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A multi-axis time domain SEM–PML for modelling wave excitation, propagation and scattering in fluid-filled pipes immersed in fluid

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ABSTRACT

Finite element models coupled with Perfectly Matched Layers (PMLs) have been used to model time-domain wave propagation in fluid-filled pipes. This is a fluid-structure interaction problem, which becomes computationally challenging when the pipes are further immersed in fluid, as the domain extends to infinity in both the radial and axial directions. The time domain excitation, propagation, and scattering of guided waves from a point source in such systems have rarely been studied theoretically. This article proposes a high-order Spectral Element Method (SEM) incorporating multi-axis PMLs that operate in both the fluid and structural domains. This development efficiently truncates the modelling domain in at least two dimensions (radial and axial) and seamlessly integrates fluid–solid coupling from unstretched regions to the PML regions. The proposed SEM–PML model is validated against an enlarged SEM model and semi-analytical finite element (SAFE) predictions, with results showing excellent agreement. It is found that when a point source is located in the inner fluid, a specific fluid-type wave mode is excited and is dominant in the inner fluid, while a structural “trapped” wave mode concentrates its energy in the pipe wall. Additionally, a radiation mode, appearing as the bulk-fluid wave, is excited and shown to couple across the inner fluid, pipe wall, and outer fluid domains. The SEM–PML model is also used to study wave scattering from a defect in the coupled system. Results indicate that both fluid and structure type guided waves can be effectively utilised for non-destructive inspection of such defects.

1. Introduction

Acoustic guided waves are widely employed for the non-destructive detection of defects in fluid-filled pipes and for diagnosing the condition of fluid-filled wells, which are often themselves immersed in fluid. In such coupled systems, guided waves exhibit modal characteristics that are frequency-dependent and dispersive. For fluid-filled pipes immersed in fluid, three primary classes of wave modes can be identified: leaky, trapped, and radiation modes [1,2]. Leaky modes propagate along the pipe while radiating a portion of their energy into the surrounding fluid. Trapped modes have their energy confined within the pipe wall or internal fluid, exhibiting no attenuation apart from intrinsic material damping. Radiation modes are standing waves in the outer fluid medium and are typically discarded in guided wave modal analysis, both because they are generally not considered to be useful for non-destructive testing (NDT) applications and also due to the infinite number of such modes arising from the unbounded surrounding medium. This study

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investigates the nature of wave modes excited by a point source located within the internal fluid of a fluid-filled pipe immersed in an external fluid. It further examines how these modes scatter in the presence of structural defects. A high-order spectral element model, coupled with a multi-axis Perfectly Matched Layer (PML), is proposed to efficiently predict wave propagation in the coupled system. Results reveal that, in addition to leaky and trapped modes, a radiation mode can also be excited by a point source in the inner fluid. This mode appears as a bulk fluid wave in the external domain, with its wavefront oriented in the radial direction, a phenomenon that has not, to the best of our knowledge, been previously documented. The numerical model is then used to explore the scattering of both leaky and trapped modes when interacting with structural defects in the system.

Analytical techniques have been proposed to investigate the modal solutions of coupled plate or pipe structures, often using transfer or global matrix methods [3,4]. However, these approaches still require numerical root-finding techniques to locate the zeros of the system matrix determinant. For more complex geometries and anisotropic materials, such as composite plates or pipes, fully numerical methods must be employed. The semi-analytical finite element (SAFE) method is widely used to compute the modal solutions of guided waves in waveguides with arbitrary cross-sectional geometries and material properties [5–8]. This technique incorporates an analytical $e^{i(\omega t - kz)}$ assumption into the modal solution in the wave propagation direction, while coupling it with finite element discretisation in the cross-sectional plane normal to the propagation direction. Here, ω is the radian frequency, $i = \sqrt{-1}$ is the imaginary unit, and k is the wavenumber in the wave propagation direction z . By analytically representing the axial dependence of the wave field, the SAFE method avoids full three-dimensional discretisation, resulting in a computationally efficient and accurate approach for modal analysis. Furthermore, the model can be reduced to a one-dimensional formulation if the sound field is assumed to be harmonic in the circumferential direction.

