

Article

Forced Vibrations of Rotating Annular Discs Under Space-Fixed Point-Force Excitation

Hilal Koç¹, Mertol Tüfekci^{2,3,*} and Ekrem Tüfekci¹

¹ Faculty of Mechanical Engineering, Istanbul Technical University, Gümüşsuyu, Istanbul 34437, Turkey; hilal.koc@itu.edu.tr (H.K.); tufekcie@itu.edu.tr (E.T.)

² Centre for Engineering Research, University of Hertfordshire, Hatfield AL10 9AB, UK

³ School of Physics, Engineering and Computer Science, University of Hertfordshire, Hatfield AL10 9AB, UK

* Correspondence: m.tufekci@herts.ac.uk

Abstract

This study investigates the forced transverse vibration of a thin rotating annular disc subjected to time-varying point forces that are fixed in space and act perpendicular to the disc surface. Earlier analytical treatments of this problem have almost always been restricted to a single support condition, most often the clamped-free disc of a hard-disk drive. The boundary conditions of the disc have not been treated as a design variable of the forced response. The contribution of this work is to remove that restriction: the same generalised Galerkin formulation is applied to clamped-clamped, clamped-free and free-clamped rotating annular discs, and the sensitivity of the forced response to the excitation parameters is compared across all three. The governing differential equation, which includes gyroscopic coupling and the membrane stresses induced by rotation, is nondimensionalised and solved by the Galerkin method with polynomial radial trial functions. The modal equations are then integrated in state-space form with light modal damping. The physical transverse response at a fixed observation point is characterised by power spectral density diagrams assembled into waterfall plots, and the corresponding steady-state harmonic responses are obtained in closed form from the frequency-domain resolvent of the same state-space model. The formulation is verified against an independent radial finite-element model for all three boundary conditions and against published rotating clamped-free natural frequencies. The central finding concerns how the excitation parameters act on the response. The excitation frequency, the radial position of a force, and the angular separation and phase of a pair of forces act as largely independent levers. The quantitative sensitivity to each lever, however, is set by the boundary conditions, as is the force placement that minimises a chosen travelling-wave family. The radial position that minimises the excitation of a chosen radial family is governed by the interior node of that mode, which lies at $r \approx 52, 68$ and 44 mm for the clamped-clamped, clamped-free and free-clamped discs, respectively, and does not in general coincide with a free edge. Angular separation, by contrast, suppresses a nodal-diameter family in a manner that is essentially boundary-condition independent. The parameter dependences are shown to be steady-state properties: the driven spectral line of the finite-duration records reproduces the resolvent solution to within 0.09 dB.

Keywords: rotating annular disc; forced vibration; Galerkin method; travelling waves; power spectral density; boundary conditions



Academic Editor: Qinkai Han

Received: 6 July 2026

Revised: 29 July 2026

Accepted: 30 July 2026

Published: 31 July 2026

Copyright: © 2026 by the authors.

Licensee MDPI, Basel, Switzerland.

This article is an open access article distributed under the terms and

conditions of the [Creative Commons](https://creativecommons.org/licenses/by/4.0/)

[Attribution \(CC BY\)](https://creativecommons.org/licenses/by/4.0/) license.

1. Introduction

Rotating discs are core components of a wide range of engineering systems: turbines, fans and flywheels in power machinery, gears, circular saws and grinding wheels in manufacturing, brake discs in vehicles and, as a more recent example, the recording discs of computer hard-disk drives (HDDs) used for data storage. In all of these applications the disc spins at high speed and interacts with space-fixed components—guides, pads, read/write heads or aerodynamic loads—so that excessive transverse vibration directly limits performance and life. The free vibration of spinning discs has therefore been studied since the classical works of [1,2], followed by the nonlinear analysis of [3], the Rayleigh–Ritz treatment of initially stressed discs by [4] and the annular finite elements of [5]. The free-vibration characteristics of rotating annular discs under various classical boundary conditions, including the configurations considered here, have been investigated with the Galerkin method [6–8]: the first of these studies [6] established the polynomial Galerkin formulation for rotating annular discs under several boundary conditions, while the later study [8] extended the same analytical framework to nine boundary-condition combinations and verified the results against finite element and experimental data. Related work on rotating blade and composite wind-turbine blade dynamics also illustrates the usefulness of reduced analytical and numerical vibration models for high-speed rotating components [9].

The forced-vibration problem has an equally long history. The response of a very flexible spinning disc to a stationary transverse load was investigated by [10]; after no bounded solution was found under the pure membrane assumption, the bending stiffness was retained and the solution obtained by Fourier series. The vibration of a rotating disc carrying an in-plane force at its rim or centre was analysed by [11], and the bifurcation of the resonance phenomena was examined by expressing the response as a sum of two counter-rotating vectors. An eigenfunction expansion was used by [12] to study analytically the forced vibration of a spinning disc under space-fixed couples, considering both the transient and the steady-state response together with the effects of external damping and rotational speed. The closed-form forced response of a damped rotating multiple disk/spindle system was obtained by [13] from Lagrange’s equations with a Rayleigh dissipation function. The nonlinear forced response of a spinning disc excited at a point fixed in space was studied by [14] using the harmonic balance method, and the results were compared with experiments. The forced vibration and critical speeds of rotating rectangularly orthotropic discs under a space-fixed load were analysed by [15,16], who emphasised that the dynamic response of the orthotropic disc depends explicitly on time because the material properties seen by the fixed load change continuously.

The free and forced vibrations of rotating disk/spindle systems with flexible housing and stator assemblies were investigated by [17] through linearised equations of motion obtained from a Lagrangian formulation, confirmed by finite element analyses and numerical simulation. The coupled in-plane and out-of-plane vibrations of a rotating Kirchhoff plate in the presence of a distributed frictional load were studied by [18], providing one of the first treatments of the coupling effect of friction pads in disc-brake squeal. The axisymmetric vibration of spinning annular plates was examined by [19] using a three-dimensional finite element method, and natural frequencies were reported for different radius ratios, thickness-to-radius ratios, rotational speeds and boundary conditions. A method based on Duhamel’s integral for the dynamic behaviour of axisymmetric rotating structures was developed by [20], in which the excitation is expressed in a time-varying space-fixed form and finite-element-based transient analyses are used to identify the impulse response. The vibration of a ferromagnetic rotating disc under an electromagnetic force was investigated by [21] using the Galerkin method, addressing the natural frequencies and the dynamic stability. Galerkin’s method was also employed by [22] to obtain the analytical solution for

a rotating disc subjected to peripheral rotating transverse loads, and an additional closed-form solution was derived by a Rayleigh integral approach. The vibrational behaviour of a rotating disc was measured by [23,24] with a non-contact technique combining optical tip-clearance sensors and a strain gauge, demonstrating that such sensors are suitable for identifying nodal diameters. The forced vibration of a spinning annular plate under a fixed transverse force was studied by [25], who compared the rotating-coordinate/fixed-load and fixed-coordinate/rotating-load modelling approaches. The primary resonance and bifurcation of a conductive circular plate rotating in air-magnetic fields were presented by [26] using Bessel-function mode shapes and the Galerkin method for the nonlinear equation of motion. Finally, the Galerkin method was used by [27] to examine the forced vibrations of a rotating annular disc with clamped–free boundary conditions, obtaining power spectral density diagrams that account for the number, frequencies, phase differences and locations of multiple space-fixed excitation forces; this study is the direct forced-response basis of the present work and is extended here to additional boundary conditions and a more systematic set of excitation-parameter sweeps. More recently, the scope of these analyses has broadened towards the mitigation and active control of travelling-wave vibration in rotating structures. A closed-form solution for the free and forced transverse vibration of a clamped–free rotating annular disk was derived by [28], together with a tuned-mass-damper strategy and the resulting acoustic-pressure mitigation, within the same homogeneous, isotropic, classical thin-plate framework adopted here. For rotating shells, piezoelectric sensor/actuator pairs and micro-fibre-composite layers were designed by [29,30] to suppress travelling-wave vibration, reflecting the current interest in deliberately tailoring the dynamic response of spinning structures. At the system level, the vibration of rotating discs and rotors is also increasingly analysed together with parameter uncertainty and with the flexibility of the surrounding gear, shaft and support components [31–34]. Both effects can shift the critical speeds and the forced-response levels predicted by an idealised model. The present study deliberately isolates the effect of the disc boundary conditions and treats the excitation as a small number of ideal space-fixed point forces, so that the boundary-condition dependence is not masked by these additional effects; their inclusion is noted as a direction for further work.

When nonlinear contact, large deformation or application-specific blade geometry must be retained, the analytical route is complemented by high-fidelity time-domain frameworks. Recent examples include time-marching frequency-sweep tools for bladed-disc and rotating-blade systems with contact nonlinearity [35]. Related semi-analytical initial-value methods have likewise been used for parameter-sensitive vibration and buckling of nonlocal nanostructures [36]. In these methods the governing equations are written in state-vector form, propagated with a transfer or principal matrix and closed by the boundary conditions. These are not rotating-disc models, but they share the philosophy adopted here: a high-order continuum equation is reduced to a compact, verifiable matrix problem suitable for systematic parametric analysis.

The survey above shows that analytical investigations of the forced response of rotating discs have, almost without exception, considered a single set of boundary conditions—most often the clamped–free disc that models an HDD. No study appears to have systematically examined how the forced response and its sensitivity to the excitation parameters change with the boundary conditions. The present paper addresses this gap. The same generalised Galerkin formulation is applied to the clamped–clamped, clamped–free and free–clamped rotating annular disc, and the effects of (i) the excitation frequency, (ii) the radial position of the excitation force, (iii) the angular separation of a pair of forces and (iv) their phase difference are investigated for all three boundary conditions. The responses are presented as PSD-based waterfall diagrams. The formulation is compared with the published natural

frequencies of the rotating clamped–free disc of [27] and with the free-vibration results of [7] for the clamped–clamped and free–clamped configurations; the published forced-response spectra are reproduced as a qualitative comparison.

The incremental contribution relative to the clamped–free forced-response study of [27] can be stated precisely. That earlier study established, for the clamped–free disc only, that the number, frequencies, locations and phases of space-fixed forces control the power spectral density of the response. The present work retains that model and forcing description but treats the disc boundary conditions as an additional variable, and its new content is threefold. First, the formulation and its numerical implementation are extended to the clamped–clamped and free–clamped discs and verified for all three cases against an independent radial finite-element model. Second, the excitation-parameter sweeps are carried out in parallel for the three boundary conditions, which isolates those features of the response that are common to all supports from those that are set by the support. Third, this comparison yields a nontrivial, boundary-condition-dependent result that is not visible in a single-support study. The radial force placement that minimises the excitation of a chosen radial family is set by the interior node of that radial family, whose position moves with the support: $r \approx 52$ mm for the clamped–clamped disc, 68 mm for the clamped–free disc and 44 mm for the free–clamped disc. The angular separation of a pair of forces, by contrast, suppresses a nodal-diameter family through the same circumferential interference law for all three supports. In other words, the radial lever is boundary-condition dependent while the circumferential lever is not—a distinction that has direct consequences for where a force or contact path should be placed on a given disc.

The paper is organised as follows. Section 2 presents the governing equation, the rotation-induced membrane stress fields for the three boundary conditions, the Galerkin discretisation and the state-space solution of the forced problem. Section 3 presents the numerical examples and discussion, starting from the verification and then considering the excitation-frequency, radial-position, angular-separation and phase-difference sweeps. Section 4 summarises the conclusions.

2. Methodology and Formulation

2.1. Problem Definition and Governing Equation

Consider a thin annular disc of inner radius R_i , outer radius R_o , uniform thickness h , density ρ , Young's modulus E and Poisson's ratio ν , rotating at the constant angular velocity Ω about its central axis. The transverse displacement is denoted by $w(r, \psi, t)$, where r is the radial coordinate, ψ is the space-fixed angular coordinate and t is time. The disc is assumed to be homogeneous, isotropic and linearly elastic, and the transverse motion is modelled using the classical Kirchhoff thin-plate theory. The in-plane stresses induced by rotation are retained in the transverse equation of motion, while transverse shear deformation and rotary inertia are neglected.

The Kirchhoff assumption is appropriate for the discs considered here. The thickness-to-outer-radius ratio is $h/R_o = 0.0125$ for the disc used in the parametric study and 0.027 for the hard-disk geometry used in the free-vibration comparison. Both values are well within the thin-plate range $h/R_o \lesssim 0.1$, and the deflections examined are small compared with the thickness, so the linear theory applies. The main quantitative consequence of the assumption is that classical plate theory omits the shear flexibility retained by Mindlin or three-dimensional models and therefore slightly overestimates the higher natural frequencies. For the radius ratios and mode numbers considered here this overestimate is a few per cent, consistent with the comparison against a shear-deformable solution reported for the same family of discs in [8], and it partly accounts for the small offset discussed later in the free-vibration comparison. Because the excitation frequencies of interest lie on the lowest

travelling-wave families, where the plate and shear-deformable predictions are closest, the thin-plate model is sufficient for the parameter-dependence study that is the object of this work. The extension to a shear-deformable disc would be required only for very thick discs or for the highest modes.

Figure 1 summarises the physical problem: the annular disc rotates with angular velocity Ω , while the transverse point force is fixed in the inertial frame and acts at the space-fixed location (r_1, ψ_1) .

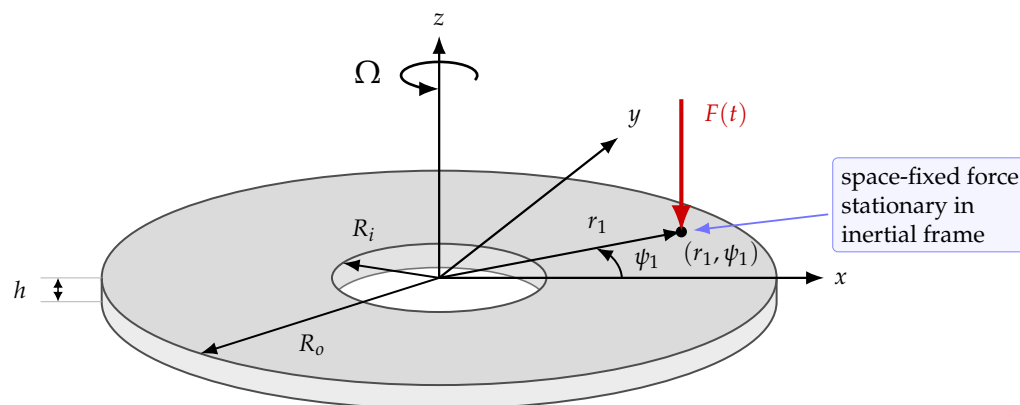


Figure 1. Schematic of the rotating annular disc under a transverse point force fixed in space. The force location is defined by the radial coordinate r_1 and the angular coordinate ψ_1 , while R_i and R_o denote the inner and outer radii of the annulus.

The forced transverse vibration of the rotating annular disc under a space-fixed distributed transverse load is governed by:

$$D\nabla^4 w + \rho h \left(\frac{\partial^2 w}{\partial t^2} + 2\Omega \frac{\partial^2 w}{\partial \psi \partial t} + \Omega^2 \frac{\partial^2 w}{\partial \psi^2} \right) - \frac{h}{r} \left[\frac{\partial}{\partial r} \left(\sigma_r r \frac{\partial w}{\partial r} \right) + \frac{\sigma_\psi}{r} \frac{\partial^2 w}{\partial \psi^2} \right] = p(r, \psi, t). \quad (1)$$

Here, $D = Eh^3/[12(1 - \nu^2)]$ is the flexural rigidity of the disc, and ρh is the mass per unit area. The quantities $\sigma_r(r)$ and $\sigma_\psi(r)$ are the radial and circumferential membrane stresses induced by rotation, respectively, and their forms depend on the boundary conditions imposed at the inner and outer radii. The function $p(r, \psi, t)$ represents the externally applied transverse load per unit area.

For compactness, the polar-coordinate Laplace operator and the corresponding biharmonic operator used to represent the bending term in Equation (1) are defined as:

$$\nabla^2(\cdot) = \frac{\partial^2(\cdot)}{\partial r^2} + \frac{1}{r} \frac{\partial(\cdot)}{\partial r} + \frac{1}{r^2} \frac{\partial^2(\cdot)}{\partial \psi^2} \quad (2)$$

In the inertia term, $\partial^2 w / \partial t^2$ is the transverse acceleration, $2\Omega \partial^2 w / (\partial \psi \partial t)$ represents the gyroscopic coupling arising from the space-fixed formulation, and $\Omega^2 \partial^2 w / \partial \psi^2$ is the centripetal contribution. The final bracketed term accounts for the effect of the rotation-induced in-plane membrane stresses on the transverse vibration response.

The space-fixed harmonic point force used in the present study is represented as:

$$p(r, \psi, t) = \frac{F}{r} \delta(r - r_1) \delta(\psi - \psi_1) \sin(\omega t - \phi_1). \quad (3)$$

Here, F is the point-force amplitude, r_1 and ψ_1 are the radial and angular coordinates of the excitation point, ω is the excitation angular frequency, ϕ_1 is the phase angle and $\delta(\cdot)$ is the Dirac delta function.

