

Duality anomalies in linearized gravity

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Classical linearized gravity admits a dual formulation in terms of a higher-rank tensor field. However, proposing a prescription for the instanton sectors of linearized gravity and its dual, we show that they may be *quantum inequivalent*. The duality anomaly is obtained by resolving the dual graviton theories into vector-valued p -form electrodynamics and is controlled by three topological invariants: the Reidemeister torsion, Ray-Singer torsion, and twisted Euler characteristic. Under the proposed instanton prescription, the duality anomaly vanishes for an odd number of spacetime dimensions as a consequence of the celebrated Cheeger-Müller theorem. In the presence of a gravitational θ -term, the partition function is a modular form in direct analogy to Abelian S-duality for Maxwell theory.

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I. INTRODUCTION

It sometimes happens that a theory admits a *dual* description that, although ostensibly distinct, is in fact physically equivalent. The most relevant example here is Montonen-Olive/S-duality in (supersymmetric) quantum field theory [1–4], but dualities have played pivotal role in the development of modern physics across diverse domains: bosonization [5] in Luttinger liquids and Majorana zero modes [7–9], Kramers-Wannier duality in the Ising model [10], particle-vortex duality in the fractional quantum Hall effect [11–15], S-duality in symmetry protected topological phases [16], and U-duality in supergravity and string/M-theory [17–21], to name but a few. Often such dualities shed new light on otherwise inaccessible facets of the phenomenology [22] and may reveal deep links to mathematics, such as mirror symmetry and the geometric Langlands correspondence [24].

An early example is the classical electric-magnetic duality of $d = 3 + 1$ Maxwell theory, a precursor to Montonen-Olive S-duality. Electric-magnetic duality requires magnetic monopoles which, as Dirac argued, would imply electric charge quantization [25]. More generally, Abelian p -form

gauge theories [26] in d spacetime dimensions enjoy a classical electric-magnetic duality [27] in which the p -form gauge potential A is dualized to a $(d - 2 - p)$ -form potential \tilde{A} via the relation $dA = \star d\tilde{A}$, where \star is the Hodge dual. Such electric-magnetic duality is a pervasive phenomenon in classical gauge theory, at least for free theories; cf. Refs. [28–30]. However, a classical electric-magnetic duality may be anomalous at the quantum level [31–35]. Rather than thinking of this as a failure, there are at least two opportunities presented by such an obstruction. First, constraining the theory to achieve anomaly freedom (covariance) frequently entails important insights [36], as in Dirac’s charge quantization condition. Second, the dual description may allow for a more favorable quantization. Historically, both takes have proven effective.

Given these successes, it is natural to seek generalizations to the more challenging context of gravitational dualities. In particular, there exists a dual formulation [44–50] of linearized gravity propagating on a background manifold (M, g) . The perturbative graviton, $h_{\mu\nu}$, is replaced by a vector-valued $(d - 3)$ -form, $\tilde{h}_{\mu_1 \dots \mu_{d-3} \nu}$, with corresponding symmetry constraints [51]. Going beyond the free theory, in Ref. [47], an action including the full dual graviton $\tilde{g}_{\mu_1 \dots \mu_{d-3} \nu}$ and graviton $g_{\mu\nu}$ together is introduced and shown to yield a theory equivalent to Einstein-Hilbert gravity [53], establishing directly the equivalence of the linearized dual graviton theories. Although standing alone, such dual formulations of gravity also inevitably arise in manifesting M-theory dualities [45–47, 56–62] and in gravitational generalized symmetries [63–69].

It is natural then to ask whether the duality anomaly persists. We address this question here. The answer rests on elegant relations among topological invariants, notably the

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Cheeger-Müller theorem [70–73], and subtle points regarding the nature of instantons in linearized gravity. Our analysis relies crucially on resolving the symmetry constraints of the linearized (dual) graviton, using the Batalin-Vilkovisky (BV) formalism [74–78], to rearticulate it as a T^*M -valued $(d-3)$ -form field. This implies that the positive-energy mode contributions to the dual partition functions combine into the Ray-Singer analytic torsion, as witnessed for dual p -form Maxwell theories [32]. In the latter case, the zero modes and instantons conspire to cancel the anomaly in odd dimensions [35]. However, the corresponding treatment in the gravitational setting is less clear since the linearization of the “instantons” is not obvious [79]. Here, we propose a prescription for the instantons as elements of $H^{p+1}(M; \mathbb{Z})$ and compute the duality anomaly for linearized gravity on a flat background metric. We find that the anomaly is

$$Z_{\text{grav}}/\tilde{Z}_{\text{grav}} = (\kappa/\tilde{\kappa})^{\frac{1}{2}\chi(M; T^*M)}, \quad (1)$$

where κ ($\tilde{\kappa}$) is the (dual) linearized gravity coupling constant and $\chi(M; T^*M)$ is the Euler characteristic for the cohomology of M twisted with respect to the local system given by T^*M [80]. The purely topological characterization of the anomaly and its absence in odd dimensions *a posteriori* justify our instanton prescription. Moreover, it suggests a number of entailments regarding the definition of the partition function for gravity and dualities in string/M-theory that we expand on in the conclusions.

II. CHEEGER-MÜLLER THEOREM

Suppose that we are given a closed oriented connected Riemannian manifold (M, g) and a vector bundle $E \rightarrow M$ equipped with a flat connection and a compatible metric.

A. Ray-Singer analytic torsion

The space of E -valued differential forms $\Omega^*(M; E)$ has canonical covariant (since E is flat) differential $d: \Omega^*(M; E) \rightarrow \Omega^{*+1}(M; E)$. The metrics on M and E induce an adjoint $d^\dagger: \Omega^*(M; E) \rightarrow \Omega^{*-1}(M; E)$, in terms of which one can define the Laplace–de Rham operator $\Delta_p^E: \Omega^p(M; E) \rightarrow \Omega^p(M; E)$ as $\Delta_p^E = dd^\dagger + d^\dagger d$. Since M is compact, Δ_p^E is positive semidefinite with pure point spectrum $\{\lambda_n\}$. Ray and Singer defined the zeta-function-regularized determinant restricted to the strictly positive spectrum,

$$\det'(\Delta_p^E) = \exp(-(\zeta_k^E)'(0)), \quad (2)$$

via the analytic continuation of $\zeta_k^E(s) = \sum_{\lambda_n > 0} \lambda_n^{-s}$. The Ray-Singer analytic torsion [81–83] is then defined as

$$\tau_{\text{RS}}(M; E) = \prod_{k=0}^d \det'(\Delta_k^E)^{-(-1)^k k/2}. \quad (3)$$

B. Reidemeister torsion

Consider the E -valued cohomology $H^k(M; E)$ (with integer coefficients). This is a finitely generated Abelian group, such that we may decompose it (noncanonically) as a direct sum of a torsion-free part $\text{Free}(H^k(M; E))$ and the torsion part $\text{Tor}(H^k(M; E))$,

$$H^k(M; E) \cong \text{Free}(H^k(M; E)) \oplus \text{Tor}(H^k(M; E)). \quad (4)$$