The SAFE technique can be coupled with a PML when the waveguide is immersed in a fluid or buried in soil, representing an infinitely extended open waveguide in the radial direction [1,2,9–12]. The PML introduces a continuous, complex-valued coordinate stretching function that ensures a smooth transition from the waveguide to the absorbing layer. Both polynomial and exponential forms of the stretching function have been proposed to achieve fast convergence to accurate solutions. In addition to modal solutions, Duan *et al.* proposed a hybrid model that divides an infinitely long waveguide into three or more sections [13]. The SAFE-PML method is employed in the uniform sections of the waveguide to reduce computational time. Mode matching is used to couple the modal and finite element solutions at the interfaces between different sections. Gallezot *et al.* proposed an alternative hybrid model for analysing wave scattering in an open elastic waveguide [14]. In their formulation, the total degrees of freedom (DoFs) in the model are partitioned into two distinct sets: one corresponding to the internal DoFs within the defect region, and the other to the DoFs located on the cross-sectional interfaces between different regions. The latter set is then expressed as a sum of wave modes calculated by SAFE-PML, thereby forming a connection between different regions. Williams *et al.* investigated breakout noise from finite and infinite length fluid-filled pipes immersed in fluid [15]. A point collocation technique was applied to match modal solutions at the interfaces between adjacent axial regions. Xu *et al.* studied the modal properties of multi-layered waveguides featuring both perfectly bonded and linear-slip boundary conditions between layers [16].

The above SAFE-PML based numerical models are formulated in the frequency domain under the assumptions of harmonic wave propagation and waveguides with uniform cross-sections. In practical NDT applications, however, acoustic signals are typically generated in the time domain, often in the form of pulses or impulse responses [17–20]. Time-domain pulse signals are preferred not only because the time-of-flight provides information about the location of potential features, such as defects in the waveguide, but also because reflections from different features appear at distinct times, allowing them to be interpreted more easily.

Transient time-domain responses at a limited number of observation points can be reconstructed from frequency-domain solutions using the inverse Fourier transform. Marzani investigated the transient propagation of ultrasonic guided waves in damped waveguides using this approach [21]. The transient excitation was first transformed into the frequency domain using the Fourier transform, and the bi-orthogonality relations between wave modes were then employed to determine the modal coefficients. After solving the SAFE-based wave equations at each frequency, the transient responses at discrete spatial locations were reconstructed using the inverse Fourier transform. Gallezot *et al.* adopted a similar frequency-domain approach to investigate the transient response of elastic embedded waveguides [22]. A PML was employed to truncate the unbounded surrounding medium, and the contributions of the three classes of wave modes to the modal solution were analysed. Claro *et al.* used the wave finite element (WFE) method to study wave propagation in fluid-filled pipes with joints and defects [23]. The transient response at the observation point was subsequently obtained by applying an inverse Fourier transform to the frequency-domain solution.

However, for large-scale fluid-solid coupled problems, reconstructing transient responses using the inverse Fourier transform requires frequency-domain solutions to be computed at a large number of discrete frequencies. Moreover, the computational cost increases significantly when transient responses are required at many observation points or throughout the entire computational domain. If the aim is to simulate and visualise time-domain wave propagation throughout the structure, it is generally more efficient to solve the governing equations directly in the time domain. Time-domain numerical modelling of wave propagation in immersed or buried pipes is particularly challenging because fine spatial and temporal discretisations are required, while the waveguide is theoretically unbounded in both the radial and axial directions.

The spectral element method (SEM) is well suited for modelling wave propagation in the time domain because it combines the high-order accuracy of spectral methods with the geometric flexibility of finite elements, while maintaining a diagonal mass matrix that enables efficient explicit time integration [24–27]. This method typically adopts high order Gauss-Lobatto-Legendre or Gauss-Lobatto-Chebyshev points as interpolation points [28]. Numerical integration over each element is performed using Gauss-Lobatto quadrature rules associated with these points. This combination produces a diagonal mass matrix, making the inversion straightforward, and this advantage can be exploited in explicit time-marching algorithms. In addition, SEM exhibits exponential

convergence in the physical domain for sufficiently smooth problems, and the required number of nodes per wavelength can be as low as π [29]. However, this exponential convergence does not necessarily extend to the PML region. Duan and Kirby compared the accuracy of quadratic finite elements with high-order spectral elements in a SAFE-PML model [11]. They observed that, for equivalent modelling accuracy, the number of nodes in the spectral element-based PML region must match that in the finite element-based PML region.

PMLs are also used in time-domain simulations to truncate infinite computational domains, although in a different form from those used in frequency-domain applications. The PML equations are typically derived through the introduction of a complex-valued coordinate stretching function. In frequency-domain models, such as SAFE-PML, the use of complex-valued functions poses no issue, as the harmonic $e^{i\omega t}$ assumption inherently incorporates complex numbers into the governing equations. However, in time-domain PML formulations, a key distinction arises: the final governing equations must remain entirely real-valued, and the explicit inclusion of complex numbers is not permissible. This creates a fundamental challenge-reconciling the inherently complex-valued coordinate stretching function with the requirement for real-valued equations in a time-domain framework. Komatitsch and Tromp proposed a split-field PML to address this issue in elastodynamics [30]. In this approach, the gradient operator is decomposed into components parallel and perpendicular to the PML interface. A special designed coordinate stretching function is employed, which ensures that the frequency-domain equations can be analytically transformed into real-valued time-domain equations, making the method compatible with time-domain SEM implementations.