The nondimensionalisation variables used in the formulation are:

$$\begin{aligned} T &= \sqrt{\frac{D}{\rho h R_o^4}}, \quad \tau = tT, \quad R = \frac{r}{R_o}, \quad R_1 = \frac{r_1}{R_o}, \\ \gamma &= \frac{R_i}{R_o}, \quad W = \frac{w}{R_o}, \quad P = \frac{p R_o^3}{D}, \quad F^* = \frac{F R_o}{D}, \\ \omega^* &= \omega/T, \quad \omega^* = \omega/T, \quad \Omega^* = \Omega/T. \end{aligned} \quad (4)$$

In Equation (4), $T = \sqrt{D/(\rho h R_o^4)}$ is the characteristic frequency that sets the time and frequency scales (so that $\tau = tT$, $\omega^* = \omega/T$ and $\Omega^* = \Omega/T$ are dimensionless), τ is the dimensionless time, R is the dimensionless radial coordinate, R_1 is the dimensionless radial coordinate of the excitation point, γ is the radius ratio, W is the dimensionless transverse displacement, P is the dimensionless transverse load, F^* is the dimensionless point-force amplitude, and ω^* , ω^* and Ω^* are the dimensionless natural frequency, excitation angular frequency and rotational speed, respectively. Substitution of these quantities into Equation (1) yields:

$$\begin{aligned} \nabla^4 W + \frac{\partial^2 W}{\partial \tau^2} + 2\Omega^* \frac{\partial^2 W}{\partial \psi \partial \tau} + \Omega^{*2} \frac{\partial^2 W}{\partial \psi^2} \\ - \Omega^{*2} \left(\bar{C}_{rr} \frac{\partial^2 W}{\partial R^2} + \bar{C}_{r\psi} \frac{\partial W}{\partial R} + \bar{C}_{\psi\psi} \frac{\partial^2 W}{\partial \psi^2} \right) = P(R, \psi, \tau). \end{aligned} \quad (5)$$

The barred coefficients $\bar{C}_{rr}$, $\bar{C}_{r\psi}$ and $\bar{C}_{\psi\psi}$ are dimensionless boundary-condition-dependent coefficients that collect the rotation-induced membrane-stress terms in Equation (5). From this point onward, the formulation is written only in terms of dimensionless quantities. These coefficients are defined as:

$$\bar{C}_{rr}(R) = \bar{\sigma}_r(R), \quad (6)$$

$$\bar{C}_{r\psi}(R) = \frac{d\bar{\sigma}_r}{dR} + \frac{\bar{\sigma}_r(R)}{R}, \quad (7)$$

$$\bar{C}_{\psi\psi}(R) = \frac{\bar{\sigma}_\psi(R)}{R^2}. \quad (8)$$

Here, $\bar{\sigma}_r(R)$ and $\bar{\sigma}_\psi(R)$ are the dimensionless radial and tangential membrane-stress functions. They are written as:

$$\begin{aligned} \bar{\sigma}_r(R) &= \frac{1}{8} \left[(1+\nu)\alpha - (1-\nu)\frac{\beta}{R^2} - (3+\nu)R^2 \right], \\ \bar{\sigma}_\psi(R) &= \frac{1}{8} \left[(1+\nu)\alpha + (1-\nu)\frac{\beta}{R^2} - (1+3\nu)R^2 \right]. \end{aligned} \quad (9)$$

The constants α and β are dimensionless and are obtained from the in-plane edge conditions. A clamped edge is taken to restrain the dimensionless radial in-plane displacement $U = u/R_o$ (with u the dimensional radial in-plane displacement), whereas a free edge is traction free, so that $\bar{\sigma}_r = 0$. The values used in the present calculations are listed in Table 1.

Table 1. Dimensionless constants of the rotation-induced membrane-stress functions for the three boundary conditions considered in the present work. The notation C–C, C–F and F–C denotes clamped–clamped, clamped–free and free–clamped conditions at the inner and outer edges, respectively.

Boundary Condition	Dimensionless In-Plane Edge Conditions	Constants
C–C	$U(\gamma) = 0, U(1) = 0$	$\alpha = 1 + \gamma^2, \quad \beta = -\gamma^2$
C–F	$U(\gamma) = 0, \bar{\sigma}_r(1) = 0$	$\alpha = \frac{(3+\nu) + (1-\nu)\gamma^4}{(1+\nu) + (1-\nu)\gamma^2}, \quad \beta = \gamma^4 - \alpha\gamma^2$
F–C	$\bar{\sigma}_r(\gamma) = 0, U(1) = 0$	$\alpha = \frac{(3+\nu)\gamma^4 + (1-\nu)}{(1+\nu)\gamma^2 + (1-\nu)}, \quad \beta = 1 - \alpha$

2.2. Assumed Displacement Field and Spatial Derivatives

The dimensionless transverse displacement is expanded as:

$$W(R, \psi, \tau) = \sum_{n=0}^N \sum_{m=0}^M [q_{mn}^s(\tau) \sin n\psi + q_{mn}^c(\tau) \cos n\psi] R_{mn}(R). \quad (10)$$

In Equation (10), q_{mn}^s and q_{mn}^c are the sine and cosine generalised coordinates, $R_{mn}(R)$ is the radial admissible function, m is the number of nodal circles and n is the number of nodal diameters. The integers M and N are the truncation limits in the radial and circumferential directions. The radial function is chosen as the five-term polynomial:

$$R_{mn}(R) = X_{mn}^1 R^m + X_{mn}^2 R^{m+1} + X_{mn}^3 R^{m+2} + X_{mn}^4 R^{m+3} + X_{mn}^5 R^{m+4}, \quad (11)$$

where the constants X_{mn}^1 – X_{mn}^5 are determined from the four transverse boundary conditions and one normalisation condition. The five coefficients are fixed by five linear conditions: two conditions at the inner edge, two at the outer edge and one normalisation condition. At a clamped edge, both the deflection and the slope vanish,

$$R_{mn} = 0, \quad \frac{dR_{mn}}{dR} = 0, \quad (12)$$

while at a free edge the radial bending moment and the Kirchhoff (effective) transverse shear force vanish,

$$\frac{d^2 R_{mn}}{dR^2} + \nu \left(\frac{1}{R} \frac{dR_{mn}}{dR} - \frac{n^2}{R^2} R_{mn} \right) = 0, \quad \frac{d}{dR} [\nabla^2 R_{mn}] + \frac{1-\nu}{R} n^2 \frac{d}{dR} \left(\frac{R_{mn}}{R} \right) = 0, \quad (13)$$

evaluated at the appropriate edge, where $\nabla^2 R_{mn} = d^2 R_{mn}/dR^2 + R^{-1} dR_{mn}/dR - n^2 R^{-2} R_{mn}$. These four conditions leave one free constant, which is removed by a normalisation condition $R_{mn}(R_{nrm}) = 1$ imposed at a point R_{nrm} that is chosen to be interior to, or on, the annulus and not to coincide with a boundary at which the deflection is forced to zero: an interior point $R_{nrm} = (1 + \gamma)/2$ for the C–C disc, the outer edge $R_{nrm} = 1$ for the C–F disc, and the inner edge $R_{nrm} = \gamma$ for the F–C disc. The normalisation only scales each admissible function and, being applied identically on both sides of the projection, has no effect on the computed natural frequencies or on the relative response levels; it merely fixes the scale of the generalised coordinates. For each (m, n) the five conditions form a 5×5 linear system that is solved once for the coefficient set $X_{mn}^1, \dots, X_{mn}^5$.

The derivatives of Equation (10) required for Equation (5) are retained explicitly. The first time derivative is:

$$\frac{\partial W}{\partial \tau} = \sum_{n=0}^N \sum_{m=0}^M \left[\frac{\partial q_{mn}^s}{\partial \tau} \sin n\psi + \frac{\partial q_{mn}^c}{\partial \tau} \cos n\psi \right] R_{mn}(R), \quad (14)$$

and the second time derivative is:

$$\frac{\partial^2 W}{\partial \tau^2} = \sum_{n=0}^N \sum_{m=0}^M \left[\frac{\partial^2 q_{mn}^s}{\partial \tau^2} \sin n\psi + \frac{\partial^2 q_{mn}^c}{\partial \tau^2} \cos n\psi \right] R_{mn}(R). \quad (15)$$

The first angular derivative is:

$$\frac{\partial W}{\partial \psi} = \sum_{n=0}^N \sum_{m=0}^M n [q_{mn}^s(\tau) \cos n\psi - q_{mn}^c(\tau) \sin n\psi] R_{mn}(R), \quad (16)$$

and the second angular derivative is:

$$\frac{\partial^2 W}{\partial \psi^2} = - \sum_{n=0}^N \sum_{m=0}^M n^2 [q_{mn}^s(\tau) \sin n\psi + q_{mn}^c(\tau) \cos n\psi] R_{mn}(R). \quad (17)$$

The mixed time–angular derivative, which forms the gyroscopic term, is:

$$\frac{\partial^2 W}{\partial \psi \partial \tau} = \sum_{n=0}^N \sum_{m=0}^M n \left[\frac{\partial q_{mn}^s}{\partial \tau} \cos n\psi - \frac{\partial q_{mn}^c}{\partial \tau} \sin n\psi \right] R_{mn}(R). \quad (18)$$

For completeness, the third angular derivative is:

$$\frac{\partial^3 W}{\partial \psi^3} = - \sum_{n=0}^N \sum_{m=0}^M n^3 [q_{mn}^s(\tau) \cos n\psi - q_{mn}^c(\tau) \sin n\psi] R_{mn}(R), \quad (19)$$

and the fourth angular derivative is:

$$\frac{\partial^4 W}{\partial \psi^4} = \sum_{n=0}^N \sum_{m=0}^M n^4 [q_{mn}^s(\tau) \sin n\psi + q_{mn}^c(\tau) \cos n\psi] R_{mn}(R). \quad (20)$$

Substitution of Equations (10)–(20) into Equation (5) gives:

$$\begin{aligned} & \sum_{n=0}^N \sum_{m=0}^M \left\{ \left[\frac{\partial^4 R_{mn}}{\partial R^4} + \frac{2}{R} \frac{\partial^3 R_{mn}}{\partial R^3} - \frac{1+2n^2}{R^2} \frac{\partial^2 R_{mn}}{\partial R^2} + \frac{1+2n^2}{R^3} \frac{\partial R_{mn}}{\partial R} + \frac{n^4-4n^2}{R^4} R_{mn} \right. \right. \\ & \quad \left. \left. - \Omega^{*2} \left(\bar{C}_{rr} \frac{\partial^2 R_{mn}}{\partial R^2} + \bar{C}_{r\psi} \frac{\partial R_{mn}}{\partial R} - \bar{C}_{\psi\psi} n^2 R_{mn} \right) \right] (q_{mn}^s \sin n\psi + q_{mn}^c \cos n\psi) \right. \\ & \quad \left. + \left(\frac{\partial^2 q_{mn}^s}{\partial \tau^2} \sin n\psi + \frac{\partial^2 q_{mn}^c}{\partial \tau^2} \cos n\psi \right) R_{mn} + 2\Omega^* n R_{mn} \left(\frac{\partial q_{mn}^s}{\partial \tau} \cos n\psi - \frac{\partial q_{mn}^c}{\partial \tau} \sin n\psi \right) \right. \\ & \quad \left. - \Omega^{*2} n^2 R_{mn} (q_{mn}^s \sin n\psi + q_{mn}^c \cos n\psi) \right\} = P(R, \psi, \tau). \end{aligned} \quad (21)$$

All radial functions in Equation (21) depend on R and all generalised coordinates depend on τ ; these dependencies are suppressed only to keep the expression readable.

The radial differential operator is abbreviated as:

$$\begin{aligned} F_{mn} = & \frac{\partial^4 R_{mn}}{\partial R^4} + \frac{2}{R} \frac{\partial^3 R_{mn}}{\partial R^3} - \frac{1+2n^2}{R^2} \frac{\partial^2 R_{mn}}{\partial R^2} + \frac{1+2n^2}{R^3} \frac{\partial R_{mn}}{\partial R} + \frac{n^4-4n^2}{R^4} R_{mn} \\ & - \Omega^{*2} \left(\bar{C}_{rr} \frac{\partial^2 R_{mn}}{\partial R^2} + \bar{C}_{r\psi} \frac{\partial R_{mn}}{\partial R} - \bar{C}_{\psi\psi} n^2 R_{mn} \right). \end{aligned} \quad (22)$$

Using Equation (22), Equation (21) becomes:

$$\sum_{n=0}^N \sum_{m=0}^M \left\{ \left[F_{mn} q_{mn}^s + \frac{\partial^2 q_{mn}^s}{\partial \tau^2} R_{mn} - 2\Omega^* n R_{mn} \frac{\partial q_{mn}^c}{\partial \tau} - \Omega^{*2} n^2 R_{mn} q_{mn}^s \right] \sin n\psi \right. \\ \left. + \left[F_{mn} q_{mn}^c + \frac{\partial^2 q_{mn}^c}{\partial \tau^2} R_{mn} + 2\Omega^* n R_{mn} \frac{\partial q_{mn}^s}{\partial \tau} - \Omega^{*2} n^2 R_{mn} q_{mn}^c \right] \cos n\psi \right\} = P(R, \psi, \tau). \quad (23)$$

2.3. Galerkin Projection

The Galerkin method is applied by projecting the residual of Equation (23) onto a set of test functions taken from the same admissible family as the trial functions but carrying an independent radial index. To keep the trial and test roles distinct, let m denote the radial index of the trial functions in the expansion, Equation (10) and let $k = 0, 1, \dots, M$ denote the radial index of the test functions. Because the circumferential functions $\sin n\psi$ and $\cos n\psi$ are mutually orthogonal on $[0, 2\pi]$, the residual of a given nodal-diameter number n is orthogonal to every circumferential function of a different nodal-diameter order n' (that is, $n' \neq n$); the projection therefore decouples the problem family by family, and each n can be treated on its own. Within the n -th family the two independent test functions are $R_{kn}(R) \sin n\psi$ and $R_{kn}(R) \cos n\psi$. Projecting the residual onto $R_{kn}(R) \sin n\psi$ and onto $R_{kn}(R) \cos n\psi$ in turn, and integrating over the annulus, yields two weighted-residual equations for each k . The test functions contain no unknown coefficients, so the operation is a standard weighted-residual projection and not a variation of the generalised coordinates. Writing the two projections together, the weighted-residual statement of the n -th family is:

$$\int_{\gamma}^1 \int_0^{2\pi} \sum_{m=0}^M \left\{ \left[F_{mn} q_{mn}^s + \frac{\partial^2 q_{mn}^s}{\partial \tau^2} R_{mn} - 2\Omega^* n R_{mn} \frac{\partial q_{mn}^c}{\partial \tau} - \Omega^{*2} n^2 R_{mn} q_{mn}^s \right] \sin n\psi \right. \\ \left. + \left[F_{mn} q_{mn}^c + \frac{\partial^2 q_{mn}^c}{\partial \tau^2} R_{mn} + 2\Omega^* n R_{mn} \frac{\partial q_{mn}^s}{\partial \tau} - \Omega^{*2} n^2 R_{mn} q_{mn}^c \right] \cos n\psi \right\} \\ \times R_{kn}(R) \begin{Bmatrix} \sin n\psi \\ \cos n\psi \end{Bmatrix} R \, d\psi \, dR = \int_{\gamma}^1 \int_0^{2\pi} P(R, \psi, \tau) R_{kn}(R) \begin{Bmatrix} \sin n\psi \\ \cos n\psi \end{Bmatrix} R \, d\psi \, dR, \quad (24)$$

where the upper entry in braces gives the sine projection and the lower entry the cosine projection. Changing the order of summation and integration and evaluating the two projections separately gives:

$$\sum_{m=0}^M \left[q_{mn}^s \int_{\gamma}^1 F_{mn} R_{kn} R \, dR \right. \\ \left. + \left(\frac{\partial^2 q_{mn}^s}{\partial \tau^2} - 2\Omega^* n \frac{\partial q_{mn}^c}{\partial \tau} - \Omega^{*2} n^2 q_{mn}^s \right) \int_{\gamma}^1 R_{mn} R_{kn} R \, dR \right] \int_0^{2\pi} \sin^2 n\psi \, d\psi \\ = \int_{\gamma}^1 \int_0^{2\pi} P(R, \psi, \tau) R_{kn} \sin n\psi R \, d\psi \, dR, \\ \sum_{m=0}^M \left[q_{mn}^c \int_{\gamma}^1 F_{mn} R_{kn} R \, dR \right. \\ \left. + \left(\frac{\partial^2 q_{mn}^c}{\partial \tau^2} + 2\Omega^* n \frac{\partial q_{mn}^s}{\partial \tau} - \Omega^{*2} n^2 q_{mn}^c \right) \int_{\gamma}^1 R_{mn} R_{kn} R \, dR \right] \int_0^{2\pi} \cos^2 n\psi \, d\psi \\ = \int_{\gamma}^1 \int_0^{2\pi} P(R, \psi, \tau) R_{kn} \cos n\psi R \, d\psi \, dR. \quad (25)$$

The sine–cosine cross terms have already dropped out at this stage because $\int_0^{2\pi} \sin n\psi \cos n\psi \, d\psi = 0$, which is why the sine projection retains only the sine coordinates and the cosine projection only the cosine coordinates. The mass-like and stiffness-like radial integrals are then introduced as:

$$\Gamma_{kmn}^* = \int_{\gamma}^1 R_{mn}(R) R_{kn}(R) R \, dR, \quad (26)$$

$$\phi_{kmn}^* = \int_{\gamma}^1 F_{mn}(R) R_{kn}(R) R \, dR. \quad (27)$$

The symbol Γ_{kmn}^* denotes the generalised mass coefficient and ϕ_{kmn}^* denotes the generalised stiffness coefficient including bending, rotational-stress and centripetal contributions; both are symmetric in the trial and test indices, $\Gamma_{kmn}^* = \Gamma_{mkn}^*$ and $\phi_{kmn}^* = \phi_{mkn}^*$.

Using Equations (26) and (27), the two projections in Equation (25) become:

$$\begin{aligned} & \sum_{m=0}^M \left[q_{mn}^s \phi_{kmn}^* + \left(\frac{\partial^2 q_{mn}^s}{\partial \tau^2} - 2\Omega^* n \frac{\partial q_{mn}^s}{\partial \tau} - \Omega^{*2} n^2 q_{mn}^s \right) \Gamma_{kmn}^* \right] \int_0^{2\pi} \sin^2 n\psi \, d\psi \\ &= \int_{\gamma}^1 \int_0^{2\pi} P(R, \psi, \tau) R_{kn} \sin n\psi \, R \, d\psi \, dR, \\ & \sum_{m=0}^M \left[q_{mn}^c \phi_{kmn}^* + \left(\frac{\partial^2 q_{mn}^c}{\partial \tau^2} + 2\Omega^* n \frac{\partial q_{mn}^c}{\partial \tau} - \Omega^{*2} n^2 q_{mn}^c \right) \Gamma_{kmn}^* \right] \int_0^{2\pi} \cos^2 n\psi \, d\psi \\ &= \int_{\gamma}^1 \int_0^{2\pi} P(R, \psi, \tau) R_{kn} \cos n\psi \, R \, d\psi \, dR. \end{aligned} \quad (28)$$

The value of the remaining angular integral depends on whether $n = 0$ or $n \neq 0$, and the two cases are treated separately.