Picking a topological basis $\{w_i\}$ on $\text{Free}(H^k(M; E))$, we may represent the metric as a $b_k^E \times b_k^E$ matrix $[\Gamma_k]_{ij} = [M] \frown ([w_i] \smile [w_j])$ [84], where b_k^E is the rank (k th Betti number) of the free \mathbb{Z} -module $\text{Free}(H^k(M; E))$. The Reidemeister torsion [86–88] is the quantity

$$\tau_{\text{Reid}}(M; E) = \prod_{k=0}^d \det \Gamma_k^{(-1)^k/2} |\text{Tor}(H^k)|^{(-1)^{k+1}}, \quad (5)$$

where $|G|$ denotes the order (cardinality) of a finite group G . (For the torsion subgroup factor, see Ref. [16], Appendix E.) It can be shown that it does not depend on the arbitrary choice of topological basis, so it is an invariant of the topological manifold M and the flat vector bundle E .

The Cheeger-Müller theorem [70–73] states that, when $\text{Tor}(H^k(M; E)) = 0$, $\tau_{\text{RS}}(M; E) = \tau_{\text{Reid}}(M; E)$, as conjectured by Ray and Singer.

III. DUALITIES FOR p -FORM ELECTRODYNAMICS

In the partition function of higher gauge theories [89] (such as p -form electrodynamics), the ghosts, ghosts-for-ghosts, and so on are important [90]. To this end, we employ the BV formalism.

The BV action for an Abelian p -form gauge potential A (valued in the trivial line bundle $E = M \times \mathbb{R}$) is

$$S = \frac{1}{q} \int dA \wedge \star dA + A^+ \wedge dc^{(0)} + c_{(-1)}^+ \wedge dc^{(-1)} \dots \\ \times \dots + c_{(2-p)}^+ \wedge dc^{(1-p)}, \quad (6)$$

where q is the coupling constant. In addition to A , this action involves the tower of $(p-1+i)$ -form field ghosts $c^{(i)}$, with ghost number $1-i$, as well as the corresponding

antifields $A^+, c_{(0)}^+, \dots, c_{(1-p)}^+$. Gauge fixing involves (see Ref. [91], Sec. 4.4) the introduction of a large number of trivial pairs of Nakanishi-Lautrup fields and antighosts, $(b^{(i,j)}, \bar{c}^{(i,j)})$, where $i \in \{1-p, \dots, 0\}$, $j \in \{i-1, i+1, \dots, -i-3, -i-1\}$, and their corresponding antifields. Here, $\bar{c}^{(i,j)}$ is a $(p-1+i)$ -form field of ghost number j , and $b^{(i,j)}$ is an auxiliary $(p-1-i)$ -form field of ghost number $j+1$. After gauge fixing in (the p -form analog of) Feynman gauge and integrating out the auxiliary Nakanishi-Lautrup fields $b^{(i,j)}$, we are left with

$$S = \frac{1}{q} \int \frac{1}{2} A \wedge \star \Delta A + \sum_{i=1-p}^0 \bar{c}^{(i,i-1)} \wedge \star \Delta c^{(i)} + \frac{1}{2} \sum_{i=1-p}^0 \sum_{\substack{j=i+1 \\ i \neq j \pmod{2}}}^{-i-1} \bar{c}^{(i,-j)} \wedge \star \Delta \bar{c}^{(i,j)}, \quad (7)$$

where $\Delta = dd^\dagger + d^\dagger d$ is the Laplace-de Rham operator.

The partition function $Z_p = \int \text{DAD}c \exp S$, where we have denoted the measure for the ghost tower by $\text{D}c$, splits into three contributions: (i) the positive eigenvalues of Δ , which involve the zeta-regularized determinant $\det' \Delta$; (ii) the zero modes of Δ , which involve $H^k(M; \mathbb{Z})$ for $k \leq p$; and (iii) a sum over the possible $U(1)$ $(p-1)$ -gerbes [89,92] on M , which involves $H^{p+1}(M; \mathbb{Z})$. In evaluating the ratio of the dual partition functions Z_p / \tilde{Z}_{d-p-2} that characterizes the duality anomaly of p -form electrodynamics, all three contributions come into play. The first contribution organizes itself into the Ray-Singer torsion [32]; the second and third contributions organize themselves into the Reidemeister torsion with some additional factors [35]. When d is odd, the two contributions precisely cancel each other out due to the Cheeger-Müller theorem; when d is even, one finds [see Ref. [35], Eq. (1.1)]

$$Z_p / \tilde{Z}_{d-p-2} = (q/\tilde{q})^{\frac{1}{2}(-1)^{p+1}\chi(M)}, \quad (8)$$

where $\chi(M)$ is the Euler characteristic of M and $\tilde{q} = 2\pi/q$ is dual the coupling constant.

IV. LINEARIZED (DUAL) GRAVITY

In this section, (M, g) is a closed oriented Riemannian manifold. We further assume g is flat so that T^*M is flat and we may employ the Cheeger-Müller theorem.

A. T^*M -valued p -form resolution of the graviton

Consider linearized gravity (with no cosmological constant) atop a background Riemannian manifold (M, g) . In terms of a perturbative metric $\sqrt{\kappa} h_{\mu\nu} = g_{\mu\nu}^{\text{dynamical}} - g_{\mu\nu}$, where $g^{\text{dynamical}}$ is the dynamical metric and $\kappa = 8\pi G$ is the gravitational constant, the massless BV Fierz-Pauli

action for linearized gravity is simply

$$S_{\text{FP}} = \frac{1}{\kappa^2} \int (\text{vol}_g \mathcal{L}_{\text{FP}}[h]) + h^{+\mu\nu} \nabla_\mu X_\nu, \quad (9)$$

where $\mathcal{L}_{\text{FP}}[h]$ is the familiar massless Fierz-Pauli Lagrangian density, $\kappa^2 = 16\pi G_{\text{N}}^{(d)} c^{-3}$, $\text{vol}_g = d^d x \sqrt{|\det g|}$ is the volume form, $h = h_{\mu\nu} g^{\mu\nu}$ is the trace, $h^{+\mu\nu}$ is the graviton antifield tensor density, and X_μ is the diffeomorphism ghost.

Harmonic gauge, $\partial_\mu h_\nu{}^\mu = \frac{1}{2} \partial_\nu h$, reduces the action to

$$S_{\text{FP}} = \frac{1}{\kappa^2} \int \text{vol}_g \left(\frac{1}{4} h^{\mu\nu} \Delta h_{\mu\nu} + \bar{X}_\mu \Delta X^\mu \right), \quad (10)$$

where \bar{X}_μ is the diffeomorphism antighost and Δ is the Laplacian with respect to g .

