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PAPER

Twisted homotopy algebras: spontaneous symmetry breaking, anomalies, localisation, and supersymmetric twists

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Abstract

Classical background fields are a foundational technique in quantum field theory, playing a central role in developments such as the Higgs mechanism. Independently, supersymmetric twisting à la Witten has emerged as a key tool underlying phenomena such as supersymmetric localisation. Although these constructions are traditionally treated as distinct, they arise on an equal footing within the homotopy-algebraic approach to quantum field theory. In this work, we formalise this connection by interpreting both supersymmetric twisting and classical backgrounds as instances of twisting curved quantum L_∞-superalgebras. Using the language of homotopy algebras and the Batalin–Vilkovisky formalism, we provide a unified algebraic framework that encompasses topological/holomorphic twists, spontaneous symmetry breaking, computation of anomalies, and supersymmetric localisation à la Festuccia–Seiberg. We examine a variety of applications and examples illustrating this perspective in supersymmetric and general quantum fields theories alike. As a byproduct, we introduce a notion of twisting for quantum L_∞-algebras and a homotopy-algebraic reformulation of the one-particle-irreducible effective action.

1. Introduction

Two of the pillars of quantum field theory are the method of introducing classical backgrounds and the method of supersymmetric twisting. The former underlies spontaneous symmetry breaking and the Higgs mechanism, where one introduces a classical background (nonzero vacuum expectation value) of a scalar field to manifest a true vacuum of the theory; when applied to the quantum theory, turning on a classical gauge field or metric background uncovers gauge and gravitational anomalies. The latter isolates topologically or holomorphically protected sectors of supersymmetric gauge theories and sheds important light on string theory [1, 2], the (2,0) theory [3, 4], M-theory [4–9], and beyond, in addition to being of fundamental importance to low-dimensional topology in the guise of Donaldson–Witten [10, 11] and Seiberg–Witten [12] theories.

It is a curious fact that the two ingredients co-occur in the approach of Festuccia and Seiberg [13] (reviewed in [14]) to supersymmetric localisation, in which one takes a supersymmetric theory, turns on some classical supergravity backgrounds, and then twists using an unbroken supersymmetry generator. This paper shows that this is no mere coincidence. It has been recently recognised [15, 16] that twisting and classical backgrounds are intimately connected—that, in a sense, twisting amounts to putting on a certain sort of global background field. We sharpen this connection using the L_∞-algebra formalism (originally proposed in [17, 18], with connections to the Feynman-diagram perturbation theory developed in [19–22] and connections to colour–kinematics duality discussed in [23–25]) for quantum field theory to cast both in terms of twisting quantum curved L_∞-superalgebras. Using this perspective, we uniformly formulate topological/holomorphic twisting, Higgsing, anomaly computations and supersymmetric localisation in the language of L_∞-algebras and the Batalin–Vilkovisky (BV) formalism [26–30].

Besides conceptual simplification, our uniform formulation clarifies various issues in the literature. Because we work consistently with homotopy algebras, it is clear that all our considerations apply straightforwardly to higher (e.g. higher-form symmetries as discussed in the recent reviews [31–35]) and/or nonstrict symmetries (e.g. on-shell representations of supersymmetry [36]). In particular, the existing literature on supersymmetric localisation [13, 14] always works with off-shell representations of local supersymmetry including auxiliary fields; our perspective makes it clear that, at least in principle, this is not necessary.

As part of our discussion, we formulate the notion of twisting for quantum L_∞-algebras, which is straightforward but nevertheless appears to be new to the literature.

Organisation of this paper. This paper is organised as follows. Section 2 reviews the twisting of (curved) L_∞-algebras and generalises this construction to the quantum case, and section 3 reformulates the construction of one-particle-irreducible actions in the language of L_∞-algebras. Then section 4 formulates perturbation theory atop a classical background—including spontaneous symmetry breaking and the Higgs mechanism—as a twist by a (quantum) Maurer–Cartan element corresponding to the background. Section 5 then applies this construction to one-particle-irreducible effective actions to detect anomalies. Section 6 reviews the physics notion of topological or holomorphic twists and their relation to the twists of L_∞-algebras as 'gauging' a global symmetry and turning on a constant 'ghost' background. Finally, section 7 discusses the supersymmetric localisation technique of Festuccia–Seiberg using supergravity backgrounds and its formulation in terms of L_∞-algebra twists.

Notation. In this paper, we use the Koszul sign convention throughout. The notation $V[i]$ indicates suspension, i.e. $(V[i])_j := V_{i+j}$; where convenient, we also denote suspension by s. The notation $\odot$ denotes graded symmetrisation.

2. Twisting classical and quantum L_∞-algebras

Twisting is a fundamental operation in the theory of classical and quantum L_∞-algebras, which has manifold manifestations in field theory, as subsequent sections endeavour to prove. The constructions are well known and classical for (classical) L_∞-algebras [37–39], but appear to be new (while straightforward) for the quantum case.

2.1. Classical L_∞-algebras and their twists

We briefly review the relevant notions of L_∞-superalgebras and their twists. For more detailed reviews about L_∞-algebras see [17, 40], and the theory of twists of L_∞-algebras are reviewed in [37–39].

A curved L_∞-superalgebra is the homotopy generalisation of the notion of a Lie superalgebra[4]. There are three equivalent standard definitions of curved L_∞-superalgebras, which are heuristically (we give definitions below):

1. A $\mathbb{Z}\times\mathbb{Z}_2$-graded vector space $\mathfrak{g}$ with multilinear maps $\mu_i\colon \mathfrak{g}^{\times i}\to\mathfrak{g}$ that obey homotopy Jacobi relations. This picture directly generalises the standard definition of a Lie superalgebra, where $\mathfrak{g}$ is a supervector space and the Lie bracket $[x_1,x_2]=\mu_2(x_1,x_2)$ is the only non-trivial bracket.
2. A cofree cocommutative $\mathbb{Z}\times\mathbb{Z}_2$-graded coalgebra with coderivation D obeying $D^2=0$. The coderivation encodes the products μ_i and $D^2=0$ imposes the homotopy Jacobi relations. This picture makes morphisms of algebras easy to state and most naturally captures the scattering amplitudes in quantum field theory.
3. A $\mathbb{Z}\times\mathbb{Z}_2$-graded manifold M that is merely a $\mathbb{Z}\times\mathbb{Z}_2$-graded vector space endowed with a degree $(1,0)$ vector field Q satisfying $Q^2=0$. This picture connects most directly to the BV formalism (as realised by symplectic $\mathbb{Z}\times\mathbb{Z}_2$-graded manifolds).

Each picture offers its own advantages and we shall freely move between them as convenience dictates. In the following we briefly introduce the key definitions and connections amongst these formulations that we shall need throughout. For a detailed account see, for example, [17].

[4] If one wants to work with theories containing fermions, working with superalgebras is necessary.

First, we have the standard definition of an L_∞-superalgebra, with explicitly stated homotopy Jacobi identities:

Definition 1. Over any commutative algebra $\mathbb{K}$ over $\mathbb{F} \supseteq \mathbb{Q}$, a *curved* L_∞*-superalgebra* $(\mathfrak{g}, (\mu_i)_{i=0}^\infty)$ consists of a free $\mathbb{Z} \times \mathbb{Z}_2$-graded[5] $\mathbb{K}$-module $\mathfrak{g}$ equipped with totally graded-antisymmetric $\mathbb{K}$-multilinear operations[6]

$$\mu_i\colon \bigwedge^i \mathfrak{g} \to \mathfrak{g} \tag{1}$$

of degree $(2-i, 0)$ that satisfy the following homotopy Jacobi identity:

$$\sum_{\substack{i+j=k\\ \sigma\in\mathrm{Sym}(k)}} \frac{(-1)^{ij}}{i!j!} \chi(\sigma)\, \mu_{j+1}\left(\mu_i\left(x_{\sigma(1)}, \dots, x_{\sigma(i)}\right), x_{\sigma(i+1)}, \dots, x_{\sigma(k)}\right) = 0, \tag{2}$$

where the sum ranges over permutations of $\{1, \dots, k\}$ and where $\chi(\sigma)$ is the graded-antisymmetric Koszul sign defined so that

$$x_{\sigma(1)} \wedge \dots \wedge x_{\sigma(n)} =: \chi(\sigma)\, x_1 \wedge \dots \wedge x_n \tag{3}$$

where

$$x \wedge y = -(-1)^{pq+rs}\, y \wedge x \tag{4}$$

for homogeneous elements x and y of degrees (p, r) and (q, s), respectively ($p, q \in \mathbb{Z}$, $r, s \in \{0, 1\}$).

Definition 2. A *flat* L_∞-superalgebra is a curved L_∞-superalgebra with $\mu_0 = 0$.

Flat L_∞-superalgebras are especially nice in the sense that the homotopy Jacobi relations imply ${\mu_1}^2 = 0$, so that they have μ_1-cohomology. (A curved L_∞-superalgebra is, in the literature, sometimes also referred to as a *weak* L_∞-superalgebra since non-trivial μ_0 implies μ_1 is *not* a differential.)

Example 1. A curved L_∞-superalgebra with $\mu_i = 0$ save for μ_1 and μ_2 is precisely equivalent to a differential graded Lie superalgebra, with μ_2 the Lie bracket and μ_1 the differential. We call such algebras *strict* L_∞-superalgebras.

Example 2. A curved L_∞-superalgebra concentrated in cohomological degree 0 is precisely equivalent to a Lie superalgebra.

Example 3. A curved L_∞-superalgebra concentrated in degree $(0, 0)$ is precisely equivalent to an (ordinary) Lie algebra.

By operadic Koszul duality [41], definition 2 is equivalent (see for example [17, 39, 40]) to a nilquadratic degree 1 coderivation D on the counital cofree cocommutative coassociative coalgebra $\bigodot \mathfrak{g}[1]$, where $\bigodot V$ is the graded-symmetric tensor coalgebra, and for a graded vector space V,

$$V[k] := [k] \otimes V, \tag{5}$$

where $[k]$ denotes the one-dimensional vector space concentrated in degree $-k$, that is, $V[k]^i \cong V^{i+k}$. The coproduct on $\bigodot \mathfrak{g}[1]$ is given by

$$\Delta\left(x_1 \odot \dots \odot x_n\right) = \sum_{\substack{k=0\\ \sigma\in\mathrm{Sh}(k,n-k)}}^{n} \chi\left(\sigma_{k,n}\right) \left(x_{\sigma(1)} \odot \dots \odot x_{\sigma(k)}\right) \otimes \left(x_{\sigma(k+1)} \odot \dots \odot x_{\sigma(n)}\right), \tag{6}$$

where the sum is over $(k, n-k)$-shuffles $\sigma \in \mathrm{Sh}(k, n-k) \subset S_n$. The products μ_i are packaged into the coderivation in the obvious manner (for details see [17]),

$$\mu_i := \pm s^{-1} \circ D_i \circ s^{\odot i}, \tag{7}$$

[5] In what follows, we refer to the $\mathbb{Z}$-grading as *cohomological degree* and the $\mathbb{Z}_2$-grading as *super degree*.

[6] Note, the tensor products are $\mathbb{F}$-linear. Furthermore, we leave implicit the completion of the tensor product required when, in particular, defining the algebra over $\mathbb{F}[[\hbar]]$ or, in general, when considering topological vector spaces.

where D_i is the restriction of the coderivation to $\bigodot^i \mathfrak{g}[1] \to \mathfrak{g}[1]$ and $s^{\odot i}\colon \bigwedge^i \mathfrak{g} \to \bigodot^i \mathfrak{g}[1]$ is the isomorphism,

$$s^{\odot i}(x_1 \wedge \ldots \wedge x_i) = (-1)^{\sum_{j=1}^{i-1}(i-j)|x_j|} sx_1 \odot \cdots \odot sx_i, \tag{8}$$

where $s\colon V \to V[1]$ denotes the suspension map. The nilquadraticity condition $D^2 = 0$ then encodes the homotopy Jacobi relations amongst the μ_i. This is equivalent to the *Chevalley–Eilenberg coalgebra* on $\mathfrak{g}$ and so also denoted $\mathrm{CE}_\bullet(\mathfrak{g}) := (\bigodot \mathfrak{g}[1], D)$.

The dual Chevalley–Eilenberg *algebra* $\mathrm{CE}^\bullet(\mathfrak{g}) := (\bigodot \mathfrak{g}[1]^*, D^*)$ is equivalent to a differential graded supermanifold[7], that is, $C^\infty(\mathfrak{g}[1])$ together with a nilquadratic homological degree 1 vector field $\mathcal{Q}$. The nilquadraticity condition $\mathcal{Q}^2 = 0$ is precisely equivalent to the classical master equation in the BV formalism: a curved L_∞-superalgebra is nothing but a differential graded supermanifold whose underlying graded supermanifold is a graded supervector space. The other key ingredient of the BV formalism is a $\mathcal{Q}$-invariant degree -1 symplectic form ω satisfying $i_\mathcal{Q}\omega = \mathrm{d}S$, where S is identified with the classical BV action.

Definition 3 (cyclic structure). A *cyclic structure* $\langle -,- \rangle_{\mathfrak{g}^*}$ on a curved L_∞-superalgebra $\mathfrak{g}$ is a degree $(3,0)$ graded symmetric bilinear pairing

$$\langle -,- \rangle_{\mathfrak{g}^*}\colon \mathfrak{g}^* \times \mathfrak{g}^* \to \mathbb{F} \tag{9}$$

such that, picking a basis $\{t_a\}$ of $\mathfrak{g}$, if the structure constants of μ_i are $f^{a_0}{}_{a_1\ldots a_i}$ and the structure constants of $\langle -,- \rangle_{\mathfrak{g}^*}$ are c^{ab}, then

$$f^{a_0\ldots a_i} := f^{a_0}{}_{b_1\ldots b_i} c^{a_1 b_1} \cdots c^{a_i b_i} \tag{10}$$

is totally graded-antisymmetric.

