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Malmquist productivity index in DEA for special cases

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ABSTRACT

The Malmquist Productivity Index (MPI) is a major development in Data Envelopment Analysis (DEA), aiming to assess productivity changes over time. The MPI estimates the total factor productivity growth of a decision-making unit (DMU) with multiple inputs and outputs. However, the special case of a single input alongside multiple outputs or multiple inputs with a single output has not been thoroughly studied. These configurations have some unique features, particularly regarding computational costs and the selection of lower bounds—commonly known as non-Ar*chimedean epsilon*—for the dual input and output weights in multiplier DEA models. Moreover, DEA-based MPI often overlooks the role of epsilon in determining efficiency measures. This paper proposes a new approach for measuring DEA-based MPI with epsilon in single-input or single-output data sets. Four scenarios are designed, each based on varying epsilon values, to address settings with a single input and multiple outputs (SIMO) as well as settings with multiple inputs and a single output (MISO), with the aim of determining the optimal epsilon values. The proposed method ensures that no input or output is deemed non-instrumental to the production technology when estimating the MPI. The methodology's effectiveness is demonstrated by an analysis of productivity growth in 18 OECD countries from 2005 to 2021.

1. Introduction

Data Envelopment Analysis (DEA), pioneered by Charnes et al. (1978), is a non-parametric linear programming (LP) method for evaluating the relative efficiency of decision-making units (DMUs) with multiple inputs and outputs. DEA constructs a non-parametric production frontier based on empirical data without assuming a specific functional form (Collier et al., 2011), adhering to the axioms of monotonicity and convexity in the production possibility set (Banker et al., 1984). Over time, DEA has been extended to various sectors and complexities (Cook & Seiford, 2009).

DEA models commonly fall into two distinct categories: multiplier and envelopment models. Multiplier models use DEA ratio-based efficiency and assign optimal weights to inputs and outputs for each DMU, ensuring favourable assessment (Cooper et al., 2007). Charnes et al. (1979) introduced a non-Archimedean epsilon parameter to enforce strictly positive weights, ensuring all inputs and outputs are considered. While the non-Archimedean epsilon can alter the implied technology, it is often employed to (i) prevent zero-weight assignment to factors, (ii) mitigate degeneracy and extreme weights, and (iii) improve discrimination and feasibility—operational criteria commonly applied to improve accuracy in DEA (Mehrabian et al., 2000; Podinovski & Bouzdine-Chameeva, 2017; Sadeghi et al., 2023).

Ali & Seiford (1993) proposed a bound for epsilon to prevent model infeasibility, but Mehrabian et al. (2000) defined a new assurance interval for epsilon using a single linear program. Amin & Toloo (2004) presented a polynomial-time algorithm to identify assurance values, later improving their method to find the most efficient DMUs without solving the model individually (Amin & Toloo, 2007). Critiques of this approach (Amin, 2009a) led to further refinements by Toloo (2014). Amin (2009b) also proposed a simple method for single-input data models, eliminating the need for solving linear programs. Podinovski & Bouzdine-Chameeva (2017) introduced a refined epsilon bound, enabling single-stage DEA models to be the same as the radial efficiency results of Ali & Seiford (1993)'s two-stage approach. Hatami-Marbini et al., (2017) discussed that the non-Archimedean ε in multiplier DEA models is a meaningful lower bound on weights to distinguish weak from strong efficiency and proposed a four-step fuzzy DEA method that reshapes weakly efficient frontiers while preserving strongly efficient ones. Most recently, Sadeghi et al. (2023) further proposed epsilon

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boundaries for measuring Pareto-Koopmans efficiency of DMUs. More recently, Papaioannou & Podinovski (2024) developed a single-stage DEA optimisation procedure that eliminates the need to specify a small ε and is exactly equivalent to the conventional two-stage radial-plus-slacks approach, yielding identical sets of primal and dual optimal solutions for any convex polyhedral technology.

1.1. Single-input or single-output technologies

Although DEA is best known for its ability to handle multiple inputs and outputs non-parametrically, single-input or single-output configurations are empirically common and methodologically meaningful. Typical situations include a single binding resource (e.g., budget) generating several outputs (Despotis, 2005; Lin et al., 2011) or multiple inputs used to produce one vital outcome (Cook & Seiford, 2009; Lovell & Pastor, 1999). The DEA literature has developed several formulations for such cases, including radial models with a single constant input or output, and no-explicit-inputs/outputs settings used in composite indicators, which delineate when standard radial measures remain valid and how to construct a proper frontier in single-input or single-output contexts (Lovell & Pastor, 1999; Despotis, 2005). These adaptations have been applied to performance evaluation in areas such as GDP per capita (Yang et al., 2014; Kohl & Brunner, 2020), banking and finance (Liu et al., 2011; Nguyen et al., 2022), and composite indicator development for healthcare systems (Despotis, 2005; Lin et al., 2011; Babaee et al., 2021). DEA models with either no inputs or no outputs are also applied as a multi-criteria decision analysis tool in applications such as inventory classification and supplier selection (Chen, 2011; Yang et al., 2014; Villanueva-Cantillo & Munoz-Marquez, 2021).

From a practical standpoint, settings with a single input and multiple outputs (SIMO) arise when performance is constrained by a single binding resource—for example, total public expenditure, staff hours, or energy use—yet organisations or countries are expected to deliver multiple outcomes, such as economic, social, and environmental outputs. Conversely, settings with multiple inputs and a single output (MISO) occur when multiple resources jointly determine a single policy-relevant outcome, such as GDP per capita, an emissions-intensity indicator, or a measurable health indicator such as life expectancy or mortality rates (Despotis, 2005; Lin et al., 2011; Babaee et al., 2021). In these contexts, productivity change is not merely about “doing more with less,” but about how frontier trade-offs across outputs (in SIMO) or the composition of resource mixes transforming into a single output (in MISO) evolve over time. These settings also make the role of multiplier weights more salient in a way that if the non-Archimedean epsilon is ignored or chosen arbitrarily, some outputs or inputs may receive near-zero weights, weakening interpretability and potentially distorting efficiency scores and productivity change measures. For example, in public sector performance measurement, a fixed budget (single input) must deliver economic, social, and environmental outputs; if some outputs receive near-zero weights due to arbitrary epsilon choices, cross-country or cross-period productivity comparisons can misleadingly suggest improvement while ignoring important outcomes.

1.2. Productivity change

Building on static applications, dynamic DEA models extend the methodology by incorporating temporal dimensions to assess changes in productivity over time. Dynamic frameworks, such as those employing the Malmquist Productivity Index (MPI), evaluate total factor productivity (TFP) shifts and have become essential tools for time-dependent analyses in numerous sectors including banking, healthcare, and aviation (Capeletti et al., 2023; Li et al., 2022; Yu & Nguyen, 2023). The MPI, introduced by Färe and Grosskopf (1992) and based on Caves et al. (1982), is a productivity measure defined via distance functions. The widely used decompositions of the MPI into the *sources of productivity change*—namely catch-up (efficiency change) (CU) and frontier-shift (technical change) (FS)—are model-based constructions proposed in the literature (Färe et al., 1994; Ray & Desli, 1997; Chen & Ali, 2004). Ray & Desli (1997) decomposed efficiency change into scale and pure technical efficiency change but faced critiques for failing to separate scale efficiency change from input/output mix effects (Balk, 2001; Lovell, 2003). Chen & Ali (2004) highlighted oversimplifications in the geometric mean approach for the DEA-based MPI, emphasising overlooked key information. Kumar & Russell (2002) analysed OECD labor productivity growth, decomposing it into technological change, catch-up, and capital accumulation, later extended by Henderson & Russell (2005) to include human capital in a four-part decomposition. Karagiannis and Lovell (2016) showed alignment between Malmquist and Hicks–Moorsteen indices with a single input, highlighting orientation-independent MPI decompositions. Although many MPIs in the literature utilise envelopment DEA models, multiplier forms have also been explored, particularly in network DEA. For instance, Kao & Liu (2016) constructed parallel production frontiers to estimate the MPI, while Kao (2017) developed a relational DEA model for estimating the biennial MPI. Recently, the MPI has gained theoretical and practical attention. Its foundations have been thoroughly summarised by Zelenyuk (2023), while Tavana et al. (2020) extended the MPI for network DEA structures to analyse sub-DMUs. Another important methodological development concerns the measurement of MPI under uncertainty (Emrouznejad et al., 2011; Hatami-Marbini et al., 2022). The application of the MPI has also grown across various sectors (Li et al., 2022; Wilson & Zhao, 2023), healthcare (Capeletti et al., 2023), and aviation (Yu & Nguyen, 2023), demonstrating its versatility and relevance in measuring efficiency and productivity. Recent contributions further illustrate the continued methodological development of MPI-based measures. Villa & Lozano (2026) proposed an intertemporal DEA framework combined with a constant-sum allocation step and a Global MPI to track nations’ productivity change in Olympic athletics, Aparicio & Santín (2026) introduced a circular TFP index based on a standard reference technology that avoids arbitrary base-period selection; relatedly, Cordero & Santín (2026) developed a synthetic reference-group approach to measure and decompose productivity gaps between groups.

1.3. Research gaps and contributions

While the MPI is formally defined via input and output distance functions, the empirical implementation of DEA–MPI in SIMO or MISO settings poses a practical measurement challenge that is less visible in fully multiple-input, multiple-output applications. In SIMO and MISO multiplier models, the frontier is supported by fewer trade-offs, so optimal solutions can collapse to extreme or near-degenerate weights. If the non-Archimedean lower bound is ignored or chosen arbitrarily, some outputs (in the SIMO case) or inputs (in the MISO case) may receive near-zero shadow values (dual prices), which can weaken discrimination among DMUs and make CU and FS decompositions difficult to interpret economically. This sensitivity is particularly consequential in adjacent-period MPI computations, where productivity changes can be driven by the imposed weights rather than by genuine technological shifts or CU.

Two research gaps emerge from the literature reviewed above. First, while single-input or single-output frontiers are discussed in static DEA work, there is limited guidance on constructing transparent and computationally efficient adjacent-period MPI decompositions (CU/FS) when the technology is naturally SIMO or MISO. Second, the influence of the non-Archimedean lower bound on multiplier-based MPI estimation has been largely overlooked; existing epsilon-bound studies focus on static feasibility and degeneracy control, but they do not address how extreme multipliers in SIMO/MISO settings can render some dimensions effectively non-instrumental and thereby distort CU and FS components in cross-period comparisons.

Existing epsilon/bound literature primarily addresses static feasibility and degeneracy control (e.g., Ali & Seiford, 1993; Mehrabian et al., 2000; Podinovski & Bouzdine-Chameeva, 2017), but it does not examine the consequences of extreme multipliers for adjacent-period MPI decompositions in SIMO/MISO contexts. We fill this gap by developing an ε-aware MPI framework that produces transparent, implementable rules (and closed-form results where available) for adjacent-period CU/FS computation in single-input or single-output environments. Building on this, the framework is extended to SIMO and MISO settings, resulting in a non-Archimedean-aware DEA–MPI approach. We make the role of epsilon explicit in MPI estimation by formulating four practically motivated epsilon scenarios and deriving scenario-specific rules (including optimal bounds and closed-form expressions where available). This approach prevents degenerate or near-zero weights, preserves the economic relevance of all inputs and outputs, and ensures that no dimension is implicitly treated as non-instrumental in productivity change measurement. The proposed procedure is both computationally light and economically interpretable. Under each scenario, CU and FS can be interpreted as efficiency change and technical change with ε-imposed strictly positive multiplier weights, rather than relying on an implicit or arbitrarily chosen weighting scheme. By stabilising multiplier weights, the framework improves discrimination among DMUs and the reliability of repeated MPI computations across many periods, scenarios, or reporting cycles — features that are especially important for policy and operational applications in data-rich environments.

Based on the discussion above, the contributions of this paper are fivefold. First, we develop four non-Archimedean scenarios for estimating the MPI in SIMO and MISO settings and use them to identify appropriate lower bounds for the multiplier weights. Second, we introduce a novel approach to estimate the MPI that avoids the need to solve optimisation problems. Third, we reduce computational complexity compared with standard DEA-based MPI implementations, which is especially valuable when MPIs are computed repeatedly across periods and scenarios. Fourth, we examine the mathematical properties of the proposed models, highlighting their applicability to real-world problems with SIMO and MISO settings. Finally, we present an empirical case study assessing productivity growth in 18 OECD countries from 2005 to 2021, demonstrating the practicality and effectiveness of the proposed framework.

1.4. *Structure of the paper*

The remainder of this paper is organised as follows. Section 2 presents the framework and methodology, presenting preliminaries and introducing the proposed approach for estimating the MPI in SIMO and MISO settings. Section 3 includes an empirical study, and a conclusion is provided in Section 4.

2. Framework and methodology

This section presents the methodological foundations and the proposed ε-aware framework. Subsection 2.1 briefly reviews the DEA production technology in the standard multi-input multi-output (MIMO) setting, the distance-function basis of MPI and its CU and FS components, and the role of non-Archimedean lower bounds in multiplier DEA models. Subsection 2.2 then develops the proposed approach, extending the construction to single-input–multiple-output (SIMO) and multiple-input–single-output (MISO) settings.

2.1. *Preliminaries*

Consider n decision-making units (DMUs) observed in periods $t \in \{1, 2\}$. In the general MIMO setting, $\mathrm{DMU}_{j\in\{1,\cdots,n\}}$ uses $x_j^t \in \mathbb{R}_+^m$ to produce $y_j^t \in \mathbb{R}_+^s$. Under constant returns to scale (CRS), free disposability, and convexity, the empirical production possibility set (PPS) in period t is the standard DEA technology $T_t = \{(x,y) \in \mathbb{R}_+^{m+s} : X^t\lambda \le x, Y^t\lambda \ge y, \lambda \ge 0\}$, where $X^t = [x_1^t, \cdots, x_n^t]$ and $Y^t = [y_1^t, \cdots, y_n^t]$ denote the observed inputs and outputs in period t (Charnes et al., 1978; Cooper et al., 2007).

2.1.1. *Distance functions and cross-period efficiencies*

Let $D_q^I(\cdot)$ and $D_q^O(\cdot)$ represent Shephard's (1970) input and output distance functions associated with technology T_q, $D_q^I(x,y) = \inf\{\theta > 0 : (\theta x, y) \in T_q\}$ and $D_q^O(x,y) = \sup\{\phi > 0 : (x, \phi y) \in T_q\}$. For any pair of periods $p, q \in \{1,2\}$, the input-oriented and output-oriented technical efficiencies of $\mathrm{DMU_o}$ observed in period p relative to the technology in period q are defined as $\theta^{pq} = 1/D_q^I(x_o^p, y_o^p)$ and $\varphi^{pq} = 1/D_q^O(x_o^p, y_o^p)$. These cross-period distance-function evaluations form the building blocks for DEA-based intertemporal productivity measurement (Caves et al., 1982; Färe & Grosskopf, 1992; Färe et al., 1994).

2.1.2. *Adjacent-period MPI and CU/FS decomposition*

The MPI analyses a DMU's performance across two periods using ratios of distance functions evaluated with respect to period-specific technologies. Using the input-distance representation (the output-distance case is analogous), the adjacent-period MPI for DMU_o between periods p and q is $\mathrm{MPI}_o = \left[D_p^I(x_o^q, y_o^q)/D_p^I(x_o^p, y_o^p) \times D_q^I(x_o^q, y_o^q)/D_q^I(x_o^p, y_o^p)\right]^{1/2}$, which is commonly decomposed as $\mathrm{MPI}_o = \mathrm{CU}_o \times \mathrm{FS}_o$ (Färe et al., 1994; Ray & Desli, 1997). In what follows, this standard adjacent-period construction will be implemented through the cross-period efficiencies θ^{pq} (in SIMO settings) and ϕ^{pq} (in MISO settings) defined above.

2.1.3. *Multiplier models and non-Archimedean ε*

Distance functions (and hence θ^{pq} and ϕ^{pq}) can be computed via DEA linear programming models either in envelopment form or in multiplier (dual) form. In multiplier formulations, the input and output weights are constrained to be nonnegative; many practical formulations also impose a non-Archimedean lower bound $\varepsilon \ge 0$ to ensure strictly positive weights when $\varepsilon > 0$ (Charnes et al., 1979; Cooper et al., 2007). For example, an input-oriented CRS multiplier formulation that evaluates $\mathrm{DMU_o}$ observed in period p against technology T_q can be written as

$$\begin{aligned}
&\max \textstyle\sum_{r=1}^{s} u_r y_{ro}^p \\
&\text{s.t.} \\
&\textstyle\sum_{i=1}^{m} v_i x_{io}^p = 1, \qquad\qquad\qquad\qquad (1)\\
&\textstyle\sum_{r=1}^{s} u_r y_{rj}^q - \sum_{i=1}^{m} v_i x_{ij}^p \le 0, \quad j = 1, \cdots, n, \\
&v_i, u_r \ge \varepsilon, \quad r = 1, \cdots, s;\ i = 1, \cdots, n,
\end{aligned}$$

where $u \in \mathbb{R}_+^s$ and $v \in \mathbb{R}_+^m$ are the output and input weights, respectively. In the subsequent analysis, attention is given to how alternative, data-driven choices of epsilon and its multiplier counterpart affect cross-period efficiency measures and the resulting CU/FS decomposition within SIMO and MISO settings.