At this point, one could express the positive-energy-mode contribution to the partition function in terms of the determinant of the Laplacian on symmetric rank-2 tensor fields, $Z_{\text{osc}} = (\det'(\Delta_0^{\text{TM} \otimes \text{M} \text{TM}}))^{-1/2} \det'(\Delta_1)$ [93]. However, we can short circuit much of this by the following trick: similar to the generalized metric in generalized geometry (see, e.g., Refs. [94,95]), we may add an extra sector corresponding to a two-form (Kalb-Ramond-type) field $B_{\mu\nu}$ and consider

$$H_{\mu\nu} = \frac{1}{\sqrt{2}} (h_{\mu\nu} + B_{\mu\nu}), \quad (11)$$

which is a 2-tensor without any symmetry properties. We may regard $H_\nu = H_{\mu\nu} dx^\mu$ as a T^*M -valued 1-form [96] so that

$$S_{\text{FP, gf}} = \frac{1}{\kappa^2} \left(S_1^{\text{T}^*M} - S_2 + \bar{S}_1 - S_0 \right), \quad (12)$$

where

$$S_1^{\text{T}^*M} = \int \frac{1}{2} H^\mu \wedge \star \Delta H_\mu + \bar{X}^\mu \wedge \star \Delta X_\mu \quad (13a)$$

$$S_2^{M \times \mathbb{R}} = \int \frac{1}{2} B \wedge \star \Delta B + \sum_{i=-1}^0 \bar{c}^{(i,i-1)} \wedge \star \Delta c^{(i)} + \frac{1}{2} \bar{c}^{(-1,0)} \wedge \star \Delta \bar{c}^{(-1,0)} \quad (13b)$$

$$\bar{S}_1^{M \times \mathbb{R}} = \int \bar{c}^{(0,-1)} \wedge \star \Delta c^{(0)} + \bar{c}^{(-1,-2)} \wedge \star \Delta c^{(-1)} + \frac{1}{2} \bar{c}^{(-1,0)} \wedge \star \Delta \bar{c}^{(-1,0)} + \frac{1}{2} \phi \wedge \star \Delta \phi \quad (13c)$$

$$S_0^{M \times \mathbb{R}} = \int \frac{1}{2} \phi \wedge \star \Delta \phi. \quad (13d)$$

Here, S_p^E is the gauge-fixed BV action for an Abelian p -form valued in the flat vector bundle E ; cf. Eq. (7). The $\tilde{S}_1^{M \times \mathbb{R}}$ term may be thought of as two copies of Maxwell theory with wrong statistics, that is, a pair of anticommuting vector fields $\tilde{c}^{(0,-1)}$ and $c^{(0)}$ together with their requisite commuting scalar (anti)ghosts $\tilde{c}^{(-1,-2)}$, $c^{(-1)}$, $\tilde{c}^{(-1,0)}$, ϕ .

In terms of the degree-of-freedom count, the identity (12) may be expressed as $\frac{1}{2}d(d-3) = (d \binom{d}{1} - 2d \binom{d}{0}) - ((\binom{d}{2} - 2\binom{d}{1} + 3\binom{d}{0}) - 2((\binom{d}{1} - 2\binom{d}{0}) - 1)$.

We therefore may write the partition function Z_{grav} for linearized gravity as

$$Z_{\text{grav}} = \frac{Z_1^{\text{T}^*M} (\tilde{Z}_1^{M \times \mathbb{R}})^2}{Z_2^{M \times \mathbb{R}} Z_0^{M \times \mathbb{R}}} = \frac{Z_1^{\text{T}^*M}}{Z_2^{M \times \mathbb{R}} (Z_1^{M \times \mathbb{R}})^2 Z_0^{M \times \mathbb{R}}}, \quad (14)$$

where we have used the fact that reversing the statistics inverts the partition function, $\tilde{Z}_p^E = 1/Z_p^E$.

B. T^*M -valued p -form resolution of the dual graviton

The dual graviton $\tilde{h}_{\mu\nu_1 \dots \nu_{d-3}}$ in Refs. [45,47,48] transforms in the $GL(d)$ -representation given by the Young diagrams

$$\overbrace{\begin{array}{|c|c|c|} \hline \square & \dots & \square \\ \hline \square & & \\ \hline \end{array}}^{d-3} = \left(\square \otimes \overbrace{\begin{array}{|c|c|c|} \hline \square & \dots & \square \\ \hline \end{array}}^{d-3} \right) - \overbrace{\begin{array}{|c|c|c|} \hline \square & \dots & \square \\ \hline \end{array}}^{d-2}. \quad (15)$$

That is, $\tilde{h}_{\mu\nu_1 \dots \nu_{d-3}} = \tilde{h}_{\mu[\nu_1 \dots \nu_{d-3}]}$ and $\tilde{h}_{[\mu\nu_1 \dots \nu_{d-3}]} = 0$. For convenience, let us call such tensor a $[1, d-3]$ -tensor.

The field \tilde{h} transforms under a gauge transformation valued in a T^*M -valued $(d-4)$ -form as

$$\delta \tilde{h}_{\mu\nu_1 \dots \nu_{d-3}} = \tilde{X}_{\mu[\nu_1 \dots \nu_{d-4}, \nu_{d-3}]}^{(0)} - \tilde{X}_{[\mu\nu_1 \dots \nu_{d-4}, \nu_{d-3}]}^{(0)}. \quad (16)$$

When $d=4$, then $\tilde{X}_\mu^{(0)}$ is merely T^*M -valued 0-form, and this reduces to the standard linearized diffeomorphism $\delta \tilde{h}_{\mu\nu} = \tilde{X}_{(\mu,\nu)}^{(0)}$, while for $d > 4$, $\tilde{X}_{\mu\nu_1 \dots \nu_{d-4}}^{(0)}$ decomposes into the irreducible $(d-3)$ -form and $[1, d-4]$ -tensor $GL(d)$ -representations; cf. Refs. [48], Eq. (3.27) and [49], Eq. (2.1). In formulating the complete BV action, it is convenient for us not to make this decomposition, however.

Additionally, there is a tower of T^*M -valued $(d-4-i)$ -form ghosts, $i \in \{0, \dots, d-4\}$, for higher-order gauge-for-gauge symmetries [97,98] given by

$$\delta \tilde{X}_{\mu\nu_1 \dots \nu_{d-4-i}}^{(-i)} = \tilde{X}_{\mu[\nu_1 \dots \nu_{d-5-i}, \nu_{d-4-i}]}^{(-i-1)}. \quad (17)$$

Thus, the BV action is

$$\tilde{S} = \tilde{S}_{\text{FP}} + \frac{1}{\tilde{\kappa}^2} \left(\int h^{+\mu\nu_1 \dots \nu_{d-3}} \tilde{X}_{\mu[\nu_1 \dots \nu_{d-4}, \nu_{d-3}]}^{(0)} + \sum_{i=5-d}^0 \tilde{X}_{(i)}^{+\mu\nu_1 \dots \nu_{d-4+i}} \tilde{X}_{\mu[\nu_1 \dots \nu_{d-5+i}, \nu_{d-4+i}]}^{(i-1)} \right), \quad (18)$$

where \tilde{S}_{FP} is the ordinary action for the dual graviton [44,97,98] and $\tilde{\kappa} = 2\pi/\kappa$.