In the above, we have permitted degenerate pairings to allow for examples such as the inner-derivation algebra (definition 12). If $\langle -,- \rangle_{\mathfrak{g}^*}$ is nondegenerate, it may be inverted to produce a degree -3 pairing on $\mathfrak{g}$, in terms of which one may phrase the definition as follows:

Definition 4 (nondegenerate cyclic structure). A *nondegenerate cyclic structure* $\langle -,- \rangle_{\mathfrak{g}}$ on a curved L_∞-superalgebra $\mathfrak{g}$ is a degree $(-3,0)$ graded symmetric non-degenerate bilinear pairing

$$\langle -,- \rangle_{\mathfrak{g}}\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{F} \tag{11}$$

satisfying

$$\langle x_0, \mu_i(x_1, \cdots, x_i) \rangle_{\mathfrak{g}} = (-1)^{i + i(|x_1| + |x_i|) + |x_i| \sum_{j=0}^{i-1} |x_j|} \langle x_i, \mu_i(x_0, \ldots, x_1) \rangle_{\mathfrak{g}} \tag{12}$$

for homogeneous $x_0, \ldots, x_i \in \mathfrak{g}$.

Note that a $\mathcal{Q}$-invariant (i.e. $\mathcal{L}_\mathcal{Q}\omega = 0$) symplectic form ω is equivalent to a nondegenerate structure on $\mathfrak{g}$.

Definition 5. Let $(\mathfrak{g}, (\mu_i^{\mathfrak{g}})_{i=0}^\infty)$ and $(\mathfrak{h}, (\mu_i^{\mathfrak{h}})_{i=0}^\infty)$ be two curved L_∞-superalgebras. An *L_∞-morphism of curved L_∞-superalgebras* $\phi\colon\ \mathfrak{g} \rightsquigarrow \mathfrak{h}$ is a morphism of counital cocommutative differential graded coalgebras

$$\phi\colon\ \mathrm{CE}_\bullet(\mathfrak{g}) \to \mathrm{CE}_\bullet(\mathfrak{h}), \qquad \phi \circ D_\mathfrak{g} = D_\mathfrak{h} \circ \phi. \tag{13}$$

Definition 6. A *Maurer–Cartan element* $Q \in \mathfrak{g}$ in a curved L_∞-superalgebra with graded supervector space $\mathfrak{g} = \bigoplus_{(i,j)\in\mathbb{Z}\times\mathbb{Z}_2} \mathfrak{g}^{i,j}$ is an element of degree $(1,0)$ such that the following sum exists and is zero:

$$0 = \sum_{i=0}^{\infty} \frac{1}{i!} \mu_i(Q, \ldots, Q). \tag{14}$$

[7] Here we gloss over the difference between smooth functions versus polynomial functions if $\mathfrak{g}[1]$ has nontrivial degree-zero elements. There are subtleties involving dualisation of infinite-dimensional vector spaces in defining the Chevalley–Eilenberg algebras, for which one should be using the pseudocompact topology; we refer to [42] for these subtleties.

Equivalently, on the counital cofree coalgebra $\bigodot \mathfrak{g}[1]$ a Maurer–Cartan element $sQ \in \mathfrak{g}[1]^{(0,0)} \cong \mathfrak{g}^{(1,0)}$ satisfies

$$0 = D(\exp(sQ))|_{\mathfrak{g}[1]} = \sum_{i=0}^{\infty} \frac{1}{i!} D_i (sQ \odot sQ \ldots \odot sQ), \tag{15}$$

where we defined the exponential $\exp(sQ) \in \bigodot \mathfrak{g}[1]$ by its Taylor expansion over $\odot$,

$$\exp(sQ) := \sum_{n=0}^{\infty} \frac{1}{i!} (sQ)^{\odot i}. \tag{16}$$

This definition immediately implies that Q is a (curved) Maurer–Cartan element if and only if $D(\exp(sQ)) = 0$ [43]. See [39, 44] for helpful reviews of such constructions and results.

If $\mathfrak{g}$ is equipped with a nondegenerate cyclic structure, then the metric $\langle -, - \rangle_{\mathfrak{g}^*}$ on $\mathfrak{g}^*$ may be inverted to yield a metric $\langle -, - \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ and the Maurer–Cartan equation is variational in the sense that it extremises the Maurer–Cartan action

$$S_{\mathrm{MC}}[x] := \sum_i \frac{1}{(i+1)!} \langle x, \mu_i(x, \ldots, x) \rangle_{\mathfrak{g}}, \tag{17}$$

where x has degree $(1,0)$. If $\langle -, - \rangle_{\mathfrak{g}^*}$ is degenerate, one may formally define $\langle -, - \rangle_{\mathfrak{g}}$ to be a (noncanonical) pseudoinverse of $\langle -, - \rangle_{\mathfrak{g}^*}$, and the expression (17) also exists, although the variational relation to the Maurer–Cartan equation may not hold.

Definition 7 (see e.g. [39]). Given a curved L_∞-superalgebra $\mathfrak{g}$ and an element $Q \in \mathfrak{g}^{1,0}$ of degree $(1,0)$, the *twist* of $\mathfrak{g}$ by Q is the curved L_∞-superalgebra $(\mathfrak{g}_Q, (\mu_i^Q)_{i=0}^{\infty})$ where $\mathfrak{g}_Q = \mathfrak{g}$ as graded supervector spaces, and

$$\mu_i^Q(x_1, \ldots, x_i) := \sum_{k=0}^{\infty} \frac{1}{k!} \mu_{i+k}(Q, Q, \ldots, Q, x_1, \ldots, x_i) \tag{18}$$

if the above sum always exists. Equivalently, on the counital cofree coalgebra $\bigodot \mathfrak{g}[1]$ the twist is given by

$$D_Q = \exp\left(-\widehat{sQ}\right) D \exp\left(\widehat{sQ}\right), \tag{19}$$

where for $x, y \in \bigodot \mathfrak{g}[1]$, we define the operator

$$\hat{x} \colon \bigodot \mathfrak{g}[1] \to \bigodot \mathfrak{g}[1], \qquad \hat{x}(y) = x \odot y. \tag{20}$$

Lemma 1 (see e.g. [39, lemma 5.14 and proposition 5.28]). *Let $\phi \colon (\mathrm{CE}_\bullet(\mathfrak{g}), D) \to (\mathrm{CE}_\bullet(\mathfrak{g}'), D')$ be an L_∞-morphism of (curved) L_∞-superalgebras and denote by ϕ^1 its restriction to $\bigodot \mathfrak{g}[1] \to \mathfrak{g}'[1]$. Let $Q \in \mathfrak{g}^{1,0}$. Then:*

1. *The twist $\mathfrak{g}_Q$ of $\mathfrak{g}$ by Q is a flat L_∞-superalgebra if and only if Q is a Maurer–Cartan element of $\mathfrak{g}$.*
2. *$\phi(\exp(sQ)) = \exp(sQ_\phi)$, where $Q_\phi := s^{-1}\phi^1(\sum_{i=1}^{\infty} \frac{1}{i!}(sQ)^{\odot i})$.*
3. *If Q is a Maurer–Cartan element of $\mathfrak{g}$, then Q_ϕ is a Maurer–Cartan element of $\mathfrak{g}'$.*
4. *The map*

$$\phi_Q := \exp\left(-\widehat{sQ_\phi}\right) \phi \exp\left(\widehat{sQ}\right) \colon \mathrm{CE}_\bullet(\mathfrak{g}_Q) \to \mathrm{CE}_\bullet\left(\mathfrak{g}'_{Q_\phi}\right) \tag{21}$$

defines an L_∞-morphism of curved L_∞-superalgebras.

5. *If $\mathfrak{g}$ and $\mathfrak{g}'$ are flat ($\mu_0 = 0$) L_∞-superalgebras, $Q \in \mathfrak{g}^{1,0}$ is a Maurer–Cartan element, and ϕ is a quasi-isomorphism, then $\phi_Q \colon \mathfrak{g}_Q \rightsquigarrow \mathfrak{g}'_{Q_\phi}$ is a quasi-isomorphism.*

A central notion for the case of flat L_∞-superalgebras is that of being minimal:

Definition 8 (minimal L_∞-superalgebras). A flat L_∞-superalgebra $(\mathfrak{g}, (\mu_i)_{i=1}^{\infty})$ is called *minimal* if $\mu_1 = 0$.

In this case, $\mathfrak{g}$ is trivially identified with the μ_1-cohomology.

Theorem 1 ([45, 46] minimal-model theorem). *Every flat L_∞-superalgebra $(\mathfrak{g}, (\mu_i)_{i=1}^{\infty})$ is quasi-isomorphic to a (representative of the L_∞-isomorphism class of) minimal model(s) $(\mathfrak{g}^\circ, (\mu_i^\circ)_{i=2}^{\infty})$, where $\mathfrak{g}^\circ \cong H^\bullet_{\mu_1}(\mathfrak{g})$ and $\mu_1^\circ = 0$.*

Mathematically, two flat L_∞-superalgebras are called *homotopy equivalent* if and only if their minimal models are related by an L_∞-isomorphism. From the perspective of perturbative quantum field theory, the minimal L_∞-superalgebras encode the tree-level S-matrix, where $H^\bullet_{\mu_1}(\mathfrak{g})$ is the space of on-shell particle states. See for example [19]. Conversely, the original L_∞-superalgebra associated to the quantum field theory provides but one off-shell realisation of the tree-level S-matrix.

The twisting by a Maurer–Cartan element Q is compatible with theorem 1 in the following sense. Shifting to the coalgebra picture, let $\mathfrak{g}$ be a flat L_∞-superalgebra, and let $\phi^\circ\colon\ \mathrm{CE}_\bullet(\mathfrak{g}) \to \mathrm{CE}_\bullet(\mathfrak{g}^\circ)$ denote the quasi-isomorphism to its minimal model $\mathfrak{g}^\circ$. By lemma 1, if $Q \in \mathfrak{g}^{1,0}$ is a Maurer–Cartan element, the map

$$\phi^\circ_Q := \exp\left(-\widehat{sQ_{\phi^\circ}}\right)\phi^\circ \exp\left(\widehat{sQ}\right), \tag{22}$$

where

$$Q_{\phi^\circ} = s^{-1}\phi^{\circ 1}\left(\sum_{i=1}^{\infty}\frac{1}{i!}(sQ)^{\odot i}\right), \tag{23}$$

defines a quasi-isomorphism of L_∞-superalgebras

$$(\mathrm{CE}_\bullet(\mathfrak{g}), D_Q) \to \left(\mathrm{CE}_\bullet(\mathfrak{g}^\circ), D^\circ_{Q_{\phi^\circ}}\right). \tag{24}$$

Since Q_{ϕ° is a Maurer–Cartan element, the twisted minimal model is flat. Moreover, $(\mathrm{CE}_\bullet(\mathfrak{g}^\circ_{Q_{\phi^\circ}}), D^\circ_{Q_{\phi^\circ}})$ is a minimal model for $(\mathrm{CE}_\bullet(\mathfrak{g}_Q), D_Q)$.

2.2. Quantum L_∞-algebras and their twists

Quantum L_∞-algebras have their origins in closed string field theory [47] and have been subsequently developed in e.g. [48, 49] (in the context of string field theory), [50–52] (in homological perturbation theory), and [20, 21] (in the context of loop Feynman diagrams), with the BV formalism [26, 27] often providing the motivation underlying the formal structures. In essence, they generalise ordinary L_∞-algebras in that they allow for an expansion in a formal parameter $\hbar$ by imposing the quantum master equation in addition to the classical master equation at zero loop order.

To identify the appropriate notions required for twists of quantum curved L_∞-algebras, the BV picture is instructive, so we briefly summarise the key points here. Detailed reviews may be found in [20, 21, 49, 51, 52].

First, recall the classical master equation $\{S_0, S_0\} = 0$ for the classical BV action S_0 implies $\mathcal{Q}^2_{\mathrm{BV}} = 0$, where $\mathcal{Q}_{\mathrm{BV}}$ is the BV differential as defined by $\mathcal{Q}_{\mathrm{BV}} = \{S_0, -\}$ and $\{-,-\}$ is the BV antibracket.

When considering the partition function in the BV formalism one must generalise the classical master equation $\{S_0, S_0\} = 0$ to the quantum master equation

$$(\mathcal{Q}_{\mathrm{BV}} - 2\mathrm{i}\hbar\Delta)S = \{S, S\} - 2\mathrm{i}\hbar\Delta S = 0, \tag{25}$$

where Δ is the BV Laplacian[8] [29]. This ensures that the expectation values of physical observables (gauge-invariant operators), which live in the cohomology of $\mathcal{Q}_{\mathrm{BV}} - 2\mathrm{i}\hbar\Delta$, are independent of the choice of gauge.

Solving the quantum master equation order-by-order in $\hbar$ then produces the required 'counterterms' of order $\hbar^g$ that are to be added to the bare classical action S_0, so that one has an $\hbar$-expansion of the quantum action

$$S = S_0 + \hbar S_1 + \hbar^2 S_2 + \cdots, \tag{26}$$

with S_0 satisfying the classical master equation $\{S_0, S_0\} = 0$. It may be that one can simply rescale the coefficients in S_0 rather than add entirely new terms; in this case, the classical action already satisfies the quantum master equation. For instance, this is always the case in the absence of gauge symmetries, simply because the classical BV action has no antifield dependence so that $\Delta S_0 = 0$ from the get-go. Then the usual L_∞-algebraic perturbation theory continues to hold provided that one replaces

[8] Properly speaking, the quantum master equation is a formal expression for local field theories due to the singular character of Δ. However, Δ can be rigorously defined in the case of a finite-dimensional field space, which can be implemented by e.g. using a periodic lattice of finite lattice spacing, which then provides ultraviolet and infrared regulators.

$D \mapsto D - \mathrm{i}\hbar\Delta^*$ where Δ^* is the corresponding *dual* BV Laplacian in the coalgebra picture [20]. Their minimal models compute the loop-level scattering amplitudes[9] of quantum field theories [20][10].