For $n = 0$, the angular integrals are:

$$\int_0^{2\pi} \cos^2 n\psi \, d\psi = 2\pi, \quad \int_0^{2\pi} \sin^2 n\psi \, d\psi = 0. \quad (29)$$

The sine projection is identically zero, since $\sin 0 = 0$ carries no axisymmetric coordinate, and only the cosine coordinates survive. Dividing the cosine projection by 2π , Equation (28) becomes:

$$\sum_{m=0}^M \left(q_{m0}^c \phi_{km0}^* + \frac{\partial^2 q_{m0}^c}{\partial \tau^2} \Gamma_{km0}^* \right) = \frac{1}{2\pi} \int_{\gamma}^1 \int_0^{2\pi} P(R, \psi, \tau) R_{k0}(R) R \, d\psi \, dR, \quad k = 0, \dots, M. \quad (30)$$

The axisymmetric family therefore has no gyroscopic or centripetal term: with $n = 0$ the mixed time–angular derivative vanishes and the forward and backward waves coincide, so the $n = 0$ modes are not split by rotation.

For $n \neq 0$, the angular integrals are:

$$\int_0^{2\pi} \cos^2 n\psi \, d\psi = \pi, \quad \int_0^{2\pi} \sin^2 n\psi \, d\psi = \pi. \quad (31)$$

Both projections now carry the same factor π , which divides out. The non-axisymmetric sine and cosine projections are therefore:

$$\begin{aligned}
& \sum_{m=0}^M \left[q_{mn}^s \phi_{kmn}^* + \left(\frac{\partial^2 q_{mn}^s}{\partial \tau^2} - 2\Omega^* n \frac{\partial q_{mn}^c}{\partial \tau} - \Omega^{*2} n^2 q_{mn}^s \right) \Gamma_{kmn}^* \right] \\
&= \frac{1}{\pi} \int_{\gamma}^1 \int_0^{2\pi} P(R, \psi, \tau) R_{kn} \sin n\psi R \, d\psi \, dR, \\
& \sum_{m=0}^M \left[q_{mn}^c \phi_{kmn}^* + \left(\frac{\partial^2 q_{mn}^c}{\partial \tau^2} + 2\Omega^* n \frac{\partial q_{mn}^s}{\partial \tau} - \Omega^{*2} n^2 q_{mn}^c \right) \Gamma_{kmn}^* \right] \\
&= \frac{1}{\pi} \int_{\gamma}^1 \int_0^{2\pi} P(R, \psi, \tau) R_{kn} \cos n\psi R \, d\psi \, dR.
\end{aligned} \tag{32}$$

Rearranging each projection so that the inertial term appears first gives the two coupled equations for the sine and cosine modal coordinates:

$$\begin{aligned}
& \sum_{m=0}^M \left[\frac{\partial^2 q_{mn}^s}{\partial \tau^2} \Gamma_{kmn}^* - 2\Omega^* n \frac{\partial q_{mn}^c}{\partial \tau} \Gamma_{kmn}^* + \left(\phi_{kmn}^* - \Omega^{*2} n^2 \Gamma_{kmn}^* \right) q_{mn}^s \right] \\
&= \frac{1}{\pi} \int_{\gamma}^1 \int_0^{2\pi} P(R, \psi, \tau) R_{kn} \sin n\psi R \, d\psi \, dR,
\end{aligned} \tag{33}$$

$$\begin{aligned}
& \sum_{m=0}^M \left[\frac{\partial^2 q_{mn}^c}{\partial \tau^2} \Gamma_{kmn}^* + 2\Omega^* n \frac{\partial q_{mn}^s}{\partial \tau} \Gamma_{kmn}^* + \left(\phi_{kmn}^* - \Omega^{*2} n^2 \Gamma_{kmn}^* \right) q_{mn}^c \right] \\
&= \frac{1}{\pi} \int_{\gamma}^1 \int_0^{2\pi} P(R, \psi, \tau) R_{kn} \cos n\psi R \, d\psi \, dR.
\end{aligned} \tag{34}$$

Equations (33) and (34) hold for each test index $k = 0, \dots, M$ and are the second-order ordinary differential equations for the sine and cosine modal coordinates of the n -th nodal-diameter family. For a given n they form a coupled set of $2(M+1)$ equations in the $2(M+1)$ unknowns q_{mn}^s and q_{mn}^c ; families of different n remain uncoupled, as anticipated above.

2.4. Matrix Form and State-Space Solution

The $2(M+1)$ scalar equations of a given family are now collected into matrices. Let Γ_n^* and ϕ_n^* be the $(M+1) \times (M+1)$ symmetric matrices whose (k, m) entries are the coefficients Γ_{kmn}^* and ϕ_{kmn}^* of Equations (26) and (27), and let $\mathbf{q}_n^s$ and $\mathbf{q}_n^c$ be the column vectors of the sine and cosine coordinates q_{mn}^s and q_{mn}^c . The sine projection, Equation (33) supplies the upper block row and the cosine projection, Equation (34) the lower block row, and the gyroscopic coupling between the two appears in the off-diagonal blocks. For $n = 1, \dots, N$, Equations (33) and (34) are thus written in the block matrix form:

$$\begin{aligned}
& \begin{bmatrix} \Gamma_n^* & \mathbf{0} \\ \mathbf{0} & \Gamma_n^* \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}}_n^s \\ \ddot{\mathbf{q}}_n^c \end{Bmatrix} + 2\Omega^* n \begin{bmatrix} \mathbf{0} & -\Gamma_n^* \\ \Gamma_n^* & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{q}}_n^s \\ \dot{\mathbf{q}}_n^c \end{Bmatrix} \\
& + \begin{bmatrix} \phi_n^* - \Omega^{*2} n^2 \Gamma_n^* & \mathbf{0} \\ \mathbf{0} & \phi_n^* - \Omega^{*2} n^2 \Gamma_n^* \end{bmatrix} \begin{Bmatrix} \mathbf{q}_n^s \\ \mathbf{q}_n^c \end{Bmatrix} = \begin{Bmatrix} \mathbf{Q}_n^{s*} \\ \mathbf{Q}_n^{c*} \end{Bmatrix}.
\end{aligned} \tag{35}$$

Here, $\mathbf{Q}_n^{s*}$ and $\mathbf{Q}_n^{c*}$ are the dimensionless generalised force components obtained by projecting the external load onto the sine and cosine basis functions. Equation (35) is expressed compactly as:

$$\mathbf{M}_n^* \ddot{\mathbf{q}}_n(\tau) + \mathbf{G}_n^* \dot{\mathbf{q}}_n(\tau) + \mathbf{K}_n^* \mathbf{q}_n(\tau) = \mathbf{Q}_n(\tau), \tag{36}$$

where an overdot denotes differentiation with respect to τ . The mass and gyroscopic matrices are:

$$\mathbf{M}_n^* = \begin{bmatrix} \mathbf{\Gamma}_n^* & \mathbf{0} \\ \mathbf{0} & \mathbf{\Gamma}_n^* \end{bmatrix}, \quad \mathbf{G}_n^* = 2\Omega^* n \begin{bmatrix} \mathbf{0} & -\mathbf{\Gamma}_n^* \\ \mathbf{\Gamma}_n^* & \mathbf{0} \end{bmatrix}, \quad (37)$$

and the stiffness matrix and force vector are:

$$\mathbf{K}_n^* = \begin{bmatrix} \boldsymbol{\phi}_n^* - \Omega^{*2} n^2 \mathbf{\Gamma}_n^* & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\phi}_n^* - \Omega^{*2} n^2 \mathbf{\Gamma}_n^* \end{bmatrix}, \quad \mathbf{Q}_n = \begin{Bmatrix} \mathbf{Q}_n^{s*} \\ \mathbf{Q}_n^{c*} \end{Bmatrix}. \quad (38)$$

The associated displacement, velocity and acceleration vectors are:

$$\mathbf{q}_n = \begin{Bmatrix} \mathbf{q}_n^s \\ \mathbf{q}_n^c \end{Bmatrix}, \quad \dot{\mathbf{q}}_n = \begin{Bmatrix} \dot{\mathbf{q}}_n^s \\ \dot{\mathbf{q}}_n^c \end{Bmatrix}, \quad \ddot{\mathbf{q}}_n = \begin{Bmatrix} \ddot{\mathbf{q}}_n^s \\ \ddot{\mathbf{q}}_n^c \end{Bmatrix}. \quad (39)$$

The matrices $\mathbf{M}_n^*$ and $\mathbf{K}_n^*$ are real and symmetric, while $\mathbf{G}_n^*$ is real and skew-symmetric. Each matrix has size $2(M+1) \times 2(M+1)$.

A light modal damping is added to the model so that the steady-state amplitude at a resonance is finite and the power spectral densities reported below have a well-defined magnitude. A damping matrix $\mathbf{C}_n^*$ is constructed from the real symmetric eigenproblem $\mathbf{K}_n^* \boldsymbol{\Phi} = \mathbf{M}_n^* \boldsymbol{\Phi} \text{diag}(\hat{\omega}_i^2)$ formed from the mass and stiffness matrices of the family at the operating speed, Equations (37) and (38), with the gyroscopic matrix omitted and the eigenvectors normalised so that $\boldsymbol{\Phi}^T \mathbf{M}_n^* \boldsymbol{\Phi} = \mathbf{I}$. The matrix $\mathbf{K}_n^*$ retains the rotation-induced membrane stiffening and the centripetal softening $-\Omega^{*2} n^2 \mathbf{\Gamma}_n^*$, so $\hat{\omega}_i$ are the rotating-frame frequencies of the family and not its non-rotating frequencies. The damping matrix is then defined as

$$\mathbf{C}_n^* = \mathbf{M}_n^* \boldsymbol{\Phi} \text{diag}(2\zeta \hat{\omega}_i) \boldsymbol{\Phi}^T \mathbf{M}_n^*, \quad (40)$$

which gives every mode of the associated non-gyroscopic system the same damping ratio ζ . A small value $\zeta = 0.01$ is used throughout; it is large enough to bound the resonant peaks but small enough to leave the natural frequencies and the travelling-wave structure unchanged (the damped natural frequencies differ from the undamped ones by a factor $\sqrt{1-\zeta^2}$, i.e., by $5 \times 10^{-3}\%$). The damping matrix enters the equations of motion, Equation (36) additively, $\mathbf{M}_n^* \ddot{\mathbf{q}}_n + (\mathbf{G}_n^* + \mathbf{C}_n^*) \dot{\mathbf{q}}_n + \mathbf{K}_n^* \mathbf{q}_n = \mathbf{Q}_n$, and is carried through the state-space form below by replacing $\mathbf{G}_n^*$ with $\mathbf{G}_n^* + \mathbf{C}_n^*$ in the system matrix. The natural-frequency tables reported later are computed from the undamped eigenproblem and are therefore unaffected by this choice.

Two properties of this construction should be stated explicitly, because they matter for the interpretation of the spectra. First, the damping is proportional, and therefore exactly modal, only for the non-gyroscopic system $(\mathbf{M}_n^*, \mathbf{K}_n^*)$. The skew-symmetric $\mathbf{G}_n^*$ is not simultaneously diagonalisable with $\mathbf{M}_n^*$ and $\mathbf{K}_n^*$, so the full rotating system is non-classically damped and the damping ratio actually realised by each travelling wave is obtained from the eigenvalues s_j of the state matrix as $\zeta_j = -\text{Re}(s_j)/|s_j|$. For the families $n = 0-6$ and the four lowest modes of each, $\zeta_j = \zeta$ exactly for $n = 0$ (where $\mathbf{G}_n^* = \mathbf{0}$) and lies within 3.7% and 4.7% of ζ for every mode of the clamped-clamped and free-clamped discs, respectively. The exception is the clamped-free disc close to its critical speed: the $n = 2$ and $n = 3$ travelling-wave pairs of its first radial family realise $\zeta_j = 0.0042$ and 0.0040 , i.e., less than half the nominal value. Second, ζ is a modelling choice made for well-posedness of the spectra and not a measured dissipation; no damping data are available for the discs analysed, and all conclusions below are drawn from level differences at a fixed ζ , whose sensitivity to ζ is quantified in Section 3.7.

Using Equation (4), the dimensional point load in Equation (3) is written in dimensionless form as:

$$P(R, \psi, \tau) = \frac{F^*}{R} \delta(R - R_1) \delta(\psi - \psi_1) \sin(\omega^* \tau - \phi_1). \quad (41)$$

Substitution of this dimensionless load into Equations (33) and (34) gives:

$$\begin{aligned} \sum_{n=0}^N \sum_{m=0}^M \left[\frac{\partial^2 q_{mn}^s}{\partial \tau^2} \Gamma_{kmn}^* - 2\Omega^* n \frac{\partial q_{mn}^c}{\partial \tau} \Gamma_{kmn}^* + \left(\phi_{kmn}^* - \Omega^{*2} n^2 \Gamma_{kmn}^* \right) q_{mn}^s \right] \\ = \frac{1}{\pi} F^* R_{kn}(R_1) \sin n\psi_1 \sin(\omega^* \tau - \phi_1), \end{aligned} \quad (42)$$

$$\begin{aligned} \sum_{n=0}^N \sum_{m=0}^M \left[\frac{\partial^2 q_{mn}^c}{\partial \tau^2} \Gamma_{kmn}^* + 2\Omega^* n \frac{\partial q_{mn}^s}{\partial \tau} \Gamma_{kmn}^* + \left(\phi_{kmn}^* - \Omega^{*2} n^2 \Gamma_{kmn}^* \right) q_{mn}^c \right] \\ = \frac{1}{\pi} F^* R_{kn}(R_1) \cos n\psi_1 \sin(\omega^* \tau - \phi_1). \end{aligned} \quad (43)$$

For several point forces, the right-hand sides of Equations (42) and (43) are summed over all excitation points, amplitudes and phases.

Equation (36) can also be expressed in a descriptor matrix form as:

$$\begin{bmatrix} \mathbf{K}_n^* & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_n^* \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{q}}_n \\ \mathbf{q}_n \end{Bmatrix} + \begin{bmatrix} \mathbf{0} & -\mathbf{K}_n^* \\ \mathbf{K}_n^* & \mathbf{G}_n^* \end{bmatrix} \begin{Bmatrix} \mathbf{q}_n \\ \dot{\mathbf{q}}_n \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{Q}_n \end{Bmatrix}. \quad (44)$$

The state and state-derivative vectors are defined as:

$$\mathbf{x} = \begin{Bmatrix} \mathbf{q}_n \\ \dot{\mathbf{q}}_n \end{Bmatrix}, \quad \dot{\mathbf{x}} = \begin{Bmatrix} \dot{\mathbf{q}}_n \\ \ddot{\mathbf{q}}_n \end{Bmatrix}. \quad (45)$$

Using these vectors, Equation (44) is written as:

$$\mathbf{M}\dot{\mathbf{x}} + \mathbf{G}\mathbf{x} = \mathbf{Q}_n^*. \quad (46)$$

The matrices and force vector in Equation (46) are:

$$\mathbf{M} = \begin{bmatrix} \mathbf{K}_n^* & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_n^* \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \mathbf{0} & -\mathbf{K}_n^* \\ \mathbf{K}_n^* & \mathbf{G}_n^* \end{bmatrix}, \quad \mathbf{Q}_n^* = \begin{Bmatrix} \mathbf{0} \\ \mathbf{Q}_n \end{Bmatrix}. \quad (47)$$

After premultiplication by $\mathbf{M}^{-1}$, the first-order state equation is obtained as:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{Q}_n^*, \quad \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}_n^{*-1}\mathbf{K}_n^* & -\mathbf{M}_n^{*-1}\mathbf{G}_n^* \end{bmatrix}, \quad \mathbf{B} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{M}_n^{*-1} \end{Bmatrix}. \quad (48)$$

Here, $\mathbf{0}$ and $\mathbf{I}$ are the zero and identity matrices of size $2(M+1) \times 2(M+1)$. The matrices $\mathbf{A}$ and $\mathbf{B}$ have sizes $4(M+1) \times 4(M+1)$ and $4(M+1) \times 2(M+1)$, respectively. When damping is included, $\mathbf{G}_n^*$ is replaced by $\mathbf{G}_n^* + \mathbf{C}_n^*$ in the lower-left and lower-right blocks of $\mathbf{A}$.

The output is taken to be the physical transverse displacement of the disc at a fixed observation point (R_o, ψ_o) in the space-fixed frame, so that the reported spectra correspond to a quantity that could be measured, for example, by a displacement probe aimed at a fixed point above the disc. Evaluating the modal expansion, Equation (10) for the n -th family at the observation point gives the scalar output

$$y(\tau) = W(R_o, \psi_o, \tau) = \mathbf{C}\mathbf{x}, \quad \mathbf{C} = \begin{bmatrix} \mathbf{c}_n^T & \mathbf{0} \end{bmatrix}, \quad (49)$$

in which the output row selects the displacement part of the state vector and evaluates the radial functions at the observation point,

$$\mathbf{c}_n^T = [R_{0n}(R_o) \sin n\psi_o, \dots, R_{Mn}(R_o) \sin n\psi_o, R_{0n}(R_o) \cos n\psi_o, \dots, R_{Mn}(R_o) \cos n\psi_o]. \quad (50)$$

The dimensionless output W is multiplied by R_o to recover the displacement in metres before the spectra are formed, so that the power spectral densities carry physical units. The observation point is placed at $r_o = 0.066$ m, $\psi_o = \pi/6$; this radius does not coincide with a nodal circle of the radial families examined, so the point responds to both families used in the parametric study. This state-space form is used to compute the physical time response for each set of excitation parameters and boundary conditions. The output row (50) is the only part of the model that depends on the choice of observation point; the system matrix, and hence the natural frequencies and damping, are independent of it.

In the free-vibration limit, $\mathbf{Q}_n = \mathbf{0}$ and the response may be interpreted in terms of travelling waves. The corresponding space-fixed frequencies satisfy the standard split:

$$\omega_{mn1,2}^* = n\Omega^* \pm \lambda_{mn}^*, \quad (51)$$

where $\lambda_{mn}^* > 0$ is the dimensionless modal frequency measured in the rotating frame. Hence, each $n \neq 0$ mode appears as a backward/forward travelling-wave pair separated by $2n\Omega^*$ in dimensionless space-fixed frequency.

