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Homotopy representations of extended holomorphic symmetry in holomorphic twists

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ABSTRACT

We argue that holomorphic twists of supersymmetric field theories naturally come with a symmetry L_∞-algebra that nontrivially extends holomorphic symmetry. This symmetry acts on spacetime fields only up to homotopy, and the extension is only visible at the level of higher components of the action. We explicitly compute this for the holomorphic twist of ten-dimensional supersymmetric Yang–Mills theory, which produces a nontrivial action of a higher L_∞-algebra on (a graded version) of five-dimensional affine space.

Contents

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1. Introduction and summary

Twisting of physical theories [1–4] has attracted great interest in the physics literature. In particular, the pure spinor formalism [5–10] (see reviews in [11,12]), which naturally describes such theories as supergravity [13,14], supersymmetric Yang–Mills theory [15,16] and M2-brane models [17–19], accommodates twisting naturally [20,21].

Physical theories come with representations of spacetime symmetry algebras, such as (super-)Poincaré algebras and (super-)conformal algebras. It has been long known that for theories with more than four supercharges it is often difficult to manifest this symmetry 'off shell', that is, without using equations of motion. The pure spinor formalism provides a means of producing off-shell supermultiplets by introducing appropriate infinite towers of auxiliary fields and furthermore shows that the on-shell supermultiplets in fact carry a *homotopy representation* of the spacetime symmetries; the higher components of the action then correspond to the equations of motion needed to make the symmetry algebra close.

In this paper, we argue that holomorphic twists of supersymmetric field theories naturally come with more than just the holomorphic symmetry but rather a certain L_∞-extension of holomorphic symmetry. The extension is not visible at the level of strict representations, but spacetime fields naturally form a homotopy representation of this extended symmetry. We shall treat in detail the example of the holomorphic twist of ten-dimensional supersymmetric Yang–Mills theory. This twisted theory is holomorphic Chern–Simons theory on $\mathbb{C}^5$ [4,20,22], which enjoys a manifest $\mathfrak{isl}(5) = \mathfrak{sl}(5) \ltimes \mathbf{5}$ symmetry. As we shall see, it naturally comes with the extended holomorphic symmetry L_∞-algebra

$$\widetilde{\mathfrak{isl}}(5) := \left(\mathfrak{sl}(5) \ltimes \mathbf{10} \xrightarrow{0} 0 \xrightarrow{0} \mathbf{5}\right) \tag{1}$$

equipped with a certain higher bracket μ_4; and this L_∞-algebra acts on $\mathbb{C}[z^1, z^2, z^3, z^4, z^5]$ (with appropriate grading) in the L_∞-algebraic sense. This may be seen as a non-strict L_∞-algebra action of $\widetilde{\mathfrak{isl}}(5)$ on the (graded version of) five-dimensional complex affine space $\mathbb{A}^5$.

We work with minimal models (of both the symmetry algebra and the field content), which canonically separates the physical information and makes clear the presence of higher-order structures (L_∞-algebras and their representations), rather than a larger strict model, which is not canonical and mixes in the physical degrees of freedom together with the unphysical auxiliary fields; this ensures that all information that we recover is physical and independent of the choice of auxiliary fields.

One way to think about this is to recall that twisting is akin to dimensional reduction [23] in which, rather than eliminating dependence on bosonic coordinates, we eliminate dependence on fermionic coordinates (restrict to Q-closed fields for a supersymmetry Q), which results in the 'pair annihilation' of bosonic and fermionic coordinates. From this perspective, we have an 'as above, so below' heuristic: the actions of twisted theories resemble those of their twistings, just like dimensional reduction preserves the forms of actions. Using the pure spinor formalism, ten-dimensional supersymmetric Yang–Mills theory may be formulated as a holomorphic Chern–Simons theory on a complex (21|16)-dimensional pure spinor superspace (with 10 complexified ordinary spacetime coordinates, 16 ordinary superspace fermionic coordinates, and 11 bosonic pure spinor coordinates). The twisted theory has the same form of a Chern–Simons theory, but this time on 5|0 dimensions, where we have killed 16|16 coordinates. Under this 'dimensional reduction', the ten-dimensional $\mathcal{N} = 1$ super-Poincaré symmetry, which is (55|16)-dimensional, reduces to a (39|0)-dimensional extended holomorphic symmetry. This dimensional reduction corresponds to twisting the supersymmetry algebra and taking the minimal models of the symmetry algebra and its homotopy representation on the field content. The additional factor **10** in (1) and the concomitant μ_4 are the 'dimensionally reduced' remnants of ten-dimensional super-Poincaré symmetry.

The discussion of the present paper is limited to the kinematics, that is, ignoring interactions and considering the linearized theory. This is not an essential restriction. A discussion of the interaction terms should make use of the L_∞-algebra formalism [24–26] for scattering amplitudes; after colour-stripping, we should get a C_∞-algebra [27], on which the extended holomorphic symmetry should act, forming an example of an open–closed homotopy algebra [28–30]. This, however, we leave to a future work.

While we focus on ten-dimensional supersymmetric Yang–Mills theory as a special case, the discussion is generic and applies, in principle, to the twists of any supersymmetric field theory. However, the twists in other dimensions often produce either a strict representation (with the $\widetilde{\mathfrak{isl}}(d)$-representation factoring through an $\mathfrak{isl}(d)$-representation) or a higher representation of an L_∞-superalgebra on affine superspace (with odd coordinates); $\mathbb{A}^5$ is one of the few nontrivial purely bosonic examples that carry a higher symmetry. (For more discussion, see section 4.)

All of our discussion is classical; there may be obstructions to quantization in the form of anomalies. For our main example of the holomorphic twist of ten-dimensional supersymmetric Yang–Mills theory, the twist (five-dimensional holomorphic Chern–Simons theory) is known to have anomalies unless it is coupled in a consistent fashion to Kodaira–Spencer gravity [31,32].

Local operators in a holomorphic theory are expected to form higher analogues of vertex algebras [33–36]. Although the additional $L^{\wedge 2}$ symmetry that we find does not seem to be part of a higher Virasoro algebra (since it is not part of holomorphic symmetries), it may arise as modes of some local operator, in which case it will be part of a higher vertex algebra, and the μ_4 that we find may be part of the higher brackets of the higher vertex algebra.

1.1. Outlook

The technical computation in this paper suggests the possibility that the holomorphic twist of ten-dimensional supersymmetric Yang–Mills theory — and, more generally, various holomorphic field theories obtained via twisting — naturally live on a broader class of 'spaces' than ordinary complex 5-manifolds. Ordinary complex manifolds may be seen as subsets of complex affine space glued together by biholomorphisms on overlaps. However, as this paper shows, the complex affine 5-space on which holomorphically twisted supersymmetric Yang–Mills theory lives enjoys an L_∞-algebra of higher symmetries, which one may try to use to glue together overlaps in a 'higher' fashion to obtain what may be a 'complex 5-manifold up to homotopy'.

1.2. Organization of this paper

This paper is organized as follows. Section 2 reviews the generalities of twisting L_∞-algebras and modules over them and the appearance of higher components of the spacetime symmetry algebras and higher components of nonstrict representations of L_∞-algebras, both in the untwisted and twisted cases. Section 3 then computes the higher components of the representation of supersymmetry for ten-dimensional supersymmetric Yang–Mills theory, the higher products of the corresponding twisted extended holomorphic algebra, and the higher components of its representation on the twisted supermultiplet. Section 4 briefly surveys phenomena that appear in dimensions other than ten.

In the body of the paper, we will usually refer to irreducible representations of $\mathfrak{sl}(5)$ using their Dynkin labels, supplemented by Young tableaux where they are helpful.

2. Mathematical background

Here we briefly review the relevant concepts of twisting of L_∞-algebras and their modules. For more detailed reviews, see [37–40].

2.1. L_∞-algebras

An L_∞-algebra is a homotopy generalization of the concept of a Lie algebra.

Definition 1. An L_∞-algebra $(\mathfrak{g}, \{\mu_k\}_{k\geq 1})$ consists of a graded vector space $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}^i$ together with skew-symmetric, multilinear maps $\mu_k\colon \mathfrak{g}^{\wedge k} \to \mathfrak{g}$ of degree $2-k$ for $k \in \{1,2,3,\dots\}$ that satisfy the identity

$$0 = \sum_{\substack{i+j=n\\ \sigma\in\mathrm{Sh}(i,j)}} (-1)^j \chi(\sigma,x)\mu_{j+1}(\mu_i(x_{\sigma(1)},\dots,x_{\sigma(i)}),\dots,x_{\sigma(i+j)}) = 0. \tag{2}$$

In the above, $\mathrm{Sh}(j_1,\dots,j_k)$ denotes the collection of *shuffles*, which are permutations σ of $\{1,\dots,j_1+\dots+j_k\}$ such that $\sigma(1) < \dots < \sigma(j_1)$ and $\sigma(j_1+1) < \dots < \sigma(j_1+j_2)$ and so on up to $\sigma(j_1+\dots+j_{k-1}+1) < \dots < \sigma(j_1+\dots+j_k)$. The symbol $\chi(\sigma,x)$ denotes the skew-symmetric Koszul sign

$$x_1 \wedge \dots \wedge x_k = \chi(\sigma,x) x_{\sigma(1)} \wedge \dots \wedge x_{\sigma(k)}, \tag{3}$$

defined for homogeneous elements $x_1,\dots,x_k \in \mathfrak{g}$ inside the exterior algebra $\bigwedge^\bullet \mathfrak{g}$.