This pictures leads one to the definition of a quantum curved L_∞-algebra [47, 48]:

Definition 9. A quantum curved L_∞-superalgebra $(\mathfrak{g}, (\mu_i^g)_{i,g=0}^\infty, \langle -,-\rangle)$ consists of

(i) a $\mathbb{Z}\times\mathbb{Z}_2$-graded $\mathbb{F}$-vector space $\mathfrak{g}$,
(ii) for each i, a degree $(2-i,0)$ graded anti-symmetric i-linear maps $\mu_i^g\colon \bigwedge^i \mathfrak{g} \to \mathfrak{g}$[11], for each 'genus' $g \geqslant 0$, and
(iii) a degree $(3,0)$ graded-symmetric bilinear form $\langle -,-\rangle\colon \mathfrak{g}^*\times\mathfrak{g}^*\to\mathbb{F}$ (or, dually, a homogeneous element of $\bigodot\mathfrak{g}$ of word length 2 and degree $(3,0)$)[12],

satisfying the following axioms ([47, 48]). For a basis $\{t_a\}$ of $\mathfrak{g}$, with the bilinear form $\langle -,-\rangle$ encoded as $c^{ab}t_a\odot t_b\in\bigodot\mathfrak{g}$ for some structure constants c^{ab}, we have, for any $n,g\geqslant 0$ and $x_1,\dots,x_n\in\mathfrak{g}$

$$0 = \sum_{\substack{k+l=n+1\\ g_1+g_2=g\\ \sigma\in\mathrm{Sym}(n)}} \frac{\chi(\sigma)(-1)^{l(k-1)}}{l!\,(k-1)!}\mu_k^{g_1}\left(\mu_l^{g_2}\left(x_{\sigma(1)},\dots,x_{\sigma(l)}\right),x_{\sigma(l+1)},\dots,x_{\sigma(n)}\right) + \frac{1}{2}\sum_a (-1)^{t_a+n}\mu_{n+2}^{g-1}\left(c^{ab}t_a,t_b,x_1,\dots,x_n\right). \tag{27}$$

Finally, the element

$$(-1)^{(n+1)t_a}c^{ab}t_a\otimes\mu_n^g(t_b,x_1,\dots,x_n), \tag{28}$$

is graded anti-symmetric for all $x_1,\dots,x_n\in\mathfrak{g}$ and $n,g\geqslant 0$. For $g=0$ this corresponds to cyclicity, see definition 3.

As for curved L_∞-superalgebras, there is an equivalent and concise coalgebra definition of quantum curved L_∞-superalgebras [48], which encodes the main identity in a codifferential. Suppose we have the pair $(\mathfrak{g},\langle -,-\rangle)$. Then, to accommodate the genera of the products μ_i^g in the coalgebra picture, one first extends the tensor coalgebra,

$$\bigodot\mathfrak{g}[1][[\hbar]] := \left(\bigodot\mathfrak{g}[1]\right)[[\hbar]] \cong \bigodot\mathfrak{g}[1]\otimes_{\mathbb{F}}\mathbb{F}[[\hbar]], \tag{29}$$

where $\hbar$ is a formal parameter (of bidegree $(0,0)$) whose powers count the genus. There is a unique degree $(1,0)$ second-order[13] coderivation

$$\theta\colon \bigodot\mathfrak{g}[1][[\hbar]] \to \bigodot\mathfrak{g}[1][[\hbar]] \tag{31}$$

such that

$$\pi_1\circ\theta = 0,\qquad \pi_2\circ\theta(x) = \begin{cases} 0, & x\in\bigodot^i\mathfrak{g}[1][[\hbar]],\, i>0\\ \frac{\hbar}{2}c^{ab}t_a\odot t_b, & x=1\end{cases} \tag{32}$$

[9] In order to define Δ^* rigorously, one needs to regularise the L_∞-algebra $\mathfrak{g}$ to be finite-dimensional, by e.g. ultraviolet and infrared cutoffs, so that the quantum minimal model computes the regularised scattering amplitudes depending on the regularisation scales. From this one may perform the usual procedure of renormalisation by sending the regularisation scales to zero/infinity while renormalising the coupling constants to compensate.

[10] For a related and less amplitudes-oriented approach to renormalisation, see [53], which works with local L_∞-algebras.

[11] The use of the term 'genus' is justified by the fact that the higher genus maps μ_i^g encode higher loop quantum contributions to the theory. In particular, the use of the term 'genus' as opposed to 'loop' has its roots in string field theory.

[12] If this is nondegenerate, we may invert it to produce a degree $(-3,0)$ graded-symmetric bilinear form $\langle -,-\rangle\colon\ \mathfrak{g}\times\mathfrak{g}\to\mathbb{F}$. We do not assume nondegeneracy here, however, since it fails in the case of the inner-derivation algebra (definition 13).

[13] A second-order coderivation θ satisfies

$$(\Delta\otimes\mathbb{1})\circ\Delta\circ\theta - (\mathbb{1}+\sigma_{231}+\sigma_{312})\circ(\Delta\otimes\mathbb{1})\circ(\theta\otimes\mathbb{1})\circ\Delta + (\mathbb{1}+\sigma_{231}+\sigma_{312})\circ(\theta\otimes\mathbb{1}\otimes\mathbb{1})\circ(\Delta\otimes\mathbb{1})\circ\Delta = 0, \tag{30}$$

where Δ is the co-product on $\bigodot\mathfrak{g}[1][[\hbar]]$ (not to be confused with the BV Laplacian Δ) and σ_{ijk} is the right action of the permutation group. This is the dual of the seven-term identity for second-order derivations as defined in [54, 55]. See [48] for details.

where π_i is the projector onto $\bigodot^i \mathfrak{g}[1][[\hbar]]$. These conditions further imply $\theta^2 = 0$. In the BV picture, θ will correspond to the dual of the BV Laplacian, so we will denote it by Δ^*.

Quantum curved L_∞-superalgebra structures on $\mathfrak{g}$ are then in bijection with degree $(1,0)$ coderivations D on $\bigodot \mathfrak{g}[1][[\hbar]]$ such that

$$(D + \hbar\Delta^*)^2 = 0 \tag{33}$$

as a formal power series in $\hbar$. Switching to the dual graded derivation (i.e. BV) picture, such coderivations are in bijection to solutions S to the quantum master equation (25), where D and Δ^* are dual to Q_{BV} and Δ.

Given this, one may straightforwardly define the quantum generalisations of Maurer–Cartan elements, twists and their compatibility with minimal models.

Definition 10. A *quantum Maurer–Cartan element* in a quantum curved L_∞-superalgebra $(\mathfrak{g}, (\mu_i^g)_{i,g=0}^\infty, \Delta^*)$ is a family $\{Q^g \in \mathfrak{g}^{(1,0)}\}_{g=0}^\infty$ of elements of bidegree $(1,0)$, such that, in the coalgebra picture, the following sum exists and is zero:

$$0 = (D + \hbar\Delta^*) \exp\left(s \sum_{g \geqslant 0} \hbar^g Q^g \right). \tag{34}$$

We will adopt the same notation as in the classical setting and let $Q := \sum_{g\geqslant 0} \hbar^g Q^g$.

Note that, since Δ^* always increases the length of words by two, when restricting (34) to words of length one, Δ^* does not contribute, so that we have

$$0 = D \exp(sQ)|_{\mathfrak{g}[1][[\hbar]]} = \sum_{i,g=0}^{\infty} \frac{1}{i!} \hbar^g \mu_i^g (Q, \ldots, Q). \tag{35}$$

This has the same form as the classical Maurer–Cartan equation (14).

Definition 11. Given a quantum curved L_∞-superalgebra $(\mathfrak{g}, (\mu_i^g)_{i,g=0}^\infty, \Delta^*)$ and

$$Q \in \mathfrak{g}^{1,0} [[\hbar]], \tag{36}$$

the *twist* by Q is the quantum curved L_∞-superalgebra whose underlying graded vector space is that of $\mathfrak{g}$ but whose operations are encoded by (recall (20))

$$D_Q = \exp\left(-\widehat{sQ}\right) D \exp\left(\widehat{sQ}\right), \tag{37}$$

where $\widehat{-}$ is defined in (20) and the BV Laplacian is unchanged.

Lemma 2. *The twist of a quantum curved L_∞-superalgebra by an element, if it exists, is indeed a quantum curved L_∞-superalgebra.*

Proof. First, we note that

$$\Delta^* = \exp\left(-\widehat{sQ}\right) \Delta^* \exp\left(\widehat{sQ}\right), \tag{38}$$

or equivalently

$$\exp\left(\widehat{sQ}\right) \Delta^* = \Delta^* \exp\left(\widehat{sQ}\right), \tag{39}$$

since the effect of Δ^* is to simply insert pairs $c^{ab} t_a t_b$, which then commute with $\exp(\widehat{sQ})$.

We must check the quantum master equation (33), the nilquadraticity of Δ^*, as well as the second-order coderivation property of Δ^*.

The first and second are immediate; given any map X such that $X^2 = 0$, it follows

$$X_Q^2 = \exp\left(-\widehat{sQ}\right) X \exp\left(\widehat{sQ}\right) \exp\left(-\widehat{sQ}\right) X \exp\left(\widehat{sQ}\right) = 0. \tag{40}$$

The third is also clear since the map

$$x \mapsto \exp\left(-\widehat{sQ}\right) x \exp\left(\widehat{sQ}\right) \tag{41}$$

is an automorphism of counital coalgebras and hence preserves the second-order property. □

Given a (non-curved) quantum L_∞-superalgebra $(\mathfrak{g},(\mu_i^g)_{i=1}^\infty,\Delta^*)$, then by the homological perturbation lemma one can define its *minimal model* [20, 52], which is a quantum L_∞-superalgebra $(\mathrm{H}(\mathfrak{g}),((\mu^\circ)_i^g)_{i=1}^\infty,(\Delta^*)^\circ)$ whose underlying graded vector space is $\mathrm{H}(\mathfrak{g}):=\mathrm{H}_{\mu_1^0}(\mathfrak{g})$, the cohomology of μ_1^0.

Note that, in the special case where $\mathfrak{g}$ describes a physical theory, the scattering amplitudes must only involve fields rather than antifields, so that, apart from the BV Laplacian Δ^* the only nonzero structure map of the quantum minimal model $\mathrm{H}(\mathfrak{g})$ is $\mu_i^g\colon\ \mathfrak{g}^1\otimes\ldots\otimes\mathfrak{g}^1\to\mathfrak{g}^2$. This means that the homotopy Maurer–Cartan action $S_{\mathrm{H}(\mathfrak{g})}$ of $\mathrm{H}(\mathfrak{g})$ does not depend on antifields, and so in particular its BV Laplacian vanishes:

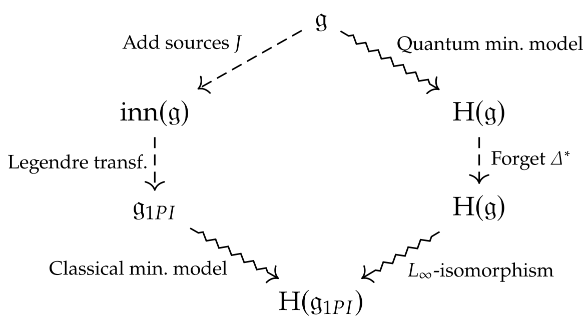
$$\Delta S_{\mathrm{H}(\mathfrak{g})}=0. \tag{42}$$

Therefore, if $S_{\mathrm{H}(\mathfrak{g})}$ satisfies the quantum master equation, it also satisfies the classical master equation. In other words, forgetting Δ^*, then $\mathrm{H}(\mathfrak{g})$ may be regarded as a cyclic (non-quantum) L_∞-superalgebra over $\mathbb{R}[[\hbar]]$.

3. Effective actions in the language of L_∞-algebras

For physics applications (e.g. Higgsing, anomalies) we need to formulate the physical concept of one-particle-irreducible effective actions in terms of L_∞-algebras. The defining quality of the one-particle-irreducible effective action is such that its tree amplitudes equal the loop amplitudes of the original theory. This may be phrased in terms of L_∞-algebras as follows. Given a quantum L_∞-superalgebra $\mathfrak{g}$ (the original theory) whose quantum minimal model is $\mathrm{H}(\mathfrak{g})$, suppose that (42) holds so that $\mathrm{H}(\mathfrak{g})$ may also be regarded as a (non-quantum) cyclic L_∞-superalgebra over $\mathbb{R}[[\hbar]]$. Then a *one-particle-irreducible effective theory* is a (non-quantum) cyclic L_∞-superalgebra $\mathfrak{g}_{\mathrm{1PI}}$, whose underlying graded supervector space coincides with that of $\mathfrak{g}$, and whose (non-quantum) minimal model $\mathrm{H}(\mathfrak{g}_{\mathrm{1PI}})$ is isomorphic as a cyclic L_∞-superalgebra over $\mathbb{R}[[\hbar]]$ to $\mathrm{H}(\mathfrak{g})$. (See [18] for a related perspective on one-particle-irreducible effective actions.)

It is not obvious that such a $\mathfrak{g}_{\mathrm{1PI}}$ exists. When $\mathfrak{g}$ represents a physical quantum field theory, however, then $\mathfrak{g}_{\mathrm{1PI}}$ is known to exist, either by an abstract construction in terms of a Legendre transformation of the free energy, or by an explicit diagrammatic construction. Both constructions may be phrased in terms of L_∞-superalgebras as follows:



(43)

In the diagram (43) the squiggly arrows represent equivalences of classical or quantum L_∞-superalgebras, and the dashed ones are operations performed on (quantum) L_∞-superalgebras, producing new ones.

3.1. Abstract construction of the effective action

Let us first recall the steps for constructing the one-particle-irreducible effective action of a quantum field theory schematically. Subsequently, this procedure will be formalised via the Weil algebra of $\mathfrak{g}$.

Schematically, the construction of the one-particle-irreducible effective action consists of the following steps:

1. To every field ϕ_i in the action $S[\phi]$, one adds a source J^i to obtain the source-extended action

$$S'[\phi,J]=S[\phi]+\int\phi_iJ^i. \tag{44}$$

2. One takes the partition function by integrating out ϕ, leaving a functional $W[J]$ of J alone:

$$\exp(-\mathrm{i}W[J])=\int\mathrm{D}\phi\exp(\mathrm{i}S'[\phi,J]). \tag{45}$$

3. Finally, one takes the Legendre transform to obtain the one-particle-irreducible effective action

$$\Gamma(\phi_{\text{cl}}) = \sup_J \left(-W[J] - \int J^i \phi_{\text{cl},i}[J] \right), \tag{46}$$

where

$$\phi_{\text{cl},i}[J] = \langle \phi \rangle_J = \int \mathrm{D}\phi \;\; \phi \exp(\mathrm{i}S'[\phi, J]). \tag{47}$$

At least formally, we may phrase the above steps, generalised to the BV setting [29], in the language of L_∞-algebras.