2.2. *Proposed approach*

This section introduces a DEA-based approach for evaluating technical efficiency and productivity across two periods under single-input (SIMO) and single-output (MISO) structures. The discussion begins with the case of input-oriented models with a single input, focusing on the evaluation of DMUs relative to technologies observed in different periods. Four scenarios, corresponding to alternative values of the non-Archimedean epsilon used as a lower bound on weights, are examined to identify an appropriate representation of efficiency. Based on these scenarios, the CU and FS effects, and consequently the MPI, are derived without requiring the solution of additional optimisation problems. The same is then extended to the case of MPI under the MISO setting.

2.2.1. *MPI in SIMO setting*

We consider the CRS framework introduced earlier and focus on the

SIMO case. Suppose there are n DMUs, each using a single strictly positive input $x_j^t \in \mathbb{R}_+$ to produce s semi-positive outputs $\boldsymbol{y}_j^t = \left(y_{1j}^t, \cdots, y_{sj}^t\right) \in \mathbb{R}_+^s$, observed over periods $t \in \{1,2\}$. The DMU under evaluation is denoted by $\left(x_o^p, y_o^p\right)$ in period $p \in \{1,2\}$. For any $p, q \in \{1,2\}$, the input-oriented cross-period technical efficiency is given by $\theta^{pq}(\varepsilon) = 1/D_q^I\left(x_o^p, y_o^p\right)$, where $D_q^I(\cdot)$ and the associated period-q technology (T_q) are as defined in Subsection 2.1.

The technical efficiency of $\mathrm{DMU}_o^p = \left(x_o^p, \boldsymbol{y}_o^p\right)$ under CRS is evaluated using the following multiplier DEA model with a non-Archimedean lower bound, defined relative to the production technology in period q, where $p, q \in \{1,2\}$:

$$
\begin{aligned}
&\theta^{pq}(\varepsilon) = \max \sum\nolimits_{r=1}^{s} u_r y_{ro}^p \\
&\text{s.t.} \\
&v x_o^p = 1, \\
&\sum\nolimits_{r=1}^{s} u_r y_{rj}^q - v x_j^q \leq 0, \quad j = 1, \cdots, n, \\
&v, u_r \geq \varepsilon, \quad r = 1, \cdots, s,
\end{aligned}
\tag{2}
$$

where v and u_r are the weights associated with the single input and the r^{th} output, respectively. The non-Archimedean parameter $\varepsilon > 0$ restricts weight flexibility inherent in multiplier models, preventing zero or near-zero shadow values and thereby improving discrimination and feasibility (Cooper et al., 2007). Using the normalisation constraint $v = 1/x_o^p$, model (2) simplifies to:

$$
\begin{aligned}
&\theta^{pq}(\varepsilon) = \max \sum\nolimits_{r=1}^{s} u_r y_{ro}^p \\
&\text{s.t.} \\
&\sum\nolimits_{r=1}^{s} u_r y_{rj}^q \leq \frac{x_j^q}{x_o^p}, \quad j = 1, \cdots, n, \\
&u_r \geq \varepsilon, \quad r = 1, \cdots, s,
\end{aligned}
\tag{3}
$$

which is equivalent to evaluating the reciprocal of the input distance function under a technology implicitly shaped by the lower-bound restriction ε.

The MPI for DMU_o between periods 1 and 2 is then defined as

$$
MPI(x_o, \boldsymbol{y}_o, \varepsilon) = CU(x_o, \boldsymbol{y}_o, \varepsilon) \times FS(x_o, \boldsymbol{y}_o, \varepsilon), \tag{4}
$$

where $CU(x_o, \boldsymbol{y}_o, \varepsilon) = \frac{\theta^{2,2}(\varepsilon)}{\theta^{1,1}(\varepsilon)}$ and $FS(x_o, \boldsymbol{y}_o, \varepsilon) = \left(\frac{\theta^{1,1}(\varepsilon)}{\theta^{1,2}(\varepsilon)} \times \frac{\theta^{2,1}(\varepsilon)}{\theta^{2,2}(\varepsilon)}\right)^{0.5}$ represent catch-up (CU) and frontier-shift (FS) components, respectively. This decomposition follows directly from the standard Malmquist index construction based on Shephard's distance functions (Caves et al., 1982; Färe & Grosskopf, 1992). The CU component measures changes in technical efficiency relative to the contemporaneous frontier, whereas the FS component captures shifts in the production frontier itself. The validity of this adjacent-period decomposition rests on the maintained assumptions that each T_q satisfies CRS, free disposability, and convexity, and that technologies are comparable across periods. In SIMO (or equivalently MISO) settings, these assumptions ensure that the MPI remains well-defined, although the reduced dimensionality of the technology amplifies the role of multiplier weights.

The choice of the non-Archimedean parameter ε as a lower bound on output (input) weights has long been debated in the DEA literature, particularly in multiplier-based models (Cooper et al., 2007; Färe & Charles, 2018; Sadeghi et al., 2023). In SIMO environments, unrestricted weight flexibility may lead to extreme or near-degenerate solutions in which some outputs receive negligible shadow values, undermining discrimination power and economic interpretation. An inappropriately specified ε can also result in infeasibility, a well-documented issue in DEA models with weight restrictions (Toloo, 2014; Salahi & Toloo, 2017). The non-Archimedean lower bound therefore plays a critical role in stabilising the implicit valuation structure underlying the output distance functions, ensuring that all outputs contribute meaningfully to efficiency and productivity change measurement.

In the multiplier SIMO model, the output weights u_r can be interpreted as implicit shadow prices (model-implied marginal valuations) that a DMU assigns to its outputs under a constructed production technology. The non-Archimedean parameter ε therefore plays the role of a lower bound that restricts how small any output's shadow value is allowed to be. Economically, changing ε alters the implicit shadow valuations assigned to outputs, which in turn affects the reference weighting used to evaluate efficiency and productivity change (e.g., the MPI).

Against this theoretical and economic background, we next consider four alternative scenarios for choosing ε in model (3) and examine their implications for the CU, FS, and MPI components in SIMO settings. Each scenario reflects a distinct economic perspective on how weight restrictions influence efficiency measurement and the interpretation of productivity changes.

Scenario i.1 (zero epsilon). We set $\varepsilon = 0$ to eliminate the role of epsilon in model (3), leading to the following model for the measurement of productivity changes:

$$
\begin{aligned}
&\theta^{pq}(0) = \max \sum\nolimits_{r=1}^{s} u_r y_{ro}^p \\
&\text{s.t.} \\
&\sum\nolimits_{r=1}^{s} u_r y_{rj}^q \leq \frac{x_j^q}{x_o^p}, \quad j = 1, \cdots, n, \\
&u_r \geq 0, \quad r = 1, \cdots, s.
\end{aligned}
\tag{5}
$$

The upside of the model is that it allows the greatest leeway in the selection of weights for the DMU under analysis to present itself in its best light. As a downside, the weights calculated from model (5) might be impractical from the decision-makers' viewpoint because some outputs may be excluded from the evaluation process due to their zero weights.

This scenario corresponds to a fully flexible, model-implied valuation setting in which the DMU is allowed to assign zero shadow value to some outputs. The frontier is therefore built under minimal weight restrictions, allowing certain outcomes to be treated as non-binding or low-priority within the efficiency assessment (e.g., environmental or social outputs receiving no weight in a composite performance measure). In addition, the CU, FS, and MPI capture productivity change under fully self-selected output valuations, which may potentially overstate progress if gains are achieved by neglecting some outputs.

Scenario i.2 (aggregate epsilon). We set a unique and aggregate epsilon value, $(\varepsilon = \varepsilon^*)$, for model (3) as the maximum value which is the lower bound of weights for all outputs for both periods p and q. To do so, Amin & Toloo (2007)'s approach provides inspiration to develop the following LP model for the purpose of an optimal value of the aggregate epsilon:

$$
\begin{aligned}
&\varepsilon^* = \max \varepsilon \\
&\text{s.t.} \\
&\sum\nolimits_{r=1}^{s} u_r y_{rj}^q \leq \min\left\{\frac{x_j^q}{x_k^p} : k = 1, \cdots, n, p = 1, 2\right\}, \quad j = 1, \cdots, n;\ q = 1, 2, \\
&\varepsilon - u_r \leq 0, \quad r = 1, \cdots, s,
\end{aligned}
\tag{6}
$$

where ε is a decision variable that presents the lower bound of u_r due to the second set of constraints. Moreover, the first constraints set aggregates the constraints related to all DMUs in all periods. The following theorems validate model (6).

A single ε^* imposes a uniform lower bound on the minimum shadow valuation assigned to each output across all DMUs over both periods. This corresponds to an evaluation setting with stable and common measurement standards, in which no output is allowed to be completely ignored in efficiency assessment. Such a specification is consistent with a regulatory or managerial perspective that requires each outcome to contribute at least minimally to performance evaluation. As a result, the resulting CU, FS, and MPI components are derived under a shared minimum-valuation convention, improving comparability across DMUs over time.

Theorem 1. *Model (6) always has a non-negative feasible solution.*

Proof. *It is easy to see that $(\varepsilon, \mathbf{u}) = (0, \mathbf{0}_s)$ is a zero feasible solution for model (6). It is sufficient to exhibit a positive feasible solution for model (6).*

Let $u'_r = \frac{1}{s \times \max\left\{y^q_{rj}; j=1,\cdots,n, q=1,2\right\}}\left(\min\left\{\frac{x^q_j}{x^p_k} : \forall j,\ k,\ p,\ q\right\}\right)$ *for* $r = 1, .., s$.

Then, $u'_r > 0, \forall r$, *and for any j and q we have* $\sum_{r=1}^{s} u'_r y^q_{rj} = \left(\min\left\{\frac{x^q_j}{x^p_k} : \forall j,\ k,\ p,\ q\right\}\right)\sum_{r=1}^{s}\frac{y^q_{rj}}{s \times \max\left\{y^q_{rj}; j=1,\cdots,n, q=1,2\right\}} \le \min\left\{\frac{x^q_j}{x^p_k} : \forall j,\ k,\ p,\ q\right\} \le \min\left\{\frac{x^q_j}{x^p_k} : k = 1, \cdots, n,\ p = 1, 2\right\}$. *Now, let* $\varepsilon' = \min\{u'_r : r = 1, \cdots, s\}$ *yields the vector* $(\varepsilon', \mathbf{u}')$ *as a positive feasible solution of model (6), which completes the proof.* ■

Theorem 2. *The optimal objective value of model (6) is positive and bounded.*

Proof. *From Theorem 1,* $(\varepsilon', \mathbf{u}')$ *is a feasible solution for model (6) where* $\varepsilon' > 0$. *Given that the model is a maximisation problem, we have* $\varepsilon^* \ge 0$. *Now consider the dual of model (6) as follows:*

$$\min \sum_{q=1}^{2}\sum_{j=1}^{n}\left(\min\left\{\frac{x^q_j}{x^p_k} : k = 1, \cdots, n, p = 1, 2\right\}\right)\lambda^q_j$$

s.t.

$$\sum_{q=1}^{2}\sum_{j=1}^{n} y^q_{rj}\lambda^q_j - \gamma_r \ge 0, \quad r = 1, \cdots, s, \tag{7}$$

$$\sum_{r=1}^{s}\gamma_r \ge 1,$$

$$\lambda^1_j, \lambda^2_j, \gamma_r \ge 0, \quad j = 1, \cdots, n;\ r = 1, \cdots, s.$$

To show feasibility of the dual, let $\bar{\lambda}^q_j = \begin{cases} \frac{1}{y^q_{r1}}, & j = 1 \\ 0, & j = 2, \cdots, n \end{cases}$, for $q = 1,\ 2$; $\bar{\gamma}_r = \begin{cases} 1, & r = 1 \\ 0, & r = 2, \cdots, s \end{cases}$. It is straightforward to verify that $(\bar{\lambda}^1, \bar{\lambda}^2, \gamma) \in \mathbb{R}^{2n+s}$ is a feasible solution of model (7). According to the fundamental theorem of duality in LP problems (see Bazaraa et al., 2010), since both models (6) and (7) are feasible, they share an identical optimal objective value, which is therefore positive and bounded. ■

It should be highlighted here that the optimal objective value of model (6) provides a suitable value for the epsilon in model (3) (for more details about a suitable value for the non-Archimedean epsilon under different return to scales, we refer the reader to Mehrabian et al., 2000).

Theorem 3. *Let* $(\varepsilon^*, \mathbf{u}^*)$ *be an optimal solution of model (6). Then, for any reference* $\mathrm{DMU}_{o\in\{1,\cdots,n\}}$ *and any reference period* $p \in \{1, 2\}$, *model (3) is feasible when* $\varepsilon = \varepsilon^*$.

Proof. *Suppose* $(\varepsilon^*, \mathbf{u}^*)$ *is an optimal solution of model (6). We multiply the second set of constraints in model (6), i.e.,* $u^*_r \ge \varepsilon^*$, *by the non-negative value scalar* y^q_{rj} *for a fixed j and q, and then sum over* $r = 1 \cdots, s$ *to yield* $\sum_{r=1}^{s}\varepsilon^* y^q_{rj} \le \sum_{r=1}^{s} u^*_r y^q_{rj}$. *By the first set of constraints of model (6) we have* $\sum_{r=1}^{s}\varepsilon^* y^q_{rj} \le \sum_{r=1}^{s} u^*_r y^q_{rj} \le \min\left\{\frac{x^q_j}{x^p_k} : k = 1, \cdots, n;\ p = 1, 2\right\}$ *for* $j = 1, \cdots, n$, $q = 1, 2$. *Now fix an arbitrary reference* DMU_o *and period p. Since* $(k, p) = (o, p)$ *is among the candidates in the minimum, it follows that* $\min\left\{\frac{x^q_j}{x^p_k} : k = 1, \cdots, n;\ p = 1, 2\right\} \le \frac{x^q_j}{x^p_o}$ *for all* $j = 1, \cdots, n$ *and* $q = 1, 2$ *(assuming* $x^p_o > 0$).

Therefore, $\sum_{r=1}^{s}\varepsilon^* y^q_{rj} \le \frac{x^q_j}{x^p_o}$ *for* $j = 1, \cdots, n$; $q = 1, 2$. *As a result, the vector* $(\mathbf{u}, u_0) = (\varepsilon^* \mathbf{1}_s, 0)$ *satisfies the constraints of model (3) with* $\varepsilon = \varepsilon^*$, *so model (3) is a feasible solution for this value of* ε. *This completes the proof.* ■

The maximum value of epsilon calculated by model (6) enhances the discriminatory power of the within and intertemporal model (3) in comparison with model (5). Put differently, determining the optimum lower bound for input and output weights often improves discriminating power and ensures realistic weights in multiplier DEA models on account of imposing more weight restrictions (Toloo 2014; Hatami-Marbini & Toloo 2017; Toloo & Tavana 2017).

Scenario i.3 (period-pair epsilon). This case obtains different epsilons when model (3) is solved for different pairs of periods. Based on Amin and Toloo (2004) approach, we suggest the following model to compute an epsilon value pertinent to the given pair of periods (p, q):

$$\varepsilon^{pq^*} = \max \varepsilon$$

s.t.

$$\sum_{r=1}^{s} u_r y^q_{rj} \le \min\left\{\frac{x^q_j}{x^p_k} : k = 1, \cdots, n\right\}, \quad j = 1, \cdots, n, \tag{8}$$

$$\varepsilon - u_r \le 0, \quad r = 1, \cdots, s.$$

Obviously, the feasible region of model (8) is a superset of that of model (6), and hence the feasibility of the former model is concluded by the feasibility of the latter model which is proved in Theorem 1.

Allowing ε to differ across period pairs reflects that the lower bound on output shadow valuations is not time-invariant. Such variation may arise from changes in regulatory frameworks, market or environmental conditions, reporting standards, or evolving societal priorities. Under this specification, intertemporal productivity comparisons are conducted using period-pair-specific lower bounds, which is appropriate when the minimum acceptable valuation of outputs changes over time rather than remaining constant.

Theorem 4. *For any fixed reference* $\mathrm{DMU}_{o\in\{1,\cdots,n\}}$, *model (3) is feasible for* $\varepsilon = \varepsilon^{pq^*}$ *when p and q are given.*

Proof. *It is clear according to the proof of Theorem 3.*

Scenario i.4 (input-oriented ε-based approach). This scenario aims to formulate the following model based on Amin (2009a) approach to obtain the maximum epsilon for model (3)

$$\varepsilon^{pq^*}_o = \max \varepsilon$$

s.t.

$$\sum_{r=1}^{s} u_r y^q_{rj} \le \frac{x^q_j}{x^p_o}, \quad j = 1, \cdots, n, \tag{9}$$

$$\varepsilon - u_r \le 0, \quad r = 1, \cdots, s.$$

A DMU- and period-pair-specific ε reflects heterogeneity in local constraints or priorities, whereby some DMUs face more binding requirements that restrict how small the shadow valuation of any output can be in a given context. Under this specification, productivity change is evaluated relative to a locally constrained benchmark technology, allowing the MPI to capture differences arising from context-specific notions of non-negotiable performance.