In what follows, we will often leave the products $\{\mu_k^{\mathfrak{g}}\}_{k\geq 1}$ implicit, and simply refer to an L_∞-algebra through its underlying graded vector space. The identities (2) imply that $\mu_1 \circ \mu_1 = 0$ so that $\mathfrak{g}$ is in particular a cochain complex.

Definition 2. A morphism of L_∞-algebras $\phi\colon (\mathfrak{g}, \{\mu_k^{\mathfrak{g}}\}_{k\geq 1}) \rightsquigarrow (\mathfrak{h}, \{\mu_k^{\mathfrak{h}}\}_{k\geq 1})$ consists of skew-symmetric, multilinear component maps

$$\phi^{(n)}\colon \mathfrak{g}^{\wedge n} \to \mathfrak{h} \tag{4}$$

of degree $1-n$ for $n \in \{1,2,\dots\}$, satisfying the following coherence relations:

$$\begin{aligned}&\sum_{\substack{j\in\{1,\dots,i\}\\ k_1+\dots+k_j=i\\ \sigma\in\mathrm{Sh}(k_1,\dots,k_j)}} \frac{\zeta(\sigma,k,x)}{j!}\mu_j^{\mathfrak{h}}\big(\phi^{(k_1)}(x_{\sigma(1)},\dots,x_{\sigma(k_1)}),\dots,\phi^{(k_j)}(x_{\sigma(k_1+\dots+k_{j-1}+1)},\dots,x_{\sigma(i)})\big)\\ &\quad= \sum_{\substack{j+k=i\\ \sigma\in\mathrm{Sh}(j,k)}} (-1)^k \chi(\sigma,x)\phi^{(k+1)}\big(\mu_j^{\mathfrak{g}}(x_{\sigma(1)},\dots,x_{\sigma(j)}),x_{\sigma(j+1)},\dots,x_{\sigma(i)}\big),\end{aligned} \tag{5}$$

where

$$\zeta(\sigma,k,x) := \chi(\sigma,x)(-1)^{\sum_{1\le m<n\le j} k_m k_n + \sum_{m=1}^{j-1} k_m(j-m) + \sum_{m=2}^{j}(1-k_m)\sum_{k=1}^{k_1+\cdots+k_{m-1}} |x_{\sigma(k)}|}. \tag{6}$$

We shall sometimes omit the μ_k and just write $\mathfrak{g} \rightsquigarrow \mathfrak{h}$. L_∞-morphisms compose associatively, so that one has the category whose objects are L_∞-algebras and whose morphisms are L_∞-morphisms between them.

An L_∞-morphism is an *L_∞-(quasi-)isomorphism* if the first component map is a (quasi-)isomorphism of the underlying cochain complexes.

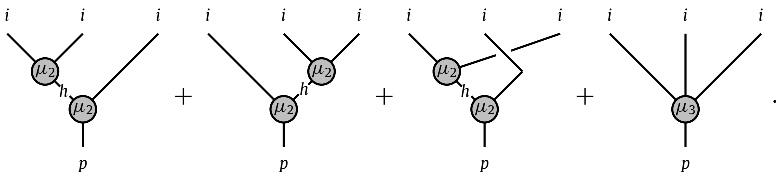
Homotopy transfer of L_∞-algebras. L_∞-algebras admit a good homotopy theory in the sense that minimal models exist and can be computed by homotopy transfer using a strong deformation retract. Let us sketch how this works. Concretely, given an L_∞-algebra $(\mathfrak{g}, \{\mu_k^{\mathfrak{g}}\}_{k\ge1})$ one can always choose a *strong deformation retract*, denoted by a triple (i,p,h), from the underlying cochain complex $(\mathfrak{g},\mu_1)$ to its cohomology $\mathrm{H}(\mathfrak{g})$:

$$h \circlearrowright (\mathfrak{g},\mu_1) \underset{i}{\overset{p}{\rightleftarrows}} (\mathrm{H}(\mathfrak{g}),0) \tag{7}$$

(i.e. $pi = \mathrm{id}_{\mathrm{H}(\mathfrak{g})}$ and $ip = \mathrm{id}_{\mathfrak{g}} - [d,h]$). Then there exists an L_∞-algebra structure on the cohomology $\mathrm{H}(\mathfrak{g})$ together with an L_∞-quasi-isomorphism

$$e\colon\ \mathrm{H}(\mathfrak{g}) \rightsquigarrow \mathfrak{g}, \tag{8}$$

whose first component is $e^{(1)} = i$; furthermore, there exist explicit formulae for the L_∞-algebra structure of $\mathrm{H}(\mathfrak{g})$ and the quasi-isomorphism e in terms of (i,p,h) [37], e.g. using the tensor trick [41], which can be interpreted as a sum over Feynman diagrams [25,42]. For example the ternary bracket $\mu_3^{\mathrm{H}(\mathfrak{g})}$, is (modulo relative signs) the sum



(9)

More generally, $\mu_k^{\mathrm{H}(\mathfrak{g})}$ is computed by a sum[1] over all rooted trees with k leaves, where one decorates the leaves with i, the $n+1$-ary vertices with μ_n, the internal edges with h, and the root with p. The L_∞-algebra structure on $\mathrm{H}(\mathfrak{g})$ is called the *minimal model* of $\mathfrak{g}$; minimal models are unique up to L_∞-isomorphisms.

Twisting L_∞-algebras L_∞-algebras also admit a notion of *twist* with respect to a Maurer–Cartan element; for reviews, see [38–40]. In the definitions below, for a L_∞-algebra $(\mathfrak{g}, \{\mu_k\}_{k\ge1})$, we assume for simplicity that $\mu_i = 0$ for sufficiently large i; this can be relaxed [39].

Definition 3 *([24,40]).* Let $(\mathfrak{g}, \{\mu_k^{\mathfrak{g}}\}_{k\ge1})$ be an L_∞-algebra such that $\mu_i = 0$ for sufficiently large i. A *Maurer–Cartan element* $Q \in \mathfrak{g}^1$ of $\mathfrak{g}$ is an element of degree 1 such that

$$\sum_{i=1}^{\infty} \frac{1}{i!}\mu_i(Q,\ldots,Q) = 0. \tag{10}$$

Definition 4 *([40]).* Let $(\mathfrak{g}, \{\mu_k^{\mathfrak{g}}\}_{k\ge1})$ be an L_∞-algebra such that $\mu_i = 0$ for sufficiently large i. Let $Q \in \mathfrak{g}^1$ be a Maurer–Cartan element. The *twist* of $\mathfrak{g}$ with respect to Q is the L_∞-algebra $\mathfrak{g}_Q$ whose underlying graded vector space is that of $\mathfrak{g}$ but whose brackets μ_k^Q are

$$\begin{aligned} \mu_k^Q:\quad & \mathfrak{g}_Q^{\wedge k} \to \mathfrak{g}_Q \\ & (x_1,\ldots,x_k) \mapsto \sum_{i\ge0} \frac{1}{i!}\mu_{i+k}(Q,\ldots,Q,x_1,\ldots,x_k). \end{aligned} \tag{11}$$

[1] The explicit relative signs between the trees can be worked out by using the aforementioned tensor trick, for example.

2.2. L_∞-representations

The notion of a representation of (or module over) a Lie algebra generalizes to the setting of homotopy algebras as follows.

Definition 5 *([24,43,44]).* An *L_∞-representation* of an L_∞-algebra $(\mathfrak{g}, \{\mu_k^{\mathfrak{g}}\}_{k\geq 1})$ on a graded vector space M is an L_∞-algebra structure $\{\mu_k^{\mathfrak{g}\ltimes M}\}_{k\geq 1}$ on the direct sum $\mathfrak{g}\oplus M$ such that

$$\mu_k^{\mathfrak{g}\ltimes M}(x_1\oplus 0,\dots,x_k\oplus 0)=\mu_k(x_1,\dots,x_k) \tag{12}$$

and $\mu_{k+1}^{\mathfrak{g}\ltimes M}(x_1,\dots x_k,m)\in 0\oplus M$ for $x_1,\dots,x_k\in\mathfrak{g}$ and $m\in M$, and such that $\mu_k^{\mathfrak{g}\ltimes M}$ vanishes whenever at least two of its arguments belong to $0\oplus M\subset \mathfrak{g}\oplus M$. We will write $\mathfrak{g}\ltimes M$ to refer to L_∞-algebras of this form. We write

$$\rho^{(k)}(x_1,\dots,x_k):=\mu_{k+1}^{\mathfrak{g}\ltimes M}(x_1,\dots,x_k,-)\colon M\to M. \tag{13}$$

Observe that, in particular, $\rho^{(k)}$ carries degree $1-k$. Note that $\rho^{(0)}$ defines a differential on M, making it a cochain complex. We call an L_∞-representation *strict* whenever $\rho^{(k)}=0$ for $k>1$.

The L_∞-algebra homotopy Jacobi identities (2) then can be written as a series of coherence relations amongst the $\rho^{(k)}$'s and $\mu_j^{\mathfrak{g}}$'s.