The first step, namely to introduce source terms into the action, corresponds to taking the inner-derivation algebra $\text{inn}(\mathfrak{g})$ [56], whose associated Chevalley–Eilenberg algebra is the Weil algebra, reviewed in e.g. [57, section 3.2]. This formally introduces sources for all elements of $\mathfrak{g}$ (including ghosts and antifields), but there are no antifields for the source, such that the cyclic structure is necessarily degenerate. (See [58] for the necessity of introducing sources for antifields in the BV formalism.) Let us recall the definition.

Definition 12. Suppose that $(\mathfrak{g}, \langle -,- \rangle_{\mathfrak{g}^*})$ is an cyclic L_∞-superalgebra with the L_∞-algebra structure given by a codifferential $D_\mathfrak{g}$ on $\bigodot \mathfrak{g}[1]$. The *inner-derivation algebra* is the cyclic L_∞-superalgebra

$$\text{inn}(\mathfrak{g}) := \mathfrak{g} \oplus \mathfrak{g}[1] \tag{48}$$

and with the differential $D_{\text{inn}(\mathfrak{g})}$ on

$$\bigodot \text{inn}(\mathfrak{g})[1] = \bigodot \mathfrak{g}[1] \otimes \bigodot \mathfrak{g}[2] \tag{49}$$

given by

$$D_{\text{inn}(\mathfrak{g})} = D_\mathfrak{g} + \delta, \tag{50}$$

where δ is the degree-shift map that maps $\mathfrak{g}[2]$ to $\mathfrak{g}[1]$ inside $\bigodot \text{inn}(\mathfrak{g})[1]$ (and annihilates $\mathfrak{g}[1]$) and extended by the graded Leibniz rule to the entirety of $\bigodot \text{inn}(\mathfrak{g})[1]$, and the action of $D_\mathfrak{g}$ is extended from generators in $\mathfrak{g}[1]$ to generators in $\mathfrak{g}[2]$ via the commutation rule

$$D_\mathfrak{g} \delta = -\delta D_\mathfrak{g}. \tag{51}$$

The cyclic structure is given by

$$\langle x + y, x' + y' \rangle_{\text{inn}(\mathfrak{g})^*} = \langle x, x' \rangle_{\mathfrak{g}^*} \tag{52}$$

for $x, x' \in \mathfrak{g}^*$ and $y, y' \in \mathfrak{g}[1]^*$.

The underlying vector space of the inner-derivation algebra may be interpreted as

$$\text{inn}(\mathfrak{g}) = \overbrace{\mathfrak{g}}^{\text{fields/antifields}} \oplus \underbrace{\mathfrak{g}[1]}_{\text{sources}}. \tag{53}$$

Note that this definition formally introduces 'sources' for every element in $\mathfrak{g}$, including for antifields, unlike the original discussion [29]. This is at least harmless for gauge-fixed actions, where antifield-containing terms have been excised from the action.

By definition the inner-derivation algebra $\mathfrak{inn}(\mathfrak{g})$ is always contractible (has trivial cohomology), due to the degree-shift isomorphism δ. This underlies the associated applications to classifying spaces for higher structure. In the present context, contractibility is essential to the definition of the one-particle-irreducible effective action; the Legrendre transform is possible if and only if the cohomology is trivial. From the physical vantage point, $\mathfrak{inn}(\mathfrak{g})$ adds sources J that remove the kernel of the free equation of motion.

Note that the cyclic structure (52) is degenerate and fails to invert to produce a pairing on $\text{inn}(\mathfrak{g})$ since there are no antifields for the sources[14]. (It is also difficult to add antifields for sources and then

[14] If one defines a degenerate inner product on $\text{inn}(\mathfrak{g})$, it will generally fail to obey the cyclic identity (12). Consider, for instance, a scalar field theory with action $\int \phi \Box \phi + J\phi$ with a source term ϕJ, and consider the degenerate metric pairing ϕ with ϕ^+ and J with nothing. One then has $\langle \phi, \mu_1(J) \rangle \sim \langle \phi, \phi^+ \rangle \neq 0$ whereas $\langle \mu_1(\phi), J \rangle \sim \langle \phi^+, J \rangle = 0$, violating (12).

extend the L_∞-superalgebra structure to sources in a suitable way.) For inn($\mathfrak{g}$), if the cyclic structure on $\mathfrak{g}$ is nondegenerate, there exists a canonical pseudoinverse $\langle -,-\rangle_{\mathrm{inn}(\mathfrak{g})}$ of $\langle -,-\rangle_{\mathrm{inn}(\mathfrak{g})^*}$ given by

$$\langle (x,y),(x',y')\rangle_{\mathrm{inn}(\mathfrak{g})} = \langle x,x'\rangle_{\mathfrak{g}} \tag{54}$$

for $x,x' \in \mathfrak{g}$ and $y,y' \in \mathfrak{g}[1]$, and with respect to this structure one can define the homotopy Maurer–Cartan action (17) for inn($\mathfrak{g}$). The result of the additional δ term then produces, in the corresponding homotopy Maurer–Cartan action, quadratic source terms $\phi_i J^i$ in (44). The additional terms coming from extending $D_{\mathfrak{g}}$ are needed for gauge invariance.

The above definition extends straightforwardly to the quantum case:

Definition 13. Suppose that $\mathfrak{g}$ is a quantum L_∞-superalgebra with the quantum L_∞-algebra structure given by the pairing $\langle -,-\rangle_{\mathfrak{g}^*}$ and a codifferential $D_{\mathfrak{g}}$ on $\bigodot \mathfrak{g}[1][[\hbar]]$. The *inner-derivation algebra* is the quantum L_∞-superalgebra

$$\mathrm{inn}(\mathfrak{g}) := \mathfrak{g} \oplus \mathfrak{g}[1] \tag{55}$$

with the degenerate cyclic pairing

$$\langle (x,y),(x',y')\rangle_{\mathrm{inn}(\mathfrak{g})^*} = \langle x,x'\rangle_{\mathfrak{g}^*} \tag{56}$$

for $x,x' \in \mathfrak{g}^*$ and $y,y' \in \mathfrak{g}[1]^*$, and with the differential $D_{\mathrm{inn}(\mathfrak{g})}$ on

$$\bigodot \mathrm{inn}(\mathfrak{g})[1][[\hbar]] = \bigodot \mathfrak{g}[1][[\hbar]] \otimes \bigodot \mathfrak{g}[2][[\hbar]] \tag{57}$$

given by

$$D_{\mathrm{inn}(\mathfrak{g})} = D_{\mathfrak{g}} + \delta, \tag{58}$$

where δ is the degree-shift map that maps $\mathfrak{g}[2][[\hbar]]$ to $\mathfrak{g}[1][[\hbar]]$ inside $\bigodot \mathrm{inn}(\mathfrak{g})[1]$ (and maps $\mathfrak{g}[1][[\hbar]]$ to zero) and extended by the graded Leibniz rule to the entirety of $\bigodot \mathrm{inn}(\mathfrak{g})[1][[\hbar]]$, and the action of $D_{\mathfrak{g}}$ is extended from generators in $\mathfrak{g}[1][[\hbar]]$ to generators in $\mathfrak{g}[2][[\hbar]]$ via the commutation rule

$$0 = D_{\mathfrak{g}}\delta + \delta D_{\mathfrak{g}} = \Delta^*\delta + \delta\Delta^*, \tag{59}$$

where Δ^* is the BV Laplacian encoding $\langle -,-\rangle$.

Thus, starting with a quantum L_∞-superalgebra $\mathfrak{g}$ describing a quantum field theory with a nondegenerate cyclic structure, we can construct the homotopy Maurer–Cartan action[15],

$$S_{\mathrm{inn}(\mathfrak{g})}[\Phi,J] = S[\Phi] + \langle \Phi,J\rangle, \tag{60}$$

of inn($\mathfrak{g}$), which is now an ($\hbar$-power-series-valued) functional on the differential graded manifold inn($\mathfrak{g}$)[1], whose underlying graded space is $\mathrm{inn}(\mathfrak{g})[1] \cong \mathfrak{g}[1] \oplus \mathfrak{g}[2]$ and $\Phi \in \mathfrak{g}[1]$ (original fields and antifields). The role of the sources for the antifields (a subset of the J) is to merely to maintain manifest BV invariance, i.e. solve the quantum master equation for inn($\mathfrak{g}$).

Given the above, we may now perform steps a and b as usual, at least formally. First we must gauge-fix; this distinguishes a choice of antifields within the L_∞-superalgebra $\mathfrak{g}$ so that we may write $\Phi = (\phi,\phi^+)$. Then, for a fixed configuration of classical expectation values of the antifields ϕ^+_{cl} as given in (47), we define the analogue of the usual generating functional of connected correlations functions with sources $J = (J_\phi, J_{\phi^+})$,

$$\exp\left(-\mathrm{i}W\left[\phi^+_{\mathrm{cl}},J\right]\right) = \int \mathrm{D}\Phi\,\delta\left(\phi^+ - \frac{\partial\left(\Psi + \phi^+_{\mathrm{cl}}\phi\right)}{\partial\phi}\right)\exp\left(\mathrm{i}S_{\mathrm{inn}(\mathfrak{g})}\right) \tag{61}$$

where Ψ is the gauge-fixing fermion (that identifies the Lagrangian submanifold of fields ϕ).

[15] In (60), the only additional terms are the source terms $\langle \Phi,J\rangle$, arising from the δ in (50). The extension of $D_{\mathrm{CE}(\mathfrak{g})}$ to $\bigodot \mathfrak{g}[2]^*$ does not yield additional terms in the action: the resulting terms in μ_i are always valued in the shifted elements $\mathfrak{g}[1] \subset \mathrm{inn}(\mathfrak{g})$ rather than $\mathfrak{g} \subset \mathrm{inn}(\mathfrak{g})$, and since $\mathfrak{g}[1]$ lies in the kernel of $\langle -,-\rangle_{\mathrm{inn}(\mathfrak{g})}$ the resulting terms in μ_i vanish inside (17).

The classical expectation value of the fields is then defined by

$$\phi_{\mathrm{cl}} = \frac{\partial W}{\partial J_\phi}\bigg|_{J_{\phi^+}=0}. \tag{62}$$

To obtain a functional (the effective action) $\Gamma(\Phi_{\mathrm{cl}})$ as a function of Φ_{cl}, one takes the Lagendre transform as usual after setting the extraneous fields J_{ϕ^+} to zero:

$$\Gamma(\Phi_{\mathrm{cl}}) = \sup_{J_\phi}\left(-W\left[\phi^+_{\mathrm{cl}}, J\right]\big|_{J_{\phi^+}=0} - \langle J_\phi, \phi_{\mathrm{cl}}\rangle\right) \tag{63}$$

This reproduces the BV effective action of [29, 59], which satisfies the classical master equation in the space of classical (anti)fields, i.e. the Zinn-Justin equation. Thus Γ is the homotopy Maurer–Cartan action of a cyclic L_∞-superalgebra over $\mathbb{R}[[\hbar]]$, which we denote as $\mathfrak{g}_{1\mathrm{PI}}$.

Example 4. Consider ϕ^4 scalar field theory:

$$S_{\mathfrak{g}} = \int \frac{1}{2}\phi\Box\phi + \frac{\lambda}{4!}\phi^4. \tag{64}$$

The corresponding quantum L_∞-algebra $\mathfrak{g}$ is concentrated in degrees 1 and 2 for the scalar field ϕ and its antifield ϕ^+ respectively and the only non-trivial brackets are

$$\mu_1^0(\phi) = \Box\phi, \qquad \mu_3^0(\phi,\phi,\phi) = \lambda\phi^3. \tag{65}$$

Then the corresponding $\mathrm{inn}(\mathfrak{g}) = \mathfrak{g}\oplus\mathfrak{g}[1]$ is concentrated in 1 (field ϕ), 2 (antifield ϕ^+), 0 (source J_{ϕ^+} for ϕ^+), and 1 (source J_ϕ for ϕ), respectively. Solutions of the classical master equation satisfy the quantum master equation[16] so the corresponding Maurer–Cartan action is

$$S_{\mathrm{inn}(\mathfrak{g})}\left[\phi,\phi^+,J_\phi,J_{\phi^+}\right] = \int \frac{1}{2}\phi\Box\phi + \frac{\lambda}{4!}\phi^4 + \phi J_\phi + \phi^+ J_{\phi^+}, \tag{66}$$

where the final two terms are the the expected source couplings.

In this case the gauge fixing is trivial and thus

$$\exp\left(-\mathrm{i}W\left[\phi^+_{\mathrm{cl}}, J\right]\right) = \int \mathrm{D}\phi\, \exp\mathrm{i}\, S_{\mathrm{inn}(\mathfrak{g})}\left[\phi,\phi^+_{\mathrm{cl}},J_\phi,J_{\phi^+}\right]. \tag{67}$$

Then

$$\Gamma\left[\phi_{\mathrm{cl}},\phi^+_{\mathrm{cl}}\right] = \sup_{J_\phi}\left(-W\left[\phi^+_{\mathrm{cl}},J_\phi\right] - \langle J_\phi,\phi_{\mathrm{cl}}\rangle\right), \tag{68}$$

where $W[\phi^+_{\mathrm{cl}},J_\phi] = W[\phi^+_{\mathrm{cl}},J]|_{J_{\phi^+}=0}$, is precisely the BV effective action solving the classical master equation. Consequently, $\Gamma[\phi_{\mathrm{cl}},\phi^+_{\mathrm{cl}}]$ defines a classical (flat) L_∞-algebra, whose minimal model encodes the loop amplitudes of the original model.

3.2. Diagrammatic construction

Rather than going through the formal construction, we may directly implement the diagrammatic construction of the quantum effective action [60, (11.63)], that is,

$$\Gamma = S + \frac{1}{2}\mathrm{i}\hbar\,\mathrm{tr}\ln\left(-\frac{\delta^2 S}{(\delta\phi)^2}\right) - \mathrm{i}\sum \hbar^L(L\text{-loop 1PI conn. diag.}). \tag{69}$$

(This does not implement the convexity property directly, but this point is not relevant for our purposes.)