This PML applies damping only in a single direction, specifically to waves propagating perpendicular to the interface, while the real part of the coordinate remains unstretched [30]. However, this method suffers from spurious reflections at grazing incidence. To address this limitation, Komatitsch and Martin proposed an unsplit convolutional PML for the seismic wave equation [31]. This technique introduces additional variables into the coordinate stretching function but presents it in a form that allows for an analytical transformation from the frequency domain to the time domain, retaining only real-valued functions in the time domain. Basu proposed a multi-axis PML to absorb outgoing waves within a finite element framework, and an explicit time-marching scheme was developed for its implementation [32]. This method allows the coordinate stretching function to be applied in all three orthogonal directions, rather than only in the direction normal to the interface. However, this approach requires strain computation and stress integration at each time step, making the numerical implementation and time-marching scheme rather complex. Kucukcoban and Kallivokas proposed an alternative multi-axis displacement-stress mixed PML formulation [33,34]. This method introduces additional auxiliary variables, and the kinematic equation governing the relationship between stretched strain and displacement is implemented separately. The separation of equations, along with the introduction of auxiliary variables, significantly reduces the complexity of both the numerical implementation and time integration compared to the method in [32]. Fathi extended the displacement-stress mixed PML formulation to three-dimensional problems in heterogeneous elastic media [35]. Duan later proposed a multi-axis displacement-strain mixed PML formulation [36], where the substitution of stress with strain in the mixed formulation further reduces the complexity of implementing separate equations based on auxiliary variables. A compatible set of equations has also been proposed for the fluid domain to enable the study of fluid-structure coupled problems [36].

The PML models presented in [33,34,36] are unsplit, mixed formulations based on the finite element method and involve a non-diagonal mass matrix. These models cannot be readily implemented with explicit time-marching schemes, which are typically employed in large-domain problems. There are many split-field mixed PML formulations exist in the literature [37,38], which typically do not require auxiliary variables. However, these approaches often treat velocity, stress, or pseudo-stress components as primary unknowns throughout the entire domain. Consequently, their integration into standard displacement-based finite element frameworks is less straightforward than that of unsplit formulations. In addition, PMLs are widely used in electromagnetic wave applications and in numerical frameworks that are not based on the finite element method; however, such models are not reviewed in this paper.

In this paper, we develop a time-domain multi-axis displacement-strain mixed PML formulation for the solid domain based on a high-order SEM with a diagonal mass matrix. A compatible formulation is also developed for the fluid domain, enabling fluid-structure coupling. The approach builds on the mixed PML formulations presented in [33,34,36], but with a key distinction: the proposed PML mass matrix is diagonal, and the entire model is implemented within the SEM framework. The proposed multi-axis SEM-PML formulation is then used to investigate the time-domain excitation, propagation, and scattering of guided waves in a fluid-filled cylindrical pipe immersed in a fluid. To the best of our knowledge, this problem has not previously been investigated theoretically. It presents a computationally challenging scenario that requires a sophisticated numerical model, motivating the development of the formulations proposed in this work.

The structure of the article is as follows. Section 2 presents the theoretical multi-axis SEM-PML framework for both fluid and structural domains, including the coupling between the fluid and elastic media within the PML region. The equations are formulated in a general form suitable for time-domain wave propagation applications. However, the applications explored in this paper are limited to wave excitation, propagation, and scattering in a fluid-filled pipe immersed in fluid, modelled under axisymmetric assumptions. Section 3 describes the high-order spectral elements, meshing strategy, and the high-order explicit time-marching scheme. In axisymmetric models, the use of Gauss-Lobatto-Legendre (GLL) nodal points and collocation-based quadrature rules along the pipe's central axis can produce zero entries in the mass matrix corresponding to the degrees of freedom on the centreline. This issue is effectively resolved by applying Gauss-Lobatto-Jacobi (GLJ) quadrature points, used only in the inner fluid domain and in the radial direction [39,40]. Section 4 presents the numerical validation of the model, followed by modal analysis and the study of the excitation and scattering of leaky, trapped, and radiation wave modes. Section 5 draws the conclusions.

2. Time-domain PML formulations

This section presents the mathematical PML formulations governing wave propagation in an ideal fluid and an elastic solid. In applications such as a fluid-filled pipe immersed in fluid, the waveguide is theoretically of infinite length (in practice, this means the waveguide length is much greater than the wavelength). Consequently, the fluid–structure coupling extends into the PML regions. An illustration of this application is shown in Fig. 1. This paper focuses primarily on the formulation of PMLs, including the implementation of coordinate stretching functions. The weak form governing equations for the unstretched regions have been presented in a previous publication [36] and are not repeated here.