Theorem 5. *Model (3) is feasible for $\varepsilon = \varepsilon_o^{pq^*}$ when p, q, and o are given.*

Proof. *It is clear according to the proof of Theorem 3.*

The aim of this scenario is to find the largest feasible region for each LP in both periods, with a lower bound on the weights, avoiding the use of an aggregate epsilon when estimating the MPI. However, the aggregate epsilon can be obtained as $\min\left\{\varepsilon_j^{pq^*}, \forall j, p, q\right\}$. The following two lemmas compare the optimal objective values of the epsilon models (8) and (9) and their effects on the optimal objective value of model (6).

Lemma 1. *Fix $p, q \in \{1, 2\}$. Let $\varepsilon^* > 0$ as in Theorem 2. Let $\varepsilon^{(pq)^*}$ and $\varepsilon_o^{(pq)^*}$ be the optimal objective values of models (8) and (9), respectively, for the same fixed (p, q) and (a fixed reference DMU$_o$ where applicable). Then, $0 < \varepsilon^* \leq \varepsilon^{pq^*} \leq \varepsilon_o^{pq^*}$.*

Proof. *Let $S_{(6)}, S_{(8)}$, and $S_{(9)}$ be the feasible region for models (6), (8), and (9), respectively, for all p, q. It is straightforward to verify that $S_{(6)} \subseteq S_{(8)} \subseteq S_{(9)}$. Since all three models share an identical objective function, this inclusion of feasible sets immediately implies $\varepsilon^* \leq \varepsilon^{pq^*} \leq \varepsilon_o^{pq^*}$. Moreover, from Theorem 2 we know that $\varepsilon^* > 0$, which completes the proof.* ■

Lemma 2. *Fix $p, q \in \{1, 2\}$ and let $\theta^{pq}(\varepsilon)$ be the optimal objective value of model (3) when the non-Archimedean lower bound is set to ε. Assume model (3) is feasible for $\varepsilon \in \left\{0, \varepsilon^*, \varepsilon^{(pq)^*}, \varepsilon_o^{(pq)^*}\right\}$. Then $0 < \theta^{pq}\left(\varepsilon_o^{pq^*}\right) \leq \theta^{pq}\left(\varepsilon^{pq^*}\right) \leq \theta^{pq}(\varepsilon^*) \leq \theta^{pq}(0)$.*

Proof. *Let the feasible region of model (3) be $S_{(3)}(\varepsilon)$. From Lemma 1, we have$0 \leq \varepsilon^* \leq \varepsilon^{pq^*} \leq \varepsilon_o^{pq^*} \leq 0$, which implies the following inclusion of feasible sets $S_{(3)}\left(\varepsilon_o^{pq^*}\right) \subseteq S_{(3)}\left(\varepsilon^{pq^*}\right) \subseteq S_{(3)}(\varepsilon^*) \subseteq S_{(3)}(0)$. Since model (3) is a maximisation problem, the optimal objective values satisfy following ordering $\theta^{pq}\left(\varepsilon_o^{(pq)^*}\right) \leq \theta^{pq}\left(\varepsilon^{(pq)^*}\right) \leq \theta^{pq}(\varepsilon^*) \leq \theta^{pq}(0)$, and positivity follows from Theorem 2.* ■

As an interesting finding, the following theorem proves that the optimal solution of model (9) can be obtained without solving any optimisation problem.

Theorem 6. *If all inputs and outputs are strictly positive,* i.e., *$x_o^p > 0$ and $y_{rj}^q > 0$ for all r, j, p, q, then the optimal objective value of model (9) is $\varepsilon_o^{pq^*} = \frac{1}{x_o^p}\min\left\{\frac{x_j^q}{\sum_{r=1}^s y_{rj}^q} : j = 1, \cdots, n\right\}$ for every p and q.*

Proof. *Let $\left(\varepsilon_o^{(pq)^*}, \boldsymbol{u}^*\right)$ be an optimal solution of model (9). We multiply the second set of constraints of model (9), i.e., $\varepsilon_o^{(pq)^*} - u_r^* \leq 0$, by the non-negative scalar y_{rj}^q for a fixed j and q, and then sum over $r = 1, \cdots, s$, to obtain $\varepsilon_o^{pq^*}\sum_{r=1}^s y_{rj}^q - \sum_{r=1}^s u_r^* y_{rj}^q \leq 0$. From the first set of constraints of model (9), $\sum_{r=1}^s u_r^* y_{rj}^q \leq \frac{x_j^q}{x_o^p}$, which implies $\varepsilon_o^{pq^*}\sum_{r=1}^s y_{rj}^q \leq \frac{x_j^q}{x_o^p}$. Since $\sum_{r=1}^s y_{rj}^q > 0$, we hence have $\varepsilon_o^{pq^*} \leq \frac{x_j^q}{x_o^p \mathbf{1}_s \boldsymbol{y}_j^q}$; $j = 1, \cdots, n$ and $q = 1, 2$. Therefore, for any feasible solution $(\overline{\varepsilon}, \overline{u})$ of model (9), $\overline{\varepsilon} \leq \frac{1}{x_o^p}\min\left\{\frac{x_j^q}{\sum_{r=1}^s y_{rj}^q} : j = 1, \cdots, n; q = 1, 2\right\}$.*

Define $\varepsilon^0 = \frac{1}{x_o^p}\min\left\{\frac{x_j^q}{\sum_{r=1}^s y_{rj}^q} : j = 1, \cdots, n; q = 1, 2\right\}$ and let $u^0 = \varepsilon^0 \mathbf{1}_s$. By construction, $u_r^0 \geq \varepsilon^0$ for all r, and the first set of constraints of model (9) holds with equality for the minimisingindex j. Hence $\left(\varepsilon^0, u^0\right)$ is feasible for model (9), which completes the proof. ■

Theorem 7 proves that there is no need to solve any LP problems to obtain the optimal solution of model (3).

Theorem 7. *Let $x_o^p > 0$ and $y_{rj}^q > 0$ for all r, j, p, q. Then, the optimal solution of model (3)* is *$\boldsymbol{u}^* = \left(\varepsilon_o^{pq^*}, \cdots, \varepsilon_o^{pq^*}\right)$.*

Proof. *By inspection, it can be shown that $\left(\varepsilon_o^{pq^*}, \cdots, \varepsilon_o^{pq^*}\right)$ is a feasible solution of model (3). Consider the following dual problem of model (3):*

$$
\begin{aligned}
&\min \frac{1}{x_o^p}\left(\sum\nolimits_{j=1}^n \lambda_j x_j^q\right) - \varepsilon_o^{pq^*}\left(\sum\nolimits_{r=1}^s s_r^-\right) \\
&\text{s.t.} \\
&\sum\nolimits_{j=1}^n \lambda_j y_{rj}^q - s_r^- = y_{ro}^p, \quad r = 1, \cdots, s, \\
&\lambda_j \geq 0, \quad j = 1, \cdots, n, \\
&s_r^- \geq 0, \quad r = 1, \cdots, s.
\end{aligned}
\tag{10}
$$

Without loss of generality, let $\frac{x_k^q}{\sum_{r=1}^s y_{rk}^q} = \min\left\{\frac{x_j^q}{\sum_{r=1}^s y_{rj}^q} : j = 1, \cdots, n\right\}$. It can be effortlessly verified that $(\lambda^*, \boldsymbol{s}^{-^*})$ is a feasible solution for model (10) where

$$
\lambda_j^* = \begin{cases} \max\left\{\frac{y_{ro}^p}{y_{rj}^q} : r = 1, \cdots, s\right\}, & \text{if} \quad j = k \\ 0, & \text{otherwise} \end{cases}
$$

$$
s_r^{-^*} = y_{rk}^q\left(\max\left\{\frac{y_{ro}^p}{y_{rj}^q} : r = 1, \cdots, s\right\}\right) - y_{ro}^p, \quad r = 1, \cdots, s, \text{and}
$$

$$
U = \max\left\{\frac{y_{ro}^p}{y_{rk}^q} : r = 1, \cdots, s\right\}
$$

Moreover, the objective function value of $(\lambda^*, \boldsymbol{s}^{-^*})$ is $\frac{x_k^q}{x_o^p}\left(\max\left\{\frac{y_{ro}^p}{y_{rj}^q} : r = 1, \cdots, s\right\}\right) - \frac{x_k^q}{x_o^p\left(\sum_{r=1}^s y_{rk}^q\right)}\left(\sum_{r=1}\left[y_{rk}^q\left(\max\left\{\frac{y_{ro}^p}{y_{rj}^q} : r = 1, \cdots, s\right\}\right) - y_{ro}^p\right]\right)$, or equivalently $\frac{x_k^q}{x_o^p\left(\sum_{r=1}^s y_{rk}^q\right)}\left(\sum_{r=1}^s y_{ro}^p\right)$. By Theorem 6, $\varepsilon_o^{qp^*} = \frac{x_k^q}{x_o^p\left(\sum_{r=1}^s y_{rk}^q\right)}$. On the other hand, $\frac{x_k^q}{x_o^p\left(\sum_{r=1}^s y_{rk}^q\right)}\left(\sum_{r=1}^s y_{ro}^p\right)$ is exactly the objective function value of model (3) given $\boldsymbol{u}^* = \left(\varepsilon_o^{qp^*}, \cdots, \varepsilon_o^{qp^*}\right) = \left(\frac{x_k^q}{x_o^p\left(\sum_{r=1}^s y_{rk}^q\right)}, \cdots, \frac{x_k^q}{x_o^p\left(\sum_{r=1}^s y_{rk}^q\right)}\right)$.

In summary, we construct a primal feasible solution $\boldsymbol{u}^* \in \mathbb{R}^s$ for model (3) and a dual feasible solution $(\lambda^*, \boldsymbol{s}^{-^*}) \in \mathbb{R}^{n+s}$ for model (10) such that $\sum_{r=1}^s u_r^* y_{ro}^p = \frac{1}{x_o^p}\left(\sum_{j=1}^n \lambda_j^* x_j^q\right) - \varepsilon_o^{qp^*}\left(\sum_{r=1}^s s_r^{-^*}\right)$. By *weak duality* in LP problems (Bazaraa et al., 2010), $(v^*, \boldsymbol{u}^*)$ and $(\theta^*, \boldsymbol{\lambda}^*, s^{+^*}, \boldsymbol{s}^{-^*})$ are optimal, which completes the proof. ■

For the evaluated DMU$_o$, the adjacent-period MPI is constructed from four efficiency scores obtained from the DEA model (3) by setting $(p, q) \in \{(1, 1), (2, 2), (1, 2), (2, 1)\}$, namely $\theta^{1,1}(\varepsilon)$, $\theta^{2,2}(\varepsilon)$, $\theta^{1,2}(\varepsilon)$ and $\theta^{2,1}(\varepsilon)$. These four values calculate the standard decompositions as

$\mathrm{CU_o}(\varepsilon) = \frac{\theta^{1,1}(\varepsilon)}{\theta^{2,2}(\varepsilon)}$, $\mathrm{FS_o}(\varepsilon) = \left(\frac{\theta^{2,1}(\varepsilon)}{\theta^{1,1}(\varepsilon)}\frac{\theta^{2,2}(\varepsilon)}{\theta^{1,2}(\varepsilon)}\right)^{\frac{1}{2}}$, and $\mathrm{MPI_o}(\varepsilon) = \mathrm{CU_o}(\varepsilon) \times \mathrm{FS_o}(\varepsilon)$.

Equivalently, using the input distance function $D_q^I(\cdot)$ associated with the technology T_q (defined at the beginning of this section), each efficiency score satisfies $\theta^{pq}(\varepsilon) = 1/D_q^I(x_o^p, y_o^p|\varepsilon)$, where the distance is computed with respect to the ε-constrained technology. Hence, the CU and FS components can be expressed in the familiar distance-function representation as $\mathrm{CU_o}(\varepsilon) = \frac{D_2^I(x_o^2,y_o^2;\varepsilon)^{-1}}{D_1^I(x_o^1,y_o^1;\varepsilon)^{-1}} = \frac{D_1^I(x_o^1,y_o^1;\varepsilon)}{D_2^I(x_o^2,y_o^2;\varepsilon)}$, and $\mathrm{FS_o}(\varepsilon) = \left(\frac{D_1^I(x_o^1,y_o^1;\varepsilon)}{D_2^I(x_o^1,y_o^1;\varepsilon)}\frac{D_1^I(x_o^2,y_o^2;\varepsilon)}{D_2^I(x_o^2,y_o^2;\varepsilon)}\right)^{1/2}$. The following equations introduce closed-form expressions for CU_o, FS_o and MPI_o that allow the MPI computation without repeatedly solving the four LPs:

$$\mathrm{CU}_o = \frac{x_o^1\left(\sum_{r=1}^s y_{ro}^2\right)\min_j\left\{x_j^2 \Big/ \sum_{r=1}^s y_{rj}^2\right\}}{x_o^2\left(\sum_{r=1}^s y_{ro}^1\right)\min_j\left\{x_j^1 \Big/ \sum_{r=1}^s y_{rj}^1\right\}},$$

$$\mathrm{FS}_o = \left[\frac{x_o^1\left(\sum_{r=1}^s y_{ro}^1\right)\min_j\left\{x_j^1 \Big/ \sum_{r=1}^s y_{rj}^1\right\}}{x_o^1\left(\sum_{r=1}^s y_{ro}^1\right)\min_j\left\{x_j^2 \Big/ \sum_{r=1}^s y_{rj}^2\right\}} \times \frac{x_o^2\left(\sum_{r=1}^s y_{ro}^2\right)\min_j\left\{x_j^1 \Big/ \sum_{r=1}^s y_{rj}^1\right\}}{x_o^2\left(\sum_{r=1}^s y_{ro}^2\right)\min_j\left\{x_j^2 \Big/ \sum_{r=1}^s y_{rj}^2\right\}}\right]^{\frac{1}{2}} = \frac{\min_j\left\{x_j^1 \Big/ \sum_{r=1}^s y_{rj}^1\right\}}{\min_j\left\{x_j^2 \Big/ \sum_{r=1}^s y_{rj}^2\right\}}, \tag{11}$$

$$\mathrm{MPI}_o = \frac{x_o^1\left(\sum_{r=1}^s y_{ro}^2\right)}{x_o^2\left(\sum_{r=1}^s y_{ro}^1\right)}.$$

These algebraic simplifications follow from the particular structure of the SIMO multiplier model under the scenario assumptions. Specifically, when the optimal multiplier vector takes the uniform form $\boldsymbol{u}^* = \left(\varepsilon_o^{(pq)^*}, \cdots, \varepsilon_o^{(pq)^*}\right)$ (or equivalently when the scenario yields the constructive bound in Theorem 7), the input-distance reciprocity collapses to the minimum benchmark ratios that appear in (11). Hence, Equations (11) are valid only under the same conditions used to derive Theorems 6–7, namely (i) all outputs sums $\sum_{r=1}^s y_{rj}^t$ are strictly positive (so denominators are defined), (ii) the computed $\varepsilon_o^{(pq)^*}$ is feasible (i.e., the implied multiplier vector satisfies all primal/dual constraints), and (iii) the CRS, convexity, and free-disposability assumptions on T_q hold so the distance-function duality is valid. In Equations (11), CU_o varies across DMUs because it depends on the evaluated DMU's input–output bundles $\left(x_o^1, x_o^2, y_o^1, y_o^2\right)$ together with period-specific benchmark terms $\min_j\left\{x_j^t/\left(\sum_{r=1}^s y_{rj}^t\right)\right\}$, $t = 1{,}2$, which are computed once from the peer set and reflect the technology T_t in each period. The FS_o term is common across DMUs in this SIMO setting because it reduces to a ratio of these period-specific benchmark terms — that is, FS captures a pure technology shift between T_1 and T_2 as represented by the precomputed minima.

Therefore, CU captures DMU-specific movement relative to the contemporaneous frontier (distance-function change), while FS summarises the pure movement of the frontier itself as measured by technology-wide benchmark ratios. Across scenarios i.1–i.4, CU/FS/MPI represent productivity change conditional on the imposed minimum-weight constraint (ε). Moving from scenarios i.1 to i.4 progressively shifts the interpretation from "best-case under self-chosen priorities" to "performance under minimum multi-outcome accountability," reducing the risk that measured progress is driven by degenerate shadow valuations. To illustrate the closed-form expressions in Equation (11) and verify the proposed MPI, CU, and FS calculations step by step, we present the following contrived numerical example.

