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Amplitubes: graph cosmohedra

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Abstract: The tree-level scattering amplitudes for $\text{tr}(\phi^3)$ theory can be interpreted as a sum over the vertices of a polytope known as the associahedron. For each graph G, there exists a natural generalisation of the associahedron, which is constructed by considering tubes and tubings of the underlying graph. This family of polytopes are called *graph associahedra*. The classical associahedra then arise as the graph associahedron for the path graphs. It is therefore natural to associate to each graph associahedron an amplitude-like object, we refer to as the *amplitube*, defined via a sum over its vertices. Recently, also in the context of $\text{tr}(\phi^3)$ theory, progress has been made towards defining a new geometric object, coined the *cosmohedron*, which computes not the amplitude, but the cosmological wavefunction as a sum over its vertices. This polytope can be constructed by consistently blowing up all boundaries of the associahedron to co-dimension one. Building on these results, in the present paper, we generalise the notion of the wavefunction for arbitrary graphs. These new expressions, which we call *cosmological amplitubes*, are defined via a sum over the vertices of a corresponding polytope, the *graph cosmohedron*. The graph cosmohedra are constructed by considering *regions* and *regional tubings* of the underlying graph which we introduce. Like the cosmohedron, the graph cosmohedra can be obtained by consistently blowing up all boundaries of the corresponding graph associahedron to co-dimension one. This new family of polytopes constitutes a vast generalisation of the cosmohedron, and we provide explicit embeddings for them, which builds upon an ABHY-like embedding for the graph associahedra.

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Contents

1 Introduction

In recent years the study of scattering amplitudes has fostered a rich interplay between mathematics and physics. As research progresses, it becomes increasingly clear that much of the structure of scattering amplitudes is governed by purely geometrical, or perhaps even combinatorial, principles. In fact, the observation that cubic Feynman diagrams can be used as labels of the vertices of a polytope, the associahedron, provides the first hint of a possible geometrical interpretation of amplitudes. In the amplitudes inspired description of the associahedron each facet corresponds to a *factorisation* channel of the n-point amplitude, and each vertex corresponds to a cubic Feynman graph. The tree-level scattering amplitudes of $\mathrm{tr}(\phi^3)$ theory for example can then be viewed as a sum over vertices of the associahedron. In this context the familiar statements of *locality* and *unitarity* of the amplitude translate into statements about factorisation properties of boundaries of the associahedron. By taking the planar dual of the tree-level Feynman diagram this can be phrased in terms of triangulations of an n-gon. In this way each boundary of the associahedron is labelled by a partial triangulation of an n-gon, with the vertices corresponding to full triangulations, and the facets corresponding to single chords. The sum over vertices of the associahedron then becomes

$$A_n = \sum_T \prod_{(ij)\in T} \frac{1}{X_{ij}}, \tag{1.1}$$

where the sum is over all triangulations of the n-gon and X_{ij} correspond to the chords of the triangulation. Upon identifying the X_{ij} with the planar Mandelstam variables, which are the squares of sums of consecutive momenta, the above expression recovers the tree-level amplitudes of $\text{tr}(\phi^3)$ theory. This combinatorial statement was made concrete by the discovery of the ABHY associahedron [1], a positive geometry [2] that provides a realisation of the associahedron in the kinematic space of n-point massless scattering whose canonical form encodes tree-level amplitudes in $\text{tr}(\phi)^3$ theory.

More recently techniques developed in the study of scattering amplitudes have started to be applied in a cosmological setting, see [3–10] and references therein for advances in this direction. The object of study in this context are the cosmological wavefunctions. At the level of combinatorics the wavefunction is described not by triangulations of an n-gon but rather by *nested polyangulations* or *Russian dolls* [3]. In terms of nested polyangulations the wavefunction takes the following form

$$\Psi_n = \sum_{\mathbf{P}} \prod_{P \in \mathbf{P}} \frac{1}{\mathcal{P}_P}. \tag{1.2}$$

Here the sum is over all maximal sets $\mathbf{P}$ of non-overlapping[1] sub-polygons of the n-gon, and the $\mathcal{P}_P$ are variables associated to the perimeter of each sub-polygon P. Much like the associahedron, the nested polyangulations can be used to define their own polytope named the *cosmohedron* [3]. The facets of the cosmohedron correspond to all partial triangulations of the n-gon, and as such are in bijection with the boundaries of the asociahedron, whereas the vertices correspond to maximally nested polyangulations.

Meanwhile, in the mathematics literature, the above developments have lead to the study of amplitude-like expressions which satisfy their own versions of locality and unitarity. These include for instance the CEGM amplitudes of [11] which provide a grassmannian generalisation of the $\text{tr}(\phi^3)$ amplitudes. Other examples of amplitude-like constructions include those provided by matroids [12] and the surface functions of [13]. To motivate the main topic of study in this paper, we consider yet another interpretation of the boundary stratification of the associahedron in terms of *tubes* and *tubings* of a path graph on $(n-3)$ vertices. A tube of a graph is defined as a subset of vertices of the graph which induce a connected subgraph. Whilst a tubing is a collection of tubes which are either nested or do not intersect and are not adjacent on the graph. In this language the facets of the associahedron are labelled by single tubes of the path graph, and the vertices are labelled by maximal tubings. The advantage of this definition is that it can immediately be applied to arbitrary graphs in which case the associahedron is replaced with the larger set of polytopes named *graph associahedra* [14]. Since the amplitude associated to the associahedron is defined via a sum over its vertices it is natural to extend this definition to arbitrary graph associahedra, where we refer to the corresponding functions as *amplitubes* defined as

$$A_G = \sum_{\tau \in \Gamma_G^{\max}} \prod_{t \in \tau} \frac{1}{X_t}, \tag{1.3}$$

where τ sums over all maximal tubings of the graph G, and X_t are formal variables associated to each tube t. In the case where we take G to be the path graph the above formula reduces to the familiar $\text{tr}(\phi^3)$ amplitudes. For the amplitube, the analogue of locality is that the variables appearing in the denominator of A_G all correspond to *connected* subgraphs, and

[1] By non-overlapping we mean that the chords defining each sub-polygon are non-overlapping.

unitarity states that upon taking a residue at $X_t = 0$ the amplitube factorises into a product of two simpler amplitubes. Furthermore, there exists an ABHY-like embedding of the graph associahedra in the *graph kinematic space* spanned by the set of X_t. By calculating the canonical form of the graph associahedron and pulling-back to an appropriate subspace one recovers the associated amplitube.

The focus of this paper will be to extend the notion of amplitubes into a cosmological setting by generalising nested polyangulations into tubing notions which can then be applied to arbitrary graphs in order to define *cosmological amplitubes*. Guided by the results for $\mathrm{tr}(\phi^3)$ theory, we will see that the cosmological amplitubes are naturally defined in terms of objects we refer to as *regional tubes* and *regional tubings* as

$$\Psi_G = \sum_{\rho \in \Phi_G^{\max}} \prod_{r \in \rho} \frac{1}{\mathcal{R}_r}. \tag{1.4}$$

Here we sum over all maximal regional tubings ρ and take the product over all regions r of ρ, which play the role of nested polyangulations and sub-polygons, respectively, for arbitrary graphs. Just as for the $\mathrm{tr}(\phi^3)$ wavefunctions, these functions also have their own geometric counterpart which we refer to as *graph cosmohedra* whose boundary structure encode the cosmological amplitubes. As such each facet of the graph cosmohedron will correspond to a boundary of the corresponding graph associahedron. As we shall explain the graph cosmohedra can be realised in the graph kinematic space by building upon the ABHY-like embedding of the graph associahedra. In the case of the empty graph, path graph and fully connected graph we show that graph cosmohedra reproduce the permutohedron, cosmohedron and permutoassociahedron respectively. In this way the graph cosmohedra can be seen as an interpolation between the permutohedron and the permutoassociahedron [15]. This suggests that, from a mathematical perspective, the graph cosmohedra constructed here might more naturally be referred to as *graph permutoassociahedra*.

This paper is organised as follows. In section 2 we introduce the notions of tubes and tubings of graphs which are needed to define the amplitubes and their associated graph associahedra. Many of the properties of the graph associahedra have already appeared in the literature and we will spend the majority of the first part of the paper collecting and rephrasing these results in term of tubings. Most importantly this will include an ABHY-like embedding, we refer to as the *tubing embedding*, for general graph associahedra which can be simply stated when phrased in terms of tubes. In section 3 we introduce the notion of regions and regional tubings of arbitrary graphs which are needed in order to define the cosmological amplitubes and their corresponding graph cosmohedra. Having defined the combinatorial object of interest, we turn to providing a geometric realisation of the graph cosmohedra. We provide an explicit embedding of the graph cosmohedra in the graph kinematic space and present explicit examples of graph cosmohedra for the empty, path and complete graph. At the end of this section we discuss the factorisation properties of graph cosmohedra on the co-dimension one facets. In section 4 we conclude and comment on possible future research directions.