Applying a suitable gauge choice [97], one obtains

$$\tilde{S}_{\text{gf}} \propto \int \frac{1}{2} \tilde{h}_\mu \wedge \star \Delta \tilde{h}^\mu + \sum_{i=4-d}^0 \tilde{X}_\mu^{(i,i-1)} \wedge \star \Delta \tilde{X}^{(i)\mu} + \frac{1}{2} \sum_{i=4-d}^0 \sum_{\substack{j=i+1 \\ i \neq j \pmod{2}}}^{-i-1} \tilde{X}_\mu^{(i,-j)} \wedge \star \Delta \tilde{X}^{(i,j)\mu}, \quad (19)$$

where we have added $(d-4-i)$ -form antighosts $\tilde{X}_\mu^{(i,j)}$ of ghost number j and used T^*M -valued p -form notation $\omega_\mu = \frac{1}{p!} \omega_{\mu\nu_1 \dots \nu_p} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_p}$. The degree-of-freedom counting is thus as expected:

$$\frac{1}{2}d(d-3) = d \underbrace{\binom{d}{d-3} - \binom{d}{d-2}}_{\tilde{h}} - \sum_{i=4-d}^0 (-1)^i (2-i) d \underbrace{\binom{d}{d-4+i}}_{X^{(i)}, \tilde{X}^{(i,j)}}. \quad (20)$$

Now, similar to (12), we may introduce an extra $(d-2)$ -form field $\tilde{B}_{\mu\nu_1 \dots \nu_{d-3}} = \tilde{B}_{[\mu\nu_1 \dots \nu_{d-3}]}$ to write

$$\tilde{H}_{\mu\nu_1 \dots \nu_{d-3}} = \tilde{h}_{\mu\nu_1 \dots \nu_{d-3}} + \frac{1}{\sqrt{d-2}} \tilde{B}_{\mu\nu_1 \dots \nu_{d-3}}, \quad (21)$$

such that $\tilde{H}_{\mu\nu_1 \dots \nu_{d-3}}$ is an arbitrary T^*M -valued $(d-3)$ -form. Then, $\tilde{S}_{\text{gf}} = \frac{1}{\tilde{\kappa}^2} (S_{d-3}^{\text{T}^*M} - S_{d-2} + \tilde{S}_{d-3} - S_{d-4})$, where

$$S_{d-3}^{\text{T}^*M} = \int \frac{1}{2} \tilde{H}^\mu \wedge \star \Delta \tilde{H}_\mu + \sum_{i=4-d}^0 \tilde{X}^{(i)\mu} \wedge \star \Delta \tilde{X}_\mu^{(i)} + \frac{1}{2} \sum_{i=4-d}^0 \sum_{\substack{j=i+1 \\ i \neq j \pmod{2}}}^{-i-1} \tilde{X}_\mu^{(i,-j)} \wedge \star \Delta \tilde{X}^{(i,j)\mu} \quad (22a)$$

$$S_{d-2} = \int \frac{1}{2} \tilde{B} \wedge \star \Delta \tilde{B} + \sum_{i=3-d}^0 \tilde{c}^{(i,i-1)} \wedge \star \Delta c^{(i)} + \frac{1}{2} \sum_{i=3-d}^0 \sum_{\substack{j=i+1 \\ i \neq j \pmod{2}}}^{-i-1} \tilde{c}^{(i,-j)} \wedge \star \Delta \tilde{c}^{(i,j)} \quad (22b)$$

$$\begin{aligned} \bar{S}_{d-3} = & \int \sum_{i=3-d}^0 \left(\bar{c}^{(i,i-1)} \wedge \star \Delta c^{(i)} + \frac{1}{2} \sum_{\substack{j=i+1 \\ i \neq j \pmod{2}}}^{-i-1} \bar{c}^{(i,-j)} \wedge \star \Delta \bar{c}^{(i,j)} \right) \\ & + \sum_{i=5-d}^0 \left(\bar{\Lambda}^{(i,i-1)} \wedge \star \Delta \Lambda^{(i)} + \frac{1}{2} \sum_{\substack{j=i+1 \\ i \neq j \pmod{2}}}^{-i-1} \bar{\Lambda}^{(i,-j)} \wedge \star \Delta \bar{\Lambda}^{(i,j)} \right) + \frac{1}{2} \tilde{\phi} \wedge \star \Delta \tilde{\phi} \end{aligned} \quad (22c)$$

$$S_{d-4} = \int \frac{1}{2} \tilde{\phi} \wedge \star \Delta \tilde{\phi} + \sum_{i=5-d}^0 \bar{\Lambda}^{(i,i-1)} \wedge \star \Delta \Lambda^{(i)} + \frac{1}{2} \sum_{i=5-d}^0 \sum_{\substack{j=i+1 \\ i \neq j \pmod{2}}}^{-i-1} \bar{\Lambda}^{(i,-j)} \wedge \star \Delta \bar{\Lambda}^{(i,j)}, \quad (22d)$$

where we have introduced the p -form BV triangles of (anti)ghosts $c^{(i)}$, $c^{(i,j)}$, $\Lambda^{(i)}$, $\Lambda^{(i,j)}$ as required. These are, in an obvious sense [99], dual to the (anti)ghosts of (12); for instance, the $(d-4)$ -form $\tilde{\phi}$ and its associated and BV triangle $\Lambda^{(i)}$, $\bar{\Lambda}^{(i,j)}$ is dual to the 2-form field B in (13b).

Then, we may interpret each term in (22) as follows: $S_{d-3}^{\text{T}^*M}$ is the action for an unconstrained T^*M -valued $(d-3)$ -form gauge field, dual to (13a); S_{d-2} is the action for a $(d-2)$ -form gauge field, dual to the (13b); \bar{S}_{d-3} is the action for a pair of wrong statistic $(d-3)$ -form fields, dual to (13b); and S_{d-4} is the action for a $(d-4)$ -form gauge field, dual to the S_2 in (13c). Again, the total degree-of-freedom count correctly matches $\frac{1}{2}d(d-3)$.