Example 1. *Consider $n = 3$ DMUs with $s = 2$ outputs and one input observed over periods $p \in \{1,2\}$. In period 1, the observations are $\left(x_1^1, y_{11}^1, y_{21}^1\right) = (10{,}6{,}4)$, $\left(x_2^1, y_{12}^1, y_{22}^1\right) = (12{,}5{,}5)$, and $\left(x_3^1, y_{13}^1, y_{23}^1\right) = (9{,}7{,}3)$. In period 2, the observations are $\left(x_1^2, y_{11}^2, y_{21}^2\right) = (11{,}7{,}5)$, $\left(x_2^2, y_{12}^2, y_{22}^2\right) = (11, 6, 5)$, and $\left(x_3^2, y_{13}^2, y_{23}^2\right) = (10, 8, 4)$. Let $Y_j^p = \sum_{r=1}^s y_{rj}^p$. Then, $Y_1^1 = Y_2^1 = Y_3^1 = 10$ and $Y_1^2 = 12$, $Y_2^2 = 11$, $Y_3^2 = 12$. To apply Equation (11), we first compute the period-specific benchmark minima, which are the smallest ratios of input to total output across all DMUs in each period: $\min_j\left\{x_j^1/Y_j^1\right\} = \min\{1, 1.2, 0.9\} = 0.9$ and $\min_j\left\{x_j^2/Y_j^2\right\} = \min\{11/12, 1, 10/12\} = 10/12$. For DMU_1, these minima determine the FS common term in the SIMO closed form representation: $FS_1 = \min_j\left\{x_j^1/Y_j^1\right\}/\min_j\left\{x_j^2/Y_j^2\right\} = 0.9/(10/12) = 1.08$. The closed-form MPI for DMU_1 using Equation (11) is $MPI_1 = \left(x_1^1 Y_1^2\right)/\left(x_1^2 Y_1^1\right) = (10 \times 12)/(11 \times 10) = 12/11 \approx 1.0909$. Consequently, the CU component is $CU_1 = MPI_1/FS_1 = (12/11)/1.08 \approx 1.0101$, which coincides with the closed-form expression in Equation (11). As shown in Table 1, this example confirms that FS_o is common across DMUs in the SIMO setting, as it depends solely on the period-specific benchmark minima $\min_j\left\{x_j^t/Y_j^t\right\}$, $t = 1, 2$.*

As mentioned, the $\mathrm{FS_o}$ effect is identical for all DUMs, i.e., $FS_1 = FS_2 = \cdots = FS_n$. In other words, we have three cases, (i) $0 < FS_j < 1$, $\forall j$, (ii) $FS_j = 1$, $\forall j$, (iii) $FS_j > 1$, $\forall j$; or equivalently $sgn(FS_j - 1)$ is constant for all $j = 1, \cdots, n$[1]*. If $sgn(FS_j - 1) = 0$, $\forall j$, then the boundaries of efficient frontiers are fully overlapped otherwise the intersection of frontiers is null.*

The $\mathrm{CU_o}$ factor for the DMU under evaluation can be obtained by simple arithmetic operations involving $2(s-1)(n+1)$ additions, $2(n+2)+1$ multiplications and, $2(n-1)$ comparisons, also the calculation of $\mathrm{FS_o}$ acquire $2n(s-1)$ additions, $2n+1$ multiplications and $2(n-1)$ comparisons. Finally, the $\mathrm{MPI_o}$ factor can be calculated by $2(s-1)$ additions and 3 multiplications. It is worth mentioning that while solving LP models is not inherently difficult, reducing computational burden remains crucial in operations research, especially when handling large-scale problems in the era of big data.

2.2.2. MPI with MISO settings

Following Subsection 2.1, consider n DMUs observed in periods $t \in \{1,2\}$, where each DMU_j uses m semi-positive inputs $\boldsymbol{x}_j^t = \left(x_{1j}^t, \cdots, x_{mj}^t\right)$ to produce a single strictly positive output $y_j^t > 0, j = 1, \cdots, n$. The DMU under evaluation in period p is denoted by $\left(x_o^p, y_o^p\right)$. For any $p, q \in \{1,2\}$, the output-oriented cross-period technical efficiency of DMU_o observed in period p relative to the period q technology T_q is defined as $\phi^{pq}(\overline{\varepsilon}) = 1/D_q^O\left(x_o^p, y_o^p\right)$, where $D_q^O(\cdot)$ represents the Shephard output distance

[1] Note that $sgn(x)$ denotes the sign function. If $x > 0$, $sgn(x) = 1$, if $x < 0$, $sgn(x) = -1$, otherwise $sgn(x) = 0$.

Table 1
MPI, CU, and FS calculations for Example 1.

DMU	x_j^1	x_j^2	Y_j^1	Y_j^2	MPI $= x_j^1 Y_j^2 / \left(x_j^2 Y_j^1\right)$	CU = MPI / FS
1	10	11	10	12	$12/11 \approx 1.0909$	$1.09091/1.08 \approx 1.0101$
2	12	11	10	11	$(12 \times 11)/(11 \times 10) = 1.2$	$1.2/1.08 \approx 1.1111$
3	9	10	10	12	$(9 \times 12)/(10 \times 10) = 1.08$	$1.08/1.08 = 1$

function. Under CRS, $\phi^{pq}(\overline{\varepsilon})$ can be obtained from the following output-oriented multiplier DEA model, in which a non-Archimedean lower bound $\overline{\varepsilon}$ is imposed on the input multipliers:

$$\begin{aligned} &\phi^{pq}(\overline{\varepsilon}) = \min \sum_{i=1}^{m} v_i x_{io}^p \\ &\text{s.t.} \\ &u y_o^p = 1, \\ &u y_j^q - \sum_{i=1}^{m} v_i x_{ij}^q \leq 0, \quad j = 1, \cdots, n, \\ &u, v_i \geq \overline{\varepsilon}, \quad i = 1, \cdots, m, \end{aligned} \tag{12}$$

where v_i and u are input and output weights, respectively, and $\overline{\varepsilon} > 0$ is the lower bound on input weights. The normalisation constraint $u y_o^p = 1$ implies $u = \frac{1}{y_o^p}$, so model (12) simplifies to the equivalent primal as follows:

$$\begin{aligned} &\phi^{pq}(\overline{\varepsilon}) = \min \sum_{i=1}^{m} v_i x_{io}^p \\ &\text{s.t.} \\ &\sum_{i=1}^{m} v_i x_{ij}^q \geq \frac{y_j^q}{y_o^p}, \quad j = 1 \cdots, n, \\ &v_i \geq \overline{\varepsilon}, \quad i = 1, \cdots, m. \end{aligned} \tag{13}$$

As in Subsection 2.1, the analysis is constructed in a CRS production technology T^q satisfying free disposability and convexity, and the resulting efficiency measures can be interpreted in terms of Shephard input distance functions. In the MISO setting considered here, the normalisation ensures that the associated distance measures are well defined whenever output is strictly positive. Consequently, the adjacent-period MPI and its CU and FS components retain their standard Malmquist interpretation as ratios and geometric means of distance functions across periods, without requiring further assumptions beyond those already stated in Subsection 2.1.

Using the four period–technology evaluations $\phi^{1,1}(\overline{\varepsilon})$, $\phi^{2,2}(\overline{\varepsilon})$, $\phi^{1,2}(\overline{\varepsilon})$, and $\phi^{2,1}(\overline{\varepsilon})$, the adjacent-period MPI and its decomposition for DMU_o are defined as $\overline{MPI}(\boldsymbol{x}_o, y_o, \overline{\varepsilon}) = \overline{CU}(\boldsymbol{x}_o, y_o, \overline{\varepsilon}) \times \overline{FS}(\boldsymbol{x}_o, y_o, \overline{\varepsilon})$ where $\overline{CU}(\boldsymbol{x}_o, y_o, \overline{\varepsilon}) = \frac{\phi^{1,1}(\overline{\varepsilon})}{\phi^{2,2}(\overline{\varepsilon})}$ and $\overline{FS}(\boldsymbol{x}_o, y_o, \overline{\varepsilon}) = \left(\frac{\phi^{1,2}(\overline{\varepsilon})}{\phi^{1,1}(\overline{\varepsilon})} \times \frac{\phi^{2,2}(\overline{\varepsilon})}{\phi^{2,1}(\overline{\varepsilon})}\right)^{0.5}$.

Similar to Subsection 2.2.1, the non-Archimedean parameter $\overline{\varepsilon}$ acts as a lower bound on the multiplier weights. In the multiplier MISO setting, these weights represent shadow costs of inputs, preventing any input from being treated as free. Changing $\overline{\varepsilon}$ therefore affects the reference weighting and the resulting MPI. The four $\overline{\varepsilon}$-scenarios described below correspond to distinct cost/constraint regimes for efficiency evaluation.

It should be noted that model (13) is always feasible provided $\overline{\varepsilon} > 0$. To examine whether a finite upper bound on $\overline{\varepsilon}$ exists, we consider the following problem:

$$\begin{aligned} &\max \overline{\varepsilon} \\ &\text{s.t.} \\ &\sum_{i=1}^{m} v_i x_{ij}^q \geq \frac{y_j^q}{y_o^p}, \quad j = 1, \cdots, n, \\ &v_i - \overline{\varepsilon} \geq 0, \quad i = 1, \cdots, m. \end{aligned} \tag{14}$$

Theorem 8. *Let $x_{ij}^q > 0$ and $y_j^q > 0$ for all i, j, q, and $y_o^p > 0$. Model (14) is unbounded. Hence, there is no finite upper bound on $\overline{\varepsilon}$, and for every $\overline{\varepsilon} > 0$ there exists a feasible $\boldsymbol{v}$ satisfying the constraints of model (13).*

Proof. See Appendix A.

Although feasibility holds for all positive $\overline{\varepsilon}$, unsuitable choices can produce economically implausible or unreliable weights. To guide practical selection, we adopt the four scenarios paralleling Subsection 2.2.1.

Scenario o.1 (zero epsilon). $\overline{\varepsilon} = 0$ yields the following unrestricted multiplier model:

$$\begin{aligned} &\phi^{pq}(0) = \min \sum_{i=1}^{m} v_i x_{io}^p \\ &\text{s.t.} \\ &\sum_{i=1}^{m} v_i x_{ij}^q \geq \frac{y_j^q}{y_o^p}, \quad j = 1, \cdots, n, \\ &v_i \geq 0, \quad i = 1, \cdots, m. \end{aligned} \tag{15}$$

Model (15) allows maximal flexibility, letting some inputs receive zero weight. While useful for descriptive benchmarking, this can yield economically unreliable multipliers when all inputs are scarce or policy-relevant.

Scenario o.2 (aggregate epsilon). A data-driven lower bound, $\overline{\varepsilon}^*$, for all inputs and periods is computed by formulating the following model which identifies the minimum epsilon (guaranteeing feasibility of all subsequent DEA evaluations) and acts as the upper bound on input weights in model (13):

$$\begin{aligned} &\overline{\varepsilon}^* = \min \overline{\varepsilon} \\ &\text{s.t.} \\ &\sum_{i=1}^{m} v_i x_{ij}^q \geq \min\left\{\frac{y_j^q}{y_k^p} : k = 1, \cdots, n, p = 1, 2\right\}, \quad j = 1, \cdots, n; q = 1, 2, \\ &\overline{\varepsilon} - v_i \geq 0, \quad i = 1, \cdots, m. \end{aligned} \tag{16}$$

The use of the minimum operator ensures that $\overline{\varepsilon}^*$ corresponds to the least restrictive normalisation across all candidate references (k, p), yielding the largest feasible set of input weights consistent with a common lower bound in model (13). This single $\overline{\varepsilon}^*$ imposes a uniform minimum weight across all inputs, DMUs, and periods, ensuring baseline cost attribution for each resource and improving the interpretability and comparability of productivity change measures. In Appendix A, Theorems 9 and 10 establish the positivity, boundedness, and closed-form computation of $\overline{\varepsilon}^*$, respectively.

Scenario o.3 (period-pair epsilon). Period-pair specific minima $\overline{\varepsilon}^{(pq)^*}$ are obtained by solving model (17) with the right-hand side aggregated only over the given pair (p, q):

$$\begin{aligned} &\overline{\varepsilon}^{(pq)^*} = \min \overline{\varepsilon} \\ &\text{s.t.} \\ &\sum_{i=1}^{m} v_i x_{ij}^q \geq \min\left\{\frac{y_j^q}{y_k^p} : k = 1, \cdots, n\right\}, \quad j = 1, \cdots, n, \\ &\overline{\varepsilon} - v_i \geq 0, \quad i = 1, \cdots, m. \end{aligned} \tag{17}$$

This allows the lower bound to adapt to intertemporal changes in

Table 2
Input and output factors for two complementary configurations.[a]

Configurations	Input	Output
Configuration 1	• Labour force (Thousand persons)	• GDP (Million US dollars/ per capita) • Net national income (NNI) (USD)
Configuration 2	• Foreign direct investment (FDI) (USD) • Labour force (Thousand persons)	• GDP (Million US dollars/ per capita)

[a] Although incorporating both volume and ratio inputs and outputs may lead to inconsistencies with the convexity assumption in DEA, the remedies proposed in the recent literature remain controversial and are not comprehensive for all cases (Hatami-Marbini & Toloo, 2019; Olesen et al., 2015; Podinovski et al., 2024). This lack of consensus highlights the complexity of addressing such issues and suggests that further research is needed.

reporting standards, factor scarcity, or policy, producing an MPI interpreted under period-specific minimum cost regimes.

Scenario o.4 (output-oriented ε-based approach). Following Amin (2009a), a DMU- and period-pair specific lower bound $\overline{\varepsilon}_o^{(pq)*}$ is computed using the following model:

$$\overline{\varepsilon}_o^{(pq)*} = \min \overline{\varepsilon}$$

s.t.

$$\sum_{i=1}^{m} v_i x_{ij}^q \geq \frac{y_j^q}{y_o^p}, \quad j = 1, \cdots, n, \tag{18}$$

$$\overline{\varepsilon} - v_i \geq 0, \quad i = 1, \cdots, m.$$

A DMU with a specific $\overline{\varepsilon}_o^{(pq)*}$ reflects local scarcity or binding constraints, ensuring that each input is assigned a meaningful minimum cost. This guarantees that MPI captures true productivity change rather than artefacts from implicitly treating scarce resources as negligible.

Theorem 11 and Lemma 3 (Appendix A) show feasibility of model (18) and establish the ordering $\overline{\varepsilon}_o^{(pq)*} \geq \overline{\varepsilon}^{(pq)*} \geq \overline{\varepsilon}^* > 0$, respectively. Monotonicity holds since stricter lower bounds yield larger (less favorable) efficiency scores (Lemma 4 in Appendix A), and Theorem 12 (Appendix A) identifies a threshold $\overline{\varepsilon}^*$ beyond which no DMU is efficient; in practice, $\overline{\varepsilon}$ should therefore be selected within $(0, \overline{\varepsilon}^*]$. In addition, Theorem 13 (Appendix A) shows that under the aggregate bound $\overline{\varepsilon}^*$, the optimal input-weight vector for model (12) is $\boldsymbol{v}^* = (\overline{\varepsilon}^*, \cdots, \overline{\varepsilon}^*)$.

Under the scenario assumptions, the output-oriented MISO model (12) admits closed-form expressions for CU, FS and MPI when the optimal input multipliers take the uniform form $\boldsymbol{v}^* = (\overline{\varepsilon}^*, \cdots, \overline{\varepsilon}^*)$, as outlined in Theorem 13. The resulting cross-period efficiency comparisons reduce to period-specific benchmark minima, leading to the following closed forms.

$$\overline{\mathrm{CU}}_o = \frac{y_o^2 \min_j \left\{1/y_j^2\right\}}{y_o^1 \min_j \left\{1/y_j^1\right\}}, \overline{\mathrm{FS}}_o = \frac{y_o^1 \sum_i x_{io}^2}{y_o^2 \sum_i x_{io}^1}, \overline{\mathrm{MPI}}_o = \frac{\sum_i x_{io}^2 \min_j \left\{1/y_j^2\right\}}{\sum_i x_{io}^1 \min_j \left\{1/y_j^1\right\}}. \tag{19}$$

These expressions show that CU is DMU-specific as it depends on the evaluated DMU's outputs and the period minima of $1/y_j^t$. FS is also DMU-specific, but it is computed solely from the evaluated DMU's aggregate input use $\sum_i x_{io}^t$ and outputs y_o^t across the two periods; hence FS comparisons are made relative to a common period benchmark, but evaluated at each DMU's own scale.

Once these period minima are computed from the peer set, CU, FS and MPI are obtained by a small number of elementary operations, substantially reducing computational burden.[2] It should be noted that across scenarios o.1–o.4, CU, FS, and MPI measure productivity change conditional on the adopted cost-attribution rule. Stronger restrictions shift the interpretation from flexible benchmarking to productivity assessment under minimum resource accountability, reducing the influence of degenerate input weights. A simple, contrived MISO example that verifies the step-by-step CU, FS and MPI calculations is provided in Appendix B (Example 2). The $\overline{\mathrm{CU}}_o$ factor for DMU under evaluation is calculated through simple elementary arithmetic operations involving $2(n+1)+1$ multiplications and, $2(n-1)$ comparisons. In addition, the $\overline{\mathrm{FS}}_o$ factor is obtained through $2(s-1)$ additions, 3 multiplications. Finally, the $\overline{\mathrm{MPI}}_o$ factor is determined using $2(s-1)$ additions and $2(n+1)+1$ multiplications and $2(n-1)$ comparisons.