The sign convention in Equation (51) requires comment, because $n\Omega^* - \lambda_{mn}^*$ may be negative. The quantity that is measured by a space-fixed probe, and that is plotted on the frequency axis of every spectrum reported below, is the positive frequency $|\omega_{mn1,2}^*|$; the sign of $n\Omega^* - \lambda_{mn}^*$ carries only the sense in which the wave pattern travels in the inertial frame. The branch $n\Omega^* - \lambda_{mn}^*$ vanishes at the critical speed of that mode and changes sign above it, so that a backward wave whose space-fixed frequency has been driven to zero reappears on the positive frequency axis and travels forward in space as the speed is increased further. Throughout this paper $f_1 < f_2 < \dots$ therefore denote the ordered moduli. A useful consequence, used below as a check on the computed spectra, is that the two members of a pair satisfy $(f_1 + f_2)/2 = \lambda_{mn}$ and $(f_2 - f_1)/2 = n\Omega$ in Hz whenever the pair is subcritical.

Two complementary solutions of the forced problem are used in this paper, and it is important to keep them apart. The first is the exact steady-state harmonic response, obtained in closed form from the frequency-domain resolvent of Equation (48). For a set of J point forces of common frequency ω^* and individual phases ϕ_j , the input of the j -th force is $u_j(\tau) = \sin(\omega^*\tau - \phi_j) = \text{Im}\{e^{i(\omega^*\tau - \phi_j)}\}$, and the particular solution of Equation (48) is

$$\mathbf{x}_{ss}(\tau) = \text{Im}\left\{ (i\omega^*\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} \sum_{j=1}^J \mathbf{Q}_{n,j} e^{-i\phi_j} e^{i\omega^*\tau} \right\}, \quad (52)$$

so that the steady output is $W_{ss}(R_o, \psi_o, \tau) = \text{Im}\{a e^{i\omega^*\tau}\}$ with the complex amplitude $a = \mathbf{C} (i\omega^*\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} \sum_j \mathbf{Q}_{n,j} e^{-i\phi_j}$. Equation (52) involves no time integration, no record length, no window and no initial condition, and its spectrum consists of the excitation line alone. Because every nodal-diameter family is driven at the same frequency, the complex amplitudes a of the families $n = 0, 1, \dots$ add coherently at the observation point, and the total steady amplitude is $R_o |\sum_n a_n|$ in metres.

The second solution, used for the waterfall diagrams, is a direct time integration of the state Equation (48) for each excitation case, followed by spectral estimation of the physical output, Equation (49). This is a finite-duration record that starts at the instant the force is switched on; it therefore contains the switch-on transient in addition to the steady state, and Section 3.7 establishes which features of the reported spectra belong to

which. The reproducibility details are as follows. The state equation is integrated with the fixed-step linear time-invariant solver of `scipy.signal.lsim`, which advances the exact matrix-exponential solution of the linear system between samples under a first-order (linear) hold on the harmonic input; this is exact for a linear time-invariant system driven by a band-limited input and therefore introduces no scheme-dependent numerical damping. The disc starts from rest, so the initial state is zero, $\mathbf{x}(0) = \mathbf{0}$: the force is switched on at $\tau = 0$ and the record contains the switch-on transient followed by the steady state. The transient is retained deliberately, because it is what populates the non-driven travelling-wave peaks of the waterfall diagrams; the steady state alone would show only the driven line. The spectra reported below are therefore finite-duration switch-on spectra, and only the driven line is a steady-state quantity. Section 3.7 quantifies the resulting dependence on the record duration, the force onset and the damping ratio, and shows that every parametric conclusion drawn in this paper is read from the driven line and is reproduced by the resolvent solution, Equation (52). Each case is integrated for a physical record of $T_{\text{rec}} = 1$ s at a sampling frequency $f_s = 65,536$ Hz, giving $N = 2^{16} = 65,536$ samples (numerically equal to f_s only because the record length is $T_{\text{rec}} = 1$ s) and a Nyquist frequency of 32,768 Hz, well above the highest mode examined. The corresponding dimensionless step is $\Delta\tau = 1/(f_s T_{\text{char}})$, where T_{char} is the characteristic time of Equation (4). The physical displacement record $w(t) = R_o W(\tau)$ is multiplied by a Hann window and transformed with a single periodogram (`scipy.signal.periodogram`, density scaling), which yields the one-sided power spectral density in $\text{m}^2 \text{Hz}^{-1}$ at a frequency resolution of $\Delta f = f_s/N = 1$ Hz. The spectra are reported in decibels relative to $1 \text{ m}^2 \text{Hz}^{-1}$, i.e., $10 \log_{10}(\text{PSD})$ with no per-case rescaling. Because the output is now a physical displacement at a single fixed point and the normalisation of the radial functions cancels in Equation (49), the decibel levels are directly comparable between traces and between boundary conditions, and the vertical axes of the waterfall diagrams carry absolute physical meaning. The waterfall diagrams are read both in terms of peak locations and in terms of these absolute levels. All computations reported in this paper, and all figures, were produced with Python 3.10.12 and the open-source libraries NumPy 2.2.6, SciPy 1.15.3 and Matplotlib 3.10.9. The notation used in the verification and parametric-response sections is collected in Table 2.

Table 2. Notation used in the verification and parametric-response sections.

Symbol	Meaning
f_1 – f_4	First four (in ascending order) space-fixed natural frequencies of the selected $n = 2$ family
r_1, ψ_1	Radial and angular coordinates of the first excitation point
$\Delta\psi$	Angular separation between two excitation points
$\Delta\phi$	Phase difference between two harmonic point forces
PSD [dB re $1 \text{ m}^2 \text{Hz}^{-1}$]	Power spectral density of the physical transverse displacement at the observation point, in decibels relative to $1 \text{ m}^2 \text{Hz}^{-1}$
r_o, ψ_o	Radial and angular coordinates of the fixed observation point ($r_o = 0.066 \text{ m}$, $\psi_o = \pi/6$)
ζ	Modal damping ratio applied to every mode of the associated non-gyroscopic system ($\zeta = 0.01$)
T_{rec}, t_r	Record duration of a simulated response and rise time of the half-cosine force ramp
λ_{mn}	Modal frequency of the (m, n) mode measured in the rotating frame
FTW, BTW	Forward and backward travelling waves

3. Results of Numerical Examples and Discussion

The numerical examples are organised to move from verification to physical interpretation. Example 1 establishes the natural-frequency benchmark and the baseline modal properties of the analysed disc. Example 2 checks the forced-response implementation against the published clamped–free benchmark. Examples 3–6 then examine how the response changes with excitation frequency, radial force position, angular separation and phase difference.

The formulation and its numerical implementation are compared with the results available in the literature at two levels: a quantitative comparison of natural frequencies and a qualitative comparison of forced-response spectra.

Two terminological points are fixed before the results are presented, because they bound what the comparisons in this section can establish. First, verification is used here in its usual computational-mechanics sense: evidence that the stated mathematical model is solved correctly. The convergence study, the independent radial finite-element solution of the same rotating Kirchhoff model, and the agreement with previously published solutions of that model are all verification exercises. Validation, that is, evidence that the model represents a physical disc, would require measurements on a real rotating disc, and no such measurements are reported here. The experimental comparison reported in the earlier free-vibration study [8] supports the free-vibration part of the underlying model, but it does not validate the forced-response predictions of this paper; the word “validation” is used below only where an experimental comparison is being referred to. Second, the spectra of Sections 3.2–3.6 are power spectral densities of a finite-duration record that begins when the force is switched on. Their driven line is a steady-state quantity; their remaining structure is switch-on transient content. Section 3.7 separates the two and demonstrates the sensitivity of the transient part to the record duration, the force onset and the damping.

3.1. Example 1: Natural-Frequency Benchmark and Baseline Modal Properties

Two literature comparisons are used, and their geometries are kept explicit in Table 3. The clamped–free forced-vibration benchmark uses the disc of [27], whereas the clamped–clamped and free–clamped checks use the hard-disk geometry reported in the free-vibration study of [7]. The primary quantitative benchmark is the rotating clamped–free disc of [27]: $R_o = 0.08$ m, $R_i = 0.024$ m, $h = 0.001$ m, $\rho = 7850$ kg m^{−3}, $E = 206$ GPa, $\nu = 0.3$, rotating at the dimensionless speed $\Omega^* = 5$, i.e., $\Omega = 192.7514$ Hz. In addition to the literature comparison, the Galerkin implementation is verified against an independent numerical solution of the same rotating Kirchhoff model: a one-dimensional finite-element discretisation of each nodal-diameter family, using C^1 cubic Hermite elements in the radial coordinate with the same rotation-induced membrane stress fields and the same gyroscopic and centripetal terms, refined to 200 elements so that its frequencies are converged to well below 0.01%. This model shares no code, basis or assembly with the polynomial Galerkin solution and therefore provides an independent reference for all three boundary conditions.

Table 3. Geometries and material parameters used for the literature comparisons and for the parametric study. The clamped–free verification case and the parametric results use the same disc as [27]; the clamped–clamped and free–clamped consistency checks use the hard-disk geometry reported by [7].

Case	R_o [m]	R_i [m]	h [m]	ρ [kg m ^{−3}]	E [GPa]	ν
C–F verification and present parametric study	0.0800	0.0240	0.0010	7850	206	0.30
C–C/F–C free-vibration consistency check	0.0475	0.01625	0.0013	7850	206	0.30

Table 4 reports the convergence of the four lowest space-fixed frequencies of the $n = 2$ family at $\Omega^* = 5$ as the number of retained radial functions is increased from three to

eight, together with the finite-element reference. The convergence is rapid and monotonic from above, as expected for an admissible-function method: with six radial functions every tabulated frequency of all three boundary conditions agrees with the finite-element reference to 0.001% or better, and with seven functions the tabulated digits coincide. All results reported in this paper therefore use six radial functions; increasing the number further changes none of the reported digits.

Table 4. Convergence of the four lowest space-fixed natural frequencies [Hz] of the $n = 2$ family at $\Omega^* = 5$ with the number of radial functions M_f , and comparison with the independent radial finite-element (FE) solution (200 Hermite elements).

	M_f	1st	2nd	3rd	4th
C–C	3	1589.52	2360.53	4819.54	5590.55
	4	1588.75	2359.76	4774.33	5545.34
	5	1588.44	2359.45	4774.32	5545.32
	6	1588.41	2359.42	4772.42	5543.42
	7	1588.41	2359.41	4772.37	5543.38
	8	1588.41	2359.41	4772.37	5543.38
	FE	1588.41	2359.41	4772.37	5543.38
C–F	3	39.79	810.79	1680.53	2451.53
	4	39.60	810.60	1679.18	2450.19
	5	39.56	810.57	1678.62	2449.62
	6	39.56	810.56	1678.56	2449.57
	7	39.56	810.56	1678.55	2449.56
	8	39.56	810.56	1678.55	2449.56
	FE	39.56	810.56	1678.55	2449.56
F–C	3	885.23	1656.23	2667.94	3438.95
	4	884.61	1655.62	2666.28	3437.28
	5	884.49	1655.49	2665.93	3436.93
	6	884.47	1655.48	2665.56	3436.57
	7	884.47	1655.48	2665.52	3436.53
	8	884.47	1655.48	2665.52	3436.53
	FE	884.47	1655.48	2665.52	3436.53

Table 5 compares the space-fixed natural frequencies with the values reported in the reference. The agreement is within 0.03% for every mode.

Table 5. Space-fixed natural frequencies of the rotating clamped–free disc ($\Omega^* = 5$, $\Omega = 192.7514$ Hz) compared with [27].

Excited Mode	[27] [Hz]	Present [Hz]	Difference
$n = 1$, 1st mode	153.0948	153.09	<0.01%
$n = 2$, 1st mode	39.5561	39.55	0.01%
$n = 2$, 3rd mode	1678.9664	1678.55	0.02%

Figure 2 shows the present forced-response spectra for the two single-force cases of the reference, with the reported modal frequencies superimposed as dashed lines. The resonant peaks of the present solution fall exactly on the reference values, and the secondary peaks appear as forward/backward travelling-wave pairs separated by the gyroscopic split $2n\Omega = 771$ Hz, as predicted by the travelling-wave split described above. The status of those secondary peaks should be stated here, since the same feature recurs throughout Section 3. A linear system driven harmonically at a single frequency responds, in the steady state, only at that frequency; the peaks seen at the other natural frequencies of the family exist because the force is switched on at $\tau = 0$ and the record therefore contains

the free decay of the family as well as its forced response. The switch-on acts, in effect, as a broadband modal probe applied at the same instant as the harmonic force, which is what makes a single record display both the driven line and the complete travelling-wave structure of the family. This is convenient for the frequency comparison made here, and it is why the switch-on records are used as the primary figures, but it means that the levels of the secondary peaks are properties of the onset, the record length and the damping rather than forced-response characteristics. Section 3.7 quantifies this and gives the corresponding steady-state solution.

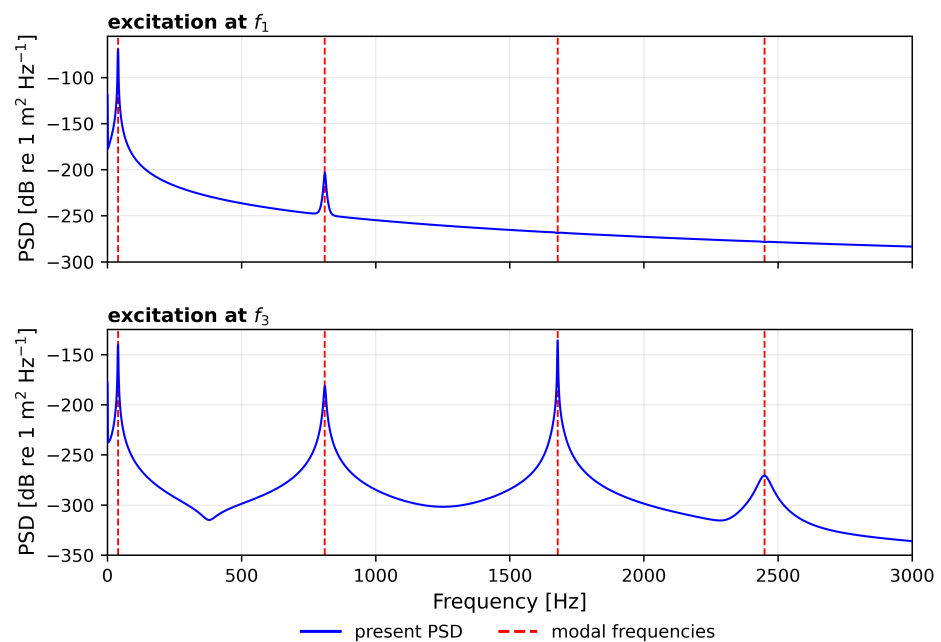


Figure 2. Computed PSD of the physical displacement at the observation point (solid) and the modal frequencies of the $n = 2$ family (dashed): **(top)** excitation at the first-mode frequency; **(bottom)** excitation at the third-mode frequency. Clamped-free disc, $n = 2$, $\Omega = 192.7514$ Hz. PSD in dB re $1 \text{ m}^2 \text{ Hz}^{-1}$; step onset at $t = 0$, $T_{\text{rec}} = 1$ s, $\zeta = 0.01$.

The clamped-clamped and free-clamped implementations are compared with the free-vibration results of [7], who analysed a steel HDD disc with $R_o = 47.5$ mm, $R_i = 16.25$ mm and $h = 1.3$ mm at rest and at 4000 and 8000 rpm. Table 6 compares the fundamental ($m = 0$) stationary frequencies and also lists the independent finite-element solution of the same rotating Kirchhoff model described above. Two observations follow. First, the present Galerkin values and the finite-element values agree to within 0.001% for every mode of both boundary conditions, which verifies that the present implementation solves the stated thin-plate model correctly. Second, the values reported in [7] lie uniformly about 2.1% below the converged Kirchhoff solution for the C–C modes, with a nearly constant ratio across all nodal diameters, and 2.1–3.0% below it for the F–C modes. A constant frequency ratio across all mode numbers is the signature of a difference in the assumed material constants rather than of a discretisation effect, because the stationary frequencies of a given geometry scale uniformly with $\sqrt{E/\rho}$; the reference does not state the elastic modulus and density it used, which were taken from the finite-element and experimental characterisation of its source study, so an exact reconciliation is not possible. A contribution from the discretisation of the reference cannot be excluded for the F–C case, where the offset varies with n . A small part of the offset with respect to measured data would additionally be expected from the shear flexibility neglected by the Kirchhoff model at these high frequencies, as discussed in Section 2.1. Since the present values are confirmed by an independent converged solution

of the same equations, the residual offset is attributed to the unreported material constants and to the numerical treatment of the reference. The offset is nearly uniform for the C–C modes and does not change the ordering of the nodal-diameter families. More importantly for the present rotating-disc formulation, the speed-dependent quantities are reproduced: at 8000 rpm the forward/backward splitting equals $2n\Omega$ to four significant figures for every mode in both studies, and the relative speed-induced frequency shifts agree to better than 0.05%. For example, the $n = 0$ free–clamped mode softens by 0.20% between 0 and 8000 rpm in the present computation against 0.21% in the reference, and the mean of the $n = 2$ travelling-wave pair stiffens by 0.03% in both. The comparison therefore verifies the rotational trends and boundary-condition implementation, while the absolute levels of the present model are anchored by the finite-element verification.

Table 6. Comparison of the clamped–clamped (C–C) and free–clamped (F–C) implementations: fundamental ($m = 0$) natural frequencies [Hz] of the stationary disc of [7], together with the independent radial finite-element (FE) solution of the same Kirchhoff model. “Diff.” is the deviation of the reference values from the present converged solution.

n	C–C				F–C			
	[7]	Present	FE	Diff.	[7]	Present	FE	Diff.
0	7150.87	7305.9	7305.9	−2.1%	1679.91	1729.9	1729.9	−2.9%
1	7320.71	7479.4	7479.4	−2.1%	2667.67	2751.2	2751.2	−3.0%
2	7893.05	8064.2	8064.2	−2.1%	4448.18	4558.9	4558.9	−2.4%
3	9005.47	9200.8	9200.7	−2.1%	6706.77	6851.0	6851.0	−2.1%
4	10,760.38	10,993.8	10,993.7	−2.1%	9424.54	9622.6	9622.6	−2.1%

Baseline Natural Frequencies of the Analysed Disc

The parametric examples consider the disc used for the clamped–free benchmark in Table 3 ($R_o = 0.08$ m, $R_i = 0.024$ m, $h = 0.001$ m, steel) rotating at $\Omega^* = 5$ ($\Omega = 192.7514$ Hz), under the three sets of boundary conditions. Table 7 lists the four lowest space-fixed natural frequencies for nodal-diameter numbers $n = 0$ –4. Three features are worth noting. First, the stiffness ordering of the boundary conditions is evident at every n : the clamped–clamped disc is by far the stiffest, the free–clamped disc is intermediate, and the clamped–free disc is the most flexible. Mechanically, the C–C condition suppresses both transverse deflection and slope at both edges, so the admissible bending shapes require larger curvature and produce higher frequencies. The C–F condition behaves as an annular cantilever, so the free outer edge permits large transverse motion and the lowest resonance range is obtained. The F–C condition is not the mirror image of the C–F response because the clamped edge is now at the outer radius, where the bending lever arm and circumferential length are larger; this raises the frequencies relative to the C–F case. Second, every $n \neq 0$ entry appears as a backward/forward travelling-wave pair separated by exactly $2n\Omega$ (771 Hz for $n = 2$), in accordance with the travelling-wave split described above. Third, for the clamped–free disc the $n = 2$ backward wave has been brought down to 39.55 Hz by the rotation; at $\Omega^* = 5$ this disc operates close to its critical speed for the (0, 2) mode, which is the classical behaviour of flexible spinning discs. In practical terms, the boundary condition therefore sets not only the modal shape but also the operating frequency band in which a space-fixed excitation can become dangerous.