2.1. Complex coordinate stretching

In an unsplit PML formulation, the wave equations retain their original form when transitioning from the physical domain to the stretched domain, with the only modification being the mapping of spatial coordinates into the complex space using complex stretching functions. This coordinate stretching is applied to the wave equations expressed in the frequency domain, and the equations can be transformed back into the time domain through careful design of the stretching function.

The original coordinate x_j ($j = 1, 2$ or 3 in an orthogonal coordinate system) is replaced by a stretched coordinate $\hat{x}_j$, defined as

$$\hat{x}_j = \int_0^{x_j} \xi_j(\tau) d\tau \quad (1)$$

This definition requires that

$$d\hat{x}_j = \xi_j dx_j, \text{ and } \frac{\partial}{\partial \hat{x}_j} = \frac{1}{\xi_j} \frac{\partial}{\partial x_j} \quad (2)$$

In this paper, a specific two-parameter form of the stretching function is used, with no summation implied over indices [35,36]:

$$\xi_j = h_j(x_j) + \frac{d_j(x_j)}{i\omega} \quad (3)$$

where, h_j and d_j are real valued scaling and damping functions, ω is the radian frequency and $i = \sqrt{-1}$. At the interface between the physical domain and the stretched PML domain, the two parameters are set as $h_j = 1$, and $d_j = 0$. This ensures that $\hat{x}_j = x_j$ at the coupling interface, thereby enforcing a smooth transition from the physical domain to the PML domain. Note that this form of the stretching function is specifically designed to facilitate an inverse Fourier transform when applied to wave equations, and is therefore different from the stretching functions typically used in frequency-domain applications [11].

The scaling parameter h_j increases the effective coordinate when $h_j > 1$, while the damping parameter d_j introduces spatial attenuation for propagating waves when $d_j > 0$. For evanescent waves, h_j is responsible for spatial attenuation, whereas d_j increases the effective coordinate. It has been observed that evanescent waves can generate spurious reflections at the PML interface in discretised numerical schemes [41]. In some PML formulations, a simplified one-parameter stretching function is used by setting $h_j = 1$, although this simplification generally reduces accuracy or efficiency, particularly in the presence of evanescent waves [42].

The stretching functions should be smooth and non-decreasing from the PML interface toward the outer boundary. Polynomial stretching functions have traditionally been used in the literature [33,34]. Duan *et al.* [2] proposed exponential stretching functions, which have demonstrated particularly strong performance in wave-propagation problems. Compared with polynomial forms, exponential stretching functions provide a smoother, lower-amplitude and more gradual transition from the unstretched region to the PML,

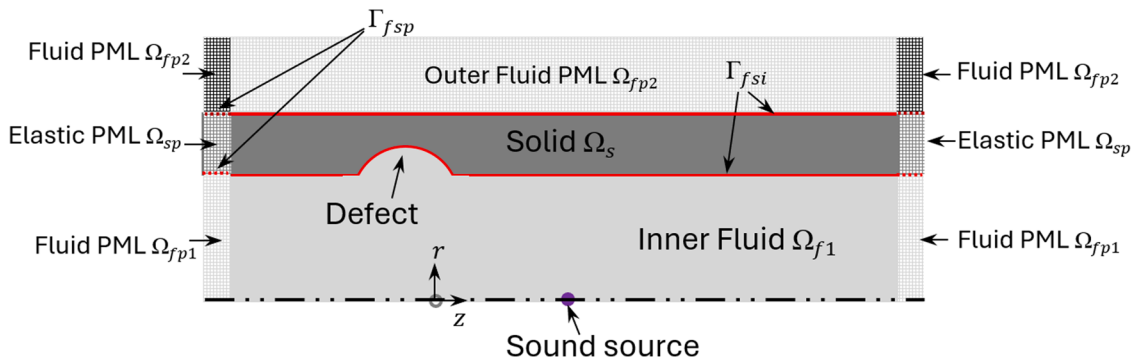


Fig. 1. Schematic of the fluid-filled pipe structure immersed in an outer fluid. PMLs enclose the domain in both axial and radial directions. The fluid inside the elastic pipe is referred to as the inner fluid, while the surrounding medium is the outer fluid. The outer fluid PML is placed adjacent to the pipe to minimise the computational domain.

while providing the strongest damping/stretching near the outer edge of the PML. In contrast, polynomial functions tend to impose larger stretching amplitudes near the entrance region but weaker stretching toward the exit. In this work, exponential stretching functions are adopted:

$$h_j = e^{\alpha_j \frac{|x_j - x_{je}|}{h_j}} \quad (4a)$$

$$d_j = e^{\beta_j \frac{|x_j - x_{je}|}{h_j}} - 1 \quad (4b)$$

Here, x_{je} is the entrance position of the PML layer and the thickness of the PML layer is h_j . α_j and β_j are real valued constants. For the cylindrical structure shown in Fig. 1, coordinate stretching can be applied in the r direction, the z direction, or both near the corner regions.