Putting these contributions together, the dual partition function \tilde{Z}_{grav} for the dual graviton may be written as

$$\tilde{Z}_{\text{grav}} = \frac{Z_{d-3}^{\text{T}^*M}}{Z_{d-4}^{M \times \mathbb{R}} (Z_{d-3}^{M \times \mathbb{R}})^2 Z_{d-2}^{M \times \mathbb{R}}}. \quad (23)$$

C. Duality anomaly for linearized gravity

Using (14) and (23), the duality anomaly is given by

$$\frac{Z_{\text{grav}}}{\tilde{Z}_{\text{grav}}} = \frac{Z_1^{\text{T}^*M}}{Z_{d-3}^{\text{T}^*M}} \cdot \frac{Z_{d-2}^{M \times \mathbb{R}}}{Z_0^{M \times \mathbb{R}}} \cdot \frac{Z_{d-4}^{M \times \mathbb{R}}}{Z_2^{M \times \mathbb{R}}} \cdot \left(\frac{Z_{d-3}^{M \times \mathbb{R}}}{Z_1^{M \times \mathbb{R}}} \right)^2. \quad (24)$$

Noting each factor is a ratio of dual E -valued p -form partition functions, upon generalizing [35] to vector-bundle-valued p -form fields, the positive-energy modes yield the Ray-Singer torsion for $E \rightarrow M$, while the zero modes and instantons combine into the Reidemeister torsion up to an anomaly given by $(\kappa/\bar{\kappa})^{\frac{1}{2}\chi(M; \text{T}^*M)}$ to yield (1).

In $d=4$ [100], one can add a gravitational θ -term, the linearization and resolution of $\theta \int R \wedge R$. In this case, the duality is extend to a modular $SL(2; \mathbb{Z})$ action on $\tau = \frac{\theta}{2} + i \frac{2\pi}{\kappa}$. The partition function is then a modular form, $\tilde{Z}(\tau) = e^{i\sigma} \tau^{-\frac{1}{4}(\chi-\sigma)} \bar{\tau}^{-\frac{1}{4}(\chi+\sigma)} Z(-1/\tau)$, up to a phase given by the twisted Hirzebruch signature $\sigma(M; \text{T}^*M)$. This is in direct analogy to Abelian S-duality [33], with the corresponding phase identified in Refs. [16,34,35]. Accordingly,

the duality interchanges the linearized first Bianchi identities [101] and equations of motion of the dual gravitons [28], which in $d=4$ can be placed into an $SL(2, \mathbb{Z})$ doublet. Finally, if we dimensionally reduce the dual gravitons to $d=4$, we obtain dual graviphotons (Maxwell gauge potentials) that are related by an Abelian S-duality.

Arriving at these conclusions relied on the Cheeger-Müller theorem equating the Ray-Singer and Reidemeister torsions. In doing so, T^*M must be assumed flat [102], but in establishing the *existence* of anomalies, this is no loss. Less trivially, the T^*M -valued p -form instantons must be taken to be in $H^{p+1}(M; \mathbb{Z})$, a matter we turn to now.

D. Instanton sector

The gravitational duality anomaly (1) depends crucially on the contributions from zero modes and instantons. Indeed, if one were to ignore them, the duality anomaly would vanish in *even* dimensions and not in odd dimensions [103], contradicting standard expectations.

The zero modes straightforwardly correspond to the zero eigenvalues of the Laplace-de Rham operator for (vector-bundle-valued) differential forms, including those for ghosts. On the other hand, the prescription for instantons is more subtle. For p -form electrodynamics, the instanton sectors are given by topologically inequivalent gerbes or, equivalently, the integer cohomology classes $H^{p+1}(M; \mathbb{Z})$; the possible gauge fields are connections on one of the possible gerbes on M . There is, however, no analog for general relativity on a fixed smooth spacetime manifold M ; all metrics are (nondegenerate) sections of the one fixed bundle, $\text{Sym}^2(\text{T}^*M)$. Rather, the gravitational partition function includes the sum over all possible topologies and smooth structures of the spacetime manifold M . Apart from being technically difficult to compute, the “linearization” appropriate for massless Fierz-Pauli gravity, which involves a *single* fixed background metric, topology, and smooth structure, is not obvious.

To some extent, this is an issue of semantics in *defining* what the quantum theory of the massless Fierz-Pauli model (and its dual) should be. What ought to be summed over in the path integral? Here, we have *defined* the instanton sectors of the dual graviton theories to be

$$\text{“} \frac{H^n(M; TM)}{H^{n+1}(M) \oplus H^n(M) \oplus H^n(M) \oplus H^{n-1}(M)} \text{”} \quad (25)$$

where $n = 2$ for the dual graviton and $n = d - 2$ for the dual graviton so that they are related via Poincaré duality. This expression is enclosed in quotes since it is not a true quotient but rather a suggestive notation for the prescription that there is a multiplicative factor corresponding to a sum over $H^n(M; TM)$ and the inverse of a factor corresponding to a sum over $H^{n+1}(M) \oplus H^n(M) \oplus H^n(M) \oplus H^{n-1}(M)$. The intuition behind (25) is that the (dual) graviton [resolved to a T^*M -valued $(d-3)/1$ -form] instantons contribute a factor of $H^n(M; TM)$, the cohomology relevant to the integral T^*M -valued curvatures. However, this overcounts due to the ghost integral curvatures, hence the denominator $H^{n+1}(M) \oplus H^n(M) \oplus H^n(M) \oplus H^{n-1}(M)$, which subtracts the ghost instanton contributions.

Intrinsically, the prescription (25) makes sense if the Fierz-Pauli model is considered as a gauge theory in its own right: we are simply summing over all ways in which the field $H_{\mu\nu}$ can have nontrivial Čech cocycles, modulo the corresponding nontrivial Čech cocycles for $B_{\mu\nu} = H_{[\mu\nu]}$ and the corresponding (anti)ghosts, and similarly for the dual graviton. This seems a natural, and almost inevitable, contribution to the path integral. Extrinsically, one can always combine a massive Fierz-Pauli model with its dual to manifest a classical $U(1)$ duality symmetry that should only be anomalous in even dimensions, since anomalies are given by certain characteristic classes of even degree. This requires the instanton sectors be given by (25). Turing this around, duality anomaly freedom (or modularity) can be used as a heuristic identifying the correct path integral, which cannot be inferred from the classical action alone.

V. CONCLUSIONS

Providing an instanton prescription and resolving $\text{Sym}^2(T^*M)$ into $\Omega^1(M, T^*M)$, we have shown that the linear graviton duality anomaly is controlled by the Euler characteristic and, when a θ -term is included, that the partition function is a modular form on $\frac{\theta}{2} + i\frac{2\pi}{\kappa^2}$. A number of implications and generalizations present themselves.

Most obviously, the resolution method should be directly applicable to the numerous exotic dualities for various spins [28,48], sufficing to compute the duality anomalies and to identify an instanton sector prescription.

More ambitiously, recall the invocation of the Cheeger-Müller theorem required T^*M be flat so as to form a representation of $\pi_1(M)$. Physically, there is no reason for this restriction: the anomaly should always exist and be an

invariant of the smooth structure of M . This suggests an extension of the Cheeger-Müller theorem that applies for nonflat vector bundles. On the “analytic” side, it would involve the zeta-regularized determinant and zero modes of the Lichnerowicz operator differing from the Laplace-de Rham operator Δ by curvature terms, $\Delta_L h_{\mu\nu} = \Delta h_{\mu\nu} - 2R_{\mu\rho\nu\sigma} h^{\rho\sigma} + 2R_{(\mu}{}^\rho h_{\nu)\rho}$, for which there exist known results [93]. Pushing this even further, one could consider the sum over background metrics, then differentiable structures and even topologies, but here one would expect the anomaly to factorize over the sum.