The forced-response examples below consider the $n = 2$ family, which contains the modes used in the clamped–free benchmark of Example 1. Unless stated otherwise, a single unit force is applied at $r_1 = 0.066$ m, $\psi_1 = \pi/6$, with zero phase, and the response is reported as the physical transverse displacement at the fixed observation point $(r_o, \psi_o) = (0.066$ m, $\pi/6)$ defined in Section 2.4. The four ordered $n = 2$ frequencies of

each boundary condition are denoted f_1 – f_4 (rows $n = 2$ of Table 7); f_1, f_2 are the backward/forward pair of the first radial family and f_3, f_4 the pair of the second.

Figure 3 shows the radial shapes of the first two radial families of the $n = 2$ modes for the three boundary conditions, computed from the eigenvectors of the associated symmetric problem at $\Omega^* = 5$ and normalised to unit maximum. These shapes are the physical key to the radial-position results of Example 4: the generalised force of a family is proportional to the value of its shape at the excitation radius, so the antinodes and interior nodes visible in Figure 3 translate directly into maxima and minima of the resonant response as the force is moved radially. The second radial family, which is the one driven at f_3 in Examples 4–6, has an interior node at $r \approx 52$ mm for the C–C disc, $r \approx 69$ mm for the C–F disc and $r \approx 44$ mm for the F–C disc; the first family has none. The dotted line marks the baseline force radius $r_1 = 0.066$ m.

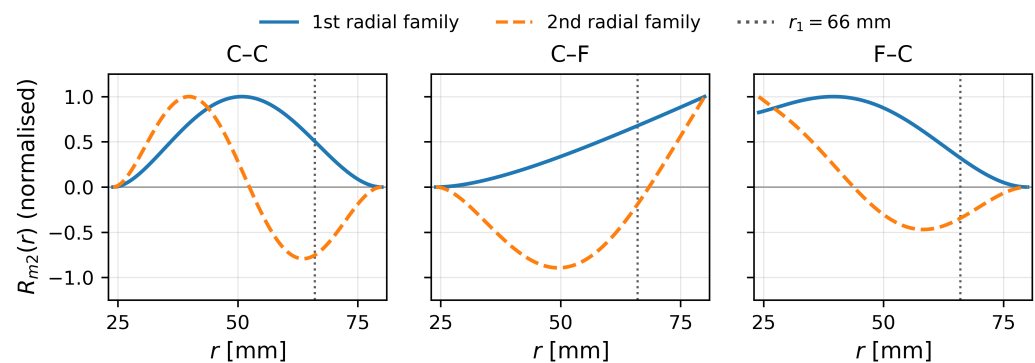


Figure 3. Radial shapes $R_{m2}(r)$ of the first (solid) and second (dashed) radial families of the $n = 2$ modes at $\Omega^* = 5$, normalised to unit maximum: clamped–clamped (**left**), clamped–free (**centre**) and free–clamped (**right**). The dotted vertical line marks the baseline excitation radius $r_1 = 0.066$ m.

Table 7. Four lowest space-fixed natural frequencies [Hz] of the analysed disc at $\Omega^* = 5$ for the three boundary conditions (six radial functions). For $n = 0$ the forward and backward waves coincide, so the four entries are the four distinct axisymmetric frequencies.

n	1st	2nd	3rd	4th
Clamped–clamped				
0	1745.95	4835.97	9505.56	15,821.17
1	1604.55	1990.06	4721.45	5106.95
2	1588.41	2359.42	4772.41	5543.42
3	1742.02	2898.53	5011.34	6167.85
4	2087.59	3629.60	5462.80	7004.82
Clamped–free				
0	333.80	1738.70	4860.35	9537.10
1	153.08	538.59	1625.04	2010.53
2	39.55	810.54	1678.55	2449.58
3	52.66	1209.11	1916.17	3072.74
4	190.94	1733.08	2343.62	3885.52
Free–clamped				
0	419.47	1978.32	5082.10	9760.96
1	560.97	946.48	2101.47	2486.97
2	884.47	1655.48	2665.56	3436.57
3	1328.34	2484.85	3446.34	4602.85
4	1884.13	3426.14	4410.60	5952.61

3.2. Example 2: Forced-Response Benchmark

Figure 4 reproduces four representative excitation cases of [27] with the present implementation: a single force tuned to the first and to the third mode of the $n = 2$ family, the third-mode excitation with a phase angle of $\pi/2$, and a pair of forces at different locations and phases acting on the $n = 1$ family. These cases correspond to the PSD diagrams reported in Figures 1–5 of [27]. The present formulation recovers the same qualitative behaviour reported in the reference: the response peaks at the excitation frequency; a force tuned to the third mode produces a dominant peak at that frequency; the phase of a single force leaves the resonant peak unchanged while redistributing the relative amplitudes of the non-resonant modes (this redistribution, Figure 4b against Figure 4c, lies entirely in the switch-on transient: the phase of a single harmonic force is a pure shift of the time origin, so it cannot alter the steady-state spectrum at all. The resolvent amplitude is identical for $\phi_1 = 0$ and $\phi_1 = \pi/2$ to machine precision, the driven line of both records is -135.8 dB, and only the non-driven transient peaks move, by 32.5 dB); and two spatially separated forces superpose into a combined response. A direct point-by-point amplitude comparison is not possible because the reference spectra are published on a relative decibel scale, the absolute PSD reference is not specified, and the published diagrams include modal contributions from more than one nodal-diameter family; in addition, the present model includes a light modal damping ($\zeta = 0.01$) that bounds the resonant peak heights, whereas the reference integrated the undamped equations, so the peak heights are not comparable while the peak locations are unaffected. Therefore, Figure 4 is used as a qualitative forced-response comparison, while the quantitative verification of the model rests on Table 5.

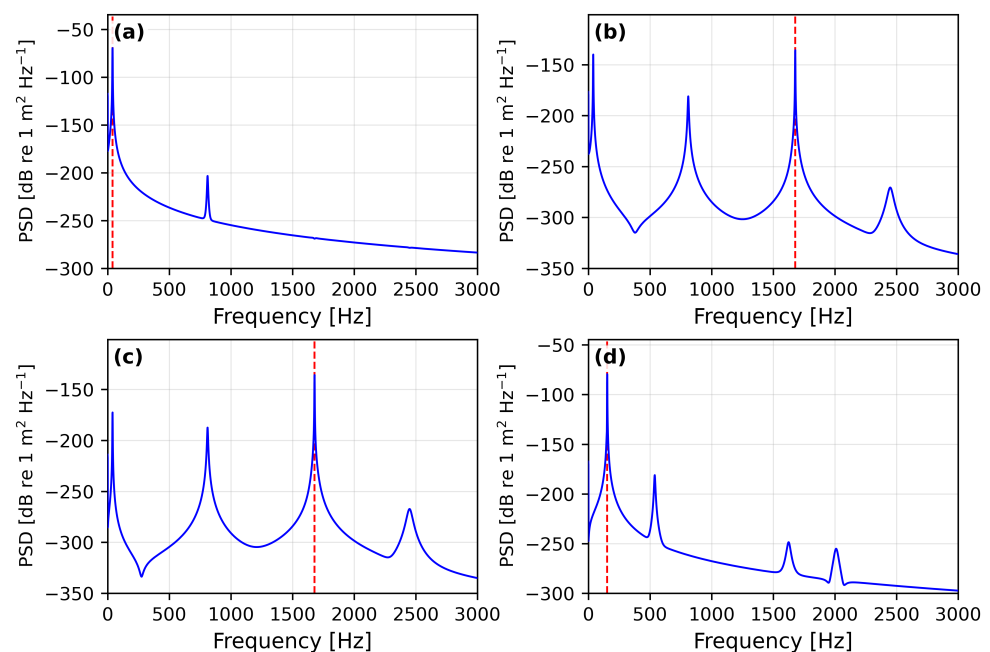


Figure 4. Reproduction of the forced-response PSD cases of [27]: (a) single force at the first-mode frequency; (b) single force at the third-mode frequency; (c) single force at the third-mode frequency with phase angle $\pi/2$; (d) two forces on the $n = 1$ family, at $(0.066 \text{ m}, \pi/6)$ and $(0.050 \text{ m}, 2\pi/3)$ with phases 0 and $\pi/4$. The dashed line marks the excitation frequency. PSD in dB re $1 \text{ m}^2 \text{ Hz}^{-1}$; step onset at $t = 0$, $T_{\text{rec}} = 1 \text{ s}$, $\zeta = 0.01$.

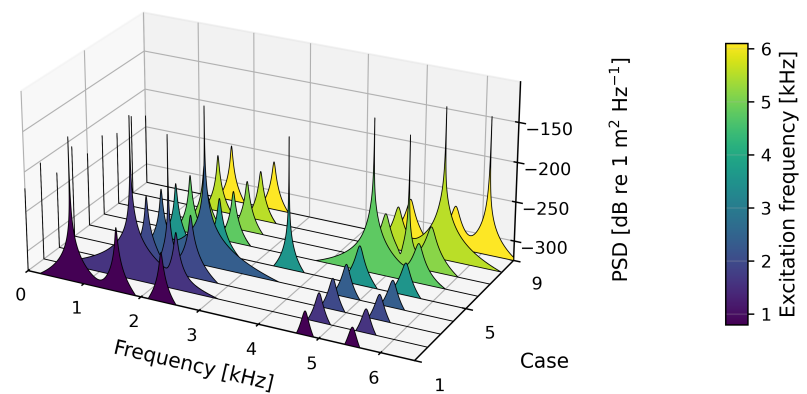
3.3. Example 3: Excitation-Frequency Sweep

Figure 5 shows waterfall diagrams of the PSD as the excitation frequency is stepped through nine values placed on and between the modes: $0.5f_1$, f_1 , $(f_1 + f_2)/2$, f_2 , $(f_2 + f_3)/2$, f_3 , $(f_3 + f_4)/2$, f_4 and $1.1f_4$ (cases 1–9; Table 8 lists the values). The behaviour is common to all three boundary conditions: whenever the excitation coincides with a natural frequency (even-numbered cases 2, 4, 6, 8), the spectrum is dominated by a single sharp peak at the driven mode, while between modes (odd-numbered cases) the overall level drops sharply: for the C–C disc the driven line falls from -120.3 dB at f_1 to -151.3 dB midway between f_1 and f_2 , and by comparable amounts for the other two supports. This resonant/anti-resonant contrast is a property of the driven line, hence of the steady state, and is unchanged by the loading onset. The residual peaks visible at the non-driven natural frequencies are, by contrast, switch-on transient content. For the C–C and F–C discs they lie 75–133 dB below the driven line and are therefore immaterial, but for the C–F disc, whose $n = 2$ first-family waves lie close to the critical speed and decay with a time constant of 0.96 s, they are of the same order as the driven line and in cases 7 and 9 exceed it by 2.7 and 5.9 dB. Applying the same force through a 50 ms half-cosine ramp instead of a step suppresses them by 71–119 dB in cases 3–9—and by only 6–46 dB in the two lowest-frequency cases, where a 50 ms rise is not slow compared with the 25 ms period of the mode concerned—while leaving the driven line unchanged to within 0.4 dB in every case. The odd-numbered C–F panels of Figure 5 should therefore be read as switch-on spectra of a lightly damped near-critical disc, not as a steady-state redistribution of energy; Section 3.7 gives the full sensitivity study. Cases 3 and 7 also fall exactly on the rotating-frame frequencies λ_{1n} and λ_{2n} of the first and second radial families, by the pairing rule noted after Equation (51). The lowest resonantly excitable frequency differs by a factor of forty between the clamped–free and clamped–clamped panels (39.6 Hz against 1588 Hz), which quantifies how strongly the boundary conditions set the frequency range in which resonant amplification can be provoked: the same disc is resonantly excitable below 50 Hz when clamped–free, but only above 1.5 kHz when clamped–clamped. This has a direct design implication. A space-fixed forcing mechanism that is harmless for a C–C mounting may fall close to a C–F backward-wave resonance after a change in support flexibility, even if the material, thickness and rotation speed are unchanged.

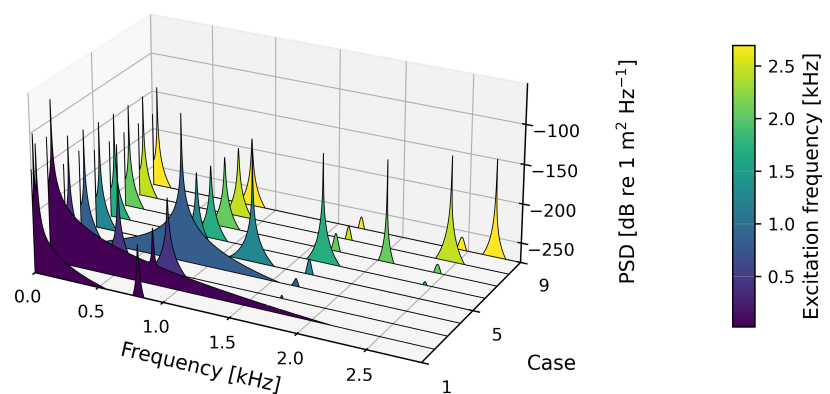
Table 8. Excitation frequencies [Hz] used in the frequency sweep of Figure 5 ($n = 2$ family of each boundary condition).

Case	1	2	3	4	5	6	7	8	9
C–C	794.2	1588.4	1973.9	2359.4	3565.9	4772.4	5157.9	5543.4	6097.8
C–F	19.8	39.6	425.1	810.6	1244.6	1678.6	2064.1	2449.6	2694.5
F–C	442.2	884.5	1270.0	1655.5	2160.5	2665.6	3051.1	3436.6	3780.2

(a) C-C



(b) C-F



(c) F-C

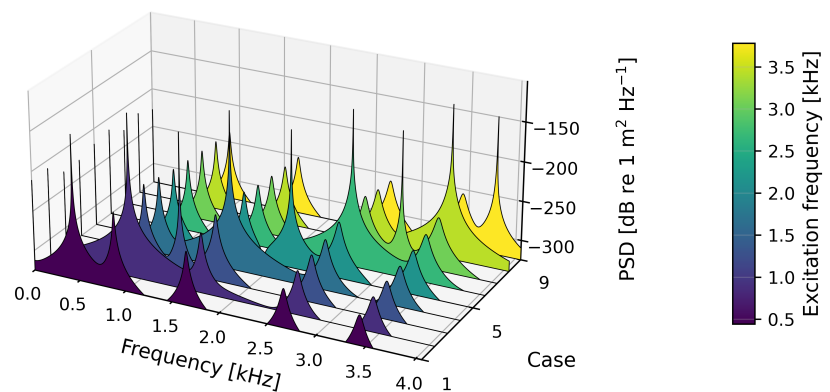


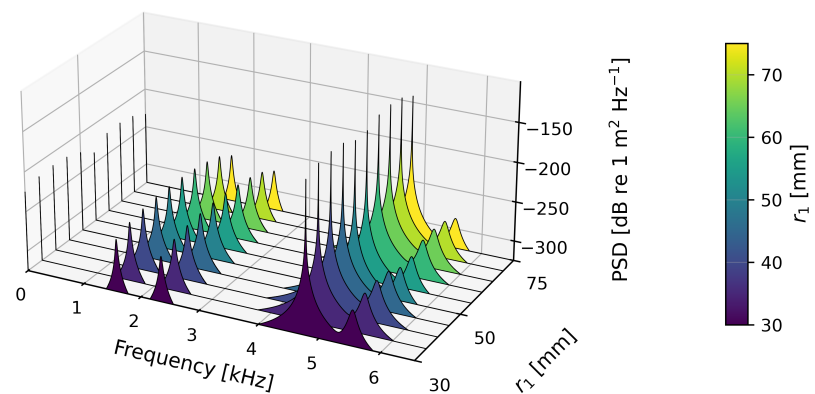
Figure 5. Waterfall diagrams of the PSD for the excitation-frequency sweep (cases 1–9 of Table 8), single unit force at $r_1 = 0.066$ m, $\psi_1 = \pi/6$, $\phi_1 = 0$, $n = 2$: (a) clamped–clamped; (b) clamped–free; (c) free–clamped. Frequency in kHz; vertical axis, PSD of the physical displacement at the observation point in dB re $1 \text{ m}^2 \text{ Hz}^{-1}$; parameter axis and colour scale, the swept parameter. Step onset at $t = 0$, $T_{\text{rec}} = 1$ s, $\zeta = 0.01$.