There is no universally accepted rule for selecting the parameters α_j , β_j and h_j in Eqs. (4a–b); these are application dependent and typically require a series of numerical tests to determine suitable values. Moreover, the appropriate parameter values differ between frequency-domain and time-domain applications. In this paper, the scaling parameter is fixed at $\alpha_j = 1$, and the damping parameter is defined as:

$$\beta = \text{ceil} \left[\log \left(1.5 * \frac{c_L}{h_j} * \log(1/R_0) \right) \right] \quad (5)$$

Here, the ceil operator rounds the value in the square brackets up to the nearest integer greater than or equal to that value. The reflection coefficient R_0 is fixed at $R_0 = 1 * 10^{-12}$.

2.2. Acoustic media

The equation of motion for wave propagation in a stationary, uniform fluid can be written in the frequency domain as [43]:

$$\nabla \cdot (\nabla \hat{\phi}) + \frac{\omega^2}{c^2} \hat{\phi} = \hat{f}_0 \quad (6)$$

where $\hat{\phi}$ is the velocity potential, c is the speed of sound in the fluid, ω is the radian frequency, and $\hat{f}_0$ is a body force in the fluid. An overhat is used in this paper to denote frequency domain variables. The frequency-domain formulation facilitates the incorporation of complex-valued coordinate stretching functions, which will later be transformed into the time domain.

In the PML region, the coordinate system is transformed, and the wave equation can be rewritten as:

$$\frac{1}{\xi_j} \frac{\partial}{\partial x_j} \left(\frac{1}{\xi_j} \frac{\partial \hat{\phi}}{\partial x_j} \right) + \frac{\omega^2}{c^2} \hat{\phi} = 0 \quad (7)$$

Here, repeated indices imply summation, and body forces are assumed to be absent in the PML region.

Auxiliary variables are often introduced to facilitate the development of governing equations. In some cases, these variables are expressed in complex forms, resulting in multiple auxiliary equations [44].

Here, we define the auxiliary variable in vector form:

$$\hat{\psi}_j = \frac{1}{\xi_j} \frac{\partial \hat{\phi}}{\partial x_j} \quad (8)$$

$\hat{\psi}$ can be physically interpreted as the fluid particle velocity within the PML region.

Eq. (8) can be substituted into Eq. (7), and the resulting expression is then multiplied by ζ , where $\zeta = \xi_1 \xi_2 \xi_3$. Considering the definition of ξ_j given in Eq. (3), the governing equation can be rewritten as:

$$\sum_{j=1,2,3} \left[\frac{\partial}{\partial x_j} \left(\mathbf{D}_3^R \hat{\psi} + \frac{1}{i\omega} \mathbf{D}_3^\epsilon \hat{\psi} - \frac{1}{\omega^2} \mathbf{D}_3^\Pi \hat{\psi} \right) \right] + \frac{\omega^2}{c^2} \mathbf{D}_3^T \hat{\phi} = 0 \quad (9)$$

The submatrices $\mathbf{D}_3^R$, $\mathbf{D}_3^\epsilon$, $\mathbf{D}_3^\Pi$, and the coefficient $\mathbf{D}_3^T$ are provided in Appendix A1-A4.

The weak form of Eq. (9) is obtained by taking the inner product with an arbitrary weighting function v , followed by integration over the fluid PML domain Ω_{fp} . Applying integration by parts and the divergence theorem yields:

$$\int_{\Omega_{fp}} \sum_{j=1,2,3} \left[\frac{\partial v}{\partial x_j} \left(\mathbf{D}_3^R \hat{\psi} + \frac{1}{i\omega} \mathbf{D}_3^\epsilon \hat{\psi} - \frac{1}{\omega^2} \mathbf{D}_3^\Pi \hat{\psi} \right) \right] d\Omega_{fp} - \int_{\Omega_{fp}} \left(\frac{\omega^2}{c^2} v \mathbf{D}_3^T \hat{\phi} \right) d\Omega_{fp} - \int_{\Gamma_{fp}} \sum_{j=1,2,3} \left[v \left(\mathbf{D}_3^R \hat{\psi} + \frac{1}{i\omega} \mathbf{D}_3^\epsilon \hat{\psi} - \frac{1}{\omega^2} \mathbf{D}_3^\Pi \hat{\psi} \right) n_{fj} \right] d\Gamma_{fp} = 0 \quad (10)$$

Here, Γ_{fp} denotes the boundary of the fluid PML domain Ω_{fp} , and n_{fj} is the outward-pointing unit normal vector on Γ_{fp} . As shown in Fig. 1, the fluid domain Ω_{fp} may refer to either the inner fluid region Ω_{fp1} or the outer fluid region Ω_{fp2} . Eq. (10) is applied separately to each fluid domain, and coupling with the solid domain is enforced at the interfaces.