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Figure 1. An illustration of the four possible configurations of tubings: intersecting, adjacent, nested and non-adjacent non-intersecting tubes. The last two pairs of tubes are compatible.

2 Graph associahedra

We start in this section by recalling the definition and basic facts about tubes and tubings of a graph. Our discussion will follow closely the definitions provided in [14]. These definitions will allow us to define the corresponding amplitube of a graph. After the combinatorial construction, we provide an embedding of the graph associahedra, which we refer as the *tube embedding*, that will be the starting point for the definition of graph cosmohedra in the next section. The tube embedding for graph associahedra we provide combines the original ABHY embedding of the associahedron [1] with the embedding in [16] for general graph associahedra.

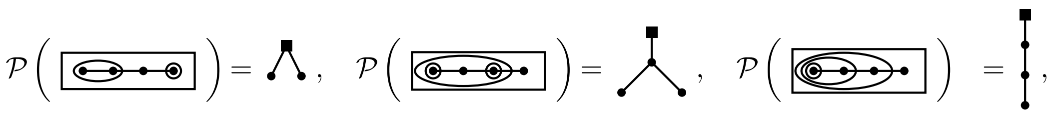
2.1 Tubes and tubings

Let G be a graph with the vertex set V_G and edge set E_G. A *tube* $t = \{v_1, v_2, \ldots, v_{|t|}\} \subset V_G$ on G is a proper non-empty subset of vertices of G such that the induced subgraph $G[t]$ is connected. The set of all tubes of G is denoted by T_G. We also define $\overline{T}_G = T_G \cup \{V_G\}$, and in this context, we call the graph vertex set V_G *the root*. We introduce the following terminology relating two tubes t_1 and t_2 on a given graph: we say that

- t_1 and t_2 *intersect* if $t_1 \cap t_2 \neq \emptyset$ and $t_1 \not\subset t_2$ and $t_2 \not\subset t_1$,
- t_1 and t_2 are *adjacent* if $t_1 \cap t_2 = \emptyset$ and $t_1 \cup t_2 \in \overline{T}_G$,
- t_1 is *nested* in t_2 if t_1 is a proper subset of t_2: $t_1 \subsetneq t_2$,
- t_1 and t_2 are *compatible* if they do not intersect and they are not adjacent.

These concepts are easily visualised as displayed in figure 1.

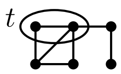
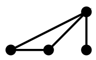
A subset of tubes $\tau = \{t_1, t_2, \ldots, t_{|\tau|}\} \subset T_G$ is called a *tubing* of G if all tubes in τ are mutually compatible and the set $V_G \setminus \bigcup_i t_i$ is not empty. A tubing is said to be *maximal* if no more compatible tubes can be added. We denote the set of all tubings of G by Γ_G and the set of all maximal tubings by $\Gamma_G^{\max}$. Given a tubing τ of G, we define $\overline{\tau} = \tau \cup V_G$. Then we can define a graded poset $\mathcal{P}_{\overline{\tau}}(G)$ on the set of tubes $t \in \overline{\tau}$ by inclusion, where the rank function is given by the number of tubes in τ between a given tube and the root. Examples of tubings τ and their corresponding posets $\mathcal{P}_{\overline{\tau}}(G)$ are


(2.1)

where the top vertex of each poset corresponds to the root, which is represented as the rectangle around the graph G. The set of tubings Γ_G for a given graph G, together with inclusion, defines a partially ordered set which describes the boundary stratification of a convex polytope called the *graph associahedron of* G and denoted as $\mathcal{A}_G$. As such all boundaries of $\mathcal{A}_G$ are labelled by tubings, with the facets labelled by tubings containing a single tube, and the vertices labelled by maximal tubings. Two facets in $\mathcal{A}_G$ are adjacent, i.e. intersect along a co-dimension-two boundary, if the tubes in their corresponding tubings are compatible. At the level of tubings the intersection of two compatible facets is given by taking the union of the tubings labelling them. The co-dimension of each boundary is specified by the cardinality of the corresponding tubing. In the following we will provide an explicit embedding of the graph associahedra that will realise the combinatorics described here. The graph associahedra contain familiar families of polytopes including:

- for the path graph with $n+1$ vertices, denoted by P_{n+1}, the graph associahedron of P_{n+1} is the classical n-dimensional associahedron,
- for the graph with $n+1$ vertices and $E_G = \emptyset$, denoted N_{n+1}, the graph associahedron is the n-dimensional simplex,
- for the complete graph on $n+1$ vertices, denoted K_{n+1}, the graph associahedron is the n-dimensional permutohedron.

An important property that we wish to emphasise is that all graph associahedra satisfy factorisation properties on their boundaries. Let the *reconnected complement* G_t^* of a tube $t \in T_G$ be the graph with the vertex set $V_G \setminus t$ such that two vertices a and b are connected by an edge in G_t^* if either of the induced graphs $G[\{a,b\}]$ or $G[\{a,b\} \cup t]$ is connected, as illustrated in the example below

$G =$  $\longrightarrow$ $G_t^* =$  . (2.2)

Then the facet of the graph associahedron of G corresponding to the tubing $\tau = \{t\}$, containing the single tube t, factorises as the Cartesian product of two simpler graph associahedra

$$\partial_t(\mathcal{A}_G) = \mathcal{A}_{G[t]} \times \mathcal{A}_{G_t^*}. \tag{2.3}$$

Similar to the definition above, we can define the *reconnected complement for a tubing* $\tau = \{t_1, \ldots, t_{|\tau|}\}$ that contains pairwise non-adjacent and non-intersecting tubes, by iterating the above procedure for each $t \in \tau$. We denote the resulting graph as $G_\tau^* = ((G_{t_1}^*)_{t_2}^* \ldots)_{t_{|\tau|}}^*$.

2.2 Amplitubes

Having defined the graph associahedron of G combinatorially we can proceed by associating the corresponding *amplitube* A_G, that is the function defined as

$$A_G = \sum_{\tau \in \Gamma_G^{\max}} \prod_{t \in \tau} \frac{1}{X_t}, \tag{2.4}$$

where the sum is over all maximal tubings of G and we have introduced a variable X_t for each tube. We collectively refer to the set of all variables $\{X_t\}_{t \in T_G}$ as the *graph kinematic space* for graph G.

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As an example of an amplitube, let us consider the path graph $G = P_{n+1}$. If we label its vertices using labels $\{1, 2, \ldots, n+1\}$ from left to right, then each tube is labelled by a string of consecutive integers $[i, j] = \{i, i+1, \ldots, j\}$ for $i \leq j$. Then, upon replacing $X_{[i,j]} \to X_{i,j+2}$, where the latter are planar Mandelstam variables defined in [1], the amplitube (2.4) recovers the familiar tree-level amplitudes in $\mathrm{tr}(\phi^3)$ theory. Under this mapping, the tubes encircling a single vertex $X_{[i,i]}$ are mapped to the Madelstam invariants $X_{i,i+2}$. For example, the amplitube for the path graph on three vertices P_3 is given by

$$A_{P_3} = \frac{1}{X_{\{1\}}X_{\{1,2\}}} + \frac{1}{X_{\{2\}}X_{\{1,2\}}} + \frac{1}{X_{\{2\}}X_{\{2,3\}}} + \frac{1}{X_{\{3\}}X_{\{2,3\}}} + \frac{1}{X_{\{1\}}X_{\{3\}}}\,. \tag{2.5}$$

Another example is provided by the amplitube for the complete graph on three vertices K_3 that takes the following form

$$A_{K_3} = \frac{1}{X_{\{1\}}X_{\{1,2\}}} + \frac{1}{X_{\{2\}}X_{\{1,2\}}} + \frac{1}{X_{\{2\}}X_{\{2,3\}}} + \frac{1}{X_{\{3\}}X_{\{2,3\}}} + \frac{1}{X_{\{1\}}X_{\{1,3\}}} + \frac{1}{X_{\{3\}}X_{\{1,3\}}}. \tag{2.6}$$

At the level of the amplitube, the factorisation property of the graph associahedra (2.3) is reflected by the following factorisation of its residue

$$\operatorname*{Res}_{X_t=0}(A_G) = A_{G[t]} \cdot A_{G_t^*}. \tag{2.7}$$

As an example, again considering the complete graph on three vertices K_3, we have

$$\operatorname*{Res}_{X_{\{1,2\}}=0}(A_{K_3}) = \frac{1}{X_{\{1\}}} + \frac{1}{X_{\{2\}}} = A_{P_2} \cdot A_{P_1}, \tag{2.8}$$

which reflects the correct factorisation of the facet $X_{\{1,2\}} = 0$ of the graph associahedron $\mathcal{A}_{K_3}$, since $K_3[\{1,2\}] = P_2$ and $(K_3)^*_{\{1,2\}} = P_1$.