As with L_∞-algebras themselves, L_∞-representations admit a good homotopy theory in that minimal models exist and homotopy transfer is possible. That is, given an L_∞-algebra $\mathfrak{g}$ and a $\mathfrak{g}$-representation M, we can always choose a strong deformation retract

$$(h,h') \circlearrowright (\mathfrak{g}\oplus M,\mu_1+\rho^{(0)}) \underset{(i,i')}{\overset{(p,p')}{\rightleftarrows}} (\mathrm{H}(\mathfrak{g})\oplus \mathrm{H}(M),0) \tag{14}$$

and perform homotopy transfer of L_∞-algebra structures along this retract to obtain an L_∞-algebra on $\mathrm{H}(\mathfrak{g})\oplus\mathrm{H}(M)$, which then defines the L_∞-representation of $\mathrm{H}(\mathfrak{g})$ on $\mathrm{H}(M)$.[2,3]

Given an L_∞-algebra $\mathfrak{g}$ and a $\mathfrak{g}$-representation M with structure maps $\rho^{(k)}$, then it is clear by inspection that a Maurer–Cartan element $Q\in\mathfrak{g}$ is also Maurer–Cartan element of $\mathfrak{g}\ltimes M$ and that the twist $(\mathfrak{g}\ltimes M)_Q$ factorizes as $(\mathfrak{g}\ltimes M)_Q=\mathfrak{g}_Q\ltimes M_Q$ [39], where M_Q comes with the structure maps

$$\rho_Q^{(k)}(x_1,\dots,x_k):=\sum_{i=0}^{\infty}\frac{1}{i!}\rho^{(i+k)}(Q,\dots,Q,x_1,\dots,x_k). \tag{15}$$

3. Higher symmetry of twisted ten-dimensional supersymmetric Yang–Mills theory

In the Batalin–Vilkovisky formalism [45–49], the field content of a perturbative gauge theory is a graded vector space $(\mathcal{F},d_{\mathcal{F}})$ that comes equipped with a differential. Field theories often respect symmetry algebras such as the super-Poincaré algebra, the (super-)conformal algebra, the (super-)(anti-)de Sitter algebra, the (super-)Galilean algebra, etc. The action of such a symmetry algebra may be *off shell* (i.e. on $\mathcal{F}$ itself) or merely *on shell* (i.e. only on the space of solutions to the equations of motion[4]). The symmetry algebras are usually ungraded or $\mathbb{Z}/2\mathbb{Z}$-graded (i.e. superalgebras), and correspondingly the field space is $\mathbb{Z}/2\mathbb{Z}$-graded into bosons and fermions (in addition to the $\mathbb{Z}$-grading corresponding to ghost number). This $\mathbb{Z}/2\mathbb{Z}$ grading may be often lifted to a $\mathbb{Z}$ grading; correspondingly, the $\mathbb{Z}/2\mathbb{Z}$ grading of the field space may also be lifted to a $\mathbb{Z}$-grading.[5] The $\mathbb{Z}$-grading enables a good homotopy theory of L_∞-algebras and L_∞-representations, and in particular off-shell realizations of symmetries can be in most cases lifted to a non-strict L_∞-representation of the corresponding symmetry [7].

We turn now to our main example of interest, which is the twist of ten-dimensional super-Yang–Mills theory. (For simplicity and convenience with twisting, we assume all symmetries and fields to be complexified.)

[2] The induced brackets on $\mathrm{H}(\mathfrak{g})\oplus\mathrm{H}(M)$ automatically satisfy the conditions given in Definition 5. Indeed, as there are no brackets in $\mathfrak{g}\oplus M$ that reduce the number of factors of M, no such brackets can arise through composition.

[3] This is the minimal model for the two-coloured operad of pairs of L_∞-algebras and their L_∞-representations, rather than the minimal model for the uncoloured operad of L_∞-representations over a fixed L_∞-algebra $\mathfrak{g}$.

[4] Since we ignore interactions, for us this is the linearized equations of motion, but in general one should consider the interacting case.

[5] In order to lift the $\mathbb{Z}/2\mathbb{Z}$ grading of field space to $\mathbb{Z}$, we work with polynomials over spacetime rather than smooth functions. This simplification can be avoided; see [7].

We first discuss the twisted super-Poincaré algebra itself. Let

$$V \cong \mathbb{C}^{10} \tag{16}$$

be a ten-dimensional complex vector space equipped with a nondegenerate symmetric bilinear form. The ten-dimensional $\mathcal{N}=(1,0)$ super-Poincaré superalgebra is the Lie superalgebra

$$\mathfrak{o}(V) \ltimes (\Pi S_+ \oplus V) \tag{17}$$

where $S_\pm$ are the two 16-dimensional Weyl spinor representations of $\mathfrak{o}(V)$ and Π denotes parity reversal. The $\mathbb{Z}/2\mathbb{Z}$ grading of the super-Poincaré superalgebra can be lifted to a $\mathbb{Z}$-grading as the graded Lie algebra

$$\mathfrak{p} := \mathfrak{o}(V) \ltimes (S_+[-1] \oplus V[-2]) \tag{18}$$

in which the elements are graded as twice the conformal dimension (i.e. rotations in degree 0, supertranslations in degree 1, translations in degree 2). For convenience, we can pick a basis $\mathtt{r}_{\mu\nu}$, $\mathtt{d}_\alpha$, $\mathtt{e}_\mu$ of $\mathfrak{p}^0$, $\mathfrak{p}^1$, $\mathfrak{p}^2$ respectively. Then the structure constants for $[\mathfrak{p}^1, \mathfrak{p}^1] \subset \mathfrak{p}^2$ are

$$[\mathtt{d}_\alpha, \mathtt{d}_\beta] = 2\gamma^\mu_{\alpha\beta}\mathtt{e}_\mu, \tag{19}$$

where $\gamma^\mu_{\alpha\beta}$ are the chiral gamma (or Pauli) matrices in ten dimensions, i.e. the branching for the $\mathfrak{o}(V)$-representation $S_+ \otimes S_+ \to V$.

3.1. The minimal model of the holomorphic twist algebra

Suppose we pick a nonzero Maurer–Cartan element of $\mathfrak{p}$, i.e. a nonzero $Q = Q^\alpha \mathtt{d}_\alpha \in S_+$ such that $[Q, Q] = 0$, that is,

$$\gamma^\mu_{\alpha\beta} Q^\alpha Q^\beta = 0. \tag{20}$$

This picks out a subspace

$$L = [Q, S_+] \subset V. \tag{21}$$

This subspace L is a maximal isotropic subspace with respect to the bilinear form on V. Indeed, using the Fierz identity

$$2\gamma^\mu_{\alpha(\beta}\gamma_{\mu|\gamma)\delta} = -\gamma^\mu_{\beta\gamma}\gamma_{\mu\alpha\delta}, \tag{22}$$

we have

$$Q^\beta Q^\gamma \gamma^\mu_{\alpha\beta}\gamma_{\mu\gamma\delta} \propto (Q^\beta Q^\gamma \gamma^\mu_{\beta\gamma})\gamma_{\mu\alpha\delta} = 0; \tag{23}$$

given now any elements $\psi, \chi \in S_+$, consider the elements $[Q, \psi], [Q, \chi] \in L$. We have

$$[Q, \psi]^\mu [Q, \chi]_\mu = (Q^\beta Q^\gamma \gamma^\mu_{\alpha\beta}\gamma_{\mu\gamma\delta})\psi^\alpha \chi^\delta = 0. \tag{24}$$

Thus L is indeed contained in its own orthogonal complement, i.e. it is an isotropic subspace. Isotropy implies $\dim L \le 5$. Furthermore, since the Maurer–Cartan condition is $\mathfrak{o}(V)$-invariant, it follows from the classification of spinors in ten dimensions [50, Prop. 2] that set of Maurer–Cartan elements of $\mathfrak{p}$ consists of two $\mathfrak{o}(V)$-orbits, namely nonzero ones and $\{0\}$; and a straightforward explicit computation shows that, when $Q \neq 0$, then $\dim L = 5$. That is, L is indeed a maximal isotropic subspace.

Thus, we have the short exact sequence of vector spaces

$$0 \to L \to V \xrightarrow{q} L^* \to 0 \tag{25}$$

where the quotient q is via the composition $V \xrightarrow{\sim} V^* \twoheadrightarrow L^*$ in which $V \xrightarrow{\sim} V^*$ is given by the bilinear form on V. Let us choose a splitting of (25) to write

$$V = L \oplus L^*. \tag{26}$$

This decomposition then fixes Lie subalgebras $\mathfrak{sl}(L) \subset \mathfrak{gl}(L) \subset \mathfrak{o}(V)$, under which the ten-dimensional representation V canonically decomposes into $\mathfrak{sl}(L)$ irreducible representations as

$$V \cong_{\mathfrak{sl}(L)} L \oplus L^* \cong_{\mathfrak{sl}(L)} (1000)_{\mathfrak{sl}(L)} \oplus (0001)_{\mathfrak{sl}(L)} \tag{27}$$

(Here and elsewhere we write $(ijkl)_{\mathfrak{sl}(L)}$ for the irreducible representation with these $\mathfrak{sl}(L)$ Dynkin labels, i.e. for the irreducible representation with highest weight $i\omega_1 + j\omega_2 + k\omega_3 + l\omega_4$ where $\omega_1, \ldots, \omega_4$ are the fundamental weights.)