The introduction of auxiliary variables introduces additional unknowns into the system, necessitating a separate independent equation to ensure the solution remains accurate and unique. This requirement can be fulfilled based on the kinematic relation in Eq. (8). Eq. (8) is first multiplied by ξ_j , then weighted with an arbitrary function $\mathbf{w}$, and integrated over the fluid PML domain Ω_{fp} . The resulting weak form of the kinematic equation is given as:

$$\int_{\Omega_{fp}} \sum_{j=1,2,3} \mathbf{w} \left(h_j \hat{\psi}_j + \frac{d_j}{i\omega} \hat{\psi}_j - \frac{\partial \hat{\phi}}{\partial x_j} \right) d\Omega_{fp} = 0 \quad (11)$$

Eqs. (10) and (11) can be transformed back into the time domain using the inverse Fourier transform. To simplify the presentation of the resulting time-domain equations, the following auxiliary variables are introduced:

$$\mathbf{I}_\psi(\mathbf{x}, t) = \int_0^t \boldsymbol{\psi}(\mathbf{x}, \tau) d\tau \quad (12a)$$

$$\mathbf{J}_\psi(\mathbf{x}, t) = \int_0^t \mathbf{I}_\psi(\mathbf{x}, \tau) d\tau \quad (12b)$$

$$I_\phi(\mathbf{x}, t) = \int_0^t \phi(\mathbf{x}, \tau) d\tau \quad (12c)$$

The corresponding time-frequency relations and differential forms of these terms are provided in [36] and are not repeated here for brevity.

The previous equations are presented in a general form. For the applications illustrated in Fig. 1, we now adopt cylindrical coordinates, where $j = 1$ denotes the radial direction, $j = 2$ the axial direction, and $j = 3$ the circumferential direction. Furthermore, for an axisymmetric model, we assume $\hat{\psi}_3 = 0$. No coordinate stretching is applied in the circumferential direction; that is, $h_3 = 1$, and $d_3 = 0$.

By substituting Eqs. (12a-c) into Eqs. (10) and (11), and applying the inverse Fourier transform, the weak form time domain equations can be written as:

$$\begin{aligned} & \int_{\Omega_{fp}} \sum_{j=1,2} \left(\frac{\partial v}{\partial x_j} \mathbf{D}_2^R \mathbf{I}_\psi \right) d\Omega_{fp} + \int_{\Omega_{fp}} \sum_{j=1,2} \left(\frac{\partial v}{\partial x_j} \mathbf{D}_2^\epsilon \mathbf{I}_\psi \right) d\Omega_{fp} + \int_{\Omega_{fp}} \left(\frac{1}{c^2} v h_1 h_2 \ddot{\phi} \right) d\Omega_{fp} + \int_{\Omega_{fp}} \left[\frac{1}{c^2} v (h_2 d_1 + h_1 d_2) \dot{\phi} \right] d\Omega_{fp} \\ & + \int_{\Omega_{fp}} \left(\frac{1}{c^2} v d_1 d_2 \dot{\phi} \right) d\Omega_{fp} - \int_{\Gamma_{fsp}} v \left[h_2 \psi_1 n_{fr} + F^{-1} \left(\frac{1}{i\omega} d_2 \hat{\psi}_1 n_{fr} \right) \right] d\Gamma_{fsp} \\ & = 0 \end{aligned} \quad (13)$$

And

$$\int_{\Omega_{fp}} \sum_{j=1,2} \mathbf{w} \left(h_j \ddot{\psi}_j \right) d\Omega_{fp} + \int_{\Omega_{fp}} \sum_{j=1,2} \mathbf{w} (d_j \dot{\psi}_j) d\Omega_{fp} - \int_{\Omega_{fp}} \sum_{j=1,2} \mathbf{w} \frac{\partial \dot{\phi}}{\partial x_j} d\Omega_{fp} = 0 \quad (14)$$

The submatrices $\mathbf{D}_2^R$ and $\mathbf{D}_2^\epsilon$ are provided in Appendix A5-A6. On the acoustic-structure interface Γ_{fsp} , an inverse Fourier transform operator F^{-1} is applied. The implementation of this interface integration is discussed in Section 2.4. Here, n_{fr} denotes the radial component of the unit normal vector on the interface Γ_{fsp} , pointing outward from the fluid domain. For an axisymmetric model, the volume element $d\Omega_{fp}$ can be expressed as $2\pi r dz dr$, and thus the meshing only needs to be performed in the $r - z$ plane.

A key distinction between the current formulation and that presented in [36] is the inclusion of a second-order differential term $\ddot{\psi}_j$ in the kinematic Eq. (14). This leads to a diagonal mass matrix when spectral elements are used. In contrast, the formulation in [36] contains no second-order differential terms in the kinematics equation, resulting in zero diagonal entries in the global mass matrix.

