D_{-\alpha_1,3}\\
&+ (X^{-\alpha_2,1}X^{\alpha_1,1} + \dots)D_{-\alpha_2,3} + (X^{\alpha_1+\alpha_2,1}X^{\alpha_1,1} + \dots)D_{\alpha_1+\alpha_2,3}\\
&+ (X^{1,1}X^{-\alpha_2,1} + X^{2,1}X^{-\alpha_2,1} + \dots)D_{-\alpha_1-\alpha_2,3}\\
&+ (-X^{-\alpha_2,1}X^{\alpha_1+\alpha_2,1} + 2X^{1,1}X^{\alpha_1,1} - X^{2,1}X^{\alpha_1,1} + \dots)D_{1,3} + (-X^{-\alpha_2,1}X^{\alpha_1+\alpha_2,1} + \dots)D_{2,3}\\
&+ (-X^{-\alpha_2,1}X^{\alpha_1+\alpha_2,1}X^{\alpha_1,1} + 2X^{1,1}X^{\alpha_1,1}X^{\alpha_1,1} - X^{2,1}X^{\alpha_1,1}X^{\alpha_1,1} + \dots)D_{\alpha_1,4}\\
&+ (-X^{-\alpha_2,1}X^{\alpha_1+\alpha_2,1}X^{\alpha_2,1} - 2X^{1,1}X^{\alpha_2,1}X^{\alpha_1,1} - 2X^{1,1}X^{1,1}X^{\alpha_1+\alpha_2,1} - X^{1,2}X^{\alpha_1+\alpha_2,1}\\
&\qquad + X^{2,1}X^{\alpha_2,1}X^{\alpha_1,1} + 2X^{2,1}X^{1,1}X^{\alpha_1+\alpha_2,1} - \frac{1}{2}X^{2,1}X^{2,1}X^{\alpha_1+\alpha_2,1} + 2X^{2,2}X^{\alpha_1+\alpha_2,1} + \dots)D_{\alpha_2,4}\\
&+ (-2X^{-\alpha_1,2}X^{\alpha_1,1} - X^{-\alpha_1-\alpha_2,2}X^{\alpha_1+\alpha_2,1} - X^{1,1}X^{-\alpha_2,1}X^{\alpha_2,1} - \frac{4}{3}X^{1,1}X^{1,1}X^{1,1}\\
&\qquad - X^{2,1}X^{-\alpha_2,1}X^{\alpha_2,1} + 2X^{2,1}X^{1,1}X^{1,1} - X^{2,1}X^{2,1}X^{1,1} + \frac{1}{6}X^{2,1}X^{2,1}X^{2,1} + \dots)D_{-\alpha_1,4}\\
&+ (X^{-\alpha_2,2}X^{\alpha_1,1} + X^{1,1}X^{-\alpha_2,1}X^{\alpha_1,1} + X^{2,1}X^{-\alpha_2,1}X^{\alpha_1,1} + \dots)D_{-\alpha_2,4}\\
&+ (X^{\alpha_2,1}X^{\alpha_1,1}X^{\alpha_1,1} + X^{\alpha_1+\alpha_2,1}X^{\alpha_1,2} - X^{-\alpha_2,1}X^{\alpha_1+\alpha_2,1}X^{\alpha_1+\alpha_2,1} + 2X^{1,1}X^{\alpha_1+\alpha_2,1}X^{\alpha_1,1}\\
&\qquad - X^{2,1}X^{\alpha_1+\alpha_2,1}X^{\alpha_1,1} + \dots)D_{\alpha_1+\alpha_2,4}\\
&+ (-X^{-\alpha_1-\alpha_2,2}X^{\alpha_1,1} + \frac{1}{2}X^{1,1}X^{1,1}X^{-\alpha_2,1} + X^{2,1}X^{1,1}X^{-\alpha_2,1} + \frac{1}{2}X^{2,1}X^{2,1}X^{-\alpha_2,1} + \dots)D_{-\alpha_1-\alpha_2,4}\\
&+ (-X^{1,1}X^{-\alpha_2,1}X^{\alpha_1+\alpha_2,1} + 2X^{1,1}X^{1,1}X^{\alpha_1,1} - X^{2,1}X^{-\alpha_2,1}X^{\alpha_1+\alpha_2,1}\\
&\qquad - 2X^{2,1}X^{1,1}X^{\alpha_1,1} + \frac{1}{2}X^{2,1}X^{2,1}X^{\alpha_1,1} + \dots)D_{1,4}\\
&+ (-X^{-\alpha_2,1}X^{\alpha_2,1}X^{\alpha_1,1} - X^{-\alpha_2,2}X^{\alpha_1+\alpha_2,1} - X^{1,1}X^{-\alpha_2,1}X^{\alpha_1+\alpha_2,1}\\
&\qquad - X^{2,1}X^{-\alpha_2,1}X^{\alpha_1+\alpha_2,1} + \dots)D_{2,4} + \dots \qquad \text{(B.4)}
\end{aligned}
$$