Similarly, the adjoint representation of $\mathfrak{o}(V)$ decomposes into irreducible $\mathfrak{sl}(L)$-representations as

$$\begin{aligned}\mathfrak{o}(V) &\cong_{\mathfrak{sl}(L)} \mathbb{C} \oplus \mathfrak{sl}(L) \oplus L^{\wedge 2} \oplus (L^*)^{\wedge 2} \\ &\cong_{\mathfrak{sl}(L)} (0000)_{\mathfrak{sl}(L)} \oplus (1001)_{\mathfrak{sl}(L)} \oplus (0100)_{\mathfrak{sl}(L)} \oplus (0010)_{\mathfrak{sl}(L)},\end{aligned} \tag{28}$$

and the spinor representations $S_\pm$ decompose as

$$S_+ \cong_{\mathfrak{sl}(L)} \bigoplus_{i=0}^{2} (L^*)^{\wedge(2i)} \cong_{\mathfrak{sl}(L)} (0000)_{\mathfrak{sl}(L)} \oplus (0010)_{\mathfrak{sl}(L)} \oplus (1000)_{\mathfrak{sl}(L)}, \tag{29}$$

$$S_- \cong_{\mathfrak{sl}(L)} \bigoplus_{i=0}^{2} (L^*)^{\wedge(2i+1)} \cong_{\mathfrak{sl}(L)} S_+^* \cong (0001)_{\mathfrak{sl}(L)} \oplus (0100)_{\mathfrak{sl}(L)} \oplus (0000)_{\mathfrak{sl}(L)}. \tag{30}$$

Our chosen element $Q \in S_+$ spans the one-dimensional $\mathfrak{sl}(L)$-submodule $\mathbb{C} \cong (L^*)^{\wedge 0}$. The twist of $\mathfrak{p}$ by Q is [20, Prop. 3.3]

$$\mathfrak{p}_Q = \left(\begin{array}{ccc} \underset{(0000)_{\mathfrak{sl}(L)}}{\mathbb{C}} \xrightarrow{\text{id}} & \underset{(0000)_{\mathfrak{sl}(L)}}{(L^*)^{\wedge 0}[-1]} & \\ \underset{(0010)_{\mathfrak{sl}(L)}}{(L^*)^{\wedge 2}} \xrightarrow{\text{id}} & \underset{(0010)_{\mathfrak{sl}(L)}}{(L^*)^{\wedge 2}[-1]} & \\ & \underset{(1000)_{\mathfrak{sl}(L)}}{(L^*)^{\wedge 4}[-1]} \xrightarrow{\text{id}} & \underset{(1000)_{\mathfrak{sl}(L)}}{L[-2]} \\ \underset{(1001)_{\mathfrak{sl}(L)}}{\mathfrak{sl}(L)} \ltimes \underset{(0100)_{\mathfrak{sl}(L)}}{L^{\wedge 2}} & & \underset{(0001)_{\mathfrak{sl}(L)}}{L^*[-2]} \end{array} \right). \tag{31}$$

We work in an explicit basis $(\mathtt{r}^i{}_j, \mathtt{r}^{ij}, \mathtt{r}_{ij}, \mathtt{d}, \mathtt{d}_{ij}, \mathtt{d}^i, \mathtt{e}^i, \mathtt{e}_i)$ of $\mathfrak{p}$ given in appendix A. In particular, the basis elements $\mathtt{r}^i{}_j$ span $\mathfrak{gl}(L)$, and we shall write

$$\tilde{\mathtt{r}}^i{}_j := \mathtt{r}^i{}_j - \frac{1}{5}\delta^i_j \mathtt{r}^k{}_k \tag{32}$$

for the basis elements of $\mathfrak{sl}(L)$.

For a reason to become apparent in the theorem below, let us note that the representation $L^* \otimes (L^{\wedge 2})^{\wedge 3}$ of $\mathfrak{sl}(L)$ decomposes into irreducibles as

$$L^* \otimes (L^{\wedge 2})^{\wedge 3} \cong (0021)_{\mathfrak{sl}(L)} \oplus (0110)_{\mathfrak{sl}(L)} \oplus (1001)_{\mathfrak{sl}(L)} \oplus (2010)_{\mathfrak{sl}(L)} \oplus (2002)_{\mathfrak{sl}(L)}. \tag{33}$$

In particular, the adjoint representation $\mathfrak{sl}(L) \cong (1001)_{\mathfrak{sl}(L)}$ occurs with multiplicity one. We shall write

$$P_{L^* \otimes (L^{\wedge 2})^{\wedge 3} \to \mathfrak{sl}(L)} \tag{34}$$

for the projector onto this irreducible component.

Theorem 1. *The minimal model of the ten-dimensional twisted $\mathbb{Z}$-graded $\mathcal{N} = (1, 0)$ super-Poincaré algebra $\mathfrak{p}_Q$ is the L_∞-algebra whose underlying graded Lie algebra is*

$$\mathrm{H}(\mathfrak{p}_Q) = \mathfrak{sl}(L) \ltimes \left(L^{\wedge 2} \oplus L^*[-2] \right), \tag{35}$$

and whose higher brackets μ_i $(i \geq 3)$ are all zero except for μ_4, whose only nonvanishing component is given by

$$\begin{aligned}\mu_4(\mathtt{e}_i, \mathtt{r}^{jk}, \mathtt{r}^{lm}, \mathtt{r}^{np}) = &-\delta_i^{[j} \varepsilon^{k]lmqr} \delta_q^{[n} \tilde{\mathtt{r}}^{p]}{}_r + \delta_i^{[j} \varepsilon^{k]npqr} \delta_q^{[l} \tilde{\mathtt{r}}^{m]}{}_r \\ &- \delta_i^{[l} \varepsilon^{m]npqr} \delta_q^{[j} \tilde{\mathtt{r}}^{k]}{}_r + \delta_i^{[l} \varepsilon^{m]jkqr} \delta_q^{[n} \tilde{\mathtt{r}}^{p]}{}_r \\ &- \delta_i^{[n} \varepsilon^{p]jkqr} \delta_q^{[l} \tilde{\mathtt{r}}^{m]}{}_r + \delta_i^{[n} \varepsilon^{p]lmqr} \delta_q^{[j} \tilde{\mathtt{r}}^{k]}{}_r \\ =: &\, P_i^{jklmnp;}{}^r_q \tilde{\mathtt{r}}^q{}_r,\end{aligned} \tag{36}$$

where $P_i^{jklmnp;}{}^r_q$ is the projector

$$L^* \otimes (L^{\wedge 2})^{\wedge 3} \to \mathfrak{sl}(L), \tag{37}$$

and where the skew-symmetrizations are unnormalized.

Proof. From eq. (31), we see that there is an evident $\mathfrak{sl}(L)$-equivariant strong deformation retract (i,p,h) of cochain complexes

$$h \circlearrowright (\mathfrak{p}_Q, \mathrm{ad}_Q) \underset{i}{\overset{p}{\rightleftarrows}} (\mathrm{H}(\mathfrak{p}_Q), 0) \tag{38}$$

from $\mathfrak{p}_Q$ to its cohomology

$$\mathrm{H}(\mathfrak{p}_Q) = (\mathfrak{sl}(L) \ltimes L^{\wedge 2} \xrightarrow{0} 0 \xrightarrow{0} L^*). \tag{39}$$

(The remaining $\mathfrak{sl}(L)$ irreducible representations present in eq. (31) participate in trivial pairs; one defines the homotopy h to act as the inverse to the differential on these.) The Lie algebra structure μ_2 on this cohomology $\mathrm{H}(\mathfrak{p}_Q)$ is given by restriction.

It remains to check what higher brackets μ_i are induced by homotopy transfer. We are to sum over rooted binary trees in which each vertex corresponds to the binary bracket $\mu_2^{\mathfrak{p}}(-,-) = [-,-]$ of $\mathfrak{p}$ (and thus of $\mathfrak{p}_Q$), each internal edge to the homotopy h, each leaf to i and the root to p [37].

We will use Feynman-diagrammatic terminology, referring to elements as 'states' (see [25,42]). Recall our notation $(\mathtt{r}^i{}_j, \mathtt{r}^{ij}, \mathtt{r}_{ij}, \mathtt{d}, \mathtt{d}_{ij}, \mathtt{d}^i, \mathtt{e}^i, \mathtt{e}_i)$ for the basis elements of $\mathfrak{p}$ (whose underlying graded vector space we identify with that of $\mathfrak{p}_Q$) as given in appendix A. We will refer to a state as *intermediate* if it lies in the image of h.

Using the strong deformation retract (i,p,h), we try to construct the possible intermediate states, keeping track of the representations under $\mathfrak{sl}(L)$. Using the embedding $i\colon \mathrm{H}(\mathfrak{p}_Q) \hookrightarrow \mathfrak{p}_Q$, we will identify the basis elements $\tilde{\mathtt{r}}^i{}_j, \mathtt{r}^{ij}, \mathtt{e}_i$ in $\mathfrak{p}_Q$ with those of $\mathrm{H}(\mathfrak{p}_Q)$. The products μ_i for $i>2$ can then be computed in a top-down recursive fashion by starting with two elements $x, y \in \mathrm{H}(\mathfrak{p}_Q)$, and then compute which intermediate states are allowed by considering

$$h[i(x), i(y)]. \tag{40}$$

The next intermediate states are then computed by plugging (40), and one element $a \in i(\mathrm{H}(\mathfrak{p}_Q)) \oplus \mathrm{Im}(h[i,i])$, into $[-,-]$. If the result lies in $\mathrm{H}(\mathfrak{p}_Q)$, we apply p, and we are done. If not, we apply h to obtain new intermediate states and then continue the procedure of pairing (using $[-,-]$) the newly obtained intermediate states with each other or with previously obtained intermediate states or states in the cohomology.