Suppose that we are given a finite-dimensional quantum L_∞-superalgebra $\mathfrak{g}$, such as one obtained from a gauge-fixed quantum field theory action with suitable infrared and ultraviolet cutoffs so as to render the field space finite-dimensional. We then consider formally an L_∞-superalgebra structure over $\mathbb{R}[[\hbar]]$ on the underlying graded supervector space of $\mathfrak{g}\otimes\mathbb{R}[[\hbar]]$ given by

[16] Since the J, J_{ϕ^+} are independent of ϕ,ϕ^+, $S_{\mathrm{inn}(\mathfrak{g})}$ is trivially in the kernel of the BV Laplacian.

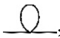
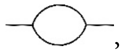
$$\tilde{\mu}_i(x_1,\ldots,x_n) = \sum_{g=0}^{\infty} \hbar^g \left(\mu_i^g(x_1,\ldots,x_n) + \tilde{\mu}_i^{g,\text{det}}(x_1,\ldots,x_n) + \tilde{\mu}_i^{g,\text{conn}}(x_1,\ldots,x_n) \right), \tag{70}$$

where $\tilde{\mu}_i^{g,\text{det}}$ and $\tilde{\mu}_i^{g,\text{conn}}$ implement the formula (69) (described below).

In detail, the term $\tilde{\mu}_i^{g,\text{det}}$ is defined by the structure constant $\tilde{c}^g_{a_1\ldots a_n}$, whose definition

$$\begin{aligned}
\kappa^g_{ab;c_1\ldots c_n} &= c^g_{abc_1\ldots c_n} - \delta_{g0} c^0_{ab} \\
\tilde{c}^g_{a_1\ldots a_n} &= \frac{1}{p}(-1)^{p+1} \binom{n}{n_1,\ldots,n_p} \\
&\quad \times \sum_{\substack{n_1+\cdots+n_p=n \\ g_1+\cdots+g_p=g-1}} \kappa^{g_1}_{b_1' b_2; a_1\ldots a_{n_1}} \hat{c}^{b_2 b_2'} \kappa^{g_2}_{b_2' b_3; a_{n_1+1}\ldots a_{n_1+2}} \hat{c}^{b_3 b_3'} \cdots \kappa^{g_p}_{b_p b_1; a_{n_1+\cdots+n_{p-1}+1}\ldots a_n} \hat{c}^{b_1 b_1'}
\end{aligned} \tag{71}$$

implements the Taylor series expansion $\ln(a+x) = \sum_{p=1}^{\infty} p^{-1}(-1)^{p+1} a^{-p} x^p + \ln(a)$ (with the cosmological-constant term $\ln(a)$ dropped), and where $c^g_{a_1\ldots a_i}$ are the structure constants of μ_i^g, and where $\hat{c}^{ab}$ is the inverse of c^0_{ab}.

The term $\tilde{\mu}_k^{g,\text{conn}}$ is implemented as follows. Given a pseudograph (i.e. undirected graph with self-loops , parallel edges , and external legs allowed) G that corresponds to a one-particle-irreducible Feynman diagram, let $\mathrm{HE}(G)$ be its set of half-edges. (A self-loop still corresponds to two different half-edges, but an external leg corresponds to only one half-edge.) Let the indices $\alpha, \beta, \ldots$ range over $\mathrm{HE}(G)$; each internal edge $e \in \mathrm{E}(G)$ may then be represented by an unordered pair of half-edges. For each pseudograph G with v vertices of degrees $d_1,\ldots,d_v$ and L loops and k external legs (so that there are $d_1+\cdots+d_v$ half-edges), define the expression

$$\begin{aligned}
\tilde{c}^{g,G;a_{d_1+\cdots+d_v+1}\ldots a_{d_1+\cdots+d_v+k}} &= -\mathrm{i} \frac{1}{|\mathrm{Aut}(G)|} \prod_{\{a_r,a_s\}\in \mathrm{E}(G)} \delta^{a_r a_s} \\
&\quad \times \sum_{g_1+\cdots+g_v=g-L} c^{g_1}_{a_1\ldots a_{d_1}} c^{g_2}_{a_{d_1+1}\ldots a_{d_1+d_2}} \cdots c^{g_v}_{a_{d_1+\cdots+d_{v-1}+1}\ldots a_{d_1+\cdots+d_v}},
\end{aligned} \tag{72}$$

where $\mathrm{E}(G)$ is the set of edges regarded as unordered pairs of half-edges; for external legs, $\mathrm{E}(G)$ pairs up the external legs with the free indices

$$a_{d_1+\cdots+d_v+1},\ldots,a_{d_1+\cdots+d_v+k}. \tag{73}$$

The factor $|\mathrm{Aut}(G)|$ is the order of the automorphism group of G (fixing external legs). Then $\tilde{\mu}_k^{g,\text{conn}}$ is defined by the structure constant

$$\tilde{c}^{g,\text{conn}}_{a_0 a_1\ldots a_k} := \sum_G \tilde{c}^{g,G}_{(a_0 a_1\ldots a_k)}, \tag{74}$$

where the parentheses indicate normalised graded symmetrisation and where the sum ranges over the connected one-particle-irreducible pseudographs G with $k+1$ external legs, and where we have lowered the indices using the cyclic structure.

3.3. One-particle-irreducible Maurer–Cartan element

Recall that, by the arguments of section 3.1, the effective action associated to a quantum L_∞-superalgebra $\mathfrak{g}$ defines a *classical* L_∞-algebra $\mathfrak{g}_{1\mathrm{PI}}$, but now taken over over $\mathbb{R}[[\hbar]]$. Hence, we may consider the set of its classical Maurer–Cartan elements. By definition these lie in the critical locus of $\delta\Gamma$ (i.e. they extremize the quantum corrected action) and thus constitute the set of *quantum-corrected* on-shell backgrounds.

Now, using (35), a quantum Maurer–Cartan element of the quantum minimal model $\mathrm{H}(\mathfrak{g})$ is automatically a (non-quantum) Maurer–Cartan element of $\mathrm{H}(\mathfrak{g})$ regarded as a (non-quantum) L_∞-superalgebra and, therefore, corresponds to a Maurer–Cartan element of $\mathfrak{g}_{1\mathrm{PI}}$ up to equivalence. To summarise, when the underlying quantum L_∞-algebra satisfies (42), its quantum Maurer–Cartan elements are classical Maurer–Cartan elements of $\mathfrak{g}_{1\mathrm{PI}}$. Thus we have passed from the picture of a quantum L_∞-algebra to an effective action over $\mathbb{R}[[\hbar]]$, which may be regarded as deformation quantization in sense developed in [61] and so may provide a bridge to factorization algebras.

For brevity, let us refer to a Maurer–Cartan element of $\mathfrak{g}_{1\mathrm{PI}}$ as a *one-particle-irreducible Maurer–Cartan element*. Since $\mathfrak{g}_{1\mathrm{PI}}$ is obtained from $\mathfrak{g}$ by $\mathcal{O}(\hbar)$ corrections, the $\mathcal{O}(\hbar^0)$ component of a one-particle-irreducible Maurer–Cartan element is a (non-quantum) Maurer–Cartan element of the (non-quantum) L_∞-algebra $(\mathfrak{g}, \mu_i^0)$ obtained by forgetting all higher-genus operations $\{\mu_i^g\}_{g>0}$. (In physics terms, this means that a stationary point of the quantum effective action is perturbatively corrected from a stationary point of the bare classical action by $\mathcal{O}(\hbar)$ terms.)

Example 5 (1PI action of scalar ϕ^4). Consider an L_∞-algebra $\mathfrak{g}$ corresponding to the classical scalar field theory

$$S = \int \mathrm{d}^4 \frac{1}{2}\phi\left(\Box - m^2\right)\phi - \frac{1}{4!}\lambda\phi^4 \tag{75}$$

The quantum effective action to one loop order is then[17] of the form [62, (16.2.15)]

$$\begin{aligned}\Gamma = &\int \mathrm{d}^4 x \frac{1}{2}\phi\left(\Box - m_{\mathrm{R}}^2\right)\phi - \frac{1}{4!}\lambda_{\mathrm{R}}\phi^4 \\ &- \frac{\hbar}{64\pi^2}\left(m_{\mathrm{R}}^2 + \frac{1}{2}\lambda_{\mathrm{R}}\phi^2\right)^2 \ln\left(m_{\mathrm{R}}^2 + \frac{1}{2}\lambda_{\mathrm{R}}\phi^2\right) + \cdots \\ = &\, S + \mathcal{O}(\hbar),\end{aligned} \tag{76}$$

where the ellipses include neglected derivative interaction terms and

$$\lambda_{\mathrm{R}} = \lambda + \mathcal{O}(\hbar) \qquad\qquad m_{\mathrm{R}} = m + \mathcal{O}(\hbar) \tag{77}$$

are the renormalised coupling constants. A Maurer–Cartan element ϕ of $\mathfrak{g}$ makes S stationary, and a one-particle-irreducible Maurer–Cartan element $\psi = \phi + \mathcal{O}(\hbar)$ makes $W = S + \mathcal{O}(\hbar)$ stationary.

4. Twisting qua classical backgrounds

Given a classical field theory defined by an action principle $S[\phi]$ whose space of fields is a linear space, one can construct the theory with respect to a classical background ϕ_0 by

$$S_{\phi_0}\left[\tilde{\phi}\right] = S\left[\tilde{\phi} + \phi_0\right]. \tag{78}$$

For arbitrary ϕ_0, the resulting action $S_{\phi_0}[\tilde{\phi}]$ will in general contain tadpole terms (i.e. terms linear in $\tilde{\phi}$), signalling the nonstationarity of the putative 'vacuum' $\tilde{\phi} = 0$ (i.e. $\phi = \phi_0$). However, when ϕ_0 is a solution to the classical equations of motion, then S_{ϕ_0} will not have tadpole terms, and the putative 'vacuum' $\tilde{\phi} = 0$ is stationary (although not necessarily stable). For a quantum field theory, the same applies except that ϕ must instead be stationary with respect to the one-particle irreducible effective action in order for the effective action with background to not have tadpole terms.

The above well known physics lore has a natural implementation in terms of L_∞-algebras. Given a perturbative classical field theory, we can formulate it as a cyclic L_∞-algebra $\mathfrak{g}$ such that the minimal model yields the scattering amplitudes [19] or CFT correlators [63, 64]. (For reviews of the L_∞-algebraic formalism to perturbative quantum field theory, see [17, 22, 57].) Now, a Maurer–Cartan element for $\mathfrak{g}$ is precisely a field (rather than an antifield, Fadeev–Popov ghost, etc) that satisfies the equations of motion. The twist corresponds to turning on a background field to obtain a new L_∞-algebra, which corresponds to the theory atop the classical backgrounds. More generally, we can twist by any field that need not fulfil the Maurer–Cartan equation (14), in which case we obtain a curved L_∞-algebra, which corresponds to a field theory action containing tadpole terms.

In the quantum case, a perturbative quantum field theory may be formulated as a cyclic quantum L_∞-algebra such that the minimal model yields the loop-level scattering amplitudes [20]. Then a one-particle-irreducible Maurer–Cartan element corresponds to a field configuration stationary with respect to the one-particle-irreducible effective action, and twisting by such a field corresponds to turning on this background.

[17] dropping a cosmological constant term, which does not enter into the (quantum) L_∞-algebra.

Example 6 (the Mexican-hat potential). Let us consider a scalar field theory with a Mexican-hat potential:

$$S[\phi] = \int \mathrm{d}^d x \left(\frac{1}{2}\phi\Box\phi + \frac{1}{2}\mu^2\phi^2 - \frac{1}{4!}\lambda\phi^4 \right). \tag{79}$$

Perturbation theory about $\phi = 0$ shows that this 'vacuum' is in fact unstable since the spectrum contains tachyons as seen by the wrong positive sign for the mass term $\frac{1}{2}\mu^2\phi^2$.

The above correspondence between perturbative field theories and (cyclic) homotopy Lie algebras allows us to recast this in terms of L_∞-algebras. The L_∞-algebra $\mathfrak{g}^{\mathrm{MH}}$ corresponding to (79) is

$$\mathfrak{g} = \left(0 \to \mathcal{C}^\infty\left(\mathbb{R}^d\right)[-1] \xrightarrow{-\Box-\mu^2} \mathcal{C}^\infty\left(\mathbb{R}^d\right)[-2] \to 0 \right), \tag{80}$$

with

$$\mu_1\left(\phi[-1]\right) = -\left(\Box + \mu^2\right)\phi[-2] \tag{81a}$$

$$\mu_3\left(\phi_1[-1], \phi_2[-1], \phi_3[-1]\right) = \left(\lambda\phi_1\phi_2\phi_3\right)[-2] \tag{81b}$$

for fields $\phi[-1], \phi_1[-1], \phi_2[-1], \phi_3[-1] \in \mathcal{C}^\infty(\mathbb{R}^d)[-1]$ with all other μ_i vanishing.

A Maurer–Cartan element of $\mathfrak{g}^{\mathrm{MH}}$ is a degree-one element (that is, a field rather than an anti-field) which is a solution to the equations of motion:

$$\left(\Box + \mu^2\right)\phi - \frac{1}{3!}\lambda\phi^3 = 0. \tag{82}$$

In particular, a constant value

$$Q = \mu\sqrt{6/\lambda} \tag{83}$$

is a solution to the equations of motion. The twist of the L_∞-algebra $\mathfrak{g}^{\mathrm{MH}}$ with respect to the Maurer–Cartan element (83) is the L_∞-algebra

$$\mathfrak{g}_Q^{\mathrm{MH}} = \left(0 \to \mathcal{C}^\infty\left(\mathbb{R}^d\right)[-1] \xrightarrow{-(\Box-2\mu^2)} \mathcal{C}^\infty\left(\mathbb{R}^d\right)[-2] \to 0 \right), \tag{84}$$

with

$$\mu_1^Q\left(\phi[-1]\right) = -\left(\Box - 2\mu^2\right)\phi[-2] \tag{85a}$$

$$\mu_2^Q\left(\phi_1[-1], \phi_2[-1]\right) = \mu\sqrt{6\lambda}\phi_1\phi_2[-2] \tag{85b}$$

$$\mu_3^Q\left(\phi_1[-1], \phi_2[-1], \phi_3[-1]\right) = \lambda\phi_1\phi_2\phi_3[-2] \tag{85c}$$

which corresponds to the action

$$S_Q\left[\tilde{\phi}\right] = S\left[Q + \tilde{\phi}\right] = \int \mathrm{d}^d x \left(\frac{1}{2}\tilde{\phi}\left(\Box - 2\mu^2\right)\tilde{\phi} - \frac{\mu\sqrt{6\lambda}}{3!}\tilde{\phi}^3 - \frac{1}{4!}\lambda\tilde{\phi}^4 \right) \tag{86}$$

corresponding to perturbation theory about the true vacuum, since now the mass term $-\mu^2\phi^2$ has the correct negative sign.