Although our focus is on single-input or single-output representations to preserve analytical transparency, the proposed ε-aware framework extends naturally to multiple-input, multiple-output (MIMO) technologies. The same scenario logic continues to control multiplier behaviour—imposing data-driven lower bounds that stabilise weights and make intertemporal comparisons more comparable and interpretable. The economic interpretation of CU and FS is preserved, but the key distinction in the MIMO case is that CU and FS effects arise from joint input–output trade-offs rather than from scalar benchmarks; as a result closed-form simplifications are generally not available and MPI components are obtained by solving the corresponding LPs under the chosen $\varepsilon/\overline{\varepsilon}$ regime. This richer structure increases the framework's applicability to realistic production systems (e.g., firms, hospitals, utilities) while retaining its core logic, providing transparent, implementable rules for choosing $\varepsilon/\overline{\varepsilon}$, clearer economic meaning for multiplier weights, and improved stability of repeated MPI computations.

3. An empirical study

Calculating productivity growth and its components for OECD countries is crucial for understanding economic performance and pinpointing areas for improvement. This analysis offers a detailed examination of productivity dynamics, shedding light on the factors that drive growth and efficiency within these economies. We focus on a set of 18 OECD countries during the period from 2005 to 2021. The dataset is directly collected from the OECD website.[3] To illustrate the proposed methods, we consider two complementary production-function configurations (see Table 2) that enable us to analyse the productivity change of the chosen countries.[4] These are not mere alternative demonstrations on the same panel; rather, they provide two distinct input–output lenses on the same country–year dataset and therefore address different—but related—policy questions. Configuration 1 (SIMO) treats labour as the binding resource and evaluates how efficiently each country converts that constrained input into a portfolio of economic outcomes (GDP per capita and net national income). This SIMO perspective aligns with multi-outcome accountability and with early MPI implementations framed via distance functions (Färe et al., 1994; Ray & Desli, 1997), making it a natural baseline for cross-country comparisons where a single scarce input is the policy focus. Configuration 2 (MISO) treats GDP per capita as a single headline outcome and evaluates how

[2] Full proofs of Theorems 8–13, Lemmas 3–4, and the derivation of (19) appear in Appendix A.

[3] https://data.oecd.org/.

[4] It should be noted that we focus on the most appropriate scenarios since the growth rate of data proportionally changes and results for other technologies are not considerably different.

efficiently a resource mix (labour and foreign direct investment) produces that output. The MISO lens emphasises resource-mix accountability and input substitution and is therefore useful when policy interest centres on which combinations of inputs best deliver a specific target metric.

We adopt Configuration 1 as the primary specification because it reflects the operational logic of our setting—labour as the binding resource driving multiple relevant outputs—and because it directly connects to the distance-function rationale underpinning many MPI analyses. Configuration 2 is retained as a complementary specification and robustness check as it recognises the multi-input nature of economic growth (for example, the role of FDI) and tests whether the main intertemporal patterns and country rankings persist when performance is defined with respect to a single headline output under a multi-input technology. Stability of results across configurations strengthens confidence in the findings; systematic divergences are informative, revealing sensitivity to the chosen production representation and highlighting distinct policy messages from each IO (input–output) perspective.

It should also be noted that the two configurations also operationalise different parts of our theoretical framework. Configuration 1 implements the mechanisms described in Subsection 2.2.1 (SIMO formulations and multi-outcome decompositions), while Configuration 2 implements the constructs of Subsection 2.2.2 (MISO formulations and resource-mix analysis). Our empirical application therefore delivers both methodological breadth—showing the proposed robust DEA machinery works across IO regimes—and substantive insight into how labour dynamics and international investment jointly shape productivity trajectories.

In the empirical analysis, the only parameters varied are the non-Archimedean lower bounds on the multiplier weights, denoted by ε for the single-input case and $\overline{\varepsilon}$ for the single-output case. We evaluate the DEA-based MPI under four ε-scenarios for the single-input setting (S i.1–S i.4) and four $\overline{\varepsilon}$-scenarios for the single-output setting (S o.1–S o.4). For the single-output case, the scenarios are defined as follows: S i.1 with $\varepsilon = 0$, S i.2 with $\varepsilon = \varepsilon^*$ obtained from model (6) ($\varepsilon^* = 2.620 \times 10^{-8}$), S i.3 with $\varepsilon = \varepsilon^{pq*}$ derived from model (8) for each period pair (p, q) (ranges from 2.512×10^{-8} to 4.970×10^{-8} across adjacent periods), and S i.4 with $\varepsilon = \varepsilon_o^{pq*}$ computed from model (9) for each DMU and period pair (p, q) or using the closed-form expression in Theorem 6 (ranges from 1.079×10^{-9} to 4.225×10^{-8}). For the single-output case, the scenarios are S o.1 with $\overline{\varepsilon} = 0$, S o.2 with $\overline{\varepsilon} = \overline{\varepsilon}^*$ determined from model (16) ($\overline{\varepsilon}^* = 8.582 \times 10^{-7}$), S o.3 with $\overline{\varepsilon} = \overline{\varepsilon}^{pq*}$ calculated from model (17) for each period pair (p, q) (ranges from 7.654×10^{-7} to 1.403×10^{-6} across adjacent periods), and S o.4 with $\overline{\varepsilon} = \overline{\varepsilon}_o^{pq*}$ obtained from model (18) for each DMU and period pair (p, q) (ranges from 1.431×10^{-7} to 1.162×10^{-6}).

In the OECD context, adopting the SIMO/MISO perspective is practically meaningful because country-level productivity at the country level is often assessed either relative to a single constrained input (a key resource) and multiple outputs (SIMO), or relative to multiple resources leading to a single key output (MISO). Reporting MPI from both perspectives therefore reveals how countries convert a constrained resource into multiple outputs (SIMO) or combine multiple inputs to produce a single target output (MISO) over time. The $\varepsilon/\overline{\varepsilon}$-based scenarios help ensure that the resulting productivity change measures remain interpretable and do not implicitly treat any factor as non-instrumental.

Table 3
Descriptive Statistics of dataset for 2005 and 2021.

		Inputs		Outputs	
Years	Statistics	LF	FDI	GDP	NNI
2005	Average	23,543.40	560,937.90	1,718,610.72	28,565.38
	Max.	149,320.3	3,637,996	13,039,197	41,711.31
	Min.	2,040.57	11,777.73	106,147.85	20,377.77
	S.D.	35,809.84	836,641.30	3,024,331.89	5,029.01
2021	Average	25,777.26	1,571,104.30	2,977,445.20	47,686.53
	Max.	161,203.9	9,765,936	22,996,100	69,639.74
	Min.	2,546.97	19,107.06	237,788.54	32,791.12
	S.D.	38,307.95	2,260,573.81	5,246,346.89	8,763.76

Table 4
Compound annual growth rate (CAGR) from 2005 to 2021.

Country	LF (%)	FDI (%)	GDP (%)	NNI (%)
Australia	1.660	6.699	4.763	3.270
Austria	0.855	7.697	3.569	3.048
Belgium	0.666	1.978	4.051	3.257
Canada	0.973	7.255	3.166	2.215
Denmark	0.321	5.640	4.319	4.048
Finland	0.359	6.900	3.525	3.216
France	0.574	4.781	3.482	2.933
Germany	0.295	5.082	3.693	3.621
Ireland	1.313	16.768	7.112	3.470
Italy	0.152	3.871	2.625	2.527
Japan	0.223	9.772	1.615	1.444
Netherlands	0.901	10.489	3.557	3.004
New Zealand	1.736	2.887	4.859	3.906
Norway	1.200	2.663	3.967	3.061
South Korea	1.046	12.847	4.163	3.574
Sweden	1.062	−1.167	4.236	3.353
UK	0.665	3.329	3.138	2.288
US	0.451	5.981	3.394	2.707
Average	0.803	6.304	3.846	3.052

Table 3 presents the descriptive statistics of the dataset for the years 2005 and 2021, focusing on inputs (labour force and FDI) and outputs (GDP and NNI). Between 2005 and 2021, the average labour force grew modestly from 23,543.40 to 25,777.26 thousand persons, while FDI increased significantly, rising from an average of $560.9 million to $1,571.1 million. Similarly, GDP and NNI showed notable growth, with GDP increasing from an average of $1,718.6 million to $2,977.4 million and NNI rising from $28,565.38 to $47,686.53. However, the data also exhibits considerable variation, as shown by the high standard deviations, reflecting disparities across countries in both inputs and outputs.

We start by reporting compound annual growth rates (CAGR)[5] of the variables as presented in Table 4. The average CAGR of GDP is 3.846% over the 2005–2021 period. Among the countries, Ireland exhibits the highest GDP CAGR at 7.112%, while Japan records the lowest at 1.615%. Regarding the labour force (LF), the average CAGR is 0.803%, indicating modest growth during the period. New Zealand and Italy show the highest (1.736%) and lowest (0.152%) growth rates, respectively. For FDI, the average CAGR is 6.304%, with Ireland achieving the highest growth rate (16.768%) and Sweden experiencing a decline of −1.167%. Finally, NNI shows an average CAGR of 3.052%, with Denmark leading at 4.048% and Japan recording the lowest growth at 1.444%. These figures provide insights into economic performance and resource utilisation across countries over the period.

Table 5 and Table 6 demonstrate the MPI, CU, and FS measures under different scenarios for Configurations 1 and 2, respectively. The analysis of Table 5 for Configuration 1 across different scenarios reveals that while most countries exhibit stable MPI values around 1, indicating consistent productivity levels, they demonstrate significant advancements in FS with values generally above 1. This suggests that countries are effectively improving their productivity frontiers, with Denmark showing particularly strong FS performance. The CU values exhibit variability, highlighting differing success rates in closing the productivity gap, with countries like Denmark and Norway making notable

[5] CAGR is a variation in the growth rate that is often used to assess the performance of an investment or company. https://www.investopedia.com/terms/c/cagr.asp.

Table 5
Average annual change for the MPI, FS, and CU for Configuration 1.

	Configuration 1											
	S i.1			S i.2			S i.3			S i.4		
Country	CU	FS	MPI	CU	FS	MPI	CU	FS	MPI	CU	FS	MPI
Australia	0.9821	1.0548	1.0330	0.9820	1.0548	1.0328	0.9823	1.0544	1.0329	0.9862	1.0494	1.0325
Austria	0.9777	1.0548	1.0289	0.9778	1.0547	1.0289	0.9778	1.0547	1.0289	0.9819	1.0494	1.0284
Belgium	0.9845	1.0548	1.0360	0.9845	1.0548	1.0360	0.9846	1.0547	1.0360	0.9886	1.0494	1.0355
Canada	0.9728	1.0548	1.0237	0.9726	1.0549	1.0235	0.9732	1.0542	1.0236	0.9770	1.0494	1.0234
Denmark	1.0101	1.0349	1.0410	1.0101	1.0349	1.0410	1.0101	1.0349	1.0410	0.9949	1.0494	1.0424
Finland	1.0025	1.0360	1.0339	1.0025	1.0359	1.0339	1.0025	1.0359	1.0339	0.9865	1.0494	1.0336
France	0.9798	1.0548	1.0311	0.9795	1.0550	1.0310	0.9806	1.0539	1.0311	0.9844	1.0494	1.0311
Germany	0.9847	1.0548	1.0364	0.9844	1.0551	1.0363	0.9859	1.0535	1.0365	0.9894	1.0494	1.0364
Ireland	1.0090	1.0447	1.0529	1.0090	1.0446	1.0529	1.0090	1.0446	1.0529	1.0099	1.0494	1.0590
Italy	0.9756	1.0548	1.0269	0.9754	1.0549	1.0268	0.9763	1.0540	1.0269	0.9802	1.0494	1.0268
Japan	0.9644	1.0548	1.0151	0.9639	1.0554	1.0150	0.9664	1.0527	1.0153	0.9691	1.0494	1.0151
Netherlands	0.9768	1.0548	1.0284	0.9768	1.0548	1.0283	0.9770	1.0545	1.0283	0.9812	1.0494	1.0281
New Zealand	1.0074	1.0236	1.0234	1.0073	1.0237	1.0235	1.0074	1.0236	1.0235	0.9857	1.0494	1.0315
Norway	1.0000	1.0274	1.0274	1.0000	1.0275	1.0275	1.0000	1.0275	1.0275	0.9824	1.0494	1.0309
South Korea	0.9815	1.0548	1.0331	0.9812	1.0549	1.0329	0.9821	1.0540	1.0330	0.9862	1.0494	1.0330
Sweden	0.9819	1.0548	1.0338	0.9819	1.0547	1.0338	0.9820	1.0546	1.0338	0.9859	1.0494	1.0331
UK	0.9753	1.0548	1.0266	0.9749	1.0550	1.0264	0.9761	1.0538	1.0265	0.9798	1.0494	1.0264
US	0.9800	1.0548	1.0314	0.9781	1.0564	1.0307	0.9847	1.0494	1.0314	0.9847	1.0494	1.0314

Table 6
Average annual change for the MPI, FS, and CU for Configuration 2.

	Configuration 2											
	S o.1			S o.2			S o.3			S o.4		
Country	CU	FS	MPI	CU	FS	MPI	CU	FS	MPI	CU	FS	MPI
Australia	0.9986	0.9785	0.9764	1.0201	0.9805	1.0000	1.0076	0.9817	0.9889	1.0148	1.0314	1.0471
Austria	1.0053	0.9785	0.9831	1.0067	0.9799	0.9861	1.0052	0.9800	0.9847	1.0020	1.0478	1.0500
Belgium	0.9970	0.9705	0.9661	1.0005	0.9853	0.9856	0.9873	0.9868	0.9742	1.0068	0.9836	0.9903
Canada	1.0102	0.9759	0.9856	1.0489	0.9873	1.0356	1.0264	0.9894	1.0159	0.9979	1.0553	1.0520
Denmark	1.0115	0.9508	0.9597	1.0271	0.9891	1.0158	1.0051	0.9917	0.9963	1.0098	1.0171	1.0274
Finland	1.0003	0.9761	0.9751	1.0019	0.9786	0.9795	1.0007	0.9787	0.9783	1.0043	1.0326	1.0373
France	1.0000	0.9924	0.9924	1.0000	0.9940	0.9940	1.0000	0.9939	0.9939	1.0003	1.0032	1.0033
Germany	0.9945	0.9783	0.9725	1.0266	0.9895	1.0157	1.0052	0.9917	0.9965	1.0025	1.0160	1.0188
Ireland	0.9953	0.9568	0.9515	1.0408	0.9840	1.0238	1.0253	0.9862	1.0106	1.0400	1.1113	1.1507
Italy	1.0032	0.9765	0.9793	1.0157	0.9824	0.9978	1.0037	0.9835	0.9871	0.9911	1.0161	1.0073
Japan	1.0199	0.9749	0.9939	1.0569	0.9879	1.0439	1.0389	0.9901	1.0281	0.9813	1.0736	1.0536
Netherlands	1.0165	0.9658	0.9807	1.0750	0.9893	1.0639	1.0473	0.9924	1.0407	1.0005	1.0857	1.0867
New Zealand	0.9970	0.9724	0.9692	0.9970	0.9724	0.9692	0.9970	0.9724	0.9692	1.0156	0.9818	0.9970
Norway	1.0008	0.9738	0.9754	0.9993	0.9784	0.9786	0.9980	0.9783	0.9773	1.0081	0.9918	0.9939
South Korea	1.0025	0.9742	0.9761	1.0155	0.9778	0.9925	1.0105	0.9786	0.9883	1.0084	1.0725	1.0816
Sweden	0.9994	0.9723	0.9701	0.9911	0.9813	0.9722	0.9838	0.9821	0.9658	1.0089	1.3809	1.3895
UK	1.0040	0.9746	0.9776	1.0270	0.9908	1.0173	1.0018	0.9938	0.9951	0.9950	1.0180	1.0127
US	1.0000	0.9780	0.9780	1.0799	1.0000	1.0799	1.0375	1.0048	1.0427	1.0000	1.0427	1.0427

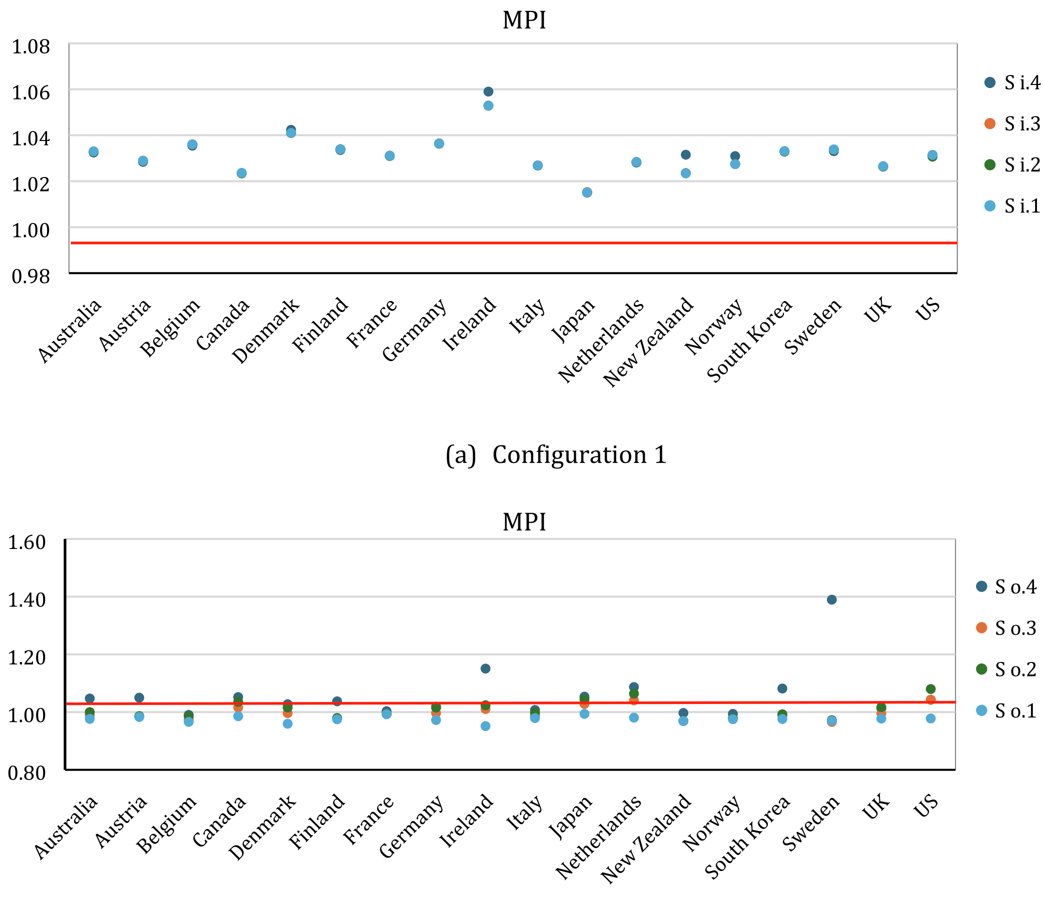
progress. Overall, the findings illustrate a robust pattern of technological advancement and the potential for countries to enhance productivity through strategic policies considering a suitable value for epsilon. Under the closed-form expressions in Equation (11), the FS component is a period-specific, technology-wide factor that depends only on benchmark scalars (e.g., period aggregates/minima) and is therefore identical across DMUs for a given adjacent-period comparison. Consequently, the identical FS values across countries in Table 5 reflect a common frontier movement, while cross-country variation is primarily captured by CU.