Starting with two elements of $\mathrm{H}(\mathfrak{p}_Q)$, applying $h[-,-]$ can only yield the intermediate states

$$h[\mathtt{e}_k, \mathtt{r}^{ij}] = \delta^i_k \mathtt{d}^j - \delta^j_k \mathtt{d}^i. \tag{41}$$

Using $\mathtt{d}^i$ together with $\tilde{\mathtt{r}}^i{}_j, \mathtt{r}^{ij}, \mathtt{e}_i$, we can only further create

$$[\mathtt{d}^i, \mathtt{d}^j] = 0, \tag{42a}$$
$$[\mathtt{d}^i, \mathtt{e}_j] = 0, \tag{42b}$$
$$h[\mathtt{r}^{ij}, \mathtt{d}^k] = -\frac{1}{2}\varepsilon^{ijklm}\mathtt{r}_{lm}, \tag{42c}$$
$$[\tilde{\mathtt{r}}^i{}_j, \mathtt{d}^k] = -\frac{1}{5}\delta^i_j \mathtt{d}^k + \delta^k_j \mathtt{d}^i. \tag{42d}$$

Among these, $[\tilde{\mathtt{r}}^i{}_j, \mathtt{d}^k]$ does not belong to the cohomology, i.e. it is not Q-closed, nor can it produce a new intermediate state since $h([\tilde{\mathtt{r}}^i{}_j, \mathtt{d}^k]) = 0$. Thus the only intermediate state we can create is given by $h[\mathtt{r}^{ij}, \mathtt{d}^k] \propto \mathtt{r}_{lm}$. Applied to (41), we obtain

$$h[h[\mathtt{e}_k, \mathtt{r}^{ij}], \mathtt{r}^{lm}] = h[\delta^i_k \mathtt{d}^j - \delta^j_k \mathtt{d}^i, \mathtt{r}^{lm}] = \frac{1}{2}\left(\delta^i_k \varepsilon^{jlmnp} - \delta^j_k \varepsilon^{ilmnp}\right)\mathtt{r}_{np}. \tag{43}$$

Using the new intermediate state $\mathtt{r}_{ij}$ together with the previously created intermediate state $\mathtt{d}^i$ and the cohomology $(\tilde{\mathtt{r}}^i{}_j, \mathtt{r}^{ij}, \mathtt{e}_i)$, we can create the following new states:

$$[\mathtt{r}_{ij}, \mathtt{e}_k] = 0, \tag{44a}$$
$$[\mathtt{r}_{ij}, \mathtt{d}^k] = 0, \tag{44b}$$
$$[\tilde{\mathtt{r}}^i{}_j, \mathtt{r}_{kl}] = -\delta^i_k \mathtt{r}_{jl} - \delta^i_l \mathtt{r}_{kj} + \frac{2}{5}\delta^i_j \mathtt{r}_{kl}, \tag{44c}$$
$$[\mathtt{r}^{ij}, \mathtt{r}_{kl}] = \delta^i_k \mathtt{r}^j{}_l - \delta^j_k \mathtt{r}^i{}_l - \delta^i_l \mathtt{r}^j{}_k + \delta^j_l \mathtt{r}^i{}_k, \tag{44d}$$
$$[\mathtt{r}_{ij}, \mathtt{r}_{kl}] = 0. \tag{44e}$$

Now, all these states sit in degree 0 and are thus killed by h, so that none of them can create further intermediate states. The only nontrivial thing we can now do is to project to the cohomology: $[\tilde{\mathtt{r}}^i{}_j, \mathtt{r}_{kl}]$ never lies in the cohomology, whereas the traceless part of $[\mathtt{r}^{ij}, \mathtt{r}_{kl}]$ does. Applied to (43), we obtain

$$\begin{aligned} p[h[h[\mathtt{e}_k, \mathtt{r}^{ij}], \mathtt{r}^{lm}], \mathtt{r}^{qr}] &= -\frac{1}{2}\left(\delta^j_k \varepsilon^{ilmnp} - \delta^i_k \varepsilon^{jlmnp}\right)[\mathtt{r}_{np}, \mathtt{r}^{qr}] \\ &= \left(\delta^j_k \varepsilon^{ilmnp} - \delta^i_k \varepsilon^{jlmnp}\right)\left(\delta^q_n \mathtt{r}^r{}_p - \delta^r_n \mathtt{r}^q{}_p\right) \\ &= \delta^{[j}_k \varepsilon^{i]lmnp} \delta^{[q}_n \tilde{\mathtt{r}}^{r]}{}_p. \end{aligned} \tag{45}$$

After graded-skew-symmetrization among the three arguments of the form $\mathtt{r}^{ij}$, this yields the only nonvanishing component of μ_4.

There are no other μ_i since we have systematically constructed all possible nonzero tree Feynman diagrams (by constructing all possible intermediate states that occur in them). □

One may doubt whether the nonstrictness and existence of a 4-bracket in $\mathrm{H}(\mathfrak{p}_Q)$ is model-independent (i.e. holds for all minimal models) or an accidental feature of the specific minimal model in question. By the general theory of minimal models, $\mathrm{H}(\mathfrak{p}_Q)$ is unique up to L_∞-isomorphisms. Concretely, we may ask whether there exists a strict minimal model of $\mathrm{H}(\mathfrak{p}_Q)$ (hence with no higher brackets). The answer is no.

Lemma 1. *Let $\mathfrak{h}$ be a minimal strict graded L_∞-algebra defined on the graded vector space $\mathrm{H}(\mathfrak{p}_Q) \cong \mathfrak{sl}(L) \oplus L^{\wedge 2} \oplus L^*[-2]$. There exists no L_∞-isomorphism $\phi\colon \mathrm{H}(\mathfrak{p}_Q) \rightsquigarrow \mathfrak{h}$.*

Proof. Suppose to the contrary that such an L_∞-isomorphism $\phi\colon \mathrm{H}(\mathfrak{p}_Q) \rightsquigarrow \mathfrak{h}$ exists with component maps $\phi^{(k)}$. Without loss of generality, we may identify the underlying graded vector spaces of $\mathrm{H}(\mathfrak{p}_Q)$ and $\mathfrak{h}$ via $\phi^{(1)}$. Since $\mathrm{H}(\mathfrak{p}_Q)$ (and hence $\mathfrak{h}$) are concentrated in even degrees, even-order components of ϕ (which have odd degree) vanish: $\phi^{(2k)} = 0$. The coherence relations eq. (5) then implies that $\phi^{(1)}$ is a Lie-algebra isomorphism of the underlying graded Lie algebra structures on $\mathrm{H}(\mathfrak{p}_Q)$ and $\mathfrak{h}$. Moreover, the coherence relation on four elements reads

$$\phi^{(1)}(\mu_4^{\mathrm{H}(\mathfrak{p}_Q)}(\mathtt{e}_p, \mathtt{r}^{ij}, \mathtt{r}^{kl}, \mathtt{r}^{mn})) = \mu_2^{\mathfrak{h}}(\phi^{(1)}(\mathtt{e}_p), \phi^{(3)}(\mathtt{r}^{ij}, \mathtt{r}^{kl}, \mathtt{r}^{mn})) + \text{permutations}. \tag{46}$$

Now, the left-hand side is nonzero and lies in the copy of $\mathfrak{sl}(L)$ inside $\mathfrak{h}$. But since $\mu_2^{\mathfrak{h}}$ agrees with $\mu_2^{\mathrm{H}(\mathfrak{p}_Q)}$, by virtue of $\phi^{(1)}$ being a Lie algebra morphism, we have that $\mu_2^{\mathfrak{h}}$ is $\mathfrak{sl}(L)$-equivariant, and hence the right-hand side cannot lie in $\mathfrak{sl}(L)$, a contradiction. □

3.2. Action on $\mathbb{A}^5$

The pure spinor formalism [7,8] associates certain sheaves on (a derived replacement of) the variety of Maurer–Cartan elements to off-shell representations of the super-Poincaré algebra. In particular, for the ten-dimensional $\mathcal{N} = (1,0)$ super-Poincaré algebra, it associates to the structure sheaf of the Maurer–Cartan variety $\operatorname{Spec}\mathbb{C}[\lambda^\alpha]/(\gamma^\mu_{\alpha\beta}\lambda^\alpha\lambda^\beta)$ the pure spinor supermultiplet [7]

$$\begin{aligned} M &:= \mathbb{C}[x^\mu, \theta^\alpha, \lambda^\alpha]/(\gamma^\mu_{\alpha\beta}\lambda^\alpha\lambda^\beta) \\ &\cong \left(\bigodot(10000)_{\mathfrak{o}(V)}\right) \otimes \left(\bigwedge(00010)_{\mathfrak{o}(V)}\right) \otimes \left(\bigoplus_{i=0}^{\infty}(000i0)_{\mathfrak{o}(V)}\right) \end{aligned} \tag{47}$$

where $\bigodot R$ denotes the symmetric algebra on R, and the $\mathfrak{o}(V)$-representation has been specified by Dynkin labels; we use the index notation where V indices are $_\mu$ and S_+ indices are $_\alpha$ (hence $S_- \cong S_+^*$ indices are $^\alpha$). The formal variables x, θ, λ transform as V, S_-, S_- respectively under $\mathfrak{o}(V)$, which in turn determines the $\mathfrak{o}(V)$-representation on M. The generators carry the degrees[6]

$$|x| = -2 \qquad |\theta| = -1 \qquad |\lambda| = 0. \tag{48}$$

The differential is [7, (3.14), (3.19)]

$$d := \lambda^\alpha\left(\frac{\partial}{\partial\theta^\alpha} - \gamma^\mu_{\alpha\beta}\theta^\beta\frac{\partial}{\partial x^\mu}\right). \tag{49}$$

6 In fact, this grading can be refined into a bigrading [7, (3.15) ff.], and the $\mathfrak{p}$-representation respects this bigrading. But we do not need this fact.