Of course, here we have only treated the linearized graviton partition functions. Ultimately, the full interacting theories should be considered. However, the linearized duality anomaly calculation is nonetheless an important step since either it persists in the full theory (which is perhaps the default expectation) or it is miraculously canceled upon including interaction (in which case knowing the linearized anomaly independent is an essential ingredient).

Finally, dual gravitons, and related generalizations thereof, arise in various approaches to quantum gravity [104], M/E-theory [45–47,56–62] and generalized symmetries [63–69]; the duality anomaly may have nontrivial consequences in such contexts. For instance, building on the argument of Ref. [105], the vanishing of the gravitational and Abelian 3-form duality anomalies in M-theory (since $d = 11$), implies anomaly freedom for type-IIA string theory. The anomaly of the IIA massless sector is canceled precisely by that of the M-theory Kaluza-Klein tower [106]. Put another way, insisting on duality anomaly freedom in type IIA implies the existence of the M-theory Kaluza-Klein spectrum. More radically, it is tempting to speculate that, just as the S-duality of $d = 4, \mathcal{N} = 4$ super-Yang-Mills theory is a consequence of $d = 6$ string/string duality [4] and the *self-dual* $d = 6, \mathcal{N} = (2, 0)$ theory [108], the modularity of the $d = 4$ linearized graviton partition function observed here is a remnant of the conjectured *self-dual* $d = 6, \mathcal{N} = (4, 0)$ “gravi-gerbe” theory [46,61,109].

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DATA AVAILABILITY

The data supporting this study’s findings are available within the article.

- [1] C. K. Montonen and D. I. Olive, Magnetic monopoles as gauge particles?, *Phys. Lett.* **72B**, 117 (1977).
- [2] E. Witten and D. I. Olive, Supersymmetry algebras that include topological charges, *Phys. Lett.* **78B**, 97 (1978).
- [3] H. Osborn, Topological charges for $N = 4$ supersymmetric gauge theories and monopoles of spin 1, *Phys. Lett.* **83B**, 321 (1979).
- [4] M. J. Duff, Strong/weak coupling duality from the dual string, *Nucl. Phys.* **B442**, 47 (1995).
- [5] Bosonization can be viewed as an instance of S-duality. Indeed, the S-duality of the Thirring and sine-Gordon models is the direct field theory analogy; the sine-Gordon model is the S-dual bosonization of the Thirring model [6].
- [6] S. R. Coleman, The quantum sine-Gordon equation as the massive Thirring model, *Phys. Rev. D* **11**, 2088 (1975).
- [7] J. M. Luttinger, An exactly soluble model of a many-fermion system, *J. Math. Phys. (N.Y.)* **4**, 1154 (1963).
- [8] D. C. Mattis and E. H. Lieb, Exact solution of a many-fermion system and its associated boson field, *J. Math. Phys. (N.Y.)* **6**, 304 (1965).
- [9] Ł. Fidkowski, R. M. Lutchyn, C. Nayak, and M. P. A. Fisher, Majorana zero modes in one-dimensional quantum wires without long-ranged superconducting order, *Phys. Rev. B* **84**, 195436 (2011).
- [10] H. A. Kramers and G. H. Wannier, Statistics of the two-dimensional ferromagnet. Part II, *Phys. Rev.* **60**, 263 (1941).
- [11] A. D. Shapere and F. A. Wilczek, Self-dual models with theta terms, *Nucl. Phys.* **B320**, 669 (1989).
- [12] M. P. A. Fisher and D.-H. Lee, Correspondence between two-dimensional bosons and a bulk superconductor in a magnetic field, *Phys. Rev. B* **39**, 2756 (1989).
- [13] S.-J. Rey and A. Zee, Self-duality of three-dimensional Chern–Simons theory, *Nucl. Phys.* **B352**, 897 (1991).
- [14] D.-H. Lee and M. P. A. Fisher, Anyon superconductivity and charge-vortex duality, *Int. J. Mod. Phys. B* **05**, 2675 (1991).
- [15] C. P. Burgess and B. P. Dolan, Particle-vortex duality and the modular group: Applications to the quantum Hall effect and other two-dimensional systems, *Phys. Rev. B* **63**, 155309 (2001).
- [16] M. A. Metlitski, S-duality of $u(1)$ gauge theory with $\theta = \pi$ on non-orientable manifolds: Applications to topological insulators and superconductors, [arXiv:1510.05663](https://arxiv.org/abs/1510.05663).
- [17] E. Cremmer and B. Julia, The $N = 8$ supergravity theory. I. The Lagrangian, *Phys. Lett.* **80B**, 48 (1978).
- [18] E. Cremmer and B. Julia, The $SO(8)$ supergravity, *Nucl. Phys.* **B159**, 141 (1979).
- [19] B. Julia, Infinite Lie algebras in physics, in *Proceedings of the Johns Hopkins Workshop on Current Problems in Particle Theory 5: Unified Field Theories and Beyond* (The Johns Hopkins University, Baltimore, Maryland, USA, 1981), pp. 23–41.
- [20] E. Cremmer, B. Julia, H. Lü, and C. N. Pope, Dualization of dualities, *Nucl. Phys.* **B523**, 73 (1998).
- [21] C. M. Hull and P. K. Townsend, Unity of superstring dualities, *Nucl. Phys.* **B438**, 109 (1995).
- [22] These duality insights have been too numerous to do justice to here, but to convey the broad scope, let us mention just two seemingly disparate examples: Kramers-Wannier duality in locating the critical point in the two-dimensional Ising model [10] and U-duality in the microscopic derivation of the Bekenstein-Hawking entropy of black holes in string theory [23].
- [23] A. E. Strominger and C. Vafa, Microscopic origin of the Bekenstein-Hawking entropy, *Phys. Lett. B* **379**, 99 (1996).
- [24] A. Kapustin and E. Witten, Electric-magnetic duality and the geometric Langlands program, *Commun. Number Theory Phys.* **1**, 1 (2007).
- [25] P. A. M. Dirac, The theory of magnetic poles, *Phys. Rev.* **74**, 817 (1948).
- [26] C. Teitelboim Weitzman, Gauge invariance for extended objects, *Phys. Lett. B* **167**, 63 (1986).
- [27] M. Henneaux and C. Teitelboim Weitzman, p -form electrodynamics, *Found. Phys.* **16**, 593 (1986).
- [28] P. de Medeiros and C. M. Hull, Exotic tensor gauge theory and duality, *Commun. Math. Phys.* **235**, 255 (2003).
- [29] S. Bansal, O. Evnin, and K. Mkrtchyan, Polynomial duality-symmetric Lagrangians for free p -forms, *Eur. Phys. J. C* **81**, 257 (2021).
- [30] Z. Avetisyan, O. Evnin, and K. Mkrtchyan, Democratic Lagrangians for nonlinear electrodynamics, *Phys. Rev. Lett.* **127**, 271601 (2021).
- [31] M. J. Duff and P. van Nieuwenhuizen, Quantum inequivalence of different field representations, *Phys. Lett. B* **94**, 179 (1980).
- [32] A. S. Schwarz and Yu. S. Tyupkin, Quantization of anti-symmetric tensors and Ray–Singer torsion, *Nucl. Phys.* **B242**, 436 (1984).
- [33] E. Witten, On S-duality in abelian gauge theory, *Sel. Math.* **1**, 383 (1995).
- [34] D. I. Olive and M. Alvarez, Spin and Abelian electromagnetic duality on four-manifolds, *Commun. Math. Phys.* **217**, 331 (2001).
- [35] W. Donnelly, B. Michel, and A. Wall, Electromagnetic duality and entanglement anomalies, *Phys. Rev. D* **96**, 045008 (2017).
- [36] Again, there are too many instances to catalog, and we merely mention three examples indicating the diverse applications: constraining the bound state spectrum of confining gauge theories via 't Hooft anomaly matching conditions [37], critical dimensions [38] and Einstein's equations [39,40] from the vanishing of Weyl anomaly [41,42] on the string world sheet, and the identification of five consistent superstring theories via the Green-Schwarz mechanism [43].
- [37] G. 't Hooft, Naturalness, chiral symmetry, and spontaneous chiral symmetry breaking, in *Recent Developments in Gauge Theories*, NATO Science Series B, Vol. 59, edited by G. 't Hooft, C. G. Itzykson, A. M. Jaffe, H. Lehmann, P. K. Mitter, I. M. Singer, and R. F. Stora (Springer-Verlag, Boston, Massachusetts, USA, 1980), pp. 135–157.
- [38] A. M. Polyakov, Quantum geometry of bosonic strings, *Phys. Lett.* **103B**, 207 (1981).
- [39] C. G. Callan, Jr., E. J. Martinec, M. J. Perry, and D. H. Friedan, Strings in background fields, *Nucl. Phys.* **B262**, 593 (1985).
- [40] E. S. Fradkin and A. A. Tseytlin, Effective field theory from quantized strings, *Phys. Lett.* **158B**, 316 (1985).