To translate to the language of physics it is easy to see that the analogue of *locality* for the amplitubes is given by the fact that the variables appearing in the denominator of A_G each correspond to *connected* subgraphs of G, whilst the analogue of *unitarity* is the fact that the residues of amplitubes are given by products of simpler amplitubes.

2.3 Associahedron embedding

We will now turn the combinatorial statements of the last section into geometric ones by providing a realisation of the graph associahedra as convex polytopes. First, we want to point out that in the case of the path graph P_{n+1}, there exists an embedding intimately connected to the kinematics of n-particle scattering in $\mathrm{tr}(\phi^3)$ theory, commonly referred to as the ABHY associahedron. It is this construction which we wish to modify and then generalise to arbitrary graph associahedra. We will refer to the embedding introduced in this section as the *tube embedding*. Our prescription follows closely the one originally laid out for the associahedron in [1] together with earlier work on realisations of graph associahedra in [16]. Similar constructions have already appeared in the literature, for instance see [17, 18].

We fix the graph G. For each tube $t \in T_G$, in addition to the kinematic variable X_t, we also introduce a *cut-parameter* c_t, which we take to be a positive real number. We impose

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that the kinematic variables satisfy the following linear constraints

$$X_t = -\sum_{t'\subset t} c_{t'}\,, \tag{2.9}$$

where the sum runs over all tubes that are subsets of t. Additionally, we impose a similar relation also for the variable X_{V_G} associated to the root:

$$X_{V_G} = -\sum_{t\in\overline{T}_G} c_t\,. \tag{2.10}$$

Importantly, in the case where the tube $t = \{v\}$ contains a single vertex v, we have $X_{\{v\}} = -c_{\{v\}}$, which allows one to re-express the linear relations (2.9) as

$$X_t = \sum_{v\in t} X_{\{v\}} - \sum_{t'\subset t, |t'|>1} c_{t'}. \tag{2.11}$$

Here the sum over c's runs over all tubes t' that are subsets of t and contain more than one vertex. The tube embedding of the graph associahedron can now simply be stated in terms of the variables $X_{\{v\}}$ as

$$\mathcal{A}_G = \{(X_{\{1\}},\ldots,X_{\{|V_G|\}}) \in \mathbb{R}^{|G|} : (\forall_{t\in T_G}\, X_t \geq 0) \text{ and } (X_{V_G} = 0)\}. \tag{2.12}$$

In the case of the classical associahedra $\mathcal{A}_{P_{n+1}}$, corresponding to the path graph $G = P_{n+1}$, the tube embedding agrees with the one obtained by ABHY in [1], after the latter is projected on an appropriate n dimensional subspace[2] of the kinematic space. Examples of the tube embedding for the graph associahedra associated to the path graph P_4, the cycle graph C_4 and the complete graph K_4 are displayed in figure 2.

To make a further comparison with the ABHY embedding, we can invert (2.9) to obtain an expression for the c's in terms of X's. The expressions for the cut parameters then take the form of an alternating sum as

$$c_{t_0} = -X_{t_0} + \sum_{t_1\subsetneq t_0} X_{t_1} - \sum_{t_2\subsetneq t_1\subsetneq t_0} X_{t_2} + \sum_{t_3\subsetneq t_2\subsetneq t_1\subsetneq t_0} X_{t_3} - \ldots\,. \tag{2.13}$$

In the case of the classical n-dimensional associahedron $\mathcal{A}_{P_{n+1}}$, the above formula reduces to the familiar conditions

$$c_{[i,j]} = X_{[i-1,j]} + X_{[i,j-1]} - X_{[i,j]} - X_{[i-1,j-1]}\,, \tag{2.14}$$

where the $X_{[i,j]}$ can again be identified with the square of sums of consecutive momenta i.e. the planar Mandelstam variables.

We emphasize that the definition (2.12) is different from the one provided in the original ABHY construction for classical associahedra. In our case, the polytope $\mathcal{A}_G$ sits on a hyperplane $X_{V_G} = 0$ inside a $|V_G|$-dimensional Euclidean space. In the original construction, the space was parametrised by $\binom{n}{2} - n$ planar Mandelstam variables, and the polytope was residing on the intersection of the conditions (2.14). Our construction also reduces to the one provided in [18] when all cut-parameters c_t are set to 1.

[2]One needs to project the ABHY associahedron on the space parametrised by the planar Mandelstam variables $X_{i,i+2}$ for $i = 1, 2, \ldots, n-2$.

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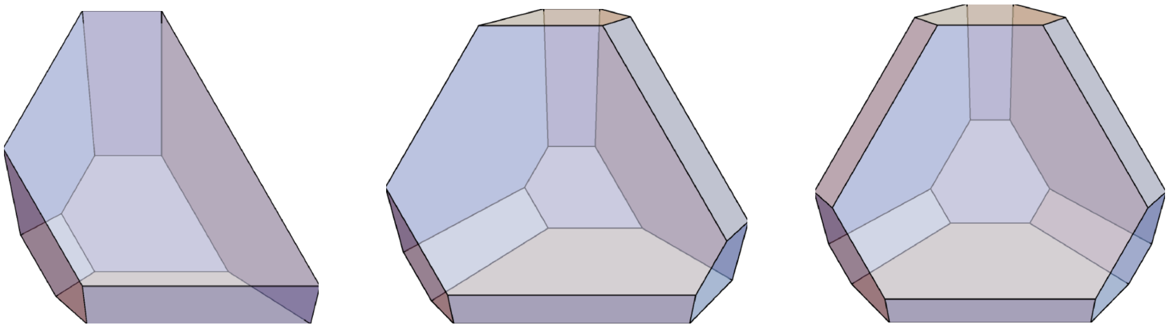

Figure 2. Embeddings of three-dimensional graph associahedra for the path graph, cycle graph, and complete graph on four vertices. These polytopes are examples of the three-dimensional associahedron, cyclohedron and permutohedron, respectively.

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Finally, every graph associahedron $\mathcal{A}_G$ is a simple convex polytope, i.e. a d-dimensional polytope with all vertices incident to exactly d facets. Therefore one can find its canonical differential form using the general formula

$$\Omega\left(\mathcal{A}_G\right)=\sum_{\tau\in\Gamma_G^{\max}}\operatorname{sign}_\tau\bigwedge_{t\in\tau}\mathrm{d}\log(X_t), \tag{2.15}$$

where the sum runs over all vertices of $\mathcal{A}_G$, and the signs can be determined by requiring that the canonical form $\Omega\left(\mathcal{A}_G\right)$ is projectively invariant. The amplitube (2.4) can then be recovered from the canonical forms as

$$\Omega(\mathcal{A}_G)=A_G\,\mathrm{d}\mu_G\,, \tag{2.16}$$

where $\mathrm{d}\mu_G=\mathrm{d}X_{\{1\}}\wedge\mathrm{d}X_{\{2\}}\wedge\ldots\wedge\mathrm{d}X_{\{|V_G|-1\}}$ and we solved the condition $X_{V_G}=0$ to eliminate $X_{\{|V_G|\}}$ from our expressions.

2.4 Examples of graph associahedra

Before moving on to discuss graph cosmohedra, we take a closer look at some two-dimensional examples of graph associahedra that correspond to graphs with three vertices, which we will return to in the next section. Consider first the empty graph N_3 on three vertices, i.e. $V_{N_3}=\{1,2,3\}$ and $E_{N_3}=\emptyset$. The inequalities defining the graph associahedron $\mathcal{A}_{N_3}$ in this case are

$$X_{\{1\}}\geq 0,\quad X_{\{2\}}\geq 0,\quad X_{\{3\}}\geq 0,\quad X_{\{1\}}+X_{\{2\}}+X_{\{3\}}=c_{\{1,2,3\}}, \tag{2.17}$$

and the resulting polytope is a two-dimensional simplex, i.e. a triangle. Next, consider the path graph P_3 on three vertices. In this case the inequalities that define the graph associahedron $\mathcal{A}_{P_3}$ are given by

$$\begin{aligned}&X_{\{1\}}\geq 0,\quad X_{\{2\}}\geq 0,\quad X_{\{3\}}\geq 0,\\&X_{\{1\}}+X_{\{2\}}\geq c_{\{1,2\}},\quad X_{\{2\}}+X_{\{3\}}\geq c_{\{2,3\}},\\&X_{\{1\}}+X_{\{2\}}+X_{\{3\}}=c_{\{1,2\}}+c_{\{2,3\}}+c_{\{1,2,3\}}.\end{aligned} \tag{2.18}$$

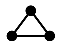
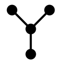
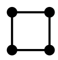
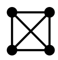
Graph G	dimension	codim-1	codim-2	codim-3
	1	2		
	1	2		
	2	3	3	
	2	4	4	
	2	5	5	
	2	6	6	
	3	9	21	14
	3	10	24	16
	3	12	30	20
	3	14	36	24

Table 1. The number of boundaries of various co-dimension of the associahedron on graph G.