The $\mathfrak{o}(V)$-representation of M extends to a strict representation of $\mathfrak{p}$ as

$$\rho_0^{(1)}(\mathtt{e}_\mu) = \frac{\partial}{\partial x^\mu}, \tag{50a}$$

$$\rho_0^{(1)}(\mathtt{d}_\alpha) = \frac{\partial}{\partial \theta^\alpha} + \gamma^\mu_{\alpha\beta}\theta^\beta \frac{\partial}{\partial x^\mu}. \tag{50b}$$

We can twist this $\mathfrak{p}$-representation to obtain a (strict) representation of $\mathfrak{p}_Q$ on M_Q; the action of $\mathfrak{p}_Q$ is through (50), but the differential on M_Q is now $d + \rho_0^{(1)}(Q)$. The cohomology of M_Q is the ring of regular functions on $\mathbb{A}^5$.

Theorem 2 *([20, Theorem 3.A]). The cohomology of M_Q is*

$$\mathrm{H}(M_Q) = \mathbb{C}[z^i], \tag{51}$$

where $z^i = (z^1, \dots, z^5)$ is a formal variable of degree -2 transforming under $\mathfrak{sl}(L)$ as the defining representation L, such that the L_∞-representation of the subalgebra $\mathfrak{isl}(L) := \mathfrak{sl}(L) \ltimes L^*[-2]$ of $\mathrm{H}(\mathfrak{p}_Q)$ is

$$\rho^{(1)}(\tilde{\mathtt{r}}^i{}_j) = z^i \frac{\partial}{\partial z^j} - \frac{1}{5}\delta^i_j z^k \frac{\partial}{\partial z^k}, \qquad \rho^{(1)}(\mathtt{e}_i) = \frac{\partial}{\partial z^i}, \tag{52}$$

with $\rho^{(k)} = 0$ for $k \geq 2$.

This corresponds to the fact that the holomorphic twist of ten-dimensional supersymmetric Yang–Mills theory is holomorphic Chern–Simons theory [4,20,22], whose space of fields is[7] the algebraic Dolbeault complex of $\mathbb{A}^5$, namely $\mathbb{C}[z^i, \bar{z}_i, \mathrm{d}\bar{z}^i]$, and whose cohomology is therefore $\mathbb{C}[z^i]$.

Furthermore, the cohomology $\mathbb{C}[z^i] = \mathrm{H}(M_Q)$ is included into M_Q as a $\mathfrak{sl}(L)$-subrepresentation [20]. Since $\mathfrak{sl}(L)$ is simple, there exists an $\mathfrak{sl}(L)$-equivariant strong deformation retract of cochain complexes

$$(h,h') \circlearrowright\ \mathfrak{p}_Q \oplus M_Q \underset{(i,i')}{\overset{(p,p')}{\rightleftarrows}} \mathrm{H}(\mathfrak{p}_Q) \oplus \mathrm{H}(M_Q), \tag{53}$$

whose restriction to $\mathfrak{p}_Q \leftrightarrow \mathrm{H}(\mathfrak{p}_Q)$ is the strong deformation-retract (i, p, h) given in (38). Thus, by taking the minimal model of $(\mathfrak{p}_Q, M_Q)$ along the strong deformation retract (53), the above $\mathfrak{isl}(L)$-representation extends into an L_∞-representation of the entirety of $\mathrm{H}(\mathfrak{p}_Q)$, and such minimal models are unique up to quasi-isomorphisms of L_∞-algebra representations. This minimal model is an L_∞-representation of the L_∞-algebra $\mathrm{H}(\mathfrak{p}_Q)$ on $\mathrm{H}(M_Q) = \mathbb{C}[z^i]$. The following theorem computes this minimal model explicitly.

Theorem 3. *The minimal model of the L_∞-representation of $\mathfrak{p}_Q$ on M_Q obtained using the strong deformation retract $(i, i'; p, p'; h, h')$ is the $\mathrm{H}(\mathfrak{p}_Q)$-representation on $\mathrm{H}(M_Q) = \mathbb{C}[z^i]$ given by*

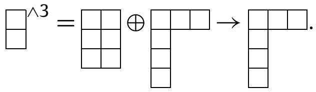
$$\begin{aligned}
\rho^{(1)}(\tilde{\mathtt{r}}^i{}_j) &= z^i \frac{\partial}{\partial z^j} - \frac{1}{5}\delta^i_j z^k \frac{\partial}{\partial z^k}, \\
\rho^{(1)}(\mathtt{e}_i) &= \frac{\partial}{\partial z^i}, \\
\rho^{(3)}(\mathtt{r}^{ij}, \mathtt{r}^{kl}, \mathtt{r}^{mn}) &= \frac{1}{2}\bigg(z^{[i}\varepsilon^{j]klp[m}z^{n]} - z^{[i}\varepsilon^{j]mnp[k}z^{l]} \\
&\quad + z^{[k}\varepsilon^{l]mnp[i}z^{j]} - z^{[k}\varepsilon^{l]ijp[m}z^{n]} \\
&\quad + z^{[m}\varepsilon^{n]ijp[k}z^{l]} - z^{[m}\varepsilon^{n]klp[i}z^{j]} \bigg)\frac{\partial}{\partial z^p} \\
&=: P^{ijklmn;r}{}_{pq} z^p z^q \frac{\partial}{\partial z^r},
\end{aligned} \tag{54}$$

with all other components vanishing (in particular, $\rho^{(1)}(\mathtt{r}^{ij}) = 0$), where $P^{ijklmn;r}{}_{pq}$ is the projection

$$(0100)^{\wedge 3}_{\mathfrak{sl}(L)} = (0020)_{\mathfrak{sl}(L)} \oplus (2001)_{\mathfrak{sl}(L)} \to (2001)_{\mathfrak{sl}(L)} \tag{55}$$

in terms of $\mathfrak{sl}(L)$ Dynkin labels or, in Young tableau notation,

[7] Up to issues such as holomorphic versus algebraic functions, which we ignore.

 (56)

Proof. First, note that for degree reasons, we can only have nonzero $\rho^{(k)}$ for odd k since $\mathbb{C}[z^i]$ and $\mathrm{H}(\mathfrak{p}_Q)$ are all concentrated in even degree and $\rho^{(k)}$ carries degree $1-k$. The leading component $\rho^{(1)}$ is fixed simply by restriction of the $\mathfrak{p}_Q$-representation ρ_0 on M_Q to $i(\mathrm{H}(\mathfrak{p}_Q)) \subset \mathfrak{p}_Q$ and $i'(\mathrm{H}(M_Q)) \subset M_Q$ as

$$\rho^{(1)}(x) = p' \circ \rho_0(i(x)) \circ i' \tag{57}$$

for $x \in \mathrm{H}(\mathfrak{p}_Q)$ (so that $i(x) \in \mathfrak{p}_Q$); in particular, $\rho^{(1)}(\mathtt{r}^{ij}) = 0$. Furthermore, the $\rho^{(k)}$ vanish whenever one of the arguments is $\tilde{\mathtt{r}}^i{}_j$ except when $k=1$ (Lemma 2).

Hence, it suffices to determine $\rho^{(3)}, \rho^{(5)}, \rho^{(7)}, \ldots$ where all arguments are either $\mathtt{r}^{ij}$ or $\mathtt{e}_i$. Now, the possibilities of $\rho^{(k)}$ are constrained by the fact that all operations μ_k, $\rho^{(k)}$, and the strong deformation retract $(i, i'; p, p'; h, h')$ are $\mathfrak{sl}(L)$-equivariant. Suppose that $\rho^{(k)}$ does not vanish when fed p arguments of the form $\mathtt{r}^{ij}$ and q arguments of the form $\mathtt{e}_i$ with $p+q=k \equiv 1 \pmod 2$. Then, representation-theoretically, it must yield a nontrivial $\mathfrak{sl}(L)$-representation that is a direct summand of

$$\left(L^{\wedge 2}\right)^{\wedge p} \otimes (L^*)^{\wedge q}. \tag{58}$$

On the other hand, it must carry the degree $(1-p-q)+2q = 1-p+q$, and hence be a sum of terms of the form

$$\begin{cases} z^n \left(\frac{\partial}{\partial z}\right)^{n+(1-p+q)/2} & \text{if } 1-p+q \geq 0 \\ z^{n-(1-p+q)/2} \left(\frac{\partial}{\partial z}\right)^n & \text{if } 1-p+q \leq 0, \end{cases} \tag{59}$$

where z^n refers to a product $z^{i_1} \cdots z^{i_n}$, and similarly for $(\frac{\partial}{\partial z})^n$. Since z^i transforms as L and $\partial/\partial z^i$ as L^*, this must transform under $\mathfrak{sl}(L)$ as a direct summand of

$$\begin{cases} L^{\odot n} \otimes (L^*)^{\odot(n+(1-p+q)/2)} & \text{if } 1-p+q \geq 0 \\ L^{\odot(n-(1-p+q)/2)} \otimes (L^*)^{\odot n} & \text{if } 1-p+q \leq 0, \end{cases} \tag{60}$$

where the superscript $\odot n$ denotes the n-th symmetric power. Thus, the two $\mathfrak{sl}(L)$-representations (58) and (60) must share some nontrivial subrepresentations if the corresponding $\rho^{(k)}$ is to not vanish. Lemma 3 shows that this is only possible for $(p,q) = (3,0)$ and $(p,q) = (4,3)$, corresponding to

$$\rho^{(3)}(\mathtt{r}^{ij}, \mathtt{r}^{kl}, \mathtt{r}^{mn}) = P^{ijklmn;r}{}_{pq} \left(\alpha_0 z^p z^q \frac{\partial}{\partial z^r} + \alpha_1 z^p z^q z^s \frac{\partial^2}{\partial z^r \partial z^s} + \cdots\right) \tag{61}$$

and

$$\rho^{(7)}(\mathtt{r}^{ij}, \mathtt{r}^{kl}, \mathtt{r}^{mn}, \mathtt{r}^{pq}, \mathtt{e}_r, \mathtt{e}_s, \mathtt{e}_t) = P^{ijklmnpq;u}_{rst}{}_{v} \left(\beta_0 z^v \frac{\partial}{\partial z^u} + \beta_1 z^v z^w \frac{\partial^2}{\partial z^u \partial z^w} + \cdots\right), \tag{62}$$

respectively, where $P^{ijklmn;r}{}_{pq}$ is the projector $(0100)^{\wedge 3}_{\mathfrak{sl}(L)} \to (2001)_{\mathfrak{sl}(L)}$ as in (55) and $P^{ijklmnpq;u}_{rst}{}_{v}$ is the projector $(0100)^{\wedge 4}_{\mathfrak{sl}(L)} \otimes (0001)^{\wedge 3}_{\mathfrak{sl}(L)} \to (1001)_{\mathfrak{sl}(L)}$.