3.4. Example 4: Radial Force-Position Sweep

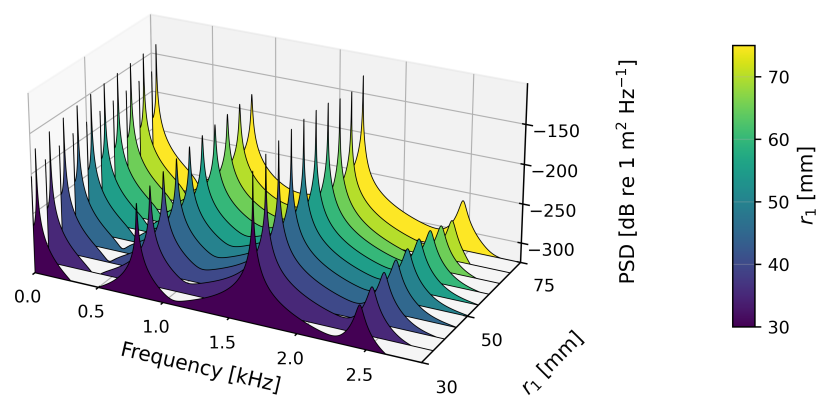
Figure 6 presents the waterfalls obtained by moving a single force, tuned to f_3 of each boundary condition, radially from $r_1 = 0.030$ m to $r_1 = 0.075$ m in steps of 5 mm. The generalised force of each radial family is proportional to the value of its shape function at the excitation radius, Equations (42) and (43), and the waterfalls map this dependence directly. The mechanism can be read off the mode shapes of Figure 3, since the excitation at f_3 drives

the second radial family. For the clamped–clamped disc both edges are nodes of every shape function, so the strongest resonant response occurs for excitation near the antinodes of the second family and decays towards either edge; the deepest local minimum of the driven peak occurs near $r_1 = 0.050\text{--}0.055$ m, precisely where the second family has its interior node ($r \approx 52$ mm in Figure 3), and the level drops by about 14 dB as the force approaches the clamped outer rim. On the 5 mm sweep grid the minimum is resolved only to 11 dB below the panel maximum; a fine steady-state sweep (Section 3.7) places the node at $r_1 = 52.5$ mm and the true minimum 27.7 dB below the maximum, which occurs at $r_1 = 40.0$ mm. For the clamped–free disc the response is strongest for excitation near $r_1 = 0.050$ m, where the second-family shape attains its interior antinode (the fine sweep gives 49.5 mm), is suppressed by about 12 dB near $r_1 = 0.070$ m, at the interior node of the second family ($r \approx 69$ mm; 68.0 mm and 15.1 dB on the fine sweep, with the node of the mode shape at 68.4 mm), and recovers towards the free outer edge, where the second-family shape rises again. The free rim is, however, not the optimal excitation point of the clamped–free disc at f_3 . The response at the rim remains 3.4 dB below the interior maximum at 49.5 mm, although it is 11.3 dB above the level at the inner end of the swept range ($r_1 = 30$ mm, close to the clamped inner edge). What the free edge provides is a region of large modal displacement for both radial families, which raises the disc’s overall excitability there, but the placement that maximises or minimises the response of the targeted family is set by that family’s own antinode and node. For the free–clamped disc the trend is the mirror image: the response is strongest when the force acts near the free inner edge, which for this disc is the antinode of the driven family within the swept range, the level at $r_1 = 30$ mm being the largest of the sweep, is suppressed by about 20 dB near $r_1 = 0.045$ m, at the interior node of its second family ($r \approx 44$ mm; the fine sweep gives 44.0 mm and 28.0 dB, with the mode-shape node at 43.8 mm), and decays by about 21 dB as the force approaches the clamped outer rim, where all shape functions vanish. In every case the minima of the driven peak occur where the projection of the point force onto the driven radial family is small, while neighbouring families remain more strongly excited, so the spectrum does not vanish. This is a steady-state effect and not a transient one: at the node of the driven family the remaining radial families of the same nodal-diameter family are still forced, and they respond off resonance at the excitation frequency, which sets the residual level of the driven line. Energy does not appear at the natural frequencies of those other families; that content is free decay from the onset and is quantified in Section 3.7. The radial position of the force is therefore an effective lever for selecting which radial family dominates the response, and its optimal placement is dictated by the boundary conditions through the node/antinode pattern of the driven family. In general terms, the position at which a chosen radial family is least excited is the interior node of that family—52.5 mm (C–C), 68.0 mm (C–F) and 44.0 mm (F–C) for the second family of the discs analysed—and the position at which it is most excited is the corresponding antinode. Whether either of these happens to fall near a free edge is incidental and differs between the three supports: it does for the free–clamped disc, whose antinode within the swept range is at the free inner edge, and it does not for the clamped–free disc, whose antinode is interior.

(a) C-C



(b) C-F



(c) F-C

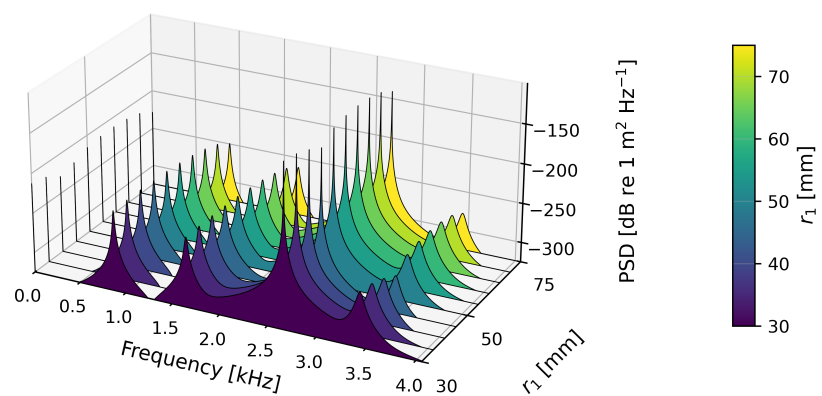


Figure 6. Waterfall diagrams of the PSD for the radial-position sweep $r_1 = 0.030\text{--}0.075$ m (single unit force at $\psi_1 = \pi/6$, $\phi_1 = 0$, excitation at f_3 , $n = 2$): (a) clamped–clamped; (b) clamped–free; (c) free–clamped. Frequency in kHz; vertical axis, PSD of the physical displacement at the observation point in dB re $1\text{ m}^2\text{ Hz}^{-1}$; parameter axis and colour scale, the swept parameter. Step onset at $t = 0$, $T_{\text{rec}} = 1$ s, $\zeta = 0.01$.

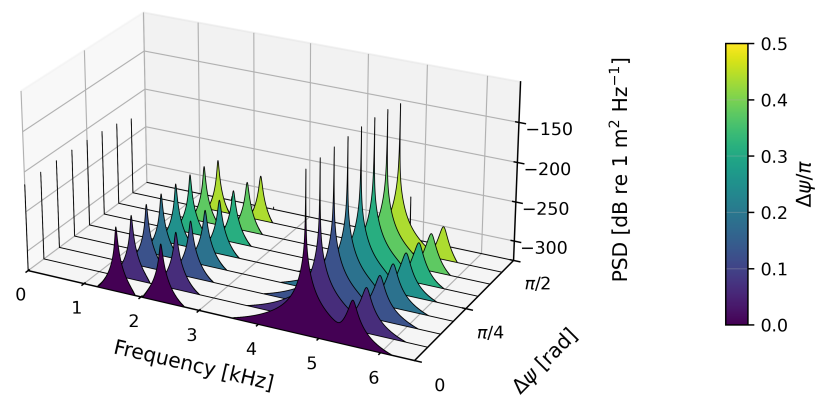
3.5. Example 5: Angular Separation of Two Space-Fixed Forces

When two unit forces of equal phase act at the same radius $r_1 = 0.066$ m but at angular positions ψ_1 and $\psi_1 + \Delta\psi$, their generalised forces add as complex phasors. The combined modal forcing of the n -diameter family is governed by the two-source interference factor $2|\cos(n\Delta\psi/2)|$, which applies equally to the forward and backward travelling components of the family, so the whole family is scaled by the same factor. For $n = 2$ this predicts

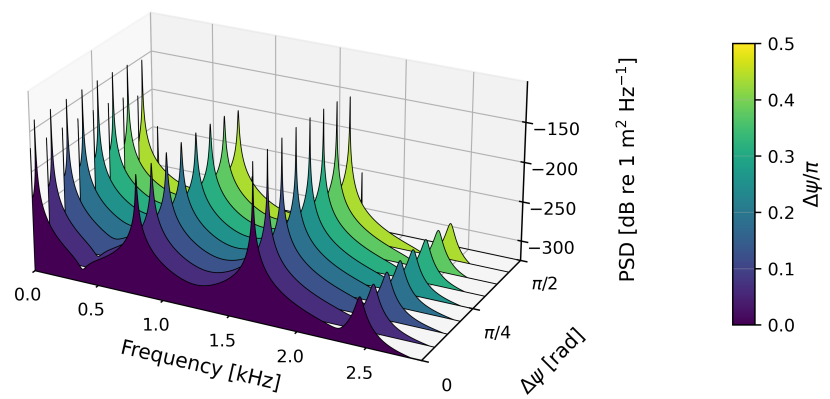
perfect reinforcement at $\Delta\psi = 0$ and complete cancellation at $\Delta\psi = \pi/2$, where the two forces act at antinodes of opposite sign of the two-nodal-diameter pattern. Figure 7 confirms this law for all three boundary conditions: as $\Delta\psi$ grows from 0 to $\pi/2$ the resonant peak at f_3 falls continuously—by 14.5, 14.1 and 14.4 dB at $\Delta\psi = 7\pi/16$ for the C–C, C–F and F–C discs, respectively, against the theoretical $20 \log_{10} |\cos(7\pi/8)|^{-1} = 14.2$ dB—and collapses into the numerical noise floor at $\Delta\psi = \pi/2$, independently of the boundary conditions, since the interference acts through the circumferential functions only. The small deviations of the C–F disc (below 1 dB at intermediate separations) arise from the off-resonant response of its neighbouring modes near the critical speed. The angular separation of multiple forces therefore offers a boundary-condition-independent means of suppressing a chosen nodal-diameter family in this idealised equal-amplitude, in-phase forcing case.

The cancellation at $\Delta\psi = \pi/n$ applies, however, only to the targeted family: a point force projects onto every nodal-diameter family, and a pair separation chosen to null the $n = 2$ projection leaves the other families forced. To quantify this residual, the steady-state response of the families $n = 0$ –6 is computed with the damped resolvent of each family model, Equation (52), so that this figure contains no transient content at all, at the same excitation frequency f_3 . The total root-mean-square transverse displacement over the circumference at the observation radius is then assembled from the family amplitudes. Figure 8a shows this total steady response as a function of $\Delta\psi$, normalised to its value at $\Delta\psi = 0$, together with the single-family law $|\cos(n\Delta\psi/2)|$. For the C–C and F–C discs the total response follows the $n = 2$ law closely up to moderate separations and then saturates at the residual floor of the non-targeted families. At $\Delta\psi = \pi/2$ the total response has dropped by only 5.6 dB (C–C) and 15.0 dB (F–C) although the $n = 2$ family itself is suppressed to the numerical floor. For the C–F disc the total response is almost unchanged (0.1 dB), because its $n = 4$ family has a mode at 1733 Hz, only 3% above the $f_3 = 1678.6$ Hz excitation, and this near-resonant neighbour dominates the residual; the pronounced dip near $\Delta\psi = \pi/4$ in Figure 8a is the partial cancellation of that $n = 4$ family, whose own interference factor $|\cos(2\Delta\psi)|$ vanishes there. Figure 8b shows the per-family amplitudes at $\Delta\psi = \pi/2$: the $n = 2$ and $n = 6$ families are nulled exactly, since $\cos(n\pi/4) = 0$ for both, while the remaining families persist at levels 8–42 dB below the $\Delta\psi = 0$ total, in a boundary-condition-dependent pattern set by the proximity of their resonances to the excitation frequency. The angular-separation lever therefore silences a chosen family, not the disc; the achievable overall reduction is set by the modal neighbourhood of the excitation frequency, which in turn depends on the boundary conditions.

(a) C-C



(b) C-F



(c) F-C

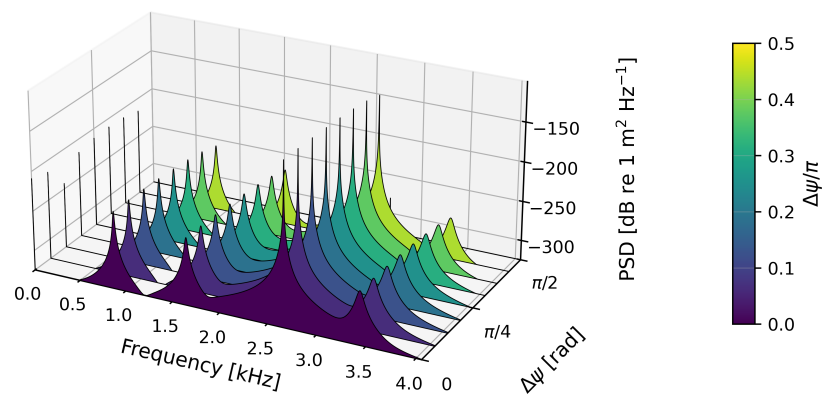


Figure 7. Waterfall diagrams of the PSD for the angular-separation sweep $\Delta\psi = 0-\pi/2$ between two equal in-phase unit forces at $r_1 = 0.066$ m (excitation at f_3 , $n = 2$): (a) clamped–clamped; (b) clamped–free; (c) free–clamped. Frequency in kHz; vertical axis, PSD of the physical displacement at the observation point in dB re 1 m² Hz⁻¹; parameter axis and colour scale, the swept parameter. Step onset at $t = 0$, $T_{\text{rec}} = 1$ s, $\zeta = 0.01$.

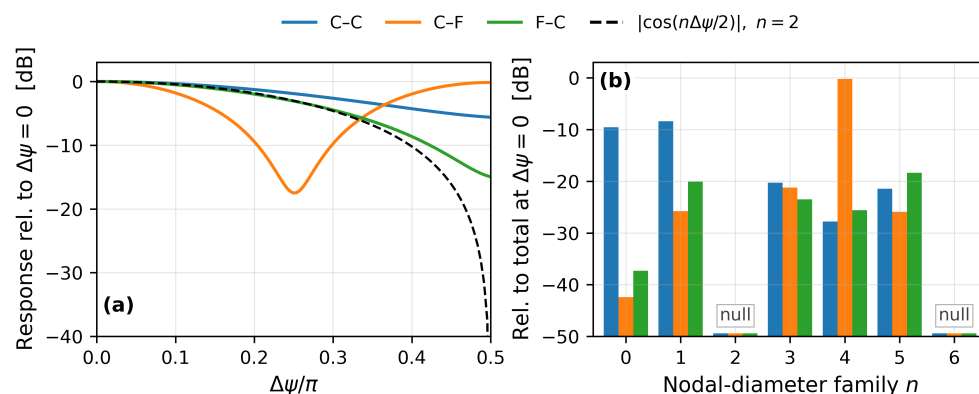


Figure 8. Residual steady-state response of the nodal-diameter families $n = 0$ – 6 for two equal in-phase forces at $r_1 = 0.066$ m separated by $\Delta\psi$, excitation at f_3 of each boundary condition, computed from the resolvent, Equation (52): (a) total RMS response at the observation radius, relative to its value at $\Delta\psi = 0$, with the single-family interference law for $n = 2$ (dashed); (b) amplitude of each family at $\Delta\psi = \pi/2$, relative to the total at $\Delta\psi = 0$. Colours and line styles are given in the legend, which applies to both panels.

3.6. Example 6: Phase Difference of Two Space-Fixed Forces

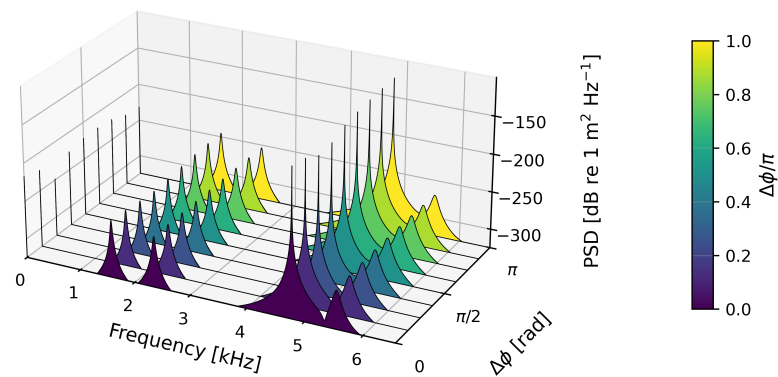
Finally, two unit forces are applied at fixed angular positions $\psi_1 = \pi/6$ and $\psi_2 = \pi/2$ at the same radius, and the phase of the second force is varied by $\Delta\phi = 0$ – π (Figure 9). The effect of the phase difference is understood by decomposing the steady response of the $n = 2$ family into its two counter-rotating waves. A force at angular position ψ_j with temporal phase ϕ_j projects onto the wave whose pattern phase is $(\omega\tau \mp n\psi)$ with the complex weight $e^{-i(\phi_j \pm n\psi_j)}$: the angular position enters the interference as an equivalent phase $n\psi_j$ added to, or subtracted from, the temporal phase. For two equal forces the amplitudes of the two waves are therefore proportional to

$$2 \left| \cos \frac{\Delta\phi \pm n \Delta\psi_{12}}{2} \right|, \quad \Delta\psi_{12} = \psi_2 - \psi_1, \quad (53)$$

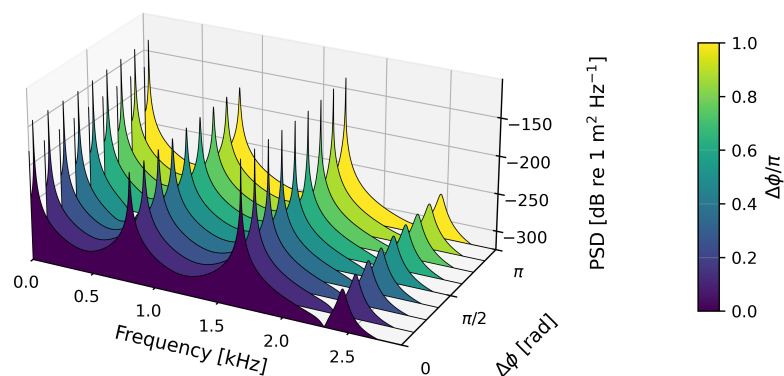
which reduces to the common factor $2|\cos(n\Delta\psi/2)|$ of Example 5 when $\Delta\phi = 0$. The phase difference thus detunes the two waves in opposite directions, whereas the in-phase angular separation scales them equally. The resonant peak at f_3 belongs to the backward wave; with $\Delta\psi_{12} = \pi/3$ and $n = 2$ its factor $2|\cos(\Delta\phi/2 + \pi/3)|$ starts at 1 for the in-phase pair, vanishes at $\Delta\phi = \pi/3$ and reaches $\sqrt{3}$ at $\Delta\phi = \pi$. It follows that the total family forcing magnitude $\sqrt{2 - \cos \Delta\phi}$, which grows monotonically with $\Delta\phi$ for these positions, does not describe the resonant peak: the projection onto the resonant travelling wave, Equation (53), is the governing quantity, and it is non-monotonic. The waterfalls confirm this law. For the clamped–clamped disc the driven peak first falls by 14.4 dB, with the minimum at the sampled separations $\Delta\phi = \pi/4$ – $3\pi/8$ that bracket the predicted null at $\pi/3$, and then rises to 5.7 dB above the in-phase level at $\Delta\phi = \pi$; the free–clamped disc behaves in the same way (13.9 dB dip, 5.4 dB rise). These dip depths are limited by the $\pi/8$ sweep grid, which steps over the null: a fine steady-state sweep (Section 3.7) locates the minima at $\Delta\phi = 0.306\pi$ and 0.317π and gives dips of 26.6 and 30.2 dB, while reproducing the 5.7 and 5.4 dB rises at $\Delta\phi = \pi$ exactly. The rise at $\Delta\phi = \pi$ slightly exceeds the 4.8 dB of the resonant-wave factor alone because the forward wave, whose factor is also $\sqrt{3}$ there, contributes coherently at the observation point. For the clamped–free disc the modulation is much weaker—a 6.5 dB dip near $\Delta\phi = 5\pi/8$ and nearly equal end levels—because this disc operates near its critical speed and the strongly compliant first radial family responds appreciably off resonance, partially masking the resonant-wave law at the observation point. The phase difference can therefore nearly silence the resonant travelling

wave at $\Delta\phi = \pi - n\Delta\psi_{12}$ (modulo 2π), but only where the resonant wave dominates the response, which again depends on the boundary conditions. The comparison with Section 3.5 makes an important practical point: the effect of the phase difference is inseparable from the angular positions of the forces—the same phase shift may amplify or attenuate the resonant response depending on where the forces act. Consistently with the observations of [27], a well-designed system of forces, with properly chosen number, locations, frequencies and phases, can therefore suppress unwanted resonant amplitudes or deliberately tailor the vibrational response of a rotating disc. The present results show that this conclusion carries over to all three boundary conditions, while the quantitative sensitivities differ.