These inequalities carve out a pentagon i.e. the classical two-dimensional associahedron. Finally, consider the complete graph K_3 on three vertices. The resulting set of inequalities is

$$\begin{aligned}
&X_{\{1\}} \geq 0, \quad X_{\{2\}} \geq 0, \quad X_{\{3\}} \geq 0, \\
&X_{\{1\}} + X_{\{2\}} \geq c_{\{1,2\}}, \quad X_{\{2\}} + X_{\{3\}} \geq c_{\{2,3\}}, \quad X_{\{1\}} + X_{\{3\}} \geq c_{\{1,3\}}, \\
&X_{\{1\}} + X_{\{2\}} + X_{\{3\}} = c_{\{1,2\}} + c_{\{2,3\}} + c_{\{1,3\}} + c_{\{1,2,3\}},
\end{aligned} \tag{2.19}$$

which produce a hexagon i.e. the two-dimensional permutohedron. We provide a list of combinatorial data associated to graph associahedra for various graphs with two, three and four vertices in table 1.

3 Graph cosmohedra

We now wish to promote the amplitubes introduced in the last section into a cosmological setting, the result of which we will refer to as *cosmological amplitubes*. As we shall explain these functions are most naturally written in terms of *regions* and *regional tubings* of the graph, which we introduce. In the case of the path graph the expression for the cosmological amplitube reduces to that of the wavefunction for $\mathrm{tr}(\phi^3)$ theory. As was the case for the graph associahedra and the amplitubes in the previous section, we will see that the cosmological amplitubes also have a geometric counterpart we refer to as *graph cosmohedra*. Graph cosmohedra are a new class of convex polytope that interpolate between the permutohedron and the permutoassociahedron. We will provide explicit embeddings for the graph cosmohedra which will rely on the already introduced variables for the graph associahedra. In the case of the path graph, the graph cosmohedron and its embedding recovers the recently introduced cosmohedron and the embedding provided in [3].

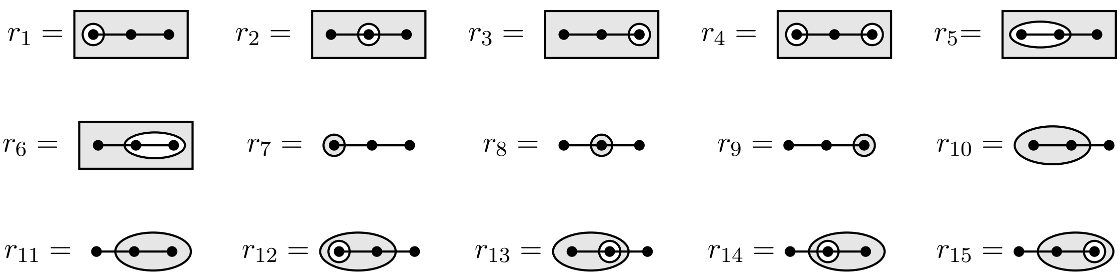


Figure 3. The fifteen regions for the graph P_3.

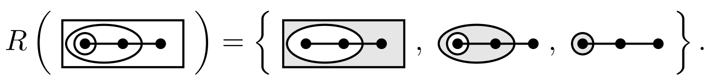
3.1 Regions and regional tubings

In the case of the path graph our goal is to reproduce the *cosmohedron* of [3] whose combinatorics is governed by sub-polygons and their nested polyangulations. Therefore, in order to extend this to an arbitrary graph we will need to generalise the notions of sub-polygons and nested polyangulations to *regions* and *nested regionalisations*. These will become the *regional tubes* and *regional tubings* we now introduce.

We say that $r = \{t_1, t_2, \ldots, t_{|r|}\} \subset \overline{T}_G$ is a *region* of the graph G if

- r is a subset of $\overline{T}_G$ such that all tubes of r are compatible as tubes,
- the poset defined by inclusion on the elements of r has a single maximal element,
- all other tubes in r are compatible with each other and not nested.

We refer to the element in r that is maximal with respect to inclusion as the parent $t_p^{(r)}$ and all other elements as the children $t_{c,i}^{(r)}$. Note, the parent will either correspond to a tube $t_p^{(r)} \in T_G$ or to the root $t_p^{(r)} = V_G$. In other words every region r is a graded poset with at most two layers, with one parent and a (possibly empty) set of children. Every tubing $\tau \in \Gamma_G$ defines a collection of regions that we denote as $R(\tau)$. To construct the set $R(\tau)$ we take for each $t \in \overline{\tau}$ the region with t as the parent together with all its children $t \supsetneq t' \in \tau$. An example of this procedure is given by the following


(3.1)

Although the notion of the regions is well defined for any graph, we can get an intuitive understanding by considering planar graphs. In this case, we can depict the regions by shading in the area between the parent and the children in the planar embedding of the graph. For example, for the path graph P_3 there are 15 possible regions, six of which have the root as the parent, as depicted in figure 3. Moreover, since the graph P_3 is related to the two-dimensional associahedron that labels triangulations of a pentagon, there is a natural bijection from the set of regions of P_3 to the set of subpolygons of a pentagon.

We say two regions r and r' of G are *compatible* if we have

$$(r \cup r') \setminus V_G \in \Gamma_G, \tag{3.2}$$

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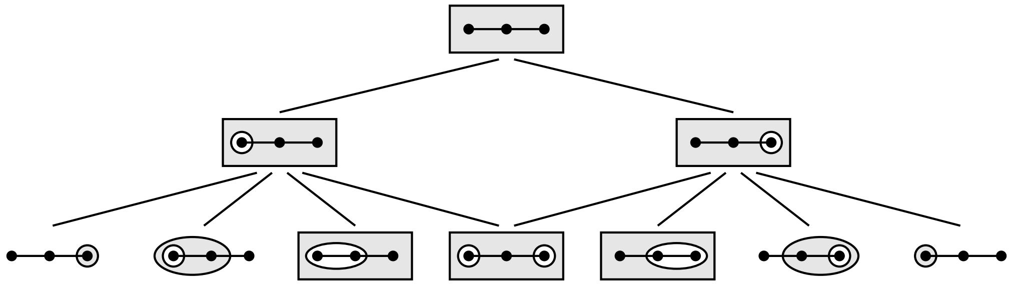
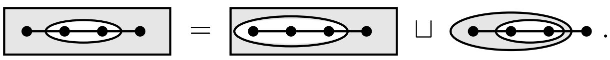


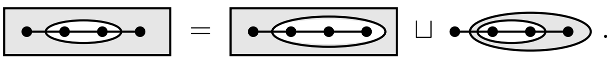
Figure 4. An example of the sub-region relation $\prec_{\text{reg}}$ for a subset of the regions of P_3.

which means that all tubes in the union of r and r' form a tubing of G. A region r' is a *sub-region* of r if it can be realised as a region on the graph $G^{(r)} := G[t_p^{(r)}]^*_{\{t_{c,i}^{(r)}\}}$, that is a region in the reconnected complement in the graph induced by the parent tube $t_p^{(r)}$ of all children tubes $t_{c,i}^{(r)}$. An example of the sub-region relation, denoted $r_1 \prec_{\text{reg}} r_2$ for r_1 is a sub-region of r_2, on a subset of the regions of P_3 is given in figure 4.

Given two compatible regions that are not sub-regions of one another, we refer to their intersection as the *border*. The border will necessary be empty or contain a single tube. The set of vertices $V^{(r)} = t_p^{(r)} \setminus \bigcup_i t_{c,i}^{(r)}$ *covered* by a region r is defined as the complement of the set of vertices in the children inside the set of vertices of the parent. A collection of sub-regions $\{r_i\}$ of r is said to *cover* r if $V^{(r)} = \cup_i V^{(r_i)}$ and $V^{(r_i)}$ are mutually disjoint. When we have such a covering, we use the notation $r = \sqcup_i r_i$. Again, these definition are much easier to visualise in a simple case, for example

 (3.3)

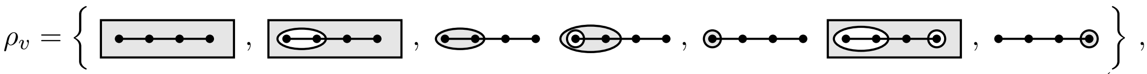
An alternative example of a covering of the same region is given by[3]

 (3.4)

Finally, we define the notion of *regional tubing* $\rho = \{r_1, \ldots, r_{|\rho|}\}$ of the graph G as a subset of mutually compatible regions for which:

- for every region $r \in \rho$ there either exists no sub-regions of r in ρ,
- or r is covered by a collection of sub-regions in ρ.