Now, we must solve the coherence relations. One L_∞-representation coherence relation states[8]

$$[\rho^{(1)}(\mathtt{e}_i), \rho^{(3)}(\mathtt{r}^{jk}, \mathtt{r}^{lm}, \mathtt{r}^{np})] = \rho^{(1)}(\mu_4(\mathtt{e}_i, \mathtt{r}^{jk}, \mathtt{r}^{lm}, \mathtt{r}^{np})). \tag{63}$$

Substituting $\rho^{(1)}(\mathtt{e}_i) = \partial/\partial z^i$ and (61) into (63) yields

$$\left[\frac{\partial}{\partial z^i}, P^{jklmnp;s}{}_{qr} \left(\alpha_0 z^q z^r \frac{\partial}{\partial z^s} + \alpha_1 z^q z^r z^t \frac{\partial^2}{\partial z^s \partial z^t} + \cdots\right)\right] = P^{jklmnp;r}_{i}{}_{q} z^q \frac{\partial}{\partial z^r}, \tag{64}$$

where $P^{jklmnp;r}_{i}{}_{q}$ is the projector (37). Solving this yields $\alpha_0 = 1$ and $\alpha_n = 0$ for $n > 0$.

Next, we have the L_∞-module coherence relation[9]

$$0 = \left[\rho^{(7)}(\mathtt{r}^{ij}, \mathtt{r}^{kl}, \mathtt{r}^{mn}, \mathtt{r}^{pq}, \mathtt{e}_{[r}, \mathtt{e}_s, \mathtt{e}_t), \rho^{(1)}(\mathtt{e}_{u]})\right]. \tag{65}$$

[8] In this coherence relation, in our case, terms of the form $[\rho^{(0)}, \rho^{(4)}]$, $[\rho^{(2)}, \rho^{(2)}]$, $\rho^{(4)}(\mu_1)$, $\rho^{(3)}(\mu_2)$, and $\rho^{(2)}(\mu_3)$ vanish.

[9] In this coherence relation, terms of the form $[\rho^{(0)}, \rho^{(8)}]$, $[\rho^{(2)}, \rho^{(6)}]$, $[\rho^{(3)}, \rho^{(5)}]$, $[\rho^{(4)}, \rho^{(3)}]$, $\rho^{(1)}(\mu_8)$, $\rho^{(2)}(\mu_7)$, …, $\rho^{(8)}(\mu_1)$ vanish.

Plugging in the ansatz (62) into (65) yields

$$0=\left[P^{ijklmnpq;\,v}_{[rst|\quad\; w}\left(\beta_0 z^w\frac{\partial}{\partial z^v}+\beta_1 z^w z^x\frac{\partial^2}{\partial z^v\partial z^x}+\cdots\right),\frac{\partial}{\partial z^{|u]}}\right]. \tag{66}$$

Solving this shows that the coefficients $\beta_0,\beta_1,\dots$ must all vanish since non-constant-coefficient differential operators do not commute with $\partial/\partial z^l$. □

Lemma 2. *In the L_∞-representation of the L_∞-algebra $\mathrm{H}(\mathfrak{p}_Q)$ on $\mathbb{C}[z^i]$ obtained by homotopy transfer from M_Q using the $\mathfrak{sl}(L)$-equivariant strong deformation retract $(i,i';p,p';h,h')$, we have*

$$\rho^{(k)}(\tilde{\mathfrak{r}}^i{}_j,\dots)=0 \tag{67}$$

if $k\geq 2$.

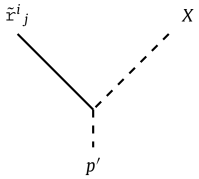
Proof. We are to perform a homotopy transfer of L_∞-algebras along

$$(h,h')\circlearrowright\ \mathfrak{p}_Q\oplus M_Q\ \underset{(i,i')}{\overset{(p,p')}{\rightleftarrows}}\ \mathrm{H}(\mathfrak{p}_Q)\oplus\mathrm{H}(M_Q). \tag{68}$$

Let us again use Feynman-diagrammatic terminology to refer to elements as 'states'. If $\rho^{(k)}(\tilde{\mathfrak{r}}^i{}_j,\dots)\neq 0$, this would mean that there is at least one tree Feynman diagram with at least one external leg corresponding to $\tilde{\mathfrak{r}}^i{}_j$. Assuming that $k\geq 2$, we have the following possibilities.

(i) The vertex connected to this leg may be directly connected to p' as

$$p'(\rho_0(\tilde{\mathfrak{r}}^i{}_j)X)= \tag{69}$$



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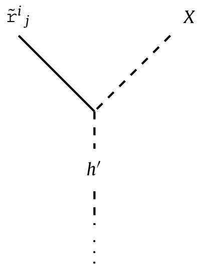
where $X\in M_Q$. In this case, we may assume X to be an intermediate state $X=h'(\tilde{X})$. (The alternative, that X lies in the cohomology, only yields $\rho^{(1)}$.)

(ii) The vertex connected to this leg may feed into h and connect to the rest of the tree as

$$p'(\cdots h'(\rho_0(\tilde{\mathfrak{r}}^i{}_j)X)\cdots)= \tag{70}$$



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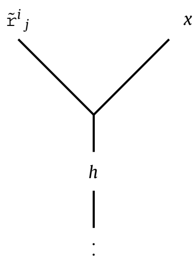
where $X\in M_Q$ may be either an intermediate state $X=h'(\tilde{X})$ or belong to the cohomology ($X\in i'(\mathrm{H}(M_Q))$). In either case, we have $h(X)=0$.

(iii) The vertex connected to this leg may feed into h and connect to the rest of the tree as

$$p'(\cdots h[\tilde{\mathfrak{r}}^i{}_j,x]\cdots)= \tag{71}$$



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where $x \in \mathfrak{p}_Q$ is either an intermediate state $x = h(\tilde{x})$ or belongs to the cohomology ($x \in i(\mathrm{H}(\mathfrak{p}_Q))$). In either case, $h(x) = 0$.

In the first case (69), since the strong deformation retract $(i, i'; p, p'; h, h')$ is $\mathfrak{sl}(L)$-equivariant, $p'(\rho_0(\tilde{\mathfrak{r}}^i{}_j)h'(\tilde{X}))$ can be nonzero only if $p'(h'(\tilde{X}))$ is already nonzero. But this cannot be the case since (i', p', h') forms a strong deformation retract, whose definition requires $p' \circ h' = 0$.

Similarly, in the latter case (70), since the strong deformation retract $(i, i'; p, p'; h, h')$ is $\mathfrak{sl}(L)$-equivariant, $h'(\rho_0(\tilde{\mathfrak{r}}^i{}_j)X)$ can be nonzero only if $h'(X)$ is already nonzero, but this is not possible.

Finally, in the last case (71), again, since the strong deformation retract $(i, i'; p, p'; h, h')$ is $\mathfrak{sl}(L)$-equivariant, $h[\tilde{\mathfrak{r}}^i{}_j, x]$ can be nonzero only if $h(x)$ is already nonzero, which is not possible. □

Lemma 3. *For $p+q$ odd and $p+q \geq 3$, the $\mathfrak{sl}(L)$-representation*

$$R_{p,q} := (L^{\wedge 2})^{\wedge p} \otimes (L^*)^{\wedge q} \tag{72}$$

has no irreducible components in common with

$$\tilde{R}_{p,q} := \begin{cases} \bigoplus_{n=0}^{\infty} L^{\odot n} \otimes (L^*)^{\odot(n+(1+q-p)/2)} & \text{if } 1+q-p \geq 0 \\ \bigoplus_{n=0}^{\infty} L^{\odot(n-(1+q-p)/2)} \otimes (L^*)^{\odot n} & \text{if } 1+q-p \leq 0 \end{cases} \tag{73}$$

except when $(p, q) = (3, 0)$ or $(4, 3)$, in which case the irreducible components in common are $(2001)_{\mathfrak{sl}(L)}$ and $(1001)_{\mathfrak{sl}(L)}$ respectively.