- [41] D. M. Capper and M. J. Duff, Trace anomalies in dimensional regularization, *Il Nuovo Cimento A* **23**, 173 (1974).
- [42] M. J. Duff, Observations on conformal anomalies, *Nucl. Phys.* **B125**, 334 (1977).
- [43] M. B. Green and J. H. Schwarz, Anomaly cancellation in supersymmetric $D = 10$ gauge theory and superstring theory, *Phys. Lett.* **149B**, 117 (1984).
- [44] T. L. Curtright, Generalized gauge fields, *Phys. Lett.* **165B**, 304 (1985).
- [45] C. M. Hull, Strongly coupled gravity and duality, *Nucl. Phys.* **B583**, 237 (2000).
- [46] C. M. Hull, Symmetries and compactifications of (4,0) conformal gravity, *J. High Energy Phys.* **12** (2000) 007.
- [47] P. C. West, E_{11} and M theory, *Classical Quantum Gravity* **18**, 4443 (2001).
- [48] C. M. Hull, Duality in gravity and higher spin gauge fields, *J. High Energy Phys.* **09** (2001) 027.
- [49] C. M. Hull, Magnetic charges for the graviton, *J. High Energy Phys.* **05** (2024) 257.
- [50] C. M. Hull, U. G. Lindström, and M. L. Velásquez Cotini Hutt, Gauge-invariant charges of the dual graviton, *J. High Energy Phys.* **02** (2025) 198.
- [51] There also exists a “double-dual” graviton, where one dualizes both indices rather than a single index; in four dimensions, this is, however, a mere algebraic relabeling [52].
- [52] M. Henneaux, V. Lekeu, and A. Léonard, A note on the double dual graviton, *J. Phys. A* **53**, 014002 (2020).
- [53] There are obstructions to going beyond the linear case [54,55]. These are evaded in Ref. [47] by the fact that the local dual graviton equation of motion includes the graviton itself. In this case, the dual graviton appears as an auxiliary field, which may be eliminated to recover the Einstein-Hilbert action. Integrating out, instead, the graviton would yield a nonlocal action, violating one of the assumptions entering [54].
- [54] X. Bekaert, N. Boulanger, and M. Henneaux, Consistent deformations of dual formulations of linearized gravity: A no-go result, *Phys. Rev. D* **67**, 044010 (2003).
- [55] R. Monteiro, No $U(1)$ ‘electric-magnetic’ duality in Einstein gravity, *J. High Energy Phys.* **04** (2024) 093.
- [56] C. M. Hull, Conformal non-geometric gravity in six dimensions and M-theory above the Planck energy, *Classical Quantum Gravity* **18**, 3233 (2001).
- [57] M. Henneaux, V. Lekeu, and A. Léonard, Chiral tensors of mixed Young symmetry, *Phys. Rev. D* **95**, 084040 (2017).
- [58] O. Hohm and H. Samtleben, The dual graviton in duality covariant theories, *Fortschr. Phys.* **67**, 1900021 (2019).
- [59] A. G. Tumanov and P. C. West, E_{11} and the non-linear dual graviton, *Phys. Lett. B* **779**, 479 (2018).
- [60] K. Glennon and P. C. West, The non-linear dual gravity equation of motion in eleven dimensions, *Phys. Lett. B* **809**, 135714 (2020).
- [61] C. M. Hull, Gravity, duality and conformal symmetry, *Proc. R. Soc. A* **478**, 20220459 (2022).
- [62] N. Boulanger, P. P. Cook, J. A. O’Connor, and P. C. West, Unfolding E_{11} , *SciPost Phys.* **18**, 149 (2025).
- [63] K. Hinterbichler, D. M. Hofman, A. Joyce, and G. Mathys, Gravity as a gapless phase and biform symmetries, *J. High Energy Phys.* **02** (2023) 151.
- [64] V. Benedetti, H. G. Casini, and J. Martínez Magán, Generalized symmetries of the graviton, *J. High Energy Phys.* **05** (2022) 045.
- [65] V. Benedetti, P. Bueno Gómez, and J. Martínez Magán, Generalized symmetries for generalized gravitons, *Phys. Rev. Lett.* **131**, 111603 (2023).
- [66] V. Benedetti, H. G. Casini, and J. Martínez Magán, Generalized symmetries and Noether’s theorem in QFT, *J. High Energy Phys.* **08** (2022) 304.
- [67] C. Gómez-Fayrén de las Heras, P. Meessen, and T. Ortín Miguel, Covariant generalized conserved charges of General Relativity, *J. High Energy Phys.* **09** (2023) 174.
- [68] C. M. Hull, M. L. Velásquez Cotini Hutt, and U. G. Lindström, Generalised symmetries in linear gravity, *J. High Energy Phys.* **04** (2025) 046.
- [69] C. M. Hull, M. L. Velásquez Cotini Hutt, and U. G. Lindström, Gauging generalised symmetries in linear gravity, *J. High Energy Phys.* **01** (2025) 145.
- [70] J. Cheeger, Analytic torsion and Reidemeister torsion, *Proc. Natl. Acad. Sci. U.S.A.* **74**, 2651 (1977).
- [71] J. Cheeger, Analytic torsion and the heat equation, *Ann. Math.* **109**, 259 (1979).
- [72] W. Müller, Analytic torsion and R -torsion of Riemannian manifolds, *Adv. Math.* **28**, 233 (1978).
- [73] W. Müller, Analytische torsion riemannscher mannigfaltigkeiten, Ph.D. thesis, Humboldt-Universität zu Berlin, Berlin, Germany, 1977.
- [74] I. A. Batalin and G. A. Vilkovisky, Relativistic S -matrix of dynamical systems with boson and fermion constraints, *Phys. Lett.* **69B**, 309 (1977).
- [75] I. A. Batalin and G. A. Vilkovisky, Gauge algebra and quantization, *Phys. Lett.* **102B**, 27 (1981).
- [76] I. A. Batalin and G. A. Vilkovisky, Quantization of gauge theories with linearly dependent generators, *Phys. Rev. D* **28**, 2567 (1983).
- [77] I. A. Batalin and G. A. Vilkovisky, Closure of the gauge algebra, generalized Lie equations and Feynman rules, *Nucl. Phys.* **B234**, 106 (1984).
- [78] I. A. Batalin and G. A. Vilkovisky, Existence theorem for gauge algebra, *J. Math. Phys. (N.Y.)* **26**, 172 (1985).
- [79] This is analogous to recovering the “linearized instanton moduli space” $H^2(M; \mathbb{Z}) \otimes \mathfrak{g}$ from the moduli space of \mathfrak{g} -valued Yang-Mills instantons on a spacetime M .
- [80] In particular, when M is parallelizable, then $\chi(M; T^*M) = \chi(M)^d$.
- [81] D. B. Ray and I. M. Singer, R -torsion and the Laplacian on Riemannian manifolds, *Adv. Math.* **7**, 145 (1971).
- [82] D. B. Ray and I. M. Singer, Analytic torsion for complex manifolds, *Ann. Math.* **98**, 154 (1973).
- [83] D. B. Ray and I. M. Singer, Analytic torsion, in partial differential equations, in *Proceedings of Symposia in Pure Mathematics*, Vol. 23, edited by D. C. Spencer (American Mathematical Society, Providence, Rhode Island, USA, 1973), pp. 167–181.
- [84] Here, \smile is the cup product between cohomology classes, and \frown is the cap product between homology and cohomology classes [85].
- [85] A. E. Hatcher, *Algebraic Topology* (Cambridge University Press, Cambridge, England, 2001).