We denote the set of tubes appearing as borders between the regions of ρ as $B(\rho)$. For example, the following collection of regions

 (3.5)

[3]Relations (3.3) and (3.4) can be interpreted as two triangulations of a particular quadrilateral inside a hexagon.

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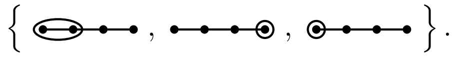
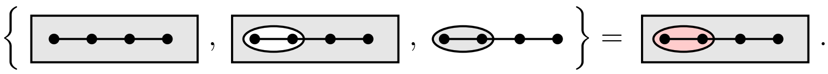
is a regional tubing of the path graph P_4 whose set of borders is given by

$$B(\rho_v) = $$  (3.6)

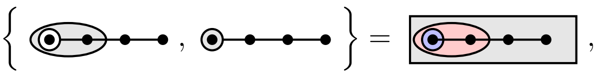
The sub-region relation on the set of regions contained in ρ_v is displayed in figure 5. In this case the regional tubing labels a non-simple vertex of the graph cosmohedron for the path graph on four vertices. This can be seen by noting that the number of regions in the bottom layer, corresponding to the facets meeting at this vertex, exceeds the dimension of the graph cosmohedron, which is three in this instance.

A regional tubing ρ is said to refine another regional tubing ρ' if $\rho' \subset \rho$. The set of regional tubings together with inclusion defines a partially ordered set which, as we will argue in the following section, gives the boundary stratification of the *graph cosmohedron* $\mathcal{C}_G$. All boundaries of the graph cosmohedron $\mathcal{C}_G$ are then labelled by regional tubings, with the vertices labelled by maximal regional tubings, i.e. the regional tubings that are maximal elements with respect to inclusion. We will denote the set of all regional tubings by Φ_G, and the set of maximal regional tubings by $\Phi_G^{\max}$.

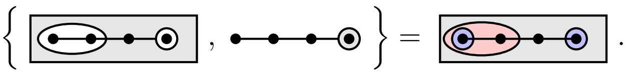
There exists an alternative diagrammatic representation of regional tubings which arises by endowing tubings with an integer label. Let ρ be a regional tubing and consider the poset on the elements of ρ defined by the partial order relation: $r_1 \prec_{\text{reg}} r_2$ if r_1 is a sub-region of r_2. We define the depth w of a region r in this poset as the maximal length of a chain $r = r_w \prec_{\text{reg}} r_{w-1} \prec_{\text{reg}} \ldots \prec_{\text{reg}} r_0 = r_G$.[4] For all regions in ρ of a fixed depth w we take the union of their pairwise borders and assign each element the label w. This results in a tubing where each tube receives an integer label. Note, the region r_G associated to the root trivially appears by itself at depth 0 for all regional tubings. To encode this information on a diagram we assign a colour to each label $\{0, 1, 2, 3, \ldots\}$ and fill in each tube with the corresponding colour. To illustrate this alternative notation, we consider the following regional tubing that corresponds to a facet of the cosmohedron on P_4

$$\rho_f = $$  (3.7)

A co-dimension two boundary i.e. an edge of the same cosmohedron is

$$\rho_e = \rho_f \cup $$  (3.8)

while a vertex is given by

$$\rho_v = \rho_e \cup $$  (3.9)

To better understand this alternative notation, one can compare this representation of ρ_v with the one in figure 5. In this case the only tube appearing as a border at level 1 is the tube $\{1, 2\}$. At level two, there are two tubes which appear as borders, namely the tubes $\{1\}$ and $\{4\}$. For more complicated graphs these diagrams no longer serve the purpose of simplifying

[4]Here $r_G = \{V_G\}$ refers to the region corresponding to the root which is the maximal element of the poset defined by the sub-region relation.

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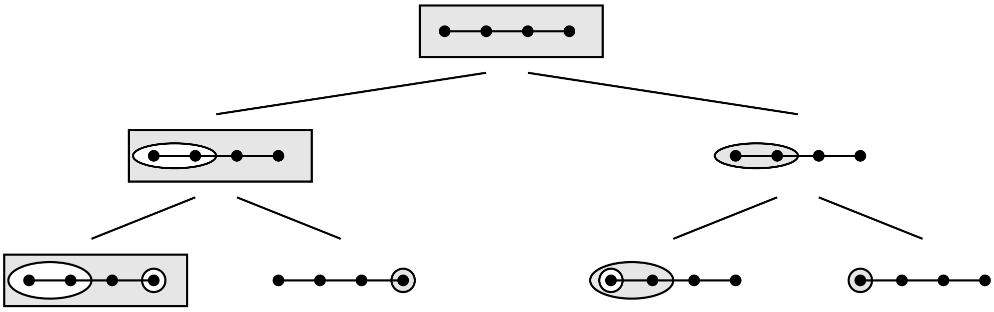

Figure 5. The sub-region relation on the set of regions contained in the regional tubing ρ_v defined in (3.5). This corresponds to a non-simple vertex of the corresponding cosmohedron for the graph P_4.

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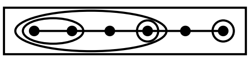
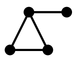
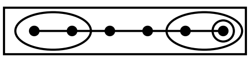
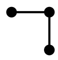
notation, in which case we revert back to the definition of regional tubings in terms of either sets of regions or tubings endowed with the additional integer labels, as previously described.

It is worth emphasising that generally the partial ordering of regional tubings described above will lead to boundary posets for non-simple polytopes, as already noted in [3]. For example, the regional tubing ρ_v from (3.9) corresponds to a non-simple vertex of the cosmohedron on P_4 which sits in the boundary of four facets. We will return to this point in detail in section 3.5.

3.1.1 Boundaries and factorisation

Similar as for graph associahedra, also graph cosmohedra have simple factorisation properties of their facets. To describe them, we introduce the notion of the *spine* of a tubing. Given a tubing τ of a graph G we define a graph $\text{spine}_\tau(G)$ whose vertices are given by the tubes $t \in \tau$ such that two vertices t_1 and t_2 are connected by an edge if t_1 and t_2 share the same parent or one is the parent of the other in the graded poset $\mathcal{P}_{\overline{\tau}}(G)$. As an example we have the following spine graphs

spine () =  , spine () =  . (3.10)

With this definition the factorisation property on the co-dimension one boundaries of the graph cosmohedra, labelled by the tubing τ, are given by

$$\partial_\tau(\mathcal{C}_G) = \mathcal{A}_{\text{spine}_\tau(G)} \times \prod_{r\in R(\tau)} \mathcal{C}_{G[t_p^{(r)}]^*_{\{t_{c,i}^{(r)}\}}}, \tag{3.11}$$

where the latter factor contains the graph cosmohedra for the graphs obtained by restricting G to one of the regions of τ. An example of the factorisation of a co-dimension one boundary of the cosmohedron for the graph P_4 is displayed in figure 6. In this case we have $\tau = \{\{1\}, \{3\}\}$ and $\text{spine}_\tau(P_4) = P_2$. Moreover, there are three regions generated by the tubing τ: two of them result in the trivial reconnected components P_1 for which the cosmohedron is a point, and the third one results in the reconnected component P_2, for which the cosmohedron is a segment. Therefore, the facet of the cosmohedron $\mathcal{C}_{P_4}$ specified by τ is the Cartesian product of two segments.

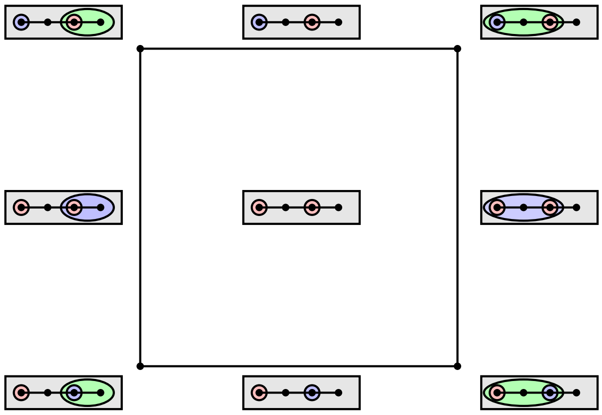


Figure 6. An illustration of the factorisation property of a facet of the cosmohedron for the graph P_4 with labels ordered as $\{0, 1, 2, 3\}$.

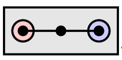
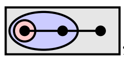
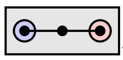
3.2 Cosmological amplitubes

Having defined the combinatorics of the graph cosmohedra we can now go ahead and define their corresponding *cosmological amplitubes* as

$$\Psi_G = \sum_{\rho \in \Phi_G^{\max}} \prod_{r \in \rho} \frac{1}{\mathcal{R}_r}\,, \tag{3.12}$$

where we assigned the variable $\mathcal{R}_r$ to the region r. We also set $\mathcal{R}_{V_g} = 1$ for the region containing the root as the parent, and no children. This formula correctly reproduces the one found in [3] since the set of regional tubing agrees with the set of non-overlapping subpolygons of an n-gon for path graphs, and the regions are in one-to-one correspondence with subpolygons.