The fluid velocity potential and auxiliary variables are discretised using high-order spectral elements within a spectral element framework. The specific spectral element shape functions employed will be described in detail in Section 3. The fluid velocity potential and auxiliary velocity are expressed as follows:

$$\phi = \mathbf{N}_\phi \boldsymbol{\Phi}_q \quad (15)$$

and

$$\boldsymbol{\psi} = \mathbf{N}_\psi \boldsymbol{\psi}_k = \begin{bmatrix} \mathbf{N}_{\psi^e} & 0 \\ 0 & \mathbf{N}_{\psi^e} \end{bmatrix} \begin{bmatrix} \boldsymbol{\psi}_{k1} \\ \boldsymbol{\psi}_{k2} \end{bmatrix} \quad (16)$$

Here, $\mathbf{N}_\phi$ and $\mathbf{N}_\psi$ are global spectral element shape functions, while $\boldsymbol{\Phi}_q$ and $\boldsymbol{\psi}_k$ denote the nodal values of the fluid velocity potential and auxiliary velocity, respectively, expressed in column vector form. The shape function $\mathbf{N}_\phi$ is a row vector of length m_f , where m_f is

the number of nodes (or degrees of freedom) in the fluid region. Similarly, $\mathbf{N}_{\psi e}$ is a row vector of length m_{fp} , with m_{fp} being the number of nodes in the fluid PML region. For the inner fluid region illustrated in Fig. 1, $m_{fp} < m_f$. In contrast, for the outer fluid region, the PML is applied to the entire domain, resulting in $m_{fp} = m_f$.

The weighting functions are chosen to be identical to the shape functions, such that $\mathbf{v} = [\mathbf{N}_\phi]^T$, and $\mathbf{w} = [\mathbf{N}_\psi]^T$.

Eqs. (15)-(16) are now substituted into Eqs. (13)-(14), and the resulting expressions can be assembled into the following global matrix form:

$$\mathbf{M}_{fpml}\ddot{\Phi} + \mathbf{C}_{fpml}\dot{\Phi} + \mathbf{K}_{fpml}\Phi = \mathbf{F}_a \quad (17)$$

Here

$$\Phi = \begin{Bmatrix} \Phi_q \\ I_\psi \end{Bmatrix} \quad (18)$$

$$\mathbf{M}_{fpml} = \begin{bmatrix} \mathbf{M}_\phi & 0 \\ 0 & \mathbf{M}_\psi \end{bmatrix} \quad (19)$$

$$\mathbf{C}_{fpml} = \begin{bmatrix} \mathbf{C}_\phi & \mathbf{C}_{\phi\psi} \\ \mathbf{C}_{\psi\phi} & \mathbf{C}_\psi \end{bmatrix} \quad (20)$$

$$\mathbf{K}_{fpml} = \begin{bmatrix} \mathbf{K}_\phi & \mathbf{K}_{\phi\psi} \\ 0 & 0 \end{bmatrix} \quad (21)$$

$$\mathbf{F}_a = \int_{\Gamma_{sp}} [\mathbf{N}_\phi]^T \left[h_2 \psi_1 n_{fr} + F^{-1} \left(\frac{1}{i\omega} d_2 \psi_1 n_{fr} \right) \right] d\Gamma_{sp} \quad (22)$$

The submatrices in these equations are defined in Appendix A7-A17. It can be observed that the global mass matrix $\mathbf{M}_{fpml}$ retains a diagonal structure, allowing seamless integration of the PML formulation with the spectral element scheme in the unstretched regions. As a result, the final global system of equations also possesses a diagonal mass matrix, enabling the use of explicit time-marching schemes for efficient computation in large-degree-of-freedom problems.

2.3. Solid media

The equation of motion for elastic wave propagation in an elastic medium can be expressed in the frequency domain as [45]:

$$-\rho\omega^2 \hat{u}_i = \frac{\partial \hat{\sigma}_{ij}}{\partial x_j} + \hat{f}_i \quad (23)$$

Here, $\hat{\sigma}_{ij}$ ($i, j = 1, 2$ or 3) is the symmetric second-rank stress tensor, ρ is the density, u_i is the displacement, and $\hat{f}_i$ represents the body force in the solid. Summation over repeated indices is implied in Eq. (23).

In the solid PML region, the elastic equation of motion in the stretched coordinate system can be expressed in the frequency domain as:

$$-\rho\omega^2 \hat{u}_i = \frac{1}{\xi_j} \frac{\partial \hat{\sigma}_{ij}}{\partial x_j} \quad (24)$$

Here, the stretching function ξ_j is defined in Eq. (3), and it is assumed that no body forces are present in the PML region.

While it is possible to directly substitute the stress-strain and strain-displacement relations into Eq. (24), doing so would complicate the formulation. To simplify the development of the equations in the PML region, auxiliary variables representing strain are introduced. The incorporation of the stretching function into the strain-displacement relation is handled through an additional kinematic equation.