(a) C-C



(b) C-F



(c) F-C

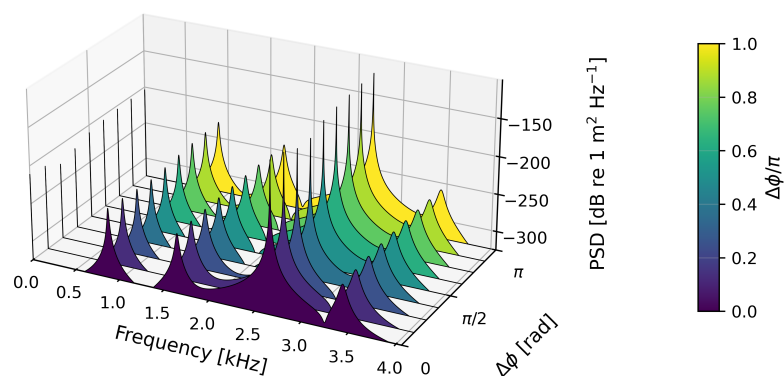


Figure 9. Waterfall diagrams of the PSD for the phase-difference sweep $\Delta\phi = 0-\pi$ between two unit forces at $\psi_1 = \pi/6$ and $\psi_2 = \pi/2$, $r_1 = 0.066$ m (excitation at f_3 , $n = 2$): (a) clamped-clamped; (b) clamped-free; (c) free-clamped. Frequency in kHz; vertical axis, PSD of the physical displacement at the observation point in dB re 1 m² Hz⁻¹; parameter axis and colour scale, the swept parameter. Step onset at $t = 0$, $T_{\text{rec}} = 1$ s, $\zeta = 0.01$.

3.7. Steady-State Harmonic Response and the Role of the Switch-On Transient

The spectra of Sections 3.2–3.6 are power spectral densities of a record of finite length T_{rec} that begins at the instant the force is switched on. They are therefore switch-on spectra, and only their driven line is a steady-state quantity. This subsection separates the two contributions, verifies the driven line against the exact steady-state solution of Equation (52), and quantifies the dependence of the transient part on the record duration, the onset of the force and the damping ratio, so that it is unambiguous which statements of this paper are properties of the forced steady state and which are properties of a particular record.

3.7.1. What Belongs to the Transient

Figure 10a shows three spectra of the same clamped–free case, the $n = 2$ family driven at $f_3 = 1678.6$ Hz. When the force is applied as an ideal step at $\tau = 0$, the record contains, besides the driven line, peaks at the other natural frequencies f_1 , f_2 and f_4 of the family. These are the free-decay content excited by the discontinuous onset. They are absent from a record taken after the transient has decayed (samples between 3 and 4 s), and they are removed by applying the same force through a half-cosine ramp of rise time $t_r = 50$ ms; in both cases the driven line is unchanged. The star marks the amplitude given by the resolvent, Equation (52), converted to the periodogram scale; it falls on the driven peak of both records.

The reason the clamped–free disc is the sensitive case is contained in the damping discussion of Section 2.4. Its $n = 2$ first-family pair sits close to the critical speed, realises an effective damping ratio of 0.0042 rather than the nominal 0.01, and decays with a time constant of 0.96 s—comparable with the whole 1 s record. For the clamped–clamped and free–clamped discs the corresponding time constants are 10.2 and 18.9 ms, so their transients are long dead before the record ends. This is precisely why, at the baseline settings ($T_{\text{rec}} = 1$ s, step onset, $\zeta = 0.01$), the strongest non-driven peak lies 119.5 dB below the driven line for the C–C disc and 106.6 dB below it for the F–C disc, but only 4.3 dB below it for the C–F disc.

3.7.2. Sensitivity of the Record

Figure 10b–d and Table 9 track the driven line and the strongest non-driven peak as the record duration, the ramp time and the damping ratio are varied. Three behaviours should be distinguished.

The driven line is completely insensitive to the onset: it changes by less than 0.1 dB as t_r is taken from an ideal step to 200 ms. Its absolute level does depend on the record length, but in a way that is a property of the estimator and not of the mechanics: a deterministic sinusoid of amplitude A in a Hann-windowed, density-scaled periodogram has $\text{PSD}(f_{\text{exc}}) = A^2 T_{\text{rec}}/3$, so the level rises by 3 dB per doubling of T_{rec} , uniformly for all traces. Every quantity reported in this paper is a level difference between traces recorded with the same T_{rec} , and such differences are exactly invariant to the record length. The driven line does depend on ζ , as any resonance must: over $\zeta = 0.002$ to 0.05 it falls by 26.9 dB for the C–C disc and 27.3 dB for the F–C disc, close to the 28.0 dB expected from the $1/\zeta$ scaling of a resonant amplitude, and by 16.6 dB for the near-critical C–F disc.

The non-driven peaks behave in the opposite way. They fall by 59.4, 120.7 and 72.8 dB for the C–C, C–F and F–C discs between a step onset and a 200 ms ramp; they fall by 14–15 dB per doubling of T_{rec} for the C–C and F–C discs, as a fixed amount of transient energy is spread over a longer record while the driven line grows; and they fall with ζ roughly as the transient decay allows. For the C–F disc the non-driven level is nearly independent of T_{rec} over the range examined, because its transient has not decayed by the end of even the longest record. Nothing about these peaks is a steady-state property, and no conclusion in this paper is drawn from them.

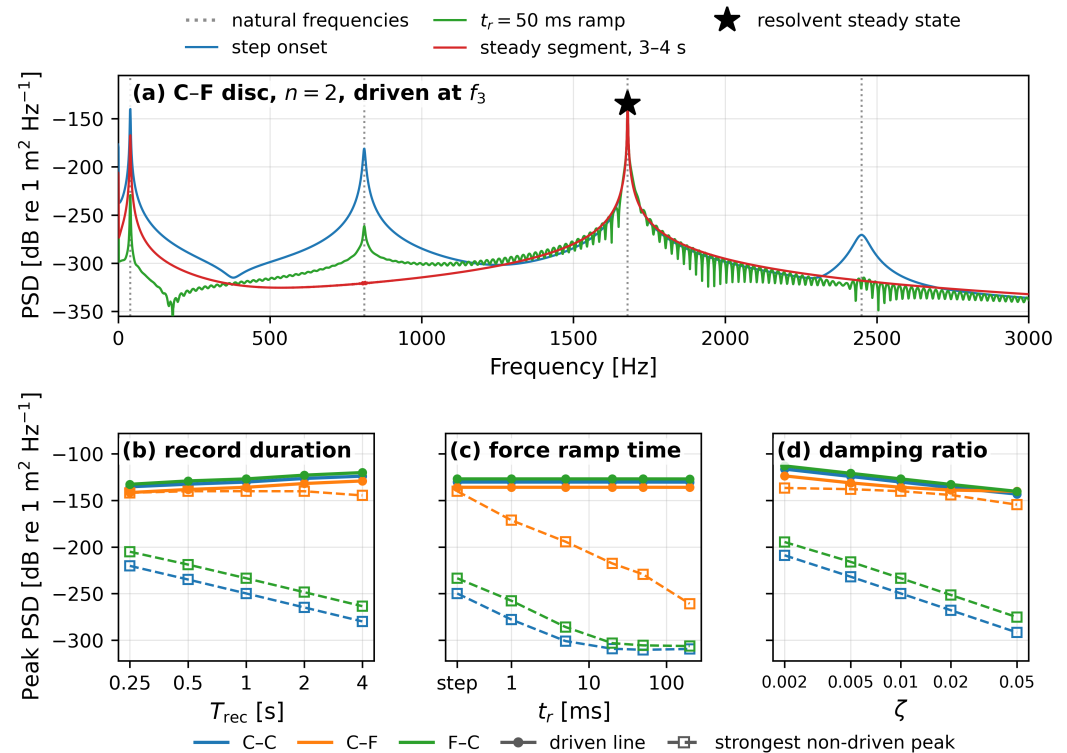


Figure 10. Separation of the switch-on transient from the steady state. (a) PSD of the clamped–free disc, $n = 2$ family driven at f_3 , for a step onset, for a 50 ms half-cosine ramp, and for a record taken after the transient has decayed; the star is the steady-state amplitude from the resolvent, Equation (52), on the periodogram scale, and the dashed lines mark the natural frequencies of the family. (b–d) Peak level, in dB re 1 m² Hz⁻¹, of the driven line (solid, filled) and of the strongest non-driven natural-frequency peak (dashed, open) against the record duration, the force ramp time and the modal damping ratio. In (c) the leftmost point of each curve is the ideal step onset, plotted off the logarithmic scale and labelled “step”.

Table 9. Sensitivity of the switch-on spectra. Levels in dB re 1 m² Hz⁻¹ of the driven line and of the strongest non-driven natural-frequency peak, for a single unit force at $r_1 = 0.066$ m, $\psi_1 = \pi/6$ tuned to f_3 of each boundary condition. Baseline: $T_{\text{rec}} = 1$ s, ideal step onset, $\zeta = 0.01$.

Parameter	Value	C–C		C–F		F–C	
		Driven	Non-Driven	Driven	Non-Driven	Driven	Non-Driven
T_{rec} [s]	0.25	−135.4	−220.2	−141.5	−141.8	−132.7	−205.0
	1	−130.3	−249.8	−135.8	−140.1	−126.9	−233.5
	4	−123.9	−279.8	−129.1	−144.6	−120.1	−263.4
t_r [ms]	0 (step)	−130.3	−249.8	−135.8	−140.1	−126.9	−233.5
	5	−130.3	−300.9	−135.8	−194.2	−126.9	−285.8
	200	−130.3	−309.2	−135.9	−260.8	−126.9	−306.3
ζ	0.002	−116.3	−208.8	−123.6	−136.6	−112.9	−194.6
	0.01	−130.3	−249.8	−135.8	−140.1	−126.9	−233.5
	0.05	−143.2	−291.7	−140.2	−154.5	−140.2	−275.3

3.7.3. Steady-State Parameter Sweeps

The three parameter sweeps of Sections 3.4–3.6 were repeated using Equation (52) alone, on grids ten to twenty times finer than the sweep grids of the waterfall diagrams. The results are collected in Figure 11; the open circles mark the coarse grid actually used for the waterfalls. Comparing these steady-state levels with the driven-line levels read from

the transient records gives the strongest available check on the whole parametric study: over the three sweeps and the three boundary conditions, at all 81 sampled parameter values at which the response is not numerically cancelled (10 radial positions, 8 angular separations and 9 phase differences for each support), the two agree to within 0.09 dB once a common offset of 1.1 dB is removed. That offset is not a discrepancy: the three excitation frequencies happen to fall 0.4 of a frequency bin off the discrete Fourier grid, for which the Hann scalloping loss of a pure sinusoid is 0.91 dB. Every parametric conclusion of this paper is therefore a statement about the forced steady state.

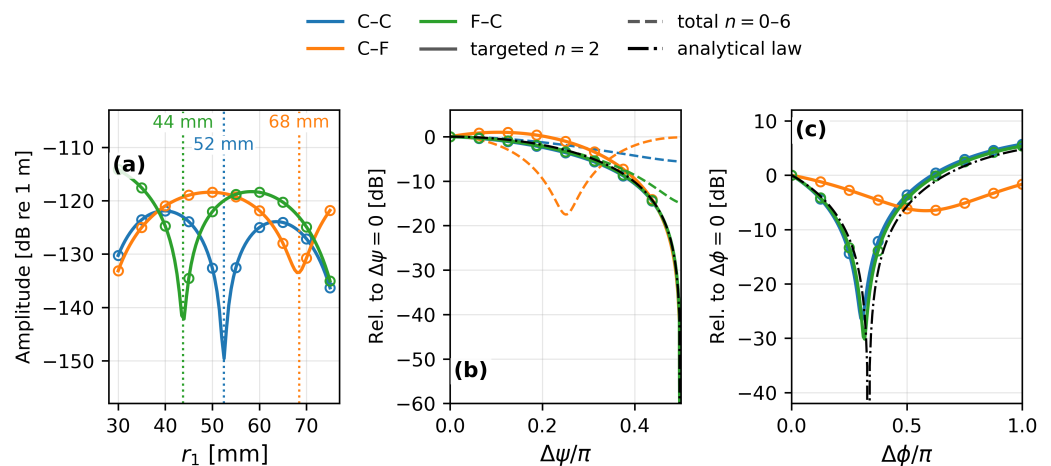


Figure 11. Steady-state harmonic response from the resolvent, Equation (52), for the three parameter sweeps of Sections 3.4–3.6, excitation at f_3 of each boundary condition. (a) Amplitude of the physical transverse displacement at the observation point against the radial force position; the dotted vertical lines are the interior nodes of the driven radial family taken from Figure 3. (b) Response of the targeted $n = 2$ family (solid) and of the total over the families $n = 0-6$ (dashed) against the angular separation of two equal in-phase forces, with the interference law $|\cos(n\Delta\psi/2)|$. (c) Response against the phase difference of two forces at $\psi_1 = \pi/6$ and $\psi_2 = \pi/2$, with the resonant-wave law of Equation (53). Open circles mark the coarse grid used for the waterfall diagrams; solid lines are the fine steady-state sweep.

The finer grids also sharpen three of the reported numbers. The radial minima of Section 3.4 are deeper than the 5 mm sweep grid could resolve: 27.7 dB at $r_1 = 52.5$ mm (C–C), 15.1 dB at 68.0 mm (C–F) and 28.0 dB at 44.0 mm (F–C), each within 0.5 mm of the interior node of the driven radial family computed from the mode shapes of Figure 3 (52.4, 68.4 and 43.8 mm). The angular-separation law is confirmed to 0.3 dB: the targeted $n = 2$ family falls by 14.3, 13.9 and 14.2 dB at $\Delta\psi = 7\pi/16$ against the theoretical 14.2 dB, and is suppressed to the level of numerical cancellation at $\Delta\psi = \pi/2$. Measured at the observation point rather than as a circumferential root-mean-square, the total over the families $n = 0-6$ falls by 13.9 dB (C–C) and 15.6 dB (F–C) at $\Delta\psi = \pi/2$ and rises by 0.9 dB (C–F); the difference from the values quoted with Figure 8 reflects the different measure of “total response”, not a different model. Finally, the phase-difference minima of Section 3.6 lie at $\Delta\phi = 0.306\pi$ (C–C) and 0.317π (F–C), within 0.03π of the null at $\pi - n\Delta\psi_{12} = \pi/3$ predicted by Equation (53), the small offset being the coherent contribution of the forward wave at the observation point; the dips are 26.6 and 30.2 dB deep, against the 14.4 and 13.9 dB seen on the coarse $\pi/8$ grid, which simply steps over the null.

3.7.4. Scope

The resulting division is a clean one. The dependence of the response on the excitation frequency, the radial force position, the angular separation and the phase difference—that

is, the entire parametric content of Sections 3.3–3.6 and of the conclusions—is carried by the driven line and is a steady-state result, reproduced independently by Equation (52). The peaks at the non-driven natural frequencies, and any reading of the waterfalls in terms of energy being redistributed among modes, belong to a 1 s record of a step-applied force with $\zeta = 0.01$ and would change with a different onset, record length or damping; they are retained in the figures because they make the modal structure of each disc visible in a single diagram, but they are not claimed as general forced-vibration characteristics.

3.8. Design Implications and Limitations

The results provide a compact design interpretation for rotating discs under space-fixed transverse forcing. The support condition first determines the frequency band of concern: C–C support moves the resonances to higher frequencies, whereas C–F support can place backward travelling waves at very low frequencies near critical speed. The radial position of a force then controls which radial family is preferentially excited through the value of $R_{mn}(r_1)$, while the angular separation and phase difference of multiple forces control the circumferential travelling-wave content through phasor interference. These mechanisms are directly relevant to engineering systems where the excitation is effectively fixed in space, such as disc-brake pads, read/write-head interactions in computer HDDs and other rotating storage media, guide/support contacts in circular saws and stationary aerodynamic forcing in rotating machinery.