As an example we consider again the path graph P_3 on three vertices. In this case the graph cosmohedron $\mathcal{C}_{P_3}$ has 10 vertices with the corresponding regional tubings

$\rho_1^{(P_3)} =$ , $\rho_2^{(P_3)} =$ , $\ldots,$ $\rho_{10}^{(P_3)} =$ , (3.13)

where the remaining regional tubings can be read off from figure 7. Then, the cosmological amplitube associated to the path graph P_3 is[5]

$$\begin{aligned}\Psi_{P_3} = & \frac{1}{\mathcal{R}_{r_1}\mathcal{R}_{r_5}\mathcal{R}_{r_7}\mathcal{R}_{r_{12}}} + \frac{1}{\mathcal{R}_{r_5}\mathcal{R}_{r_7}\mathcal{R}_{r_{10}}\mathcal{R}_{r_{12}}} + \frac{1}{\mathcal{R}_{r_5}\mathcal{R}_{r_8}\mathcal{R}_{r_{10}}\mathcal{R}_{r_{13}}} + \frac{1}{\mathcal{R}_{r_2}\mathcal{R}_{r_5}\mathcal{R}_{r_8}\mathcal{R}_{r_{13}}} \\ & + \frac{1}{\mathcal{R}_{r_2}\mathcal{R}_{r_6}\mathcal{R}_{r_8}\mathcal{R}_{r_{14}}} + \frac{1}{\mathcal{R}_{r_6}\mathcal{R}_{r_8}\mathcal{R}_{r_{11}}\mathcal{R}_{r_{14}}} + \frac{1}{\mathcal{R}_{r_6}\mathcal{R}_{r_9}\mathcal{R}_{r_{11}}\mathcal{R}_{r_{15}}} + \frac{1}{\mathcal{R}_{r_3}\mathcal{R}_{r_6}\mathcal{R}_{r_9}\mathcal{R}_{r_{15}}} \\ & + \frac{1}{\mathcal{R}_{r_3}\mathcal{R}_{r_4}\mathcal{R}_{r_7}\mathcal{R}_{r_9}} + \frac{1}{\mathcal{R}_{r_1}\mathcal{R}_{r_4}\mathcal{R}_{r_7}\mathcal{R}_{r_9}}\,, \end{aligned} \tag{3.14}$$

whose terms are in one-to-one correspondence to that of the wavefunction of $\mathrm{tr}(\phi^3)$ theory [3]. Note, the expression for the cosmological amplitube can be re-organised into a sum over

[5] By convention we have chosen to drop an overall factor coming from the region $\mathcal{R}_{V_G}$ associated to the entire graph.

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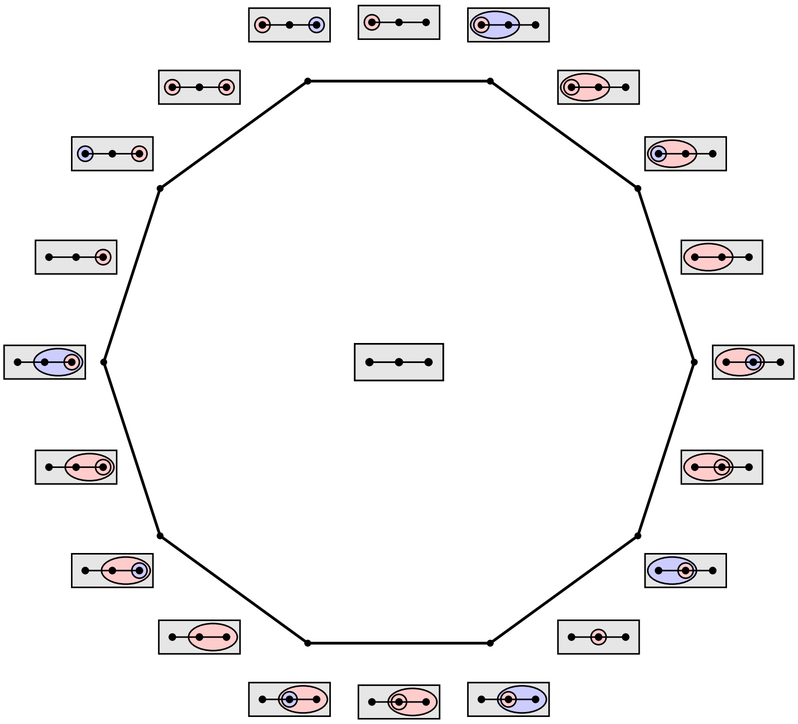

Figure 7. The graph cosmohedron for the path graph on three vertices which reproduces the two dimensional cosmohedron.

vertices of the corresponding graph associahedron as

$$\begin{aligned}\Psi_{P_3} = &\frac{1}{\mathcal{R}_{r_5}\mathcal{R}_{r_7}\mathcal{R}_{r_{12}}}\left(\frac{1}{\mathcal{R}_{r_1}}+\frac{1}{\mathcal{R}_{r_{10}}}\right)+\frac{1}{\mathcal{R}_{r_5}\mathcal{R}_{r_8}\mathcal{R}_{r_{13}}}\left(\frac{1}{\mathcal{R}_{r_{10}}}+\frac{1}{\mathcal{R}_{r_2}}\right)\\ &+\frac{1}{\mathcal{R}_{r_6}\mathcal{R}_{r_8}\mathcal{R}_{r_{14}}}\left(\frac{1}{\mathcal{R}_{r_2}}+\frac{1}{\mathcal{R}_{r_{11}}}\right)+\frac{1}{\mathcal{R}_{r_6}\mathcal{R}_{r_9}\mathcal{R}_{r_{15}}}\left(\frac{1}{\mathcal{R}_{r_{11}}}+\frac{1}{\mathcal{R}_{r_3}}\right)\\ &+\frac{1}{\mathcal{R}_{r_4}\mathcal{R}_{r_7}\mathcal{R}_{r_9}}\left(\frac{1}{\mathcal{R}_{r_3}}+\frac{1}{\mathcal{R}_{r_1}}\right). \end{aligned} \tag{3.15}$$

In the physics literature this decomposition is refered to as a sum over 'channels'. Generally, each vertex of the graph associahedron is labelled by a maximal tubing τ and contributes $|\Gamma_{\mathrm{spine}_\tau(G)}|$ many terms to the cosmological amplitube.

3.3 Cosmohedron embedding

We now move on to provide an embedding for the graph cosmohedron for any graph G. We will start with the coordinate system that we introduced for the tube embedding of the graph associahedron $\mathcal{A}_G$ described in section 2.3, and provide additional inequalities for each tubing which will carve off facets of this polytope in order to arrive at the corresponding graph cosmohedron $\mathcal{C}_G$. This construction will reproduce the cosmohedra defined in [3] for path graphs, but will provide previously unknown polytopes for a general graph.

First, for each tubing τ on G we introduce a new variable Y_τ and a cut parameter ϵ_τ and set

$$Y_\tau = \sum_{t\in\tau} X_t - \epsilon_\tau\,. \tag{3.16}$$

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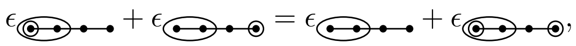
As in the case of path graphs in [3], the ϵ parameters must satisfy certain conditions to correctly cut out the graph cosmohedron. In particular, we demand that

$$\epsilon_\tau + \epsilon_{\tau'} = \epsilon_{\tau\cup\tau'} + \epsilon_{\tau\cap\tau'}, \tag{3.17}$$

for every pair of compatible tubings τ and τ' such that

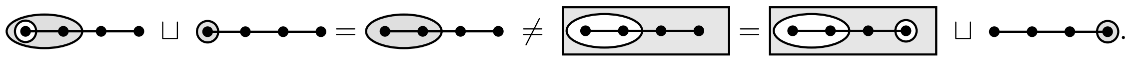
- the regions in $R(\tau) \setminus R(\tau \cap \tau')$ are all nested inside a single region r_1 of $R(\tau \cap \tau')$,
- the regions in $R(\tau') \setminus R(\tau \cap \tau')$ are all nested inside a single region r_2 of $R(\tau \cap \tau')$,
- and $r_1 \neq r_2$.