Table 6 reports the results across different scenarios and reveals varying trends in the MPI, FS, and CU for the countries examined for Configuration 2. In the initial scenarios (S o.1 and S o.2), several countries exhibit reduced productivity (MPI<1), alongside FS values below 1, indicating limited outward movement of the frontier on average. In S o.3, some countries (e.g., Canada and the Netherlands) display modest recovery with MPI values above 1, reflecting a combination of CU and mild FS improvement. In S o.4, the results indicate a markedly stronger FS component for several countries; in particular, Sweden exhibits a noticeable FS (FS=1.3809) and the highest productivity growth (MPI=1.3895), while Ireland also shows substantial productivity improvement (MPI=1.1507). Overall, these scenarios highlight the dynamic nature of productivity change under different epsilon conventions and emphasise the importance of selecting a suitable lower bound for weights when estimating MPI and its components.

Fig. 1 demonstrates the scattering of MPIs for Configurations 1 and 2 under four different scenarios for each configuration. Fig. 1(a) for Configuration 1 displays a range of values from 0.99 to 1.07, with a notable concentration around 1. The values are organised in a descending order from Si.1 to Si.4, indicating a systematic increase in the measurements. This configuration suggests a stable or slightly increasing trend, with Si.1 being the lowest and Si.4 the highest, reflecting a gradual progression in the observed parameters.

In Fig. 1 (b) for Configuration 2, the MPI values range from 0.94 to 1.39, showing wider spread compared to Configuration 1. Scenario S o.4 shifts the distribution upward (many MPI values exceed 1), whereas S o.1–S o.3 remain more tightly clustered around 1 along with several values below 1. This pattern indicates greater scenario sensitivity in Configuration 2 and highlights that the choice of the epsilon-based scenario can materially affect cross-country productivity comparisons.

We test the consistency (correlation) and goodness of fit among the four scenarios in both Configurations 1 and 2 by various non-parametric statistical tests (Siegel & Castellan, 1988). The first non-parametric test



(a) Configuration 1

(b) Configuration 2

Fig. 1. Scatterplots of the MPIs for both configurations.

Table 7
Wilcoxon-Mann-Whitney test for Configuration 1 at the country level.

	MPI					
	Si.1/Si.2	Si.1/Si.3	Si.1/Si.4	Si.2/Si.3	Si.2/Si.4	Si.3/Si.4
z	0.28475	0.15819	−0.09491	−0.34802	−0.37966	−0.14237
p-value	0.77583	0.87430	0.92438	0.72782	0.70419	0.88678
	CU					
	Si.1/Si.2	Si.1/Si.3	Si.1/Si.4	Si.2/Si.3	Si.2/Si.4	Si.3/Si.4
z	0.14237	−0.45875	−0.94915	−0.61695	−1.07571	−0.64859
p-value	0.88678	0.64640	0.34254	0.53726	0.28205	0.51660
	FS					
	Si.1/Si.2	Si.1/Si.3	Si.1/Si.4	Si.2/Si.3	Si.2/Si.4	Si.3/Si.4
z	−0.60113	2.68928	2.27797	2.72091	2.27797	1.99323
p-value	0.54775	0.00716*	0.02272*	0.00651*	0.02272*	0.04623*

* Indicates significant at the 5%

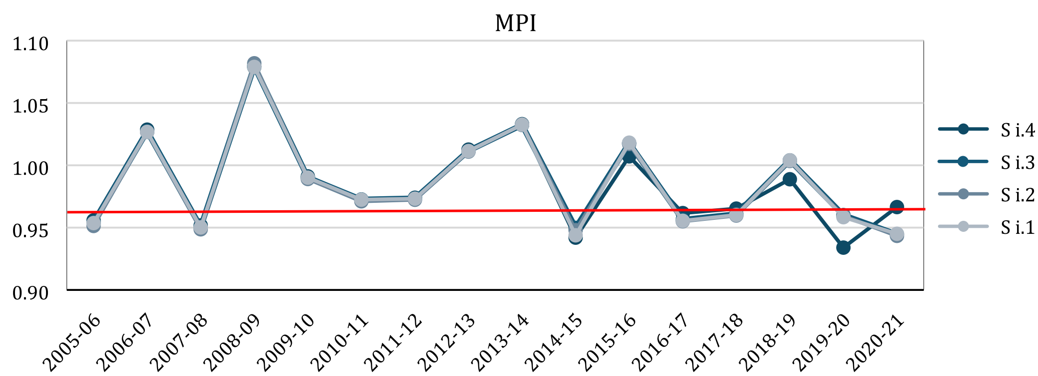
is the *Wilcoxon–Mann–Whitney* test, which is used to compare differences between four independent scenarios.[6] This approach follows prior DEA studies that have used rank-based tests to compare estimated scores across groups (Banker et al., 2010; Simpson, 2005). It should be noted that we implement this test for the CU, FS, and MPI. The results of the *Wilcoxon–Mann–Whitney* test at the 5% significance level for Configurations 1 and 2 are presented in Table 7 and Table 8, respectively. Focusing on Configuration 1′s results in Table 7, we find that for the MPI, the *z*-values across all pairwise comparisons are close to zero, and the corresponding *p*-values exceed 0.7, indicating no statistically significant differences between the subsets. Similarly, for the CU, the *z*-values remain small, and the *p*-values are all above 0.2820, confirming the absence of statistically significant differences between the subsets. In contrast, the FS reveals significant differences in several pairwise comparisons. These results suggest that while the MPI and CU exhibit consistency across the subsets, whereas the estimated FS scores exhibit variability with statistically significant differences in several

[6] Because DEA efficiency scores and MPIs are non-parametric estimators that may exhibit finite-sample bias and slow convergence (Daraio & Simar, 2007; Simar & Wilson, 1998), we treat the Wilcoxon–Mann–Whitney results as exploratory evidence on the distribution of estimated scores rather than as formal inference on the underlying *true* efficiencies.

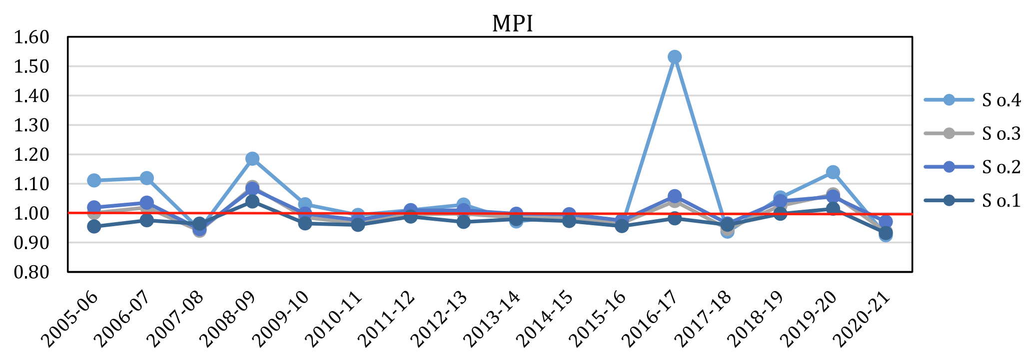
Table 8
Mann-Whitney test for Configuration 2 at the country level.

	MPI					
	So.1/So.2	So.1/So.3	So.1/So.4	So.2/So.3	So.2/So.4	So.3/So.4
z	-3.71753	-3.05312	-5.04668	1.15480	-2.59437	-3.70219
p-value	0.00020*	0.002264*	0.000001*	0.24816	0.00947*	0.00021*
	CU					
	So.1/So.2	So.1/So.3	So.1/So.4	So.2/So.3	So.2/So.4	So.3/So.4
z	-2.49944	-1.26554	-0.66471	1.48701	2.07273	0.52210
p-value	0.01243*	0.20567	0.50624	0.13701	0.03820*	0.60160
	FS					
	So.1/So.2	So.1/So.3	So.1/So.4	So.2/So.3	So.2/So.4	So.3/So.4
z	-4.06556	-4.22375	-5.03086	-0.68022	-4.39777	-4.27149
p-value	0.00004*	0.000024*	0.000000*	0.49635	0.000010*	0.000019*

*Indicates significant at the 5%.



(a) Configuration 1



(b) Configuration 2

Fig. 2. The MPI and its components for two scenarios.

comparisons.

Let us focus on the results for Configuration 2 in Table 8. For MPI, several pairwise comparisons exhibit statistically significant differences, with the smallest p-values observed for S o.1/S o.4 ($p \approx 0.000001$) and S o.3/S o.4 (p=0.00021), indicating significant variations in productivity change estimates involving S o.4. For CU, significant differences are detected for S o.1/S o.2 (p = 0.01243) and S o.2/S o.4 (p=0.03820), whereas S o.1/S o.4 is no longer significant (p=0.50624). For FS, significant differences are consistently observed across most pairwise comparisons, including S o.1/S o.4, S o.2/S o.4, and S o.1/S o.3, while S o.2/S o.3 remains non-significant (p=0.49635). Overall, these results suggest that FS displays the highest degree of variability among the components, followed by MPI and CU, with several pairwise comparisons achieving significance at the 5% level.

It could be beneficial to assess the goodness of fit between each pair of methods using Spearman's rank correlation test. For this purpose, the ranks derived from the four scenarios for Configurations 1 and 2 are considered, with focus on the MPI, to evaluate the correlation between them through Spearman's correlation coefficient (r) and the corresponding p-values.

For Configuration 1, all pairwise comparisons exhibit exceptionally strong positive correlations (values close to 1), indicating a near-perfect monotonic relationship among the subsets. Specifically, the correlation coefficients r range from 0.9195 to 1. All p-values are $p < 0.05$, confirming the statistical significance of these correlations at the 5% level. This suggests a high level of consistency in MPI values across the subsets for Configuration 1.

In Configuration 2, the correlations are more varied, indicating differences in the monotonic relationships between subsets. The correlation coefficients r range from 0.1290 to 0.9732. Notably, the highest correlation is observed between So.2 and So.3 ($r = 0.9731, p < 0.05$), which is statistically significant. Other pairwise comparisons, such as So.1/So.3 ($r = 0.4055$) and So.1/So.2 ($r = 0.3952$), demonstrate moderate correlations, but their p-values ($p > 0.05$) indicate a lack of statistical significance. The weakest correlation is observed between So.1 and So.4 ($r = 0.1290, p = 0.6091$), reflecting minimal monotonicity and no statistical significance.

Overall, the results indicate that MPI values under Configuration 1 are highly consistent across subsets, while those under Configuration 2 show greater variability, with significant monotonic relationships only for select comparisons. This highlights potential differences in the performance patterns among the different scenarios for both Configurations 1 and 2.

Fig. 2 illustrates the MPI values for Configurations 1 and 2, focusing on subsets S i.1 to S i.4 and S o.1 to S o.4.

In Configuration 1 (Fig. 2a), the MPI values for S i.1 to S i.4 range from approximately 0.90 to 1.10, indicating relatively stable productivity changes across scenarios. The highest MPI is observed for S i.4, reaching about 1.08, while the lowest score is recorded for S i.1, just below 0.94. Scenarios S i.2 and S i.3 show scores clustered around 0.96 to 0.98, suggesting minor differences in productivity changes among these scenarios.

In Configuration 2 (Fig. 2b), the MPI values exhibit greater variability over time compared to Configuration 1. The series for S o.4 displays the highest peaks (up to 1.53) and the largest dispersion, including a noticeable spike (consistent with the Sweden-driven frontier expansion). In contrast, S o.1–S o.3 remain closer to 1, with smaller deviations and several periods below 1. These results confirm that productivity dynamics in Configuration 2 are highly scenario-dependent, highlighting the importance of selecting a suitable lower bound for the multiplier weights.

These results suggest that productivity changes are more uniform across scenarios in Configuration 1, whereas Configuration 2 exhibits significant variability in productivity improvements across its scenarios. This indicates potential differences in performance dynamics under the two configurations under different scenarios and the impact of selecting a suitable lower bound for the inputs and outputs weights.

4. Concluding remarks

In conventional DEA and productivity analysis, production units are often characterised by multiple inputs and multiple outputs, with weights constrained to be greater than or equal to an exceedingly small bound referred to as *non-Archimedean epsilon*. However, relatively limited attention has been paid to efficiency and productivity analysis in special yet practically relevant situations in which a single input produces multiple outputs (SIMO) or multiple inputs are consumed to produce a single output (MISO). These SIMO and MISO configurations arise naturally in many empirical contexts, particularly at the aggregate and country levels.

This study presents a new decomposition framework for measuring efficiency and productivity growth in SIMO and MISO settings without the need to solve optimisation models. The framework introduces four epsilon-based scenarios, supported by theorems that define and justify the selection of the most appropriate non-Archimedean epsilon value. An empirical analysis of 18 OECD countries over the period 2005–2021 demonstrates that the proposed framework yields meaningful productivity change measures while significantly reducing the computational burden. This feature is particularly valuable in data-intensive environments, where efficiency and productivity analysis must be conducted repeatedly or in near real time.

Despite these contributions, the study has several limitations. The proposed framework is developed specifically for SIMO/MISO settings and relies on multiplier-based MPI estimation, and the empirical results may be influenced by data quality and modelling choices such as the assumed returns to scale and orientation. Moreover, the OECD application focuses on adjacent-period MPI computations, and the findings may not directly generalise to other datasets or institutional contexts. In addition, the analysis adopts a non-parametric DEA-based perspective and does not include a direct methodological comparison with parametric benchmarking approaches such as Stochastic Frontier Analysis (SFA). While such a comparison could provide further insights, it would require reconciling fundamentally different assumptions regarding functional form, noise, and inefficiency, and would substantially extend the scope and length of the present study.

These limitations point naturally to several directions for future research. First, extending the analysis to alternative dynamic constructions, such as global MPI (Pastor & Lovell, 2005), biennial MPI (Pastor et al., 2011), or window-based approaches, warrants further investigation. Second, the proposed framework could be extended beyond the constant returns to scale (CRS) assumption to accommodate alternative technologies, such as variable returns to scale (VRS). Third, future studies may explore the application of the framework to other productivity indices, including non-oriented Hicks-Moorsteen productivity measures, which could provide complementary insights into productivity dynamics. Fourth, integrating the proposed approach with other DEA-based productivity frameworks, such as network DEA, may further broaden its applicability and practical relevance. Finally, a dedicated comparative assessment of the proposed framework against parametric methods such as SFA represents a promising avenue for future research. Addressing these extensions would strengthen the generality of the framework and broaden its usefulness for efficiency and productivity analysis across diverse empirical settings.