Proof. We must compute the tensor product appearing in $\tilde{R}_{p,q}$. For $1+q-p \geq 0$ and any nonnegative integer $n \geq 0$, it is easy to see that

$$\big(n000\big) \otimes \big(000(n+(1+q-p)/2)\big) = \bigoplus_{i=0}^{n} \big(i00(i+(1+q-p)/2)\big). \tag{74}$$

Similarly, for $1+q-p \leq 0$ we have

$$\big((n-(1+q-p)/2))000\big) \otimes \big(000n\big) = \bigoplus_{i=0}^{n} \big((i-(1+q-p)/2))00i\big). \tag{75}$$

Thus, $\tilde{R}_{p,q}$ only contains irreducible representations of the form

$$\begin{cases} \big(i00(i+(1+q-p)/2)\big) & \text{if } 1+q-p \geq 0 \\ \big((i-(1+q-p)/2)00i\big) & \text{if } 1+q-p \leq 0 \end{cases} \qquad (i \in \{0, 1, 2, \dots\}). \tag{76}$$

Given this, iterating over[10] $p \in \{0, 1, \dots, 10\}$ and $q \in \{0, 1, \dots, 5\}$ and verifying whether an irreducible representation of the above form appears in $R_{p,q}$ (using e.g. a computer algebra system), one can see that $(p, q) \in \{(3, 0), (4, 3)\}$ are the only possible solutions. □

4. Other dimensions and amounts of supersymmetry

The above construction works for general supersymmetry algebras and general supermultiplets. This raises the question of how special (or generic) the ten-dimensional $\mathcal{N} = (1, 0)$ super-Poincaré algebra analysed above is. Without claiming an exhaustive analysis, let us remark in this section that there do not seem to be very many examples (besides the example analysed above) that nontrivial and purely bosonic higher products, at least if one considers those superalgebras that commonly arise in physics; a glance at [4] shows that this is the only case in which the cohomology of the twisted supermultiplet is simply a polynomial ring in bosonic variables.

In general, for sufficiently large dimension n, the number of spinorial components in a super-Poincaré algebra increases as $\mathcal{O}(2^n)$ whereas bosonic components increase as $\mathcal{O}(n^2)$. Indeed, already at 14 dimensions, the minimal spinor has 128 components while $\mathfrak{io}(14)$ has 105 components. So we are restricted to 13 or fewer dimensions even if one did not take into consideration no-go theorems about higher-spin theories (since we ignore dynamics here). Similarly, the 11-dimensional case (starting with the supergravity multiplet) is discussed in [20,21]. There are two possible cases. In one case [20], $\mathrm{H}(\mathfrak{p}_Q)$

[10] Recall that $(L^{\wedge 2})^{\wedge k}$ and $(L^*)^{\wedge l}$ are zero for $k > 10$ and $l > 5$.

contains fermionic elements. Then we expect the action of $\mathrm{H}(\mathfrak{p}_Q)$ to contain a $\rho^{(2)}$ involving the remaining fermionic elements. In the other case [21], however, we expect to see a higher action of

$$\mathrm{H}(\mathfrak{p}_Q) = (\mathfrak{g}_2 \oplus \mathfrak{sl}(L) \oplus V_7 \otimes L \oplus \mathbb{C}) \ltimes L \tag{77}$$

(which should carry nontrivial μ_2 and μ_4) on

$$\mathbb{A}(L^*) = \operatorname{Spec}\mathbb{C}[z_1, z_2], \tag{78}$$

where L is a two-dimensional vector space and V_7 is a seven-dimensional vector space; here, $\mathbb{C}[z_1, z_2] = \mathrm{H}(\Omega^{0,\bullet}(L) \otimes \Omega^{\bullet}(V))$ is the cohomology of the Dolbeault–de Rham complex on two complex and seven real dimensions.

On the other hand, if there are too few dimensions, empirically it appears that higher products often simply vanish. For example, for the four-dimensional $\mathcal{N} = 1$ super-Poincaré algebra, the twist gives a decomposition of four-dimensional complexified spacetime V as $V = L \oplus L^*$, where L is a two-dimensional vector space, with the twisted super-Poincaré algebra being [20]

$$\mathfrak{p}_Q = \begin{pmatrix} & (L^*)^{\wedge 2} \longrightarrow L^{\wedge 0} & \\ (\mathfrak{sl}(L) \ltimes L^{\wedge 2}) \oplus \mathfrak{gl}(1)_{\mathrm{R}} & L^{\wedge 1} \longrightarrow L & \\ & \mathfrak{gl}(1)_{\mathrm{tr}} \longrightarrow L^{\wedge 2} & L^* \end{pmatrix}, \tag{79}$$

where $\mathfrak{gl}(1)_{\mathrm{R}}$ is the R-symmetry and $\mathfrak{gl}(1)_{\mathrm{tr}}$ is the trace part of $\mathfrak{gl}(L)$. Following the proof of Theorem 1, we see that there are no higher brackets for $\mathrm{H}(\mathfrak{p}_Q)$ by constructing all possible intermediate states: apply $\mu_2(L^{\wedge 2}, -)$ to $L^*[-2]$ to get $L[-2]$; applying the homotopy h yields $L^{\wedge 1}$; but now applying another $\mu_2(L^{\wedge 2}, -)$ simply kills everything.

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Appendix A. Conventions

In a basis adapted to the choice of pure spinor $Q \in S_+$, the super-Poincaré algebra $\mathfrak{p} = \mathfrak{p}^0 \oplus \mathfrak{p}^1 \oplus \mathfrak{p}^2$ in ten dimensions has the basis elements

$$\mathrm{r}^i{}_j, \mathrm{r}^{ij}, \mathrm{r}_{ij} \in \mathfrak{p}^0, \qquad \mathrm{d}, \mathrm{d}_{ij}, \mathrm{d}^i \in \mathfrak{p}^1, \qquad \mathrm{e}^i, \mathrm{e}_i \in \mathfrak{p}^2, \tag{80}$$

with $\mathrm{r}^{ij} = -\mathrm{r}^{ji}$, $\mathrm{r}_{ij} = -\mathrm{r}_{ji}$, and $\mathrm{d}_{ij} = -\mathrm{d}_{ji}$. The graded-skew-symmetric Lie brackets among these basis elements are

$$\begin{aligned}
&\left[\mathrm{r}^i{}_j, \mathrm{r}^k{}_l\right] = \delta^k_j \mathrm{r}^i{}_l - \delta^i_l \mathrm{r}^k{}_j && \left[\mathrm{r}^i{}_j, \mathrm{r}^{kl}\right] = \delta^k_j \mathrm{r}^{il} + \delta^l_j \mathrm{r}^{ik}\\
&\left[\mathrm{r}^i{}_j, \mathrm{r}_{kl}\right] = -\delta^i_k \mathrm{r}_{jl} - \delta^i_l \mathrm{r}_{kj} && \left[\mathrm{r}^{ij}, \mathrm{r}_{kl}\right] = \delta^i_k \mathrm{r}^j{}_l - \delta^j_k \mathrm{r}^i{}_l - \delta^i_l \mathrm{r}^j{}_k + \delta^j_l \mathrm{r}^i{}_k\\
&\left[\mathrm{r}^i{}_j, \mathrm{e}^k\right] = \delta^k_j \mathrm{e}^i && \left[\mathrm{r}^i{}_j, \mathrm{e}_k\right] = -\delta^i_k \mathrm{e}_j\\
&\left[\mathrm{r}^{ij}, \mathrm{e}_k\right] = \delta^j_k \mathrm{e}^i - \delta^i_k \mathrm{e}^j && \left[\mathrm{r}_{ij}, \mathrm{e}^k\right] = \delta^k_i \mathrm{e}_j - \delta^k_j \mathrm{e}_i\\
&\left[\mathrm{r}^{ij}, \mathrm{d}\right] = 0 && \left[\mathrm{r}_{ij}, \mathrm{d}\right] = -\mathrm{d}_{ij}\\
&\left[\mathrm{r}^{ij}, \mathrm{d}_{kl}\right] = (\delta^i_l \delta^j_k - \delta^i_k \delta^j_l)\mathrm{d} && \left[\mathrm{r}_{ij}, \mathrm{d}_{kl}\right] = -\varepsilon_{ijklm} \mathrm{d}^m\\
&\left[\mathrm{r}^{ij}, \mathrm{d}^k\right] = -\frac{1}{2}\varepsilon^{ijklm} \mathrm{d}_{lm} && \left[\mathrm{r}_{ij}, \mathrm{d}^k\right] = 0\\
&\left[\mathrm{r}^i{}_j, \mathrm{d}\right] = \frac{1}{2}\delta^i_j \mathrm{d} && \left[\mathrm{r}^i{}_j, \mathrm{d}_{kl}\right] = \frac{1}{2}\delta^i_j \mathrm{d}_{kl} - \delta^i_k \mathrm{d}_{jl} - \delta^i_l \mathrm{d}_{kj}\\
&\left[\mathrm{r}^i{}_j, \mathrm{d}^k\right] = -\frac{1}{2}\delta^i_j \mathrm{d}^k + \delta^k_j \mathrm{d}^i && \left[\mathrm{d}, \mathrm{d}^i\right] = \mathrm{e}^i\\
&\left[\mathrm{d}^i, \mathrm{d}_{jk}\right] = \delta^i_k \mathrm{e}_j - \delta^i_j \mathrm{e}_k && \left[\mathrm{d}_{ij}, \mathrm{d}_{kl}\right] = -\varepsilon_{ijklm} \mathrm{e}^m
\end{aligned} \tag{81}$$

with all remaining brackets of basis elements vanishing. Here the indices $i, j, \ldots$ run over $\{1, 2, 3, 4, 5\}$, and we employ the Einstein summation convention. Here $\mathfrak{r}^i{}_j$ span $\mathfrak{gl}(5)$. The basis elements of the subalgebras $\mathfrak{sl}(5)$ are

$$\tilde{\mathfrak{r}}^i{}_j := \mathfrak{r}^i{}_j - \frac{1}{5}\delta^i_j \mathfrak{r}^k{}_k. \quad (82)$$

Data availability

No data was used for the research described in the article.

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