- [86] K. W. F. Reidemeister, Homotopieringe und Linsenräume, *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg* **11**, 102 (1935).
- [87] W. Franz, Über die Torsion einer Überdeckung, *J. Reine Angew. Math.* **173**, 245 (1935).
- [88] G. de Rham, Sur les nouveaux invariants topologiques de M. Reidemeister, *Mat. Sb.*, **43**, 737 (1936).
- [89] L. Borsten, M. Jalali Farahani, B. Jurčo, H. Kim, J. Nárožný, D. Rist, C. Saemann, and M. Wolf, Higher gauge theory, in *Encyclopedia of Mathematical Physics*, 2nd ed. edited by R. J. Szabo and M. Bojowald (Academic Press, Boston, Massachusetts, USA, 2024), Vol. 4, pp. 159–185.
- [90] W. Siegel, Hidden ghosts, *Phys. Lett.* **93B**, 170 (1980).
- [91] B. Jurčo, L. Raspollini, C. Sämann, and M. Wolf, L_∞ -algebras of classical field theories and the Batalin-Vilkovisky formalism, *Fortschr. Phys.* **67**, 1900025 (2019).
- [92] L. Alfonsi, Higher geometry in physics, in *Encyclopedia of Mathematical Physics*, 2nd ed., edited by R. J. Szabo and M. Bojowald (Academic Press, Boston, Massachusetts, USA, 2024), Vol. 4, pp. 39–61.
- [93] S. M. Christensen and M. J. Duff, Quantizing gravity with a cosmological constant, *Nucl. Phys.* **B170**, 480 (1980).
- [94] G. R. Cavalcanti and M. Gualtieri, Generalized complex geometry and T-duality, in *A Celebration of the Mathematical Legacy of Raoul Bott*, CRM Proceedings and Lecture Notes, Vol. 50, edited by P. R. Kotiuga (American Mathematical Society, Providence, Rhode Island, USA, 2010), pp. 341–365.
- [95] M. Zabzine, Lectures on generalized complex geometry and supersymmetry, *Archivum Math.* **42**, 119 (2006).
- [96] Indeed, this is equivalent to starting from the frame-bundle picture and linearizing. The addition symmetries and ghosts that arise here correspond, in this picture, to the local Lorentz ghosts.
- [97] C. S. Aulakh, I.-G. Koh, and S. Ouvry, Higher spin fields with mixed symmetry, *Phys. Lett.* **173B**, 284 (1986).
- [98] J. M. Fernández de Labastida y del Olmo and T. R. Morris, Massless mixed-symmetry bosonic free fields, *Phys. Lett. B* **180**, 101 (1986).
- [99] That is, as dual p -form and $(d-p-2)$ -form theories. On the other hand, B and \tilde{B} , for example, should be regarded as dual fields once one recalls that the $(d-2)$ -form is a T^*M -valued $(d-3)$ -form with a symmetry constraint and, so, dual to a T^*M -valued 1-form with a symmetry constraint.
- [100] More generally, we can consider any $\frac{d}{2} = p+1$ for arbitrary T^*M -valued p -form and $(d-2-p)$ -form dual formulations of linearized gravity.
- [101] The dual second Bianchi identities are rotated among themselves [28].
- [102] That is, M is a flat Riemannian manifold.
- [103] The same observation was made for p -forms in Ref. [35].
- [104] J. N. Borissova, B. Dittrich, and K. Krasnov, Area-metric gravity revisited, *Phys. Rev. D* **109**, 124035 (2024).
- [105] L. Borsten, M. J. Duff, and S. Nagy, Odd dimensional analogue of the Euler characteristic, *J. High Energy Phys.* **12** (2021) 178.
- [106] The details will be given in Ref. [107].
- [107] L. Borsten, M. J. Duff, D. Kanakaris, and H. Kim (to be published).
- [108] E. Witten, Geometric Langlands from six dimensions, [arXiv:0905.2720](https://arxiv.org/abs/0905.2720).
- [109] L. Borsten, $D = 6$, $\mathcal{N} = (2, 0)$ and $\mathcal{N} = (4, 0)$ theories, *Phys. Rev. D* **97**, 066014 (2018).