CRediT authorship contribution statement

Mehdi Toloo: Writing – original draft, Validation, Software, Methodology, Funding acquisition, Conceptualization. **Adel Hatami-Marbini:** Writing – review & editing, Supervision, Methodology, Formal analysis, Conceptualization. **Aliasghar Arabmaldar:** Validation, Software, Methodology, Data curation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A

The proof of Theorem 8.

We show that model (14) is feasible and its dual problem is infeasible; by the fundamental duality theorem of LP (Bazaraa et al., 2010), this implies model (14) has no finite optimal solution and is therefore unbounded. Without loss of generality, assume that $\frac{y_k^q}{\sum_{i=1}^m x_{ik}^q} = \max\left\{\frac{y_j^q}{\sum_{i=1}^m x_{ij}^q} : j = 1, \cdots, n\right\}$. *A trivial computation reveals that the vector* $(\varepsilon^0, \boldsymbol{v}^0) = \left(\frac{y_k^q}{y_o^p\left(\sum_{i=1}^m x_{ik}^q\right)}, \frac{y_k^q}{y_o^p\left(\sum_{i=1}^m x_{ik}^q\right)}, \cdots, \frac{y_k^q}{y_o^p\left(\sum_{i=1}^m x_{ik}^q\right)}\right)$ *is a feasible solution for model (14). Consider the following dual of model (14):*

$$
\begin{aligned}
& \min \frac{1}{y_o^p} \sum\nolimits_{j=1}^n y_j^q \lambda_j \\
& \text{s.t.} \\
& \sum\nolimits_{j=1}^n \lambda_j x_{ij}^q + s_i \geq 0, \quad i = 1, \cdots, m, \\
& -\sum\nolimits_{i=1}^m s_i \geq 1, \\
& s_i \leq 0, \quad i = 1, \cdots, m, \\
& \lambda_j \leq 0, \quad j = 1, \cdots, n.
\end{aligned}
\tag{A1}
$$

Given that $\lambda_j \leq 0$, *for all* j, *and* $x_{ij}^q > 0$, *we have* $\lambda_j x_{ij}^q \leq 0$, *for all* j, i. *Summing over* i *and* j *yields* $\sum_{i=1}^m \sum_{j=1}^n \lambda_j x_{ij}^q \leq 0$. *From the first and second constraints of model* (A1), *it follows that* $1 \leq -\sum_{i=1}^m s_i \leq \sum_{i=1}^m \sum_{j=1}^n \lambda_j x_{ij}^q \leq 0$, *which is a contradiction. Hence, model* (A1) *is infeasible, which completes the proof.* ■

Theorem 9. *Let* $x_{ij}^q \geq 0$ *for all* i, j, q, *and* $y_j^q > 0$ *for all* j, q. *If* $(\overline{\varepsilon}^*, \boldsymbol{v}^*)$ *is an optimal solution of model (15), then,* $\overline{\varepsilon}^*$ *is (i) strictly positive, and (ii) bounded.*

Proof. **(*i*)** *Suppose on the contrary, that* $\overline{\varepsilon}^* = 0$. *From the second set of constraints of model (16) we have* $v_i^* \leq 0$ *for* $i = 1, \cdots, m$. *Then, considering the first set of constraints for* $j = 1, \cdots, n$, $q = 1, 2$, *and using the fact that* $x_{ij}^q \geq 0$, *it follows that* $\sum_{i=1}^m v_i^* x_{ij}^q \leq 0$. *However, the same constraints require* $\sum_{i=1}^m v_i^* x_{ij}^q \min\left\{\frac{y_j^q}{y_k^p} : k = 1, \cdots, n, p = 1, 2\right\}$. *Since* $y_j^q > 0$ *for all* j, q, *the right-hand side is strictly positive, implying* $\min\left\{\frac{y_j^q}{y_k^p} : k = 1, \cdots, n, \ p = 1, 2\right\} > 0$, *which yields a contradiction. Therefore,* $\overline{\varepsilon}^* > 0$.

Proof. **(*ii*)** *To prove that the optimal objective value* $\overline{\varepsilon}^*$ *of model (16) is bounded, it is a need to present a feasible solution with a positive objective value. For a fixed* j, *multiplying the second set of constraints in model (16) by the non-negative value scalar* x_{ij}^q *and sum over* $i = 1, \cdots, m$. *This gives* $\sum_{i=1}^m \overline{\varepsilon}^* x_{ij}^q \geq \sum_{i=1}^m v_i^* x_{ij}^q$. *Next, by the first set of constraints of model (16), we have* $\sum_{i=1}^m \overline{\varepsilon}^* x_{ij}^q \geq \sum_{i=1}^m v_i^* x_{ij}^q \geq \min\left\{\frac{y_j^q}{y_k^p} : k = 1, \cdots, n, \ p = 1, 2\right\}$ *for* $j = 1, \cdots, n$, $q = 1, 2$. *As a result,* $\overline{\varepsilon}^* \geq \max\left\{\frac{1}{\sum_{i=1}^m x_{ij}^q}\left(\min\left\{\frac{y_j^q}{y_k^p} : k = 1, \cdots, n, \ p = 1, 2\right\}\right) : \ j = 1, \cdots, n; \ q = 1, 2\right\} > 0$, *which completes the proof.* ■

Theorem 10. *Let* $x_{ij}^q \geq 0$ *for all* i, j, q, *and* $y_j^q > 0$ *for all* j, q. *Then, the optimal objective value of model (16) is* $\overline{\varepsilon}^* = \frac{y_j^q}{\sum_{i=1}^m x_{ij}^q} \min\left\{\frac{1}{y_k^p} : k = 1, \cdots, n; p = 1, 2\right\}$.

Proof. *We first multiply the last set of constraints of model (16), namely* $\overline{\varepsilon}^* - v_i^* \geq 0$, *by the non-negative scalar* x_{ij}^q *for a fixed* j *and* q, *and then sum the resulting constraints over* $i = 1, \cdots, m$, *which yields* $\overline{\varepsilon}^* \sum_{i=1}^m x_{ij}^q - \sum_{i=1}^m v_i^* x_{ij}^q \geq 0$. *Considering the first set of constraints of model (16), i.e.,* $\sum_{i=1}^m v_i^* x_{ij}^q \geq \min\left\{\frac{y_j^q}{y_k^p} : k = 1, \cdots, n; p = 1, 2\right\}$, $j = 1, \cdots, n$; $q = 1, 2$, *it follows that* $\overline{\varepsilon}^* \sum_{i=1}^m x_{ij}^q \geq \min\left\{\frac{y_j^q}{y_k^p} : k = 1, \cdots, n; p = 1, 2\right\}$, $j = 1, \cdots, n$; $q = 1, 2$. *Rearranging, and using* $\sum_{i=1}^m x_{ij}^q > 0$, *we obtain* $\overline{\varepsilon}^* \geq \frac{y_j^q}{\sum_{i=1}^m x_{ij}^q} \min\left\{\frac{1}{y_k^p} : k = 1, \cdots, n; \ p = 1, 2\right\}$, $j = 1, \cdots, n$; $q = 1, 2$, *and hence for any feasible solution of model (16)* $(\overline{\varepsilon}, \overline{\boldsymbol{u}})$ *we have* $\overline{\varepsilon} \geq \frac{y_j^q}{\sum_{i=1}^m x_{ij}^q} \min\left\{\frac{1}{y_k^p} : k = 1, \cdots, n; \ p = 1, 2\right\}$, $j = 1, \cdots, n$; $q = 1, 2$. *It is therefore sufficient to exhibit a feasible solution of model (16) achieving this bound. Let* $\varepsilon^0 = \frac{y_j^q}{\sum_{i=1}^m x_{ij}^q} \min\left\{\frac{1}{y_k^p} : k = 1, \cdots, n; \ p = 1, 2\right\}$, *and set* $\boldsymbol{v}^0 = \overline{\boldsymbol{\varepsilon}}^0 = (\overline{\varepsilon}^0, \cdots, \overline{\varepsilon}^0) \in \mathbb{R}^m$. *A direct verification shows that* $(\overline{\varepsilon}^0, \boldsymbol{v}^0)$ *satisfies all constraints of model (16). Therefore, the lower bound equals the optimal value, which completes the proof.* ■

Theorem 11. *Let* $x_{ij}^q \geq 0$ *for all* i, j, q, *and* $y_j^q > 0$ *for all* j, q. *Then, model (18) is always feasible.*

Proof. *It is sufficient to present a feasible solution model (18). Let* $v_i' = \frac{1}{m \times \min\left\{x_{ij}^q ; j = 1, \cdots, n\right\}}\left(\max\left\{\frac{y_j^q}{y_o^p} : \forall j, \ p, \ q\right\}\right)$, $i = 1, .., m$. *It is easy to verify that* $v_i' > 0$, $\forall r$. *Moreover,* $\sum_{i=1}^m v_i' x_{ij}^q = \max\left\{\frac{y_j^q}{y_o^p} : \forall j, \ p, \ q\right\} \sum_{i=1}^m \frac{x_{ij}^q}{m \times \min\left\{x_{ij}^q ; j = 1, \cdots, n\right\}} \geq \max\left\{\frac{y_j^q}{y_o^p} : \forall j, \ p, \ q\right\} \geq \frac{y_j^q}{y_o^p}$ *for* $j = 1, \cdots, n$ *and* $q = 1, 2$. *Let* $\varepsilon' = \max\{v_i' : i = 1, \cdots, m\}$. *Then, the vector* $(\varepsilon', \boldsymbol{v}')$ *satisfies all constraints of model (18), confirming feasibility, which completes the proof.* ■

Lemma 3. *Fix* $p, q \in \{1, 2\}$. *Let* $\overline{\varepsilon}^*$, $\overline{\varepsilon}^{(pq)*}$, *and* $\overline{\varepsilon}_o^{(pq)*}$ *be the optimal objective values of models (16), (17),* and *(18), respectively, for the same fixed* (p, q) *(and* *a fixed reference o where applicable). Then,* $\bar{\varepsilon}_o^{pq^*} \ge \bar{\varepsilon}^{pq^*} \ge \bar{\varepsilon}^* > 0$.

Proof. *Let* $S_{(16)}$, $S_{(17)}$, *and* $S_{(18)}$ *be the feasible region of models* (16), (17), *and* (18), *respectively, for given* p, q. *It is plain to verify that* $S_{(18)} \subseteq S_{(17)} \subseteq S_{(16)}$. *Since all three models minimise the same scalar variable* $\bar{\varepsilon}$, *this nesting of feasible regions directly implies* $\bar{\varepsilon}^* \le \bar{\varepsilon}^{(pq)^*} \le \bar{\varepsilon}_o^{(pq)^*}$. *Moreover, unlike traditional DEA models where their objective function is* $\max\min\{\boldsymbol{v}, \boldsymbol{u}\}$, *model* (16) *is formulated as* $\min\max\{\boldsymbol{v}, \boldsymbol{u}\}$ *in which* $\bar{\varepsilon}$ *acts as a common lower bound on all input weights. As a result, we have* $\bar{\varepsilon}^* = \max\{\boldsymbol{v}^*\}$ *rather than* $\min\{\boldsymbol{v}^*\}$. *Accordingly, the zero vector* $(\bar{\varepsilon}, \bar{\boldsymbol{v}}) = (0, 0_m)$ *cannot satisfy the first set of constraints of model* (16) *and is therefore infeasible. Hence,* $\bar{\varepsilon}^* = \max\{\boldsymbol{v}^*\} > 0$, *which completes the proof.* ■

Lemma 4. *Fix* $p, q \in \{1, 2\}$, *and let* $\phi^{pq}(\bar{\varepsilon})$ be the optimal objective value of model (13) *when the lower bound on the relevant multipliers is set to* $\bar{\varepsilon}$. *Assume model* (13) *is feasible for* $\bar{\varepsilon} \in \{0, \bar{\varepsilon}^*, \bar{\varepsilon}^{pq^*}, \bar{\varepsilon}_o^{pq^*}\}$. *Then,* $\phi^{pq}(\bar{\varepsilon}_o^{pq^*}) \ge \phi^{pq}(\bar{\varepsilon}^{pq^*}) \ge \phi^{pq}(\bar{\varepsilon}^*) > \phi^{pq}(0)$.

Proof. *Let* $S_{(13)}(\bar{\varepsilon})$ *denote the feasible region of model* (13) *under the lower bound* $\bar{\varepsilon}$. *By* Lemma 3, *we have* $\bar{\varepsilon}_o^{(pq)^*} \ge \bar{\varepsilon}^{(pq)^*} \ge \bar{\varepsilon}^* > 0$, *which implies the nesting* $S_{(13)}\left(\bar{\varepsilon}_o^{(pq)^*}\right) \subseteq S_{(13)}\left(\bar{\varepsilon}^{(pq)^*}\right) \subseteq S_{(13)}(\bar{\varepsilon}^*) \subseteq S_{(13)}(0)$. *Since model* (13) *is a minimisation problem, shrinking the feasible region weakly decreases the optimal objective value, yielding the stated inequalities. The strict inequality* $\phi^{pq}(\bar{\varepsilon}^*) > \phi^{pq}(0)$ *follows from* $\bar{\varepsilon}^* > 0$, *which imposes a strictly tighter lower bound than the case* $\bar{\varepsilon} = 0$. ■

Theorem 12. *Let* $\bar{\varepsilon}^* = \max\left\{\frac{1}{\sum_{i=1}^m x_{ij}^p} : j = 1, \cdots, n; p = 1, 2\right\}$. *Then, for* $\bar{\varepsilon} \in (\bar{\varepsilon}^*, \infty)$, *no DMU is efficient. In other words,* $\phi^{1,1}(\boldsymbol{u}, \bar{\varepsilon}) > 1$ *and* $\phi^{2,2}(\boldsymbol{u}, \bar{\varepsilon}) > 1$ *for all* $\bar{\varepsilon} \in (\bar{\varepsilon}^*, \infty)$.

Proof. *Without loss of generality, assume that* $\max\left\{\frac{1}{\sum_{i=1}^m x_{ij}^1} : j = 1, \cdots, n\right\} = \max\left\{\frac{1}{\sum_{i=1}^m x_{ij}^p} : j = 1, \cdots, n;\ p = 1, 2\right\}$. *If* $\bar{\varepsilon} > \max\left\{\frac{1}{\sum_{i=1}^m x_{ij}^1} : j = 1, \cdots, n\right\}$, *then* $\bar{\varepsilon} > \frac{1}{\sum_{i=1}^m x_{ij}^1}$ *for* $j = 1, \cdots, n$ *or equivalently* $1 < \bar{\varepsilon}\sum_{i=1}^m x_{ij}^1$ *for* $j = 1, \cdots, n$. *On the other hand, by multiplying the second constraint set* $\bar{\varepsilon} \le v_i$ *of model* (13) *by the non-negative values* x_{ij}^1 *for a fixed* j, *and summing over* $i = 1, \cdots, m$, *we obtain* $\bar{\varepsilon}\sum_{i=1}^m x_{ij}^1 \le \sum_{i=1}^m v_i x_{ij}^1$. *Combining the above inequalities yields* $1 < \bar{\varepsilon}\sum_{i=1}^m x_{ij}^1 \le \sum_{i=1}^m v_i x_{ij}^1 \le \sum_{i=1}^m v_i x_{ij}^2$. *Hence, the objective function value of model* (13) *for any feasible solution is greater than* 1. *Since model* (13) *is a minimisation problem, its optimal value* $\sum_{i=1}^m v_i^* x_{ij}^p$ *must also exceed* 1, *implying that no DMU is efficient for* $\bar{\varepsilon} \in (\bar{\varepsilon}^*, \infty)$. ■

Theorem 13. *The optimal solution of model* (13) *is* $\boldsymbol{v}^* = (\bar{\varepsilon}^*, \cdots, \bar{\varepsilon}^*)$.