An example of a configuration satisfying (3.17) for the graph P_4 is given by



(3.18)

since

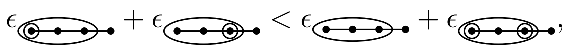


(3.19)

Additionally, for all other pairs of compatible tubings τ and τ' we require

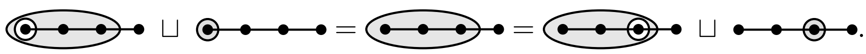
$$\epsilon_\tau + \epsilon_{\tau'} < \epsilon_{\tau\cup\tau'} + \epsilon_{\tau\cap\tau'}. \tag{3.20}$$

An example of a configuration where the inequality is not saturated, again for the graph P_4, is given by



(3.21)

since we have the following



(3.22)

The set of equalities in (3.17) are trivially satisfied if the cut parameters ϵ_τ are written as a sum over *regions* of the tubing τ as

$$\epsilon_\tau = \sum_{r\in R(\tau)} \delta_r, \tag{3.23}$$

where we introduced new parameters δ_r for each region of G. Finally, the inequalities (3.20) are satisfied whenever δ_r is a convex function of $|V_r|$ that vanishes on the region specified by the root: $r = \{V_G\}$. A particular example of such a function is

$$\delta_r = \delta \sum_{I\subset\overline{V_r}} p_I, \tag{3.24}$$

where $\overline{V_r}$ is the complement of the set of vertices of r in V_G, $p_I > 0$ for all $I \subset V_G$ with $|I| > 1$, and $0 < \delta \ll 1$ is a small positive parameter. We also set $p_{\{v\}} = 0$ for all $v \in V_G$. With this notation, we define the graph cosmohedron $\mathcal{C}_G$ as the intersection of the collection of half spaces defined by $Y_\tau \geq 0$ for all tubings on G:

$$\mathcal{C}_G = \{(X_{\{1\}}, \ldots, X_{\{|G|\}}) \in \mathbb{R}^{|G|} : (\forall_{\tau\in\Gamma_G} Y_\tau \geq 0) \wedge (X_{V_G} = 0)\}, \tag{3.25}$$

restricted to the $X_{V_G} = 0$ hyperplane. Explicit examples of the embedding of the graph cosmohedra for the path, cycle and complete graphs on four vertices are displayed in figure 8.

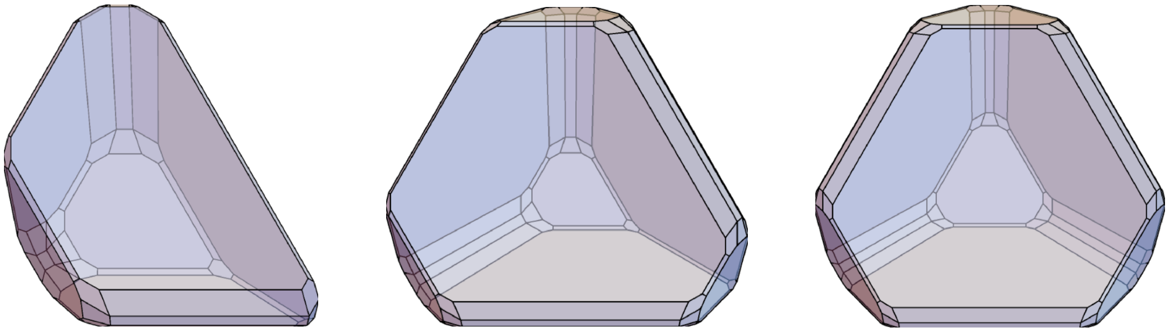


Figure 8. Embeddings of three-dimensional graph cosmohedra for the path graph, cycle graph, and complete graph on four vertices. In the case of the path graph and complete graph we recover the cosmohedron and the permutoassociahedron respectively.

3.4 Examples of graph cosmohedra

Let us demonstrate the graph cosmohedron inequalities in some simple examples. We begin with the empty graph on three vertices N_3. In this case the inequalities of the graph cosmohedron are given by a modification of those of (2.17)

$$X_1 \geq \delta\, p_{\{2,3\}}, \quad X_2 \geq \delta\, p_{\{1,3\}}, \quad X_3 \geq \delta\, p_{\{1,2\}}, \tag{3.26}$$

together with the additional inequalities

$$\begin{aligned} X_1 + X_2 &\geq \delta(p_{\{1,2\}} + p_{\{1,3\}} + p_{\{2,3\}}), \\ X_1 + X_3 &\geq \delta(p_{\{1,2\}} + p_{\{1,3\}} + p_{\{2,3\}}), \\ X_2 + X_3 &\geq \delta(p_{\{1,2\}} + p_{\{1,3\}} + p_{\{2,3\}}). \end{aligned} \tag{3.27}$$

Then the cosmohedron $\mathcal{C}_{N_3}$ coincides with the two-dimensional permutohedron displayed in figure 9. More generally, the graph cosmohedron for the empty graph on $(d+1)$ vertices reproduces the d-dimensional permutohedron. An explicit embedding of the three-dimensional example is displayed in figure 8.

Next consider the path graph P_3 on three vertices. The inequalities carving out the cosmohedron in this case are given by the modified version of (2.18), namely

$$\begin{aligned} &X_1 \geq \delta\, p_{\{2,3\}}, \quad X_2 \geq \delta\, p_{\{1,3\}}, \quad X_3 \geq \delta\, p_{\{1,2\}}, \\ &X_1 + X_2 \geq c_{\{1,2\}} + \delta\, p_{\{1,2\}}, \quad X_2 + X_3 \geq c_{\{2,3\}} + \delta\, p_{\{2,3\}}. \end{aligned} \tag{3.28}$$

In addition, we have five new inequalities

$$\begin{aligned} X_1 + X_3 &\geq \delta(p_{\{1,2\}} + p_{\{1,3\}} + p_{\{2,3\}}), \\ 2X_1 + X_2 &\geq c_{\{1,2\}} + \delta(p_{\{1,2\}} + p_{\{1,3\}} + p_{\{2,3\}}), \\ X_1 + 2X_2 &\geq c_{\{1,2\}} + \delta(p_{\{1,2\}} + p_{\{1,3\}} + p_{\{2,3\}}), \\ 2X_2 + X_3 &\geq c_{\{2,3\}} + \delta(p_{\{1,2\}} + p_{\{1,3\}} + p_{\{2,3\}}), \\ X_2 + 2X_3 &\geq c_{\{2,3\}} + \delta(p_{\{1,2\}} + p_{\{1,3\}} + p_{\{2,3\}}). \end{aligned} \tag{3.29}$$

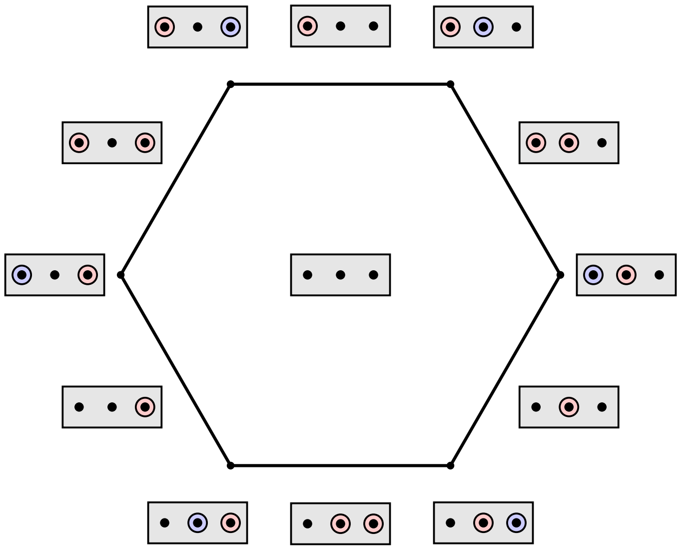


Figure 9. The graph cosmohedron for the empty graph on three vertices which reproduces the two dimensional permutohedron.

As the result, the graph cosmohedron $\mathcal{C}_{P_3}$ is the decagon displayed in figure 7. For arbitrary path graphs, one recovers the family of cosmohedra defined in [3]. A three dimensional example is shown in figure 8.

Finally, we consider the complete graph on $(d+1)$ vertices. In this case the graph cosmohedron produces the family of polytopes known as permutoassociahedra introduced in [15]. The two-dimensional permutoassociahedron with boundaries labelled by regional tubings is displayed in figure 10. An explicit embedding of the three-dimensional example is displayed in figure 8. We provide a list of combinatorial data associated to graph cosmohedra for various graphs with two, three and four vertices in table 2.

3.5 Canonical forms for graph cosmohedra

Since a generic graph cosmohedron is not a simple polytope, its canonical form cannot be obtained from the formula (2.15) that we used for the graph associahedra. However, one can use its modification, inspired by the recent results in [19], where the canonical form of any convex polytope can be written as a linear combination of terms coming from all vertices. For a d-dimensional graph cosmohedron, the formula is

$$\Omega(\mathcal{C}_G) = \sum_{\rho\in\Phi_G^{\max}} \sum_{I\in\binom{[F_\rho]}{d}} \alpha_{\rho,I} \bigwedge_{\tilde{\rho}\in I} \mathrm{d}\log(Y_{\tilde{\rho}}), \tag{3.30}$$

where the first sum is over all vertices of the cosmohedron, the second sum is over all d-element subsets of faces meeting at the vertex specified by ρ. The coefficients $\alpha_{\rho,I}$ can be fixed on the case-by-case basis by demanding that the form $\Omega_{\mathcal{C}_G}$ is projectively invariant.