We can, furthermore, consider quantum corrections. Consider (76) but with the wrong-sign mass term:

$$\begin{aligned} W &= \int \mathrm{d}^4 x \frac{1}{2}\phi\left(\Box + \mu_{\mathrm{R}}^2\right)\phi - \frac{1}{4!}\lambda_{\mathrm{R}}\phi^4 - \frac{\hbar}{64\pi^2}M^2(\phi)\ln M^2(\phi) + \cdots \\ &= S + \mathcal{O}(\hbar), \end{aligned} \tag{87}$$

where

$$\mu_{\mathrm{R}} = \mu + \mathcal{O}(\hbar) \qquad \lambda_{\mathrm{R}} = \lambda + \mathcal{O}(\hbar) \tag{88}$$

$$M^2(\phi) = -\mu_{\mathrm{R}}^2 + \frac{1}{2}\lambda_{\mathrm{R}}\phi^2 \tag{89}$$

are the renormalised quantities. Then a one-particle-irreducible Maurer–Cartan element Q is a stationary point of the quantum effective action W. One choice is the constant value

$$Q = \mu_{\mathrm{R}}\sqrt{6/\lambda_{\mathrm{R}}} + \mathcal{O}(\hbar). \tag{90}$$

Twisting by this, we obtain the quantum L_∞-algebra whose effective action is

$$\Gamma_Q\left[\tilde{\phi}\right]=\Gamma\left[Q+\tilde{\phi}\right]=\int \mathrm{d}^d x\left(\frac{1}{2}\tilde{\phi}\left(\Box-2\mu_{\mathrm{R}}^2\right)\tilde{\phi}-\frac{\mu_{\mathrm{R}}\sqrt{6\lambda_{\mathrm{R}}}}{3!}\tilde{\phi}^3-\frac{1}{4!}\lambda_{\mathrm{R}}\tilde{\phi}^4\right.$$
$$\left.+M^2\left(Q+\tilde{\phi}\right)\ln M^2\left(Q+\tilde{\phi}\right)+\cdots\right). \tag{91}$$

5. Classical backgrounds and anomalies

In the previous section, we introduced a classical background for a field that was already present in the L_∞-algebra. In this section, we instead enlarge the L_∞-algebra first by coupling it to nondynamical fields and then twist to put the theory on nontrivial backgrounds. If we apply this procedure to one-particle-irreducible effective actions, the twist may generate tadpoles sourcing violations of the classically conserved currents, signalling the presence of a quantum anomalies. (For reviews of anomalies, see e.g. [65–67]. Put another way, anomalies are witnessed as twists that induce curved quantum L_∞-superalgebras from flat quantum L_∞-superalgebras.

There is a long tradition characterising anomalies in the BV-BFV, AKSZ, L_∞-algebra and factorisation algebra formalisms, with various relations amongst the diverse perspectives [61, 68–84]. For example, in [78] the chiral anomalies are understood as obstructions encoded in cohomology classes of an obstruction-deformation complex. Our presentation in terms of twists very directly connects L_∞-algebras to the most familiar incarnation of anomalies as violations of classically conserved currents induced by polygonal Feynman diagrams.

5.1. Background gauge fields and gauge anomalies

We start with a field theory that has a classical global symmetry H that may be anomalous, described by a quantum L_∞-superalgebra $\mathfrak{g}$. Now, we may couple this theory to a nondynamical H-gauge field A, but with neither a kinetic term for A nor ghosts for the gauge transformations, to produce a larger quantum L_∞-superalgebra $\tilde{\mathfrak{g}}$. (This procedure is possible even if H is anomalous, since there are no ghosts or gauge transformations, but not always unique since there are always curvature ambiguities (improvement terms).) Since A has no dynamics, it is not constrained by any equations of motion.

We may then twist the enlarged algebra $\tilde{\mathfrak{g}}$ by a background field A_0 to obtain $\tilde{\mathfrak{g}}_{A_0}$. This will in general not be a one-particle-irreducible Maurer–Cartan element since the corresponding one-particle-irreducible effective theory $(\tilde{\mathfrak{g}}_{A_0})_{\mathrm{1PI}}$ will have a tadpole term (corresponding to nullary operations $\mu_0=\sum_g\hbar^g\mu_0^g$), which represents the current j induced by the background field A_0; one detects an anomaly when $\partial\cdot j\neq 0$. The same discussion applies to abelian p-form symmetries except that the corresponding connection A is then a $(p+1)$-form potential [57].

To illustrate this procedure concretely, we consider the well-understood case of the chiral anomaly. This has been treated in detail using the BV formalism by Rabinovich in [78], where the anomaly is understood as an obstruction (witnessed by a non-trivial top-degree cohomology class which equivalent to a non-trivial Dirac index) to a well-defined fermionic partition function. This obstruction can then be related directly to a violation of axial current conservation, which is the interpretation that our perspective naturally takes.

For example, in $d=2n$ spacetime dimensions, consider a Dirac field Ψ with action

$$S[\phi]:=\int \mathrm{d}^d x\,\bar{\Psi}\partial_\mu\gamma^\mu\Psi \tag{92}$$

corresponding to the L_∞-superalgebra

$$\mathfrak{g}:=\left(0\to\underbrace{\Omega^0(M;\Pi E)[-1]}_{\Psi}\to\underbrace{\Omega^d(M;\Pi E^*)[d-2]}_{\Psi^+}\to 0\right), \tag{93}$$

where ΠE is the Dirac spinor bundle with the $\mathbb{Z}_2$ parity reversed. This has the global $\mathrm{U}(1)\times\mathrm{U}(1)$ vector and axial symmetries

$$\Psi\mapsto\exp\left(\mathrm{i}\alpha+\mathrm{i}\beta\gamma^{d+1}\right)\Psi \tag{94}$$

for any real numbers $\alpha,\beta\in\mathbb{R}$. It is well known that if we gauge one of the two $\mathrm{U}(1)$ symmetries, the other becomes anomalous and hence cannot be consistently gauged—if we try to do so, longitudinal modes of the other gauge field will not decouple, and we lose unitarity. However, we may 'nearly' gauge both in the sense of introducing non-dynamical gauge fields A^{vect} and A^{axial} (without kinetic terms or gauge transformations) and changing the derivative to the covariant derivative:

$$S_{\text{BV}} := \int \mathrm{d}^d x \,\bar{\Psi}\gamma^\mu \left(\partial_\mu + \mathrm{i}A^{\text{vect}}_\mu + \mathrm{i}A^{\text{axial}}_\mu \gamma^{d+1}\right)\Psi, \tag{95}$$

corresponding to the L_∞-superalgebra

$$\begin{aligned}\tilde{\mathfrak{g}} := \Bigg(0 \to &\underbrace{\Omega^1(M)}_{A^{\text{vect}}} \oplus \underbrace{\Omega^1(M)}_{A^{\text{axial}}} \oplus \underbrace{\Omega^0(M;\Pi E)[-1]}_{\Psi} \\ &\to \underbrace{\Omega^{d-1}(M)[d-3]}_{A^{\text{vect}+}} \oplus \underbrace{\Omega^{d-1}(M)[d-3]}_{A^{\text{axial}+}} \oplus \underbrace{\Omega^d(M;\Pi E)[d-2]}_{\Psi+} \to 0\Bigg).\end{aligned} \tag{96}$$

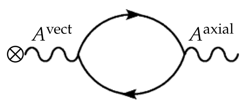
Note that we do *not* include ghosts for any would-be gauge transformations or kinetic terms for the gauge fields, which remain nondynamical.

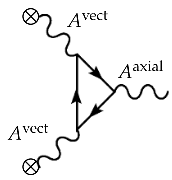
This now admits a classical local $\mathrm{U}(1)\times\mathrm{U}(1)$ symmetry

$$\Psi \mapsto \exp\left(\mathrm{i}\alpha(x) + \mathrm{i}\beta(x)\gamma^{d+1}\right)\Psi, \qquad A^{\text{vect}}_\mu \mapsto A_\mu - \partial_\mu\alpha, \qquad A^{\text{axial}}_\mu \mapsto A_\mu - \partial_\mu\beta, \tag{97}$$

which is broken at the quantum level. If one sets the background field to be $A^{\text{vect}}_\mu = A^{\text{axial}}_\mu = 0$, one recovers the original action (92) as a special case of (95).

Now, we may twist $\tilde{\mathfrak{g}}$ by background fields $A_0 := (A^{\text{vect}}_0, A^{\text{axial}}_0)$ to obtain the curved L_∞-superalgebra $\tilde{\mathfrak{g}}_{A_0}$. If we then pass to the one-particle-irreducible effective action represented by the *curved* L_∞-superalgebra $(\tilde{\mathfrak{g}}_{A_0})_{\text{1PI}}$, this in general has a tadpole term, corresponding to (anomalously) non-conserved currents. For simplicity, let us suppose that $A^{\text{axial}}_0 = 0$, that is, we only turn on a background vector gauge field. In that case, there are tadpole terms in the one-particle-irreducible effective action generated by one-point one-loop diagrams such as



(if $d = 2$) or


(if $d = 4$), (98)

where $\otimes$ refers to contraction with the background A^{vect}_0. These tadpoles $\mu_0^{(\tilde{\mathfrak{g}}_{A_0})_{\text{1PI}}}$ correspond to the axial current, i.e.

$$\langle A^{\text{axial}}_\mu, \mu_0^{(\tilde{\mathfrak{g}}_{A_0})_{\text{1PI}}}\rangle = j^{\text{axial}}_\mu, \tag{99}$$

induced by the background gauge field A^{vect}_0.

In this example, the Adler–Bell–Jackiw chiral anomaly may then be detected by the non-vanishing of the expression

$$\partial_\mu \langle j^\mu_{\text{axial}}\rangle = \partial_\mu \langle A^{\text{axial},\mu}, \mu_0^{(\tilde{\mathfrak{g}}_{A_0})_{\text{1PI}}}\rangle \propto \star\left(\overbrace{F^{\text{vect}}_0 \wedge \ldots \wedge F^{\text{vect}}_0}^{d/2}\right), \tag{100}$$

where $F^{\text{vect}}_0 = \mathrm{d}A^{\text{vect}}_0$ is the background field strength (and, as usual, we insist that the vector current is anomaly free).

5.2. Curved background spacetime and mixed anomalies

Similar considerations apply for mixed anomalies except that we must introduce a background metric g on the spacetime manifold M in addition to a background gauge field A. Given a theory with a global symmetry given by a Lie group H (with Lie algebra $\mathfrak{h}$) that is described by a quantum L_∞-superalgebra $\mathfrak{g}$, we may then couple it to a non-dynamical metric g and a background gauge field A (and their anti-fields g^+ and A^+) to produce the larger quantum L_∞-superalgebra

$$\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \left(0 \to \overbrace{\Omega^0\left(M; \mathrm{T}^{*\odot 2}M\right)[-1]}^{g} \oplus \overbrace{\Omega^1\left(M;\mathfrak{h}\right)}^{A} \right.$$

$$\left. \to \underbrace{\Omega^d\left(M; \mathrm{T}^{\odot 2}M\right)[d-2]}_{g^+} \oplus \underbrace{\Omega^{d-1}\left(M;\mathfrak{h}\right)[d-3]}_{A^+} \to 0 \right), \tag{101}$$

where the antifields are tensor densities, or equivalently (after picking an orientation of spacetime) top-degree differential forms valued in powers of the (co)tangent bundle.

As before, we may twist $\tilde{\mathfrak{g}}$ by background fields (g_0, A_0) to obtain the curved L_∞-superalgebra $\tilde{\mathfrak{g}}_{(g_0,A_0)}$. If we then pass to the one-particle-irreducible effective action represented by the curved L_∞-superalgebra $(\tilde{\mathfrak{g}}_{(g_0,A_0)})_{\text{1PI}}$, mixed gauge–gravitational anomalies may be detected by the non-vanishing of the expression

$$\nabla_\mu \langle j^\mu_a \rangle = \partial_\mu \frac{\delta \mu_0^{(\tilde{\mathfrak{g}}_{(g_0,A_0)})_{\text{1PI}}}}{\delta A^a_{0\mu}}, \tag{102}$$

where a is an adjoint index for $\mathfrak{h}$.

6. Twisting qua twisting

The seminal work of Witten [11] introduced the notion of twisting a supersymmetry field theory by nilquadratic supersymmetry generators. Using the formulation of twisting developed in [76], which marries with definition 7 applied to the supersymmetry algebra of the theory, it is possible to generate twisted theories that are topological, holomorphic, or somewhere in between [15, 85]. This section reviews how this notion dovetails with the L_∞-algebra formalism for scattering amplitudes.

In particular, we shall explain how supersymmetric twists and introducing classical backgrounds, as described in section 4, are actually instances of one and the same operation. Let us unpack the basic intuition underlying this claim here. Suppose we have a perturbative supersymmetric theory characterised by an L_∞-algebra $\mathfrak{g}$ with a supersymmetry algebra $\mathfrak{p}$. Then, using the action of $\mathfrak{p}$ on $\mathfrak{g}$, we may form an L_∞-superalgebra $\mathfrak{p} \ltimes \mathfrak{g}$, where the $\mathfrak{p}$ term may be regarded as the space of 'global ghosts' for the supersymmetry. Then a supersymmetric twist corresponds to a choice of Maurer–Cartan Q_{twist} element that lives only in $\mathfrak{p}$, while a classical background corresponds to a choice of Maurer–Cartan $Q_{\text{c.b.}}$ element that lives only in $\mathfrak{g}$. Both are realised by giving fields non-trivial expectation values, where for the twist the field is a global ghost for the supersymmetry (as in twisted supergravity [5], but global). More generally, a choice of Maurer–Cartan element Q living in $\mathfrak{p} \ltimes \mathfrak{g}$ will implement a compatible supersymmetric twist on a classical background. Thus, supersymmetric twists and classical backgrounds are extremal points in the moduli space of all L_∞-twists of $\mathfrak{p} \ltimes \mathfrak{g}$. From this perspective, one can regard the set of twists as a parameter space for a family of theories.