These examples should be interpreted as design analogies rather than complete industrial models. Brake squeal, for example, involves friction-induced instability and distributed pad contact [18], and gas-turbine excitation is usually distributed and aerodynamic rather than a small number of ideal point forces. Nevertheless, the present results show how the angular placement or phasing of multiple space-fixed excitation paths can be used to avoid feeding a selected nodal-diameter family. For instance, in a multi-piston brake-caliper idealisation, placing two equal force paths at approximately $\Delta\psi = \pi/n$ would minimise the direct excitation of the n -nodal-diameter transverse disc mode—with the important qualification of Example 5 that this silences the targeted family only, and that the overall response reduction is limited by the other nodal-diameter families, whose residual level depends on the boundary conditions. In an HDD or optical-disc idealisation, a secondary actuator could be placed and phased to apply a counter-force to the dominant travelling-wave component, exploiting the wave-selective interference law of Equation (53). Such extensions would require application-specific modelling of contact, damping, actuator bandwidth and closed-loop stability; they are therefore outside the scope of the present analytical benchmark.

The design interpretation of the four excitation parameters can be summarised as follows. The boundary condition sets the resonance frequency band and the stiffness ordering of the disc, so the support flexibility should be established before any force-placement optimisation is attempted. The excitation frequency selects the travelling-wave pair that dominates the spectrum, and operation near backward-wave critical speeds should be avoided or controlled. The radial force position scales the modal forcing of each radial family through $R_{mn}(r_1)$, so force or contact paths should be placed away from the antinodes of undesirable radial families—and, where suppression of a family is sought, near its interior node, whose location is boundary-condition dependent (Figure 3). The angular separation of two equal in-phase forces produces phasor cancellation or reinforcement of an n -diameter family and can suppress the targeted family when $\Delta\psi \approx \pi/n$, subject to the residual response of the remaining families. Finally, the phase difference between two forces acts on the two counter-rotating waves in opposite directions according to Equation (53), so active or multi-source excitation requires the joint tuning of phase and

angular position; a phase difference alone can nearly null the resonant travelling wave only where that wave dominates the response. These statements are implications of the present idealised space-fixed point-force model and are not intended as complete industrial design rules.

4. Conclusions

The forced transverse vibration of a thin rotating annular disc under space-fixed harmonic point forces has been analysed with a generalised Galerkin formulation for three sets of boundary conditions—clamped–clamped, clamped–free and free–clamped—and the effects of the excitation parameters have been investigated systematically through PSD waterfall diagrams of the physical response at a fixed observation point, using a lightly damped state-space model. The main conclusions are as follows.

First, the formulation is verified and the results agree closely with the literature. A convergence study shows that six radial functions converge the tabulated frequencies of all three boundary conditions, and an independent radial finite-element solution of the same rotating Kirchhoff model reproduces the converged Galerkin frequencies to within 0.001% for every case examined, including the stationary hard-disk comparison. On the forced side, the driven spectral line of the simulated records reproduces the exact steady-state response obtained from the frequency-domain resolvent to within 0.09 dB over all three parameter sweeps and all three boundary conditions. These are verification results; no experimental validation of the forced response is claimed. The computed space-fixed natural frequencies of the rotating clamped–free disc agree with the published values of [27] to within 0.03%, and the residual 2–3% offset with respect to the stationary hard-disk values of [7] is shown to be uniform across the nodal diameters and attributable to the unreported material constants and numerical treatment of that reference rather than to the present formulation. The published forced-response spectra are reproduced on a consistent PSD basis, and the clamped–clamped and free–clamped implementations reproduce the rotational behaviour (gyroscopic splitting and speed-induced frequency shifts) of [7].

Second, the boundary conditions govern the frequency range of the forced response. At the same rotation speed the clamped–clamped disc responds resonantly only in the kilohertz range, while the clamped–free disc, whose $n = 2$ backward wave approaches zero frequency near $\Omega^* = 5$, can be driven resonantly below 50 Hz. Every resonance of the spinning disc appears as a backward/forward travelling-wave pair separated by exactly $2n\Omega$.

Third, the excitation parameters act as largely independent levers on the response, but the quantitative action of each lever is set by the boundary conditions. The excitation frequency selects the dominant mode. The radial position of the force selects the dominant radial family, with the optimal placement dictated by the boundary conditions through the node and antinode pattern of the targeted family rather than through proximity to a free edge: the radial position at which a chosen radial family is least excited is that family's interior node, at 52.5, 68.0 and 44.0 mm for the second family of the clamped–clamped, clamped–free and free–clamped discs, and the position at which it is most excited is the corresponding antinode, at 40.0 mm (clamped–clamped), 49.5 mm (clamped–free) and at the inner end of the swept range for the free–clamped disc, whose response rises steadily towards the free inner edge. Only in the last of these three cases does the optimum lie at a free edge: for the clamped–free disc the free rim remains 3.4 dB below the interior antinode, and the local minima of the driven response coincide with the interior nodes of the driven radial family, whose positions move with the boundary conditions. The angular separation of a pair of equal in-phase forces modulates the resonant amplitude according to the interference law $2|\cos(n\Delta\psi/2)|$, providing complete suppression of the

targeted nodal-diameter family in the ideal equal-amplitude case; the residual response of the non-targeted families, however, limits the overall reduction to between 0.1 and 15 dB in the cases examined as a circumferential root-mean-square, and to between -0.9 and 15.6 dB measured at the observation point, in a boundary-condition-dependent manner set by the modal neighbourhood of the excitation frequency. Finally, the phase difference of a pair of forces acts on the two counter-rotating waves of a family in opposite directions, with wave amplitudes proportional to $2|\cos[(\Delta\phi \pm n\Delta\psi_{12})/2]|$: the angular position of each force enters as an equivalent phase, so a phase difference can nearly null the resonant travelling wave at $\Delta\phi = \pi - n\Delta\psi_{12}$, and the effect of phasing is inseparable from the force positions. This is confirmed by the waterfall diagrams for all three boundary conditions.

Fourth, the scope of the reported spectra is stated explicitly. The waterfall diagrams are power spectral densities of a 1 s record that begins when the force is switched on. Their driven line is a steady-state quantity, invariant to the onset to within 0.1 dB and reproduced by the resolvent solution; their peaks at the non-driven natural frequencies are switch-on transient content, which falls by 59–121 dB when the same force is applied through a 200 ms ramp and which is significant relative to the driven line only for the clamped-free disc, whose near-critical waves decay with a time constant comparable with the record. All parametric conclusions above are read from the driven line.

These results indicate that unwanted resonant amplitudes of a rotating disc can be suppressed, or its vibrational characteristics deliberately tailored, by an appropriate design of the excitation force system, and that such a design must account for the boundary conditions of the disc. Several limitations bound the scope of these conclusions. The excitation is idealised as a small number of harmonic point forces fixed in space; real excitation paths—brake-pad patches, aerodynamic loading, read/write-head interactions—are distributed, may involve friction and follower-force effects, and can add stiffness, damping and instability mechanisms that the present linear model does not contain. The supports are idealised as perfectly clamped or perfectly free edges, whereas real mountings have finite flexibility that interpolates between these bounds; the damping is a uniform light modal damping introduced for well-posedness of the spectra rather than a measured dissipation model, is exactly modal only for the associated non-gyroscopic system, and departs from the nominal ratio by more than a factor of two for the near-critical clamped-free waves; and the plate model is a classical Kirchhoff plate, adequate for the thin discs analysed but not for thick discs or very high modes. Finally, all results are numerical verification of the stated model; experimental validation of the forced response on a physical rotating disc remains to be done. The extension of the formulation to the remaining classical and elastic boundary conditions, distributed frictional and aerodynamic loads, measured damping models, multiple nodal-diameter families excited simultaneously, and closed-loop control applications are left for future work.

Author Contributions: Conceptualisation, E.T. and M.T.; methodology, E.T., H.K. and M.T.; software, H.K. and M.T.; validation, H.K. and M.T.; formal analysis, H.K.; investigation, H.K. and M.T.; resources, E.T.; data curation, H.K.; writing—original draft preparation, H.K.; writing—review and editing, M.T. and E.T.; visualisation, H.K. and M.T.; supervision, E.T. and M.T.; project administration, E.T. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The data and code used to generate the results are available from the corresponding author upon reasonable request.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Lamb, H.; Southwell, R.V. The vibrations of a spinning disk. *Proc. R. Soc. Lond. Ser. A* **1921**, *99*, 272–280. <https://doi.org/10.1098/rspa.1921.0041>.
2. Southwell, R.V. On the free transverse vibrations of a uniform circular disc clamped at its centre, and on the effects of rotation. *Proc. R. Soc. Lond. Ser. A* **1922**, *101*, 133–153. <https://doi.org/10.1098/rspa.1922.0032>.
3. Nowinski, J.L. Nonlinear transverse vibrations of a spinning disk. *J. Appl. Mech.* **1964**, *31*, 72–78. <https://doi.org/10.1115/1.3629573>.
4. Mote, C.D. Free vibration of initially stressed circular disks. *J. Eng. Ind.* **1965**, *87*, 258–264. <https://doi.org/10.1115/1.3670808>.
5. Kirkhope, J.; Wilson, G.J. Vibration and stress analysis of thin rotating discs using annular finite elements. *J. Sound Vib.* **1976**, *44*, 461–474. [https://doi.org/10.1016/0022-460X\(76\)90088-2](https://doi.org/10.1016/0022-460X(76)90088-2).
6. Oldaç, O.; Tüfekci, M.; Genel, Ö.E.; Tüfekci, E. Free bending vibrations of rotating disks with a concentric hole. *Int. J. Technol. Eng. Stud.* **2017**, *3*, 253–263. <https://doi.org/10.20469/ijtes.3.40005-6>.
7. Tüfekci, M.; Koç, H.; Genel, Ö.E.; Oldaç, O. Free transverse vibrations of rotating annular disks under various boundary conditions. *Sigma J. Eng. Nat. Sci.* **2018**, *9*, 243–255.
8. Koç, H.; Genel, Ö.E.; Tüfekci, M.; Tüfekci, E. Analysis of the dynamical behaviour of spinning annular disks with various boundary conditions. *Mech. Based Des. Struct. Mach.* **2023**, *51*, 5427–5451. <https://doi.org/10.1080/15397734.2021.1999269>.
9. Tüfekci, M.; Rendu, Q.; Yuan, J.; Dear, J.P.; Salles, L.; Cherednichenko, A.V. Stress and modal analysis of a rotating blade and the effects of nonlocality. In *Proceedings of the ASME Turbo Expo 2020: Turbomachinery Technical Conference and Exposition*; American Society of Mechanical Engineers (ASME): New York, NY, USA, 2020; Volume 10B, p. V10BT26A008. <https://doi.org/10.1115/GT2020-14821>.
10. Benson, R.C.; Bogy, D.B. Deflection of a very flexible spinning disk due to a stationary transverse load. *J. Appl. Mech.* **1978**, *45*, 636–642. <https://doi.org/10.1115/1.3424374>.
11. Leung, R.C.N.; Pinnington, R.J. Vibration of a rotating disc subjected to an in-plane force at its rim, or at its centre. *J. Sound Vib.* **1987**, *114*, 281–295. [https://doi.org/10.1016/S0022-460X\(87\)80154-2](https://doi.org/10.1016/S0022-460X(87)80154-2).
12. Chen, J.S.; Hsu, C.M. Forced response of a spinning disk under space-fixed couples. *J. Sound Vib.* **1997**, *206*, 627–639. <https://doi.org/10.1006/jsvi.1997.1126>.
13. Shen, I.Y. Closed-form forced response of a damped, rotating, multiple disks/spindle system. *J. Appl. Mech.* **1997**, *64*, 343–352. <https://doi.org/10.1115/1.2787313>.
14. Torii, T.; Yasuda, K.; Toyoda, T. Nonlinear forced oscillation of a rotating disk excited at a point fixed in space. *JSME Int. J. Ser. C* **1998**, *41*, 592–598. <https://doi.org/10.1299/jsmec.41.592>.
15. Phylactopoulos, A.; Adams, G.G. Transverse vibration of a rectangularly orthotropic spinning disk, Part 1: Formulation and free vibration. *J. Vib. Acoust.* **1999**, *121*, 273–279. <https://doi.org/10.1115/1.2893976>.
16. Phylactopoulos, A.; Adams, G.G. Transverse vibration of a rectangularly orthotropic spinning disk, Part 2: Forced vibration and critical speeds. *J. Vib. Acoust.* **1999**, *121*, 280–285. <https://doi.org/10.1115/1.2893977>.
17. Shen, J.Y.; Tseng, C.W.; Shen, I.Y. Vibration of rotating disk/spindle systems with flexible housing/stator assemblies. *J. Sound Vib.* **2004**, *271*, 725–756. [https://doi.org/10.1016/S0022-460X\(03\)00413-9](https://doi.org/10.1016/S0022-460X(03)00413-9).
18. Spelsberg-Korspeter, G.; Hochlenert, D.; Kirillov, O.N.; Hagedorn, P. In- and out-of-plane vibrations of a rotating plate with frictional contact: Investigations on squeal phenomena. *J. Appl. Mech.* **2009**, *76*, 041006. <https://doi.org/10.1115/1.3112734>.
19. Liu, C.F.; Lee, J.F.; Lee, Y.T. Axisymmetric vibration analysis of rotating annular plates by a 3D finite element. *Int. J. Solids Struct.* **2000**, *37*, 5813–5827. [https://doi.org/10.1016/S0020-7683\(99\)00256-5](https://doi.org/10.1016/S0020-7683(99)00256-5).
20. Carmignani, C.; Forte, P.; Melani, G. A Duhamel approach for forced vibrations of rotating axisymmetric bodies. *Int. J. Numer. Methods Eng.* **2012**, *90*, 928–938. <https://doi.org/10.1002/nme.3365>.
21. Pei, Y.C.; Lu, H.; Chatwin, C. Vibration of a rotating flexible ferromagnetic disc subjected to a solenoid electromagnet. *Proc. Inst. Mech. Eng. Part K J. Multi-Body Dyn.* **2011**, *225*, 195–203. <https://doi.org/10.1177/1464419311417752>.
22. Younesian, D.; Aleghafourian, M.H.; Esmailzadeh, E. Vibration analysis of circular annular plates subjected to peripheral rotating transverse loads. *J. Vib. Control* **2015**, *21*, 1443–1455. <https://doi.org/10.1177/1077546313499178>.
23. García, I.; Zubia, J.; Beloki, J.; Arrue, J.; Villatoro, J. Evaluation of the vibrational behaviour of a rotating disk by optical tip-clearance measurements. In *Proceedings of the Proceedings of SPIE, Optical Measurement Systems for Industrial Inspection IX*; SPIE: Bellingham, WA, USA, 2015; Volume 9525, p. 95251Z. <https://doi.org/10.1117/12.2185273>.
24. García, I.; Zubia, J.; Beloki, J.; Arrue, J.; Durana, G.; Aldabaldetrek, G. Optical tip clearance measurements as a tool for rotating disk characterization. *Sensors* **2017**, *17*, 165. <https://doi.org/10.3390/s17010165>.
25. Norouzi, H.; Younesian, D. Forced vibration analysis of spinning disks subjected to transverse loads. *Int. J. Struct. Stab. Dyn.* **2015**, *15*, 1450049. <https://doi.org/10.1142/S0219455414500497>.
26. Hu, Y.D.; Li, W.Q. Study on primary resonance and bifurcation of a conductive circular plate rotating in air-magnetic fields. *Nonlinear Dyn.* **2018**, *93*, 671–687. <https://doi.org/10.1007/s11071-018-4217-y>.

27. Tüfekci, M.; Genel, Ö.E.; Koç, H.; Oldaç, O.; Tüfekci, E. Vibrations of a rotating disk under perpendicular spacefixed forces. *Gazi Univ. J. Sci.* **2019**, *32*, 273–284.
28. Norouzi, H.; Younesian, D. Analytical modeling of transverse vibrations and acoustic pressure mitigation for rotating annular disks. *Math. Probl. Eng.* **2022**, *2022*, 3722410. <https://doi.org/10.1155/2022/3722410>.
29. Sun, S.; Zhao, C.; Cao, D. Traveling wave vibration control of rotating functionally graded conical shells via piezoelectric sensor/actuator pairs. *Arch. Appl. Mech.* **2024**, *94*, 2769–2791. <https://doi.org/10.1007/s00419-024-02614-5>.
30. Hao, Y.; Sun, L.; Zhang, W.; Li, H. Active traveling wave vibration control of elastic supported conical shells with smart micro fiber composites based on the differential quadrature method. *Appl. Math. Mech. (Engl. Ed.)* **2025**, *46*, 305–322. <https://doi.org/10.1007/s10483-025-3216-7>.
31. Sinou, J.J.; Nechak, L.; Besset, S. Kriging metamodeling in rotordynamics: Application for predicting critical speeds and vibrations of a flexible rotor. *Complexity* **2018**, *2018*, 1264619. <https://doi.org/10.1155/2018/1264619>.
32. Fu, C.; Sinou, J.J.; Zhu, W.; Lu, K.; Yang, Y. A state-of-the-art review on uncertainty analysis of rotor systems. *Mech. Syst. Signal Process.* **2023**, *183*, 109619. <https://doi.org/10.1016/j.ymssp.2022.109619>.
33. Zhang, X.; Yang, Y.; Deng, W.; Zhao, S.; Zhu, Q.; Ma, H.; Han, Q.; Qin, Z.; Liu, W. Vibration energy analysis of rub-impact rotor system: Nonlinear fault-induced energy response and transfer state. *Mech. Syst. Signal Process.* **2025**, *236*, 113035. <https://doi.org/10.1016/j.ymssp.2025.113035>.
34. Ma, Z.; Zhao, X.; Wu, Y.; Ma, H.; Cao, P.; Guan, H.; Liu, Z.; Fan, C.; Zhu, J.; Yan, P.; et al. Two novel indicators for gear crack diagnosis based on vibration responses: Experiment and simulation. *Mech. Mach. Theory* **2025**, *205*, 105905. <https://doi.org/10.1016/j.mechmachtheory.2024.105905>.
35. Tüfekci, M. Development and application of a high-fidelity numerical tool for dynamic analysis of bladed disc systems with underplatform dampers in aircraft engine turbines. In *Proceedings of the ASME Turbo Expo 2024: Turbomachinery Technical Conference and Exposition*; American Society of Mechanical Engineers (ASME): New York, NY, USA, 2024; Volume 10B, p. V10BT27A008. <https://doi.org/10.1115/GT2024-123318>.
36. Artan, R.; Tepe, A. In-plane static behavior of nonlocal curved nanobeams resting on an elastic foundation determined using the initial value method. *J. Elast.* **2025**, *157*, 49. <https://doi.org/10.1007/s10659-025-10140-w>.

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.