After pulling back to the appropriate subspace and stripping off the measure, the last step needed in order to recover the cosmological amplitube is to make the following substitution

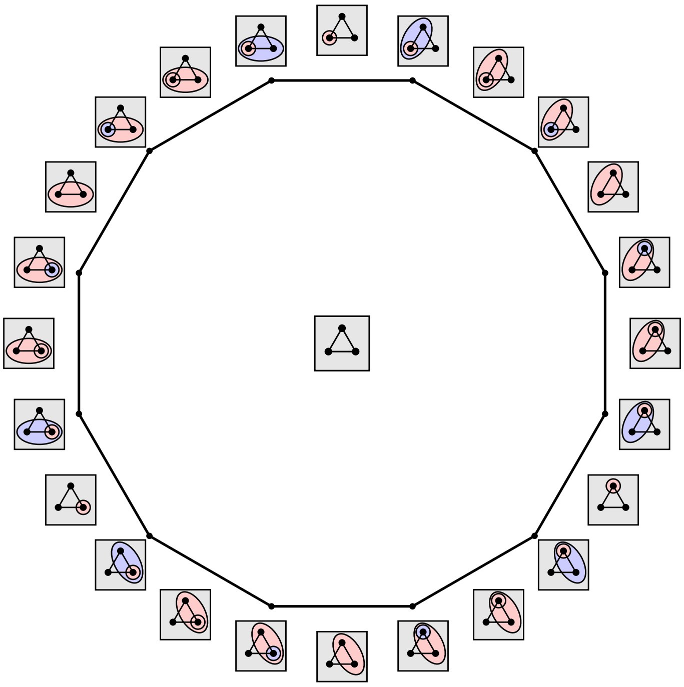


Figure 10. The graph cosmohedron for the complete graph on three vertices which reproduces the two dimensional permutoassociahedron.

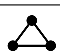
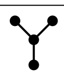
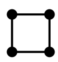
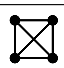
Graph G	dimension	codim-1	codim-2	codim-3
	1	2		
	1	2		
	2	6	6	
	2	8	8	
	2	10	10	
	2	12	12	
	3	44	114	72
	3	50	132	84
	3	62	164	104
	3	74	192	120

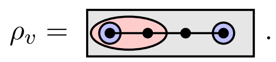
Table 2. The number of boundaries of various codimension of the cosmohedron on graph G.

for the Y variables

$$Y_\rho = \sum_{t\in B(\rho)} \frac{1}{\mathcal{R}^{(1)}_{t,\rho}\mathcal{R}^{(2)}_{t,\rho}}, \tag{3.31}$$

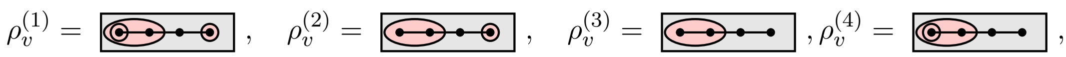
where $\mathcal{R}^{(i)}_{t,\rho}$, $i = 1, 2$, correspond to the unique pair of regions in ρ which share the tube t as a border. Having made the replacements (3.31) and keeping only terms with simple poles we arrive at the expression for the corresponding cosmological amplitube Ψ_G.

We demonstrate the above procedure with an example. The first interesting case where the graph cosmohedron is not simple is for the path graph P_4 on four vertices. In this case the graph cosmohedron $\mathcal{C}_{P_4}$ has 12 non-simple vertices, an example of which is given by



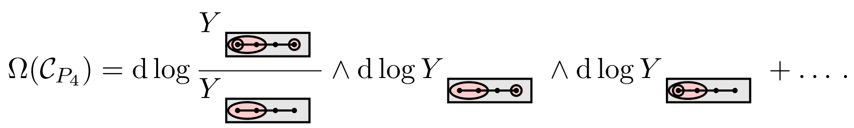
(3.32)

This vertex is defined by the intersection of the following four facets



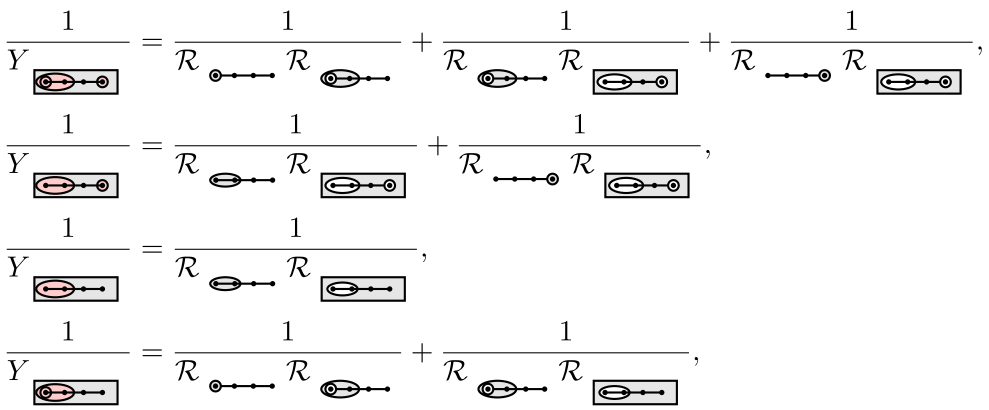
(3.33)

and the contribution of this vertex to the canonical form $\Omega(\mathcal{C}_{P_4})$ is



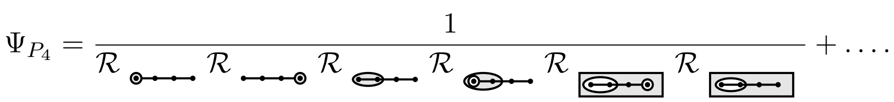
(3.34)

After pulling back to the appropriate subspace and stripping off the measure, we make the following substitutions



(3.35)

and keep only terms with simple poles. The contribution to the canonical form (3.34) from this vertex provides the correct contribution to the cosmological amplitube given by



(3.36)

4 Outlook and conclusions

In this paper we introduced a new class of polytopes, that we call the graph cosmohedra, that are of interest to mathematicians and physicists alike. From the physics point of view, they generalise the recently introduced cosmohedra that capture the combinatorics of singularities of the wavefunctions for $\mathrm{tr}(\phi^3)$ theory. For mathematicians, we provide a combinatorial and geometric realisation of a new class of polytope that generalise associahedra and permutohedra. The combinatorial stratification of the graph cosmohedra is given by regional tubings that generalise polyangulations of n-gons. We also provide an explicit embedding of these polytopes (3.25) that generalises the ABHY-like embedding (2.12) initially studied for associahedra. To each graph associahedron, we associate a function — amplitube — that possesses interesting factorisation properties reminiscent of the factorisations of scattering amplitudes in high energy physics. Whilst, for each graph cosmohedron, we associate a function — cosmological amplitube — that possesses factorisation properties reminiscent of the cosmological wavefunctions of $\mathrm{tr}(\phi^3)$ theory.

There are a number of future research direction which our work suggests. The graph associahedra studied here are a special case of *generalised permutohedra* introduced and studied in [18, 20, 21]. As such it would be interesting to extend the construction presented here to generalised permutohedra in order to define their corresponding 'cosmological' polytopes. We expect the resulting construction to be closely related to the work of [22]. In a similar direction the CEGM amplitudes introduced in [11] provide a vast generalisation of the familiar $\mathrm{tr}(\phi^3)$ amplitudes, and it would be interesting to see whether a cosmological version of these generalised amplitudes exists as well.

The authors of [3] introduced yet another new family of polytopes referred to as *correlahedra*, which encode the geometry of cosmological correlation functions. The correlahedra have two special facets, one which takes the form of the associahedron, and one which takes the form of the cosmohedron. As such the correlahedra live in one higher dimension than these polytopes. It is natural to extend the notion of correlahedra to *graph correlahedra* which have two special facets corresponding to the graph associahedron and graph cosmohedron. Since the embedding of the graph associahedra/cosmohedra presented here are already described in a one higher dimensional space we expect a simple extension of our inequalities to recover the graph correlahedron.

Finally, in the case of the graph P_{n+1}, the regional tubings introduced here provide a way to encode all sub-polygons of the n-gon on a single graph. Since the recently discovered *kinematic flow*, which governs the structure of differential equations satisfied by FRW correlators [4, 5], is naturally phrased in terms of sub-polygons, it would be interesting to see whether the regional tubings can be generalised to also describe the kinematic flow.

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Data Availability Statement. This article has no associated data or the data will not be deposited.

Code Availability Statement. This article has no associated code or the code will not be deposited.

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