6.1. General construction and off/on-shell supersymmetry

For many supersymmetric models, one finds that the supersymmetry algebra is only realised on shell. That is, the algebra closes only modulo the equations of motion. Oftentimes the failure of the supersymmetry algebra to close can be explained through the language of homotopy algebra; the would-be Lie algebra morphism ρ from the supersymmetry algebra to the endomorphisms of the model is in fact an L_∞-algebra morphism and there are higher maps, correcting the failure of ρ to close off shell [36].

To set the stage, we are interested in theories which have a (not necessarily strict) symmetry given by a super Lie algebra $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$. (That is, a strict flat L_∞-superalgebra concentrated in cohomological degree 0.) Usually these supersymmetry algebras are of the form:

$$\mathfrak{p} = \overline{\mathfrak{der}(\mathfrak{t})} \ltimes \mathfrak{t}, \tag{103}$$

where $\mathfrak{t} = \mathfrak{t}^+ \oplus \mathfrak{t}^-$ is a Lie superalgebra whose only nontrivial bracket is defined by a map $\mathfrak{t}^- \otimes \mathfrak{t}^- \to \mathfrak{t}^+$, and $\overline{\mathfrak{der}(\mathfrak{t})}$ is a subalgebra of the (bidegree-preserving) Lie algebra of derivations of $\mathfrak{t}$.

Remark 1. Physically, $\mathfrak{t}^+$ is to be thought of as the translations of the underlying space of the theory while $\mathfrak{t}^-$ are the supersymmetries and $\mathfrak{der}(\mathfrak{t})$ are the Lorentz and R-symmetries of the theory. Lie superalgebras of this form encode supersymmetry algebras in any dimension with any amount of supersymmetry.

Remark 2. Algebras of this form furthermore allow for a lift of the $\mathbb{Z}\times\mathbb{Z}_2$-grading to a $\mathbb{Z}\times\mathbb{Z}$-grading by putting $\overline{\mathfrak{der}(\mathfrak{t})}$, $\mathfrak{t}^-$, and $\mathfrak{t}^+$ in bidegrees $(0,0)(0,1)$, and $(0,2)$, respectively. This lift allows for a more tractable treatment of the algebraic structures and homotopy transfer.

Let $\mathfrak{g}$ be a cyclic L_∞-superalgebra corresponding to a perturbative field theory enjoying a (not necessarily strict) global supersymmetry given by a Lie superalgebra $\mathfrak{p}$ of the form (103)[18], regarded as an L_∞-superalgebra concentrated in the $(\mathbb{Z}\times\mathbb{Z}_2)$-bidegrees $(0,0)$ and $(0,1)$.

Since the global symmetry acts on the fields, there must be an action of $\mathfrak{p}$ on $\mathfrak{g}$. In the language of homotopy algebras there then exists a morphism of L_∞-superalgebras

$$\mathfrak{p}\rightsquigarrow\left(\mathrm{Coder}\left(\mathrm{CE}_\bullet\left(\mathfrak{g}\right)\right)\right)^1 \tag{104}$$

where $\mathrm{CE}_\bullet(\mathfrak{g})$ is the Chevalley–Eilenberg cosuperalgebra of $\mathfrak{g}$, and $\mathrm{Coder}(X)$ is the differential graded Lie superalgebra of $(\mathbb{Z}\times\mathbb{Z}_2)$-graded coderivations on a differential graded cosuperalgebra X (whose elements need not be chain maps), and whose differential is given by the commutator with the differential on X, and the notation $(-)^1$ denotes the degree-one part [46, 86, 87].

This is equivalent to a family of morphisms of degree $(2-p-q,0)$ for $p\geqslant 1$ and $q\geqslant 0$

$$\rho^{(p,q)}:\mathfrak{p}^{\wedge p}\otimes\mathfrak{g}^{\wedge q}\to\mathfrak{g}, \tag{105}$$

with $\rho^{(0,k)}=\mu_k^{\mathfrak{g}}$, satisfying some compatibility relations with the L_∞-algebra structure on $\mathfrak{p}$ and $\mathfrak{g}$ [46, 86, 87] of the form:

$$\begin{aligned}0=&\sum_{p+r=n}\sum_{\sigma\in\mathrm{Sym}(n)}\frac{\pm1}{p!r!}\rho^{(1+r,m)}\left(\mu_p^{\mathfrak{p}}\left(x_{\sigma(1)},\cdots,x_{\sigma(p)}\right),x_{\sigma(p+1)}\cdots,x_{\sigma(n)};\phi_1,\cdots,\phi_m\right)\\&+\sum_{\substack{\sigma,\tilde\sigma\in\mathrm{Sym}(n)\\p+r=n\\i+s+j=m}}\frac{\pm1}{p!r!i!s!j!}\rho^{(p,i+1+j)}\left(x_{\sigma(1)},\ldots,x_{\sigma(p)};\phi_{\tilde\sigma(1)},\ldots,\phi_{\tilde\sigma(i)},\right.\\&\left.\rho^{(r,s)}\left(x_{\sigma(p+1)},\ldots,x_{\sigma(n)};\phi_{\tilde\sigma(i+1)},\ldots,\phi_{\tilde\sigma(i+s)}\right),\phi_{\tilde\sigma(i+s+1)},\ldots,\phi_{\tilde\sigma(m)}\right).\end{aligned} \tag{106}$$

Remark 3. Algebraic structures of this form are essentially a symmetrised version of an A_∞-algebra over an L_∞-algebra (or open–closed homotopy algebra), introduced by Kajiura and Stasheff in [46, 86, 87], where an L_∞-algebra acts on an A_∞-algebra through homotopy derivations [88–91]. See also [92] for open–closed homotopy algebras in supersymmetry.

Given the two L_∞-superalgebras $\mathfrak{p}$ and $\mathfrak{g}$ and the maps $\rho^{(p,q)}$, we may form an L_∞-superalgebra whose Chevalley–Eilenberg coalgebra is of the following form. There is an isomorphism of differential graded cosuperalgebras

$$\left(\mathrm{CE}_\bullet\left(\mathfrak{p}\right),d^{\mathfrak{g}}_{\mathrm{CE}_\bullet}\right)\otimes\left(\mathrm{CE}_\bullet\left(\mathfrak{g}\right),d^{\mathfrak{g}}_{\mathrm{CE}_\bullet}\right)\cong\left(\mathrm{CE}_\bullet\left(\mathfrak{p}\oplus\mathfrak{g}\right),d^{\mathfrak{p}}_{\mathrm{CE}_\bullet}+d^{\mathfrak{g}}_{\mathrm{CE}_\bullet}\right). \tag{107}$$

We extend the L_∞-map (104) to a degree $(1,0)$-coderivation $d^\rho_{\mathrm{CE}_\bullet}$ on $\mathrm{CE}_\bullet(\mathfrak{p}\oplus\mathfrak{g})$. Then, $d^\rho_{\mathrm{CE}_\bullet}$ can be added to form the full algebra

$$\left(\mathrm{CE}_\bullet\left(\mathfrak{p}\oplus\mathfrak{g}\right),d^{\mathfrak{p}\ltimes\mathfrak{g}}_{\mathrm{CE}_\bullet}:=d^{\mathfrak{p}}_{\mathrm{CE}_\bullet}+d^{\mathfrak{g}}_{\mathrm{CE}_\bullet}+d^{\rho}_{\mathrm{CE}_\bullet}\right) \tag{108}$$

The nilquadraticity of $d^{\mathfrak{p}\ltimes\mathfrak{g}}_{\mathrm{CE}_\bullet}$ is then equivalent to the homotopy Jacobi identities on $\mathfrak{p}$ and $\mathfrak{g}$, and the compatibility (106). The proof of this is straightforward and similar to the ones in [46, 86, 87]. The resulting L_∞-superalgebra is then

$$\mathfrak{p}\ltimes\mathfrak{g}:=\left(\mathfrak{p}\oplus\mathfrak{g},\left(\left\{\mu_i^{\mathfrak{p}},\rho^{(p,q)},\mu_j^{\mathfrak{g}}\right\}\right)_{\substack{i,j\geqslant0\\p,q\geqslant1}}\right). \tag{109}$$

(This may be formally thought of as 'gauging' the global super-Poincaré symmetry $\mathfrak{p}$ since $\mathfrak{p}$ sits in degree zero, which normally corresponds to Fadeev–Popov ghosts [17].)

[18] The construction goes through for any Lie superalgebra $\mathfrak{p}$, but we will focus on this case.

We would like to twist by a nilquadratic element of $\mathfrak{p} \subset \mathfrak{p} \ltimes \mathfrak{g}$. Naïvely, this cannot correspond to a Maurer–Cartan element of $\mathfrak{p} \ltimes \mathfrak{g}$ since $\mathfrak{p}$ is concentrated in degree zero rather than one. For this purpose, following [85], we introduce a formal parameter u of degree $(1,-1)$ and take

$$\mathfrak{G}[u] := (\mathfrak{p} \ltimes \mathfrak{g}) \otimes \mathbb{C}[u]. \tag{110}$$

(Complexification is convenient and often necessary for twisting [15].) Then $\mathfrak{p}[u] \subset \mathfrak{G}[u]$ contains a ready supply of bidegree $(1,0)$ elements of the form uQ where $Q \in \mathfrak{p}^-$ satisfies $[Q,Q]_{\mathfrak{p}} = 0$, and these form Maurer–Cartan elements inside $\mathfrak{G}[u]$ with which we may twist.

More generally, we may twist by a Maurer–Cartan element of $\mathfrak{G}[u]$ that is not purely restricted to $\mathfrak{p}[u]$; this corresponds to a combination of taking a classical background and twisting, a possibility that will be fully exploited in section 7.

For supersymmetric models where supersymmetry is realised only on shell—that is, after imposing the equations of motion—the projective superspace (reviewed in [93, 94]), harmonic superspace (reviewed in [95, 96]) or the pure spinor (reviewed in [36, 97–99]) formalisms often succeed in producing an off-shell representation at the cost of introducing infinite towers of auxiliary fields.

However, since such infinite towers are often unwieldy, it is sometimes convenient to be able to work directly with the on-shell supersymmetry representation. For this, one can often lift the $\mathbb{Z} \times \mathbb{Z}_2$ bigrading to $\mathbb{Z} \times \mathbb{Z}$ [36, 100] and apply a minimal-model construction to obtain an on-shell supersymmetry representation with finitely many components.

Supersymmetry algebras arising in physics usually admit such lifts (cf remark 2). Schematically, let V be an n-dimensional (real) vector space, $\mathfrak{o}(V)$ the Lie algebra of orthogonal transformations of V, and S a (linear combination of) spin representations of $\mathfrak{o}(V)$. Then, a (lifted) supersymmetry algebra (without R-symmetry) is of the form

$$\mathfrak{p} = \mathfrak{o}(V) \ltimes (S[0,-1] \oplus V[0,-2]). \tag{111}$$

We then introduce a formal parameter u of bidegree $(1,-1)$:

$$\mathfrak{p}[u] = \mathfrak{o}(V) \ltimes (S[0,-1] \oplus V[0,-2]) \otimes \mathbb{C}[u]. \tag{112}$$

Then $\mathfrak{p}[u]$ contains a ready supply of degree 1 elements of the form uQ where $Q \in S[0,-1]$ is such that $[Q,Q] = 0$.

6.2. Example with off-shell supersymmetry

Here we provide a simple example with off-shell supersymmetry, that is, where the supersymmetry algebra is represented on the nose. Let us consider a free theory containing a $\mathcal{N}=1$ chiral supermultiplet (sometimes called the Wess–Zumino multiplet) on Minkowski space $\mathbb{R}^{1,3}$ (or, rather, complexified Minkowski space $\mathbb{C}^4$). The supersymmetry algebra is

$$\mathfrak{p} = \mathfrak{o}\left(\mathbb{C}^4\right) \ltimes \left(\Pi S \oplus \mathbb{C}^4\right), \tag{113}$$

where $S = S_+ \oplus S_-$ is the Dirac spin representation, $S_\pm \cong \mathbb{C}^2$ are the Weyl spinor representations, and Π denotes shift in the $\mathbb{Z}_2$ degree. In addition to the brackets defined by the action of $\mathfrak{o}(\mathbb{C}^4)$ on $\mathbb{C}^4$ and S, the only nontrivial brackets are defined by the isomorphism

$$S_+ \otimes S_- \cong \mathbb{C}^4. \tag{114}$$

The L_∞-superalgebra defining the supermultiplet is

$$\begin{aligned}\mathfrak{g} = \Big(0 \to \Omega^0\left(\mathbb{C}^4; \mathbb{C} \oplus \mathbb{C} \oplus \Pi S_+ \oplus \overline{\mathbb{C} \oplus \mathbb{C} \oplus \Pi S_+}\right)[-1] \\ \times \xrightarrow{\mu_1} \Omega^4\left(\mathbb{C}^4; \mathbb{C} \oplus \mathbb{C} \oplus \Pi S_- \oplus \overline{\mathbb{C} \oplus \mathbb{C} \oplus \Pi S_-}\right)[2] \to 0\Big),\end{aligned} \tag{115}$$

corresponding to the supermultiplet (ϕ, ψ, F) with a complex scalar field ϕ, a Weyl fermion ψ, and an auxiliary field F, and with the complex conjugate fields explicitly separated out due to the complexification (see e.g. [15, §10.1]). The only nontrivial bracket is μ_1, corresponding to the free equations of motion.

Gauging supersymmetry in the sense of adjoining global supersymmetry ghosts, we obtain the L_∞-superalgebra

$$(\mathfrak{p} \ltimes \mathfrak{g})[u] = \Big(\mathfrak{p} \xrightarrow{0} \Omega^0\left(\mathbb{C}^4; \mathbb{C} \oplus \mathbb{C} \oplus S_+\right)[-1] \to \Omega^4\left(\mathbb{C}^4; \mathbb{C} \oplus \mathbb{C} \oplus S_-\right)[2] \to 0\Big)[u], \tag{116}$$

where we have also adjoined a formal variable u of degree $(1,-1)$, with differential μ_1, and μ_2 corresponding to the Lie algebra structure on $\mathfrak{p}$ and its action on $\mathfrak{g}$.