Proof. *It is easy to verify that* $(\bar{\varepsilon}^*, \cdots, \bar{\varepsilon}^*)$ *is a feasible solution of model* (13). *Consider the dual problem of model* (13) *as follows:*

$$\begin{aligned}
&\max \frac{1}{y_o^p}\left(\sum_{j=1}^n \lambda_j y_j^q\right) + \bar{\varepsilon}^*\left(\sum_{i=1}^m s_i^+\right) \\
&\text{s.t.} \\
&\sum_{j=1}^n \lambda_j x_{ij}^q + s_i^+ = x_{io}^p, \quad i = 1, \cdots, m, \\
&\lambda_j \ge 0, \quad j = 1, \cdots, n, \\
&s_i^+ \ge 0, \quad i = 1, \cdots, m.
\end{aligned} \tag{A2}$$

Without loss of generality, let $\frac{1}{y_l^p} = \min\left\{\frac{1}{y_k^p} : k = 1, \cdots, n;\ p = 1, 2\right\}$. *Then, define the feasible dual solution* $(\lambda^*, \boldsymbol{s}^{+^*})$ *of model* (A2) *as*

$$\lambda_j^* = \begin{cases} \min\left\{\dfrac{x_{io}^p}{x_{ij}^q} : i = 1, \cdots, m\right\}, & \text{if } j = l \\ 0, & \text{otherwise} \end{cases}$$

$$s_i^{+^*} = x_{io}^p - x_{il}^q\left(\min\left\{\frac{x_{io}^p}{x_{ij}^q} : i = 1, \cdots, m\right\}\right), \quad i = 1, \cdots, m, \text{ and } U = \min\left\{\frac{x_{io}^p}{x_{ij}^q} : i = 1, \cdots, m\right\}.$$

The corresponding dual objective function value is $\frac{y_j^q}{y_o^p}\left(\min\left\{\frac{x_{io}^p}{x_{ij}^q} : i = 1, \cdots, m\right\}\right) + \frac{y_j^q}{\left(\sum_{i=1}^m x_{ij}^q\right)y_l^p}\left(\sum_{i=1}^m\left[x_{io}^p - x_{il}^q\left(\min\left\{\frac{x_{io}^p}{x_{ij}^q} : i = 1, \cdots, m\right\}\right)\right]\right)$, *which simplifies to* $\frac{y_j^q}{\left(\sum_{i=1}^m x_{ij}^q\right)y_l^p}\left(\sum_{i=1}^m x_{io}^p\right)$. *By* Theorem 10, $\bar{\varepsilon}^* = \frac{y_j^q}{\left(\sum_{i=1}^m x_{ij}^q\right)y_l^p}$. *On the other hand,* $\frac{y_j^q}{\left(\sum_{i=1}^m x_{ij}^q\right)y_l^p}\left(\sum_{i=1}^m x_{io}^p\right)$ *is exactly the objective function value of model* (13) *for* $\boldsymbol{v}^* = (\bar{\varepsilon}^*, \cdots, \bar{\varepsilon}^*) = \left(\frac{y_j^q}{\left(\sum_{i=1}^m x_{ij}^q\right)y_l^p}, \cdots, \frac{y_j^q}{\left(\sum_{i=1}^m x_{ij}^q\right)y_l^p}\right)$. *Hence, the constructed feasible primal–dual pair* $\boldsymbol{v}^* \in \mathbb{R}^m$ and $(\lambda^*, \boldsymbol{s}^{+^*}) \in \mathbb{R}_+^n \times \mathbb{R}_+^m$ *are feasible and satisfy the weak duality property of LPs* (see Bazaraa et al., 2010), *implying that* $(\boldsymbol{v}^*, u^*)$*and* $(\theta^*, \lambda^*, \boldsymbol{s}^{+^*}, s^{-^*})$ *are indeed optimal solutions to their respective problems.* ■

Appendix B

Example 2. Consider $n = 3$ DMUs with $m = 2$ inputs and one output observed over periods $p \in \{1,2\}$. Period 1 observations are $(x_{11}^1, x_{21}^1, y_1^1) = (6,4,20)$, $(x_{12}^1, x_{22}^1, y_2^1) = (4,5,25)$, and $(x_{13}^1, x_{23}^1, y_3^1) = (8,3,18)$. Period 2 observations are $(x_{11}^2, x_{21}^2, y_1^2) = (7,5,22)$, $(x_{12}^2, x_{22}^2, y_2^2) = (6,6,28)$, and $(x_{13}^2, x_{23}^2,$

$y_3^2) = (10, 3, 20)$. Let $X_j^p = \sum_{i=1}^{m} x_{ij}^p$; then $X_1^1 = 10, X_2^1 = 9, X_3^1 = 11$ and $X_1^2 = 12, X_2^2 = 12, X_3^2 = 13$. The period minima of reciprocal outputs are $\min_j\left\{1/y_j^1\right\} = 1/25$ and $\min_j\left\{1/y_j^2\right\} = 1/28$. Applying the closed forms in model (19) gives, for DMU_1, $\overline{\text{CU}}_1 = (22\times 1/28)/(20\times 1/25) \approx 0.9821$, $\overline{\text{FS}}_1 = (20\times 12)/(22\times 10) \approx 1.0909$, and $\overline{\text{MPI}}_1 \approx 1.0714$, with $\overline{\text{MPI}}_1 = \overline{\text{CU}}_1 \times \overline{\text{FS}}_1$ holding exactly. For DMU_2 and DMU_3, $\overline{MPI}_o$ differs because X_o^2/X_o^1 varies across DMUs (see Table B1), illustrating the closed-form computation and the DMU-specific nature of both $\overline{CU}_o$ and $\overline{FS}_o$ under (19).

Table B1
MPI, CU, and FS calculations for Example 2

DMU	y_j^1	y_j^2	X_j^1	X_j^2	$\overline{\text{CU}}$	$\overline{\text{FS}}$	$\overline{\text{MPI}} = \overline{\text{CU}} \times \overline{\text{FS}}$
1	20	22	10	12	$(22\times 1/28)/(20\times 1/25) \approx 0.9821$	$(20\times 12)/(22\times 10) \approx 1.0909$	$0.9821 \times 1.0909 \approx 1.0714$
2	25	28	9	12	$(28\times 1/28)/(25\times 1/25) = 1$	$(25\times 12)/(28\times 9) = 25/21 \approx 1.1905$	$1 \times 1.1905 \approx 1.1905$
3	18	20	11	13	$(20\times 1/28)/(18\times 1/25) = 125/126 \approx 0.9921$	$(18\times 13)/(20\times 11) = 117/110 \approx 1.0636$	$0.9921 \times 1.0636 \approx 1.0552$

Data availability

Data will be made available on request.

References

Ali, A. I., & Seiford, L. M. (1993). Computational accuracy and infinitesimals in data envelopment analysis. *INFOR: Information Systems and Operational Research, 31*(4), 290–297.

Amin, G. R. (2009a). Comments on finding the most efficient DMUs in DEA: An improved integrated model. *Computers & Industrial Engineering, 56*(4), 1701–1702.

Amin, G. R. (2009b). Optimal solution of technology selection model: A computational efficient form. *International Journal of Advanced Manufacturing Technology, 43*(9–10), 1046–1050.

Amin, G. R., & Toloo, M. (2007). Finding the most efficient DMUs in DEA: An improved integrated model. *Computers & Industrial Engineering, 52*(1), 71–77.

Amin, G. R., & Toloo, M. (2004). A polynomial-time algorithm for finding ε in DEA models. *Computers and Operations Research, 31*(5), 803–805.

Aparicio, J., & Santín, D. (2026). The standard total factor productivity index and its decomposition. *European Journal of Operational Research, 329*(3), 981–1003.

Babaee, S., Toloo, M., Hermans, E., & Shen, Y. (2021). A new approach for index construction: The case of the road user behavior index. *Computers and Industrial Engineering, 152*.

Balk, B. M. (2001). Scale efficiency and productivity change. *Journal of Productivity Analysis, 15*, 159–183.

Banker, R. D., Charnes, A., & Cooper, W. W. (1984). Some models for estimating technical and scale inefficiencies in data envelopment analysis. *Management Science, 30*(9), 1078–1092.

Banker, R. D., Zheng, Z. (Eric), & Natarajan, R. (2010). DEA-based hypothesis tests for comparing two groups of decision-making units. *European Journal of Operational Research, 206*(1), 231–238.

Bazaraa, M. S., Jarvis, J. J., & Sherali, H. D. (2010). *Linear programming and network flows* (4th ed.). John Wiley & Sons.

Capeletti, N. M., Amado, C. A. F., & Santos, S. P. (2023). Performance assessment of primary health care services using data envelopment analysis and the quality-adjusted malmquist index. *Journal of the Operational Research Society*, 1–17.

Caves, D. W., Christensen, L. R., & Diewert, W. E. (1982). The economic theory of index numbers and the measurement of input, output, and productivity. *Econometrica, 50*(6), 1393.

Charnes, A., Cooper, W. W., & Rhodes, E. (1978). Measuring the efficiency of decision-making units. *European Journal of Operational Research, 2*, 429–444.

Charnes, A., Cooper, W. W., & Rhodes, E. (1979). Measuring the efficiency of decision-making units, short communication. *European Journal of Operational Research, 3*, 339.

Chen, J.-X. (2011). Peer-estimation for multiple criteria ABC inventory classification. *Computers & Operations Research, 38*(12), 1784–1791.

Chen, Y., & Ali, A. I. (2004). DEA Malmquist productivity measure: New insights with an application to computer industry. *European Journal of Operational Research, 159*(1), 239–249.

Collier, T., Johnson, A. L., & Ruggiero, J. (2011). Technical efficiency estimation with multiple inputs and multiple outputs using regression analysis. *European Journal of Operational Research, 208*(2), 153–160.

Cook, W. D., & Seiford, L. M. (2009). Data envelopment analysis (DEA) - Thirty years on. *European Journal of Operational Research, 192*(1), 1–17.

Cooper, W. W., Seiford, L. M., & Tone, K. (2007). *Data Envelopment Analysis: A Comprehensive Text with Models, applications, References and Dea-solver Software* (2nd ed.). US: Springer.

Cordero, J. M., & Santín, D. (2026). A new approach for measuring productivity gaps across groups: An application with historic footballers from FC Barcelona and Real Madrid. *Omega, 138*, Article 103421.

Daraio, C., & Simar, L. (2007). *Advanced Robust and Nonparametric Methods in Efficiency Analysis*. Methodology and Applications: Springer.

Despotis, D. K. (2005). A reassessment of the human development index via data envelopment analysis. *Journal of the Operational Research Society, 56*(8), 969–980.

Emrouznejad, A., Rostamy-Malkhalifeh, M., Hatami-Marbini, A., Tavana, M., & Aghayi, N. (2011). An overall profit Malmquist productivity index with fuzzy and interval data. *Mathematical and Computer Modelling, 54*(11–12), 2827–2838.

Färe, R., & Charles, V. (2018). A directional distance function approach to void the non-Archimedean in DEA. *Journal of the Operational Research Society, 69*(5), 772–775.

Färe, R., & Grosskopf, S. (1992). Malmquist Productivity Indexes and Fisher Ideal Indexes. *The Economic Journal, 102*(410), 158–160.

Färe, R., Grosskopf, S., Norris, M., & Zhang, Z. (1994). Productivity growth, technical progress, and efficiency change in industrialized countries. *The American Economic Review, 84*(1), 66–83.

Hatami-Marbini, A., Agrell, P. J., Fukuyama, H., Gholami, K., & Khoshnevis, P. (2017). The role of multiplier bounds in fuzzy data envelopment analysis. *Ann. Oper. Res., 250*(1), 249–276.

Hatami-Marbini, A., Arabmaldar, A., & Asu, J. O. (2022). Robust productivity growth and efficiency measurement with undesirable outputs: Evidence from the oil industry. *OR Spectrum, 44*(4), 1213–1254.

Hatami-Marbini, A., & Toloo, M. (2017). An extended multiple criteria data envelopment analysis model. *Expert Systems with Applications, 73*, 201–219.

Hatami-Marbini, A., & Toloo, M. (2019). Data envelopment analysis models with ratio data: A revisit. *Computers & Industrial Engineering, 133*, 331–338.

Henderson, D. J., & Russell, R. R. (2005). Human capital and convergence: A production-frontier approach. *International Economic Review, 46*(4), 1167–1205.

Kao, C. (2017). Measurement and decomposition of the Malmquist productivity index for parallel production systems. *Omega, 67*, 54–59.

Kao, C., & Liu, S.-T. (2016). A parallel production frontiers approach for intertemporal efficiency analysis: The case of Taiwanese commercial banks. *European Journal of Operational Research, 255*(2), 411–421.

Karagiannis, G., & Lovell, C. A. K. (2016). Productivity measurement in radial DEA models with a single constant input. *European Journal of Operational Research, 251*(1), 323–328.

Kohl, S., & Brunner, J. O. (2020). Benchmarking the benchmarks – Comparing the accuracy of data envelopment analysis models in constant returns to scale settings. *European Journal of Operational Research, 285*(3), 1042–1057.

Kumar, S., & Russell, R. R. (2002). Technological change, technological catch-up, and capital deepening: Relative contributions to growth and convergence. *The American Economic Review, 92*(3), 527–548.

Li, Z., Feng, C., & Tang, Y. (2022). Bank efficiency and failure prediction: A nonparametric and dynamic model based on data envelopment analysis. *Ann. Oper. Res., 315*(1), 279–315.

Lin, M.-I., Lee, Y.-D., & Ho, T.-N. (2011). Applying integrated DEA/AHP to evaluate the economic performance of local governments in China. *European Journal of Operational Research, 209*(2), 129–140.

Liu, W. B., Zhang, D. Q., Meng, W., Li, X. X., & Xu, F. (2011). A study of DEA models without explicit inputs. *Omega, 39*(5), 472–480.

Lovell, C. A. K. (2003). The decomposition of Malmquist productivity indexes. *Journal of Productivity Analysis, 20*(3), 437–458.

Lovell, C. A. K., & Pastor, J. T. (1999). Radial DEA models without inputs or without outputs. *European Journal of Operational Research, 118*(1), 46–51.

Mehrabian, S., Jahanshahloo, G. R., Alirezaee, M. R., & Amin, G. R. (2000). An Assurance Interval for the Non-Archimedean Epsilon in DEA Models. *Operations Research, 48*(2), 344–347.

Nguyen, T. T. T., Pham, B. T., Prior, D., & van Hemmen, S. (2022). Performance of tax simplification around the world: A panel frontier analysis. *Socio-Economic Planning Sciences, 80*.

Olesen, O. B., Petersen, N. C., & Podinovski, V. V. (2015). Efficiency analysis with ratio measures. *European Journal of Operational Research, 245*(2), 446–462.

Papaioannou, G., & Podinovski, V. V. (2024). A single-stage optimization procedure for data envelopment analysis. *European Journal of Operational Research, 313*(3), 1119–1128.

Pastor, J. T., Asmild, M., & Lovell, C. A. K. (2011). The biennial Malmquist productivity change index. *Socio-Economic Planning Sciences, 45*(1), 10–15.

Pastor, J. T., & Lovell, C. A. K. (2005). A global Malmquist productivity index. *Economics Letters, 88*(2), 266–271.
Podinovski, V. V., & Bouzdine-Chameeva, T. (2017). Solving DEA models in a single optimization stage: Can the non-Archimedean infinitesimal be replaced by a small finite epsilon? *European Journal of Operational Research, 257*(2), 412–419.
Podinovski, V. V., Wu, J., & Argyris, N. (2024). Production trade-offs in models of data envelopment analysis with ratio inputs and outputs: An application to schools in England. *European Journal of Operational Research, 313*(1), 359–372.
Ray, S. C., & Desli, E. (1997). Productivity growth, technical progress, and efficiency change in industrialized countri'es: Comment. *American Economic Review, 87*(5), 1033–1039.
Sadeghi, J., Begen, M. A., & Ødegaard, F. (2023). Refined bounds for the non-Archimedean ∊ in DEA. *Computers & Operations Research, 154*, Article 106163.
Salahi, M., & Toloo, M. (2017). In the determination of the most efficient decision making unit in data envelopment analysis: A comment. *Computers and Industrial Engineering, 104*, 216–218.
Shephard, R. W. (1970). *Theory of cost and production function*. Princeton University Press.
Siegel, S., & Castellan, N. J. J. (1988). *Nonparametric statistics for the behavioral sciences*. McGraw-Hill.
Simar, L., & Wilson, P. W. (1998). Sensitivity analysis of efficiency scores: How to bootstrap in nonparametric frontier models. *Management Science, 44*(1), 49–61.
Simpson, G. (2005). Programmatic efficiency comparisons between unequally sized groups of DMUs in DEA. *Journal of the Operational Research Society, 56*(12), 1431–1438.
Tavana, M., Khalili-Damghani, K., Arteaga, F. J. S., & Hashemi, A. (2020). A Malmquist productivity index for network production systems in the energy sector. *Annals of Operations Research, 284*, 415–445.
Toloo, M. (2014). The role of non-Archimedean epsilon in finding the most efficient unit: With an application of professional tennis players. *Applied Mathematical Modelling, 38*(21–22), 5334–5346.
Toloo, M., & Tavana, M. (2017). A novel method for selecting a single efficient unit in data envelopment analysis without explicit inputs/outputs. *Ann. Oper. Res., 253*(1), 657–681.
Villa, G., & Lozano, S. (2026). A global malmquist productivity index of athletics performance in olympic games. *European Journal of Operational Research, 332*(3), 880–896.
Villanueva-Cantillo, J., & Munoz-Marquez, M. (2021). Methodology for calculating critical values of relevance measures in variable selection methods in data envelopment analysis. *European Journal of Operational Research, 290*(2), 657–670.
Wilson, P. W., & Zhao, S. (2023). Investigating the Performance of Chinese Banks over 2007–2014. *Annals of Operations Research, 321*, 663–692.
Yang, G., Shen, W., Zhang, D., & Liu, W. (2014). Extended utility and DEA models without explicit input. *Journal of the Operational Research Society, 65*(8), 1212–1220.
Yu, M.-M., & Nguyen, M.-A.-T. (2023). Productivity changes of Asia-Pacific airlines: A Malmquist productivity index approach for a two-stage dynamic system. *Omega, 115*, Article 102774.
Zelenyuk, V. (2023). Productivity analysis: Roots, foundations, trends and perspectives. *Journal of Productivity Analysis, 60*(3), 229–247.