A supercharge $Q \in \mathfrak{p}$ squares to zero precisely when $Q \in S_+$ or $Q \in S_-$. Thus, by picking such a Q we can twist $\mathfrak{p} \ltimes \mathfrak{g}$, to obtain the twisted theory [85]. Moreover, since $\mu_i|_{\mathfrak{g}\otimes\cdots\otimes\mathfrak{g}} = 0$ for $i \neq 1$[19], any element $(\phi, \bar{\phi}) \in \Omega^0(\mathbb{C}^4; \mathbb{C} \oplus \bar{\mathbb{C}})[-1]$ that satisfies the Klein–Gordon equation defines a Maurer–Cartan element. Furthermore, if we pick $Q \in S_+$, the holomorphic field ϕ is Q-closed, so we may twist $(\mathfrak{p} \ltimes \mathfrak{g})[u]$ by a linear combination

$$\mathcal{Q}_\phi = \phi + uQ \tag{117}$$

to obtain a twisted L_∞-superalgebra with differential

$$\mu_1^{(\mathfrak{p}\ltimes\mathfrak{g})_{\mathcal{Q}_\phi}}(-) = \mu_1(-) + \mu_2(\phi, -) + \mu_2(uQ, -) . \tag{118}$$

6.3. Example with on-shell supersymmetry

As a less trivial example, we consider the holomorphic twist of ten-dimensional $\mathcal{N} = (1,0)$ supersymmetric Yang–Mills theory [15, 101] into holomorphic Chern–Simons theory in five complex-dimensions, which can be naturally formulated in the pure-spinor superfield formalism [85]. The ten-dimensional $\mathcal{N} = (1,0)$ super-Poincaré Lie superalgebra is (after lifting the $\mathbb{Z}_2$-grading to a $\mathbb{Z}$-grading)

$$\mathfrak{p} = (\mathfrak{o} \ltimes (V[0,-2] \oplus S[0,-1])) \otimes \mathbb{C}[u] , \tag{119}$$

where $V \cong \mathbb{C}^{10}$ and S is one of the Weyl spin representations of $\mathfrak{o}(V)$.

Gauging supersymmetry in the sense of adjoining global ghosts to the L_∞-superalgebra (and adjoining the formal variable u), we obtain the L_∞-superalgebra

$$(\mathfrak{p} \ltimes (\mathfrak{g} \otimes X)) \otimes \mathbb{C}[u] , \tag{120}$$

where

$$X = \mathbb{C}[x^\mu, \theta^\alpha, \lambda^\alpha] / \left(\lambda^\alpha \gamma^\mu_{\alpha\beta} \lambda^\beta\right) \tag{121}$$

is a differential graded-commutative algebra encoding the pure spinor formulation of the ten-dimensional (colour-stripped) $\mathcal{N} = (1,0)$ vector supermultiplet with

$$\mathrm{d}x := \lambda^\alpha \left(\frac{\partial}{\partial\theta^\alpha} - \gamma^\mu_{\alpha\beta}\theta^\beta \frac{\partial}{\partial x^\mu}\right) x, \tag{122}$$

(and other differentials vanishing), and $\mathfrak{g}$ is the colour Lie algebra (concentrated in bidegree $(0,0)$). The coordinates carry the bidegrees [36, (3.15) ff.][20]

$$|x| = (0,-2) \qquad |\theta| = (0,-1) \qquad |\lambda| = (1,-1) \qquad |u| = (1,-1) . \tag{123}$$

Now, a Maurer–Cartan element here can be either purely in $\mathfrak{p}[u]$, or purely in $\mathfrak{g} \otimes M[u]$, or in some combination. An example of the former is

$$uQ \tag{124}$$

where Q is a pure spinor, i.e. $Q^\alpha \gamma^\mu_{\alpha\beta} Q^\beta = 0$. An example of the latter is any bosonic solution to the classical equations of motion, such as

$$a_\mu (\lambda \gamma^\mu \theta) , \tag{125}$$

19 This is because $\mathfrak{g}$, describing a free theory, did not have a μ_2. Adjoining $\mathfrak{p}$ produces new brackets between elements of $\mathfrak{p}$ and elements of $\mathfrak{g}$ but never amongst elements of $\mathfrak{g}$.

20 Strictly speaking, this only works if we regard spacetime as a supervariety (equipped with the structure sheaf of polynomial superfunctions) rather than a supermanifold (equipped with the structure sheaf of smooth superfunctions). This difficulty can be avoided by introducing a formal variable u that compensates for the nontrivial degrees [36].

where $a_\mu \in \mathfrak{g} \times \mathbb{C}^{10}$ is a fixed Lie-algebra-valued 10-vector. This corresponds to a constant vacuum expectation value of the gluon field A_μ (which breaks part of Lorentz symmetry). More generally, however, one can have a combination of both, as long as they are compatible. In the above example, suppose that $a_\mu(\lambda\gamma^\mu\theta)$ is annihilated by Q[21]. In that case, we can twist by

$$uQ + a_\mu(\lambda\gamma^\mu\theta), \tag{126}$$

which corresponds to twisting by Q plus turning on a background field A that survives twisting. This produces a strict L_∞-superalgebra (that is, $\mu_{i>2} = 0$), whose underlying graded supervector space and μ_2 coincides with that of (120), and whose differential is of the form

$$\mu_1^Q = \lambda\left(\frac{\partial}{\partial\theta} - \gamma^\mu\theta\frac{\partial}{\partial x^\mu}\right) + \mu_2(uQ, -) + \mu_2(a_\mu(\lambda\gamma^\mu\theta), -). \tag{127}$$

In other words, supersymmetric twisting and classical backgrounds correspond to two extreme regions of a single moduli space of possible twisting backgrounds; equivalently, twisting corresponds to giving an expectation value of a certain global 'ghost' (similar to twisted supergravity [5] but 'global'). This presents a natural question: what facet of quantum field theory are twistings by Maurer–Cartan elements in the interior of the moduli space implementing? In the following section, we give one (not necessarily unique) answer: supersymmetric localization on curved spacetimes. (At least, when this is possible.)

7. Supersymmetric backgrounds and localisation

In the preceding section 6 we unified supersymmetric twists and classical backgrounds as twists belonging to L_∞-superalgebra $\mathfrak{p} \ltimes \mathfrak{g}$. The classical backgrounds correspond to vacuum expectations values of fields within $\mathfrak{g}$, while the supersymmetric twists correspond to vacuum expectations values of fields within $\mathfrak{p}$. On the other hand, in section 5 the background fields were external (nondynamical) to $\mathfrak{g}$, which is first enlarged as vector space before twisting to introduce the background[22]. This section now combines the two procedures to show that the resulting notion of twisting reproduces the setup for localisation of supersymmetric field theories on curved spacetimes using supergravity background fields [13], as reviewed in [14]. From this perspective, supersymmetric localization on a curved spacetime is yet another instance of twisting, where the Maurer–Cartan elements belong to the interior of the moduli space of Maurer–Cartan elements.

One important up-shot of this perspective is that an off-shell closed formulation of supergravity is not necessary for localization. Indeed, using the L_∞-algebraic formalism, it is clear that an on-shell multiplet suffices if we do homotopy transfer. In principle, this opens the door to a variety of unexplored localization schemes. The observation that on-shell supersymmetry suffices for localisation was already made in [102] in the case of finite-dimensional BV theories. See also [103] for the interaction of the BV formalism with localisation, see [103]. Here we limit ourselves to a schematic discussion of how a localization scheme for a supersymmetric theory on a curved background can be realised via a twisting; detailed computations will be given in [104].

For simplicity, let us consider the case of four-dimensional $\mathcal{N} = 1$ supersymmetry as in [13]. Recall that there are multiple possible supermultiplets containing the stress–energy tensor. Two of these are the Ferrara–Zumino multiplet,

$$\left(j_\mu, S_{\mu\alpha}, x, T_{\mu\nu}\right), \tag{128}$$

which consists of a nonconserved current, the supercharge, a complex scalar, and the stress–energy, respectively, and (in the cases where there is a conserved U(1) R-symmetry current) the R-current supermultiplet, which is given by

$$\left(j_\mu^{(\mathrm{R})}, S_{\mu\alpha}, T_{\mu\nu}, C_{\mu\nu}\right), \tag{129}$$

corresponding to the conserved R-current, the supercharge, the stress–energy, and a conserved two-form current, respectively. These correspond to the old minimal supermultiplet [105–107] $(A_\mu, \psi_{\mu\alpha}, M, g_{\mu\nu})$

[21] There are indeed nontrivial such elements, see [15, section 4.1.1] for details.

[22] This is really a distinction without a difference; the external fields could be included in $\mathfrak{g}$ from the get-go, as long as μ_1 is trivial on this subsector.

and the new minimal supermultiplet [108] $(A_\mu^{(\mathrm{R})}, \psi_{\mu\alpha}, g_{\mu\nu}, B_{\mu\nu})$, respectively; gauging local diffeomorphisms and supersymmetries corresponds to introducing the fields of supergravity (for reviews, see [109–112]) but without introducing kinetic terms for the graviton and gravitino, or equivalently a suitable $M_{\mathrm{Planck}} \to \infty$ limit [13].

For definiteness, let us consider localisation with an old-minimal supergravity background. Consider chiral superfields taking values in a Kähler manifold (N, ω) with Kähler potential $K \in \Omega^{0,0}(N)$ and superpotential $W \in \Omega^{0,0}(N)$. After fake-gauging super-diffeomorphisms, the L_∞-algebra is concentrated in degrees $0, 1, 2, 3$ as

$$\begin{aligned}\Omega^0(M, \mathrm{T}M \oplus S) &\to \Omega^0(M, E)[-1] \\ &\to \Omega^d(M, E^*)[d-2] \to \Omega^d(M, \mathrm{T}M \oplus S)[d-3],\end{aligned} \tag{130}$$

where

$$E = N \oplus \cdots \oplus \mathrm{T}^{*\odot 2}M \oplus \mathrm{T}^*M \oplus \mathbb{C} \oplus \mathbb{C} \tag{131}$$

is the fibre bundle over M whose section is $(\phi, \psi, F, g_{\mu\nu}, b_\mu, m, \bar{m})$, and where the ghosts and ghost antifields correspond to diffeomorphisms and super-diffeomorphisms. Putting the theory on a curved manifold corresponds to twisting by a Maurer–Cartan element $(g, b, m, \bar{m})$.

Now, for localisation, we adjoin a new formal coordinate u of degree $+1$, so that we get

$$\begin{aligned}\Omega^0(M, \mathrm{T}M \oplus S)[u] &\to \Omega^0(M, E)[-1][u] \\ &\to \Omega^d(M, E^*)[d-2][u] \to \Omega^d(M, \mathrm{T}M \oplus S)[d-3][u].\end{aligned} \tag{132}$$

(The inner $[i]$ denotes suspension while the outer $[u]$ denotes a polynomial ring.) If the background $(g, b, m, \bar{m})$ that we twisted by is annihilated by a nilquadratic supersymmetry generator $Q \in \Gamma(S)$, then the sum

$$uQ + (g, b, m, \bar{m}) \tag{133}$$

is also a Maurer–Cartan element, and then localisation corresponds to twisting by this Maurer–Cartan element.

The L_∞-algebraic formulation makes it clear that none of this discussion depends on having an off-shell strict realisation of supersymmetry—an on-shell realisation that can be completed into a non-strict L_∞-algebra representation of supersymmetry suffices for localisation [36, 102].

7.1. Old- versus new-minimal supergravity for localisation

In the localisation literature, it is well known [13, 14] that localisations using old-minimal versus new-minimal supergravities are not in general equivalent: for instance, the new-minimal localisation is only applicable to cases with an unbroken R-symmetry. This is superficially in tension with our claim that in fact on-shell supersymmetry suffices in general for localisation, since old-minimal and new-minimal supergravities are but different off-shell formulations of the one and the same four-dimensional $\mathcal{N} = 1$ supergravity theory.

However, there are in general inequivalent ways to couple a given matter theory to supergravity, such that the resulting theories are different on shell. For a simpler situation, consider coupling a matter theory with a stress–energy tensor $T_{\mu\nu}$ to ordinary gravity:

$$S = \frac{1}{2\kappa} \int \sqrt{|\det g|} \left(R - \frac{1}{2} g^{\mu\nu} T_{\mu\nu} \right) + \cdots. \tag{134}$$

The choice of a stress–energy tensor is in general not unique (see e.g. [14]), since one can always add improvement terms

$$T'_{\mu\nu} = T_{\mu\nu} + \left(\partial_\mu \partial_\nu - \delta_{\mu\nu} \partial^2\right) O \tag{135}$$

for a scalar operator O. Then one can instead take

$$S' = \frac{1}{2\kappa} \int \sqrt{|\det g|} \left(R - \frac{1}{2} g^{\mu\nu} T'_{\mu\nu} \right) + \cdots, \tag{136}$$

such that S and S' are physically inequivalent theories with different scattering amplitudes.

A similar situation obtains in supergravity. Suppose that one has a four-dimensional non-gravitational $\mathcal{N}=1$ supersymmetric theory $\mathcal{T}$ with a conserved R-symmetry, and suppose that we couple it to $\mathcal{N}=1$ old-minimal and new-minimal supergravities (assuming both couplings are possible). In that case, $\mathcal{T}$ has two different supermultiplets—the Ferrara–Zumino supermultiplet and the $\mathcal{R}$-supermultiplet (see e.g. [14]) — that each contain a stress–energy tensor, but the two stress–energy tensors T_{FZ}, $T_{\mathcal{R}}$ are related by improvement terms and in general differ. Coupling to old-minimal supergravity entails using the Ferrara–Zumino supermultiplet while coupling to new-minimal supergravity entails using the $\mathcal{R}$-supermultiplet. Therefore, even when auxiliary fields have been integrated out (thus obtaining an action with only on-shell local supersymmetry), the theories $\mathcal{T}+$old-minimal and $\mathcal{T}+$new-minimal in general are physically inequivalent. The inequivalence persists when one takes a rigid $M_{\mathrm{Planck}}\to\infty$ limit.

The situation is even clearer for those theories for which one of the two supermultiplets (Ferrara–Zumino and $\mathcal{R}$) does not exist: in this case, one is forced to couple to either old-minimal or new-minimal supergravity depending on which supermultiplet exists, and this can be seen in principle even after auxiliary fields have been integrated out.

Data availability statement

No new data were created or analysed in this study.

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