Eq. (24) can be multiplied by ζ , where $\zeta = \xi_1 \xi_2 \xi_3$. Since each ξ_j is a function of x_j only, the ratio $\frac{\zeta}{\xi_j}$ is independent of x_j . Therefore, the governing equation can be rewritten as:

$$\sum_{j=1,2,3} \left(\frac{\partial \left[\hat{\sigma} \mathbf{D}_3^R + \frac{1}{i\omega} \hat{\sigma} \mathbf{D}_3^\epsilon - \frac{1}{\omega^2} \hat{\sigma} \mathbf{D}_3^\Pi \right]}{\partial x_j} \right) + \rho\omega^2 \mathbf{D}_3^T u_i = 0 \quad (25)$$

The submatrices $\mathbf{D}_3^R$, $\mathbf{D}_3^\epsilon$, $\mathbf{D}_3^\Pi$, and the coefficient $\mathbf{D}_3^T$ are provided in Appendix A1-A4.

Note that Eq. (25), which applies to the solid, is similar in form to Eq. (9) in the fluid. However, Eq. (25) is a vector equation, whereas Eq. (9) is a scalar equation.

The weak form of Eq. (25) is derived using Galerkin's principle by taking the inner product with an arbitrary weighting function v_i and integrating over the elastic PML domain Ω_{sp} . Applying integration by parts and the divergence theorem yields:

$$\int_{\Omega_{sp}} \sum_{i=1,2,3} \left\{ \sum_{j=1,2,3} \left[\frac{\partial v_i}{\partial x_j} \left(\hat{\mathbf{c}} \mathbf{D}_3^R + \frac{1}{i\omega} \hat{\mathbf{c}} \mathbf{D}_3^\epsilon - \frac{1}{\omega^2} \hat{\mathbf{c}} \mathbf{D}_3^\Pi \right) \right] - \rho \omega^2 v_i \mathbf{D}_3^T u_i \right\} d\Omega_{sp} - \int_{\Gamma_{sp}} \sum_{i=1,2,3} \sum_{j=1,2,3} \left[v_i \left(\hat{\mathbf{c}} \mathbf{D}_3^R + \frac{1}{i\omega} \hat{\mathbf{c}} \mathbf{D}_3^\epsilon - \frac{1}{\omega^2} \hat{\mathbf{c}} \mathbf{D}_3^\Pi \right) n_{sj} \right] d\Gamma_{sp} = 0 \quad (26)$$

Here, Γ_{sp} denotes the boundary of the elastic PML domain Ω_{sp} , and n_{sj} is the outward-pointing unit normal vector on Γ_{sp} . The stress-strain constitutive relation in the PML region is given by:

$$\hat{\sigma}_{ij} = c_{ijkl}^E : \hat{\epsilon}_{kl} \quad (27)$$

Here, the double dot product of the stiffness and strain tensors implies summation over the repeated indices k and l . c_{ijkl}^E are the components of the fourth-order elasticity tensor, and $\hat{\epsilon}$ is the strain tensor.

The strain-displacement kinematic relation in the stretched coordinate system is given by:

$$\hat{\epsilon} = \frac{1}{2} \left(\frac{1}{\xi_j} \frac{\partial u_i}{\partial x_j} + \frac{1}{\xi_i} \frac{\partial u_j}{\partial x_i} \right) = \frac{1}{2} \left((\nabla \hat{\mathbf{u}}) \mathbf{J}^{-1} + \mathbf{J}^{-T} (\nabla \hat{\mathbf{u}})^T \right) \quad (28)$$

Here, $\hat{\mathbf{u}}$ is the displacement vector and ∇ is the gradient operator. The stretching tensor $\mathbf{J}$ is defined in Appendix B1.

In this paper, $\hat{\epsilon}$ is introduced as an auxiliary unknown variable. Rather than substituting both Eqs. (27) and (28) directly into Eq. (26), which would result in an overly complex expression, Eqs. (26) and (28) are instead solved separately, facilitated by the use of the auxiliary variable $\hat{\epsilon}$.

Eq. (28) is now pre- and post-multiplied by $i\omega \mathbf{J}^T$ and $\mathbf{J}$, respectively, and the following expression is obtained after regrouping [36]:

$$i\omega \mathbf{H}^T \hat{\epsilon} \mathbf{H} + \mathbf{D}^T \hat{\epsilon} \mathbf{H} + \mathbf{H}^T \hat{\epsilon} \mathbf{D} + \frac{1}{i\omega} \mathbf{D}^T \hat{\epsilon} \mathbf{D} = \frac{1}{2} i\omega (\mathbf{H}^T (\nabla \hat{\mathbf{u}}) + (\nabla \hat{\mathbf{u}})^T \mathbf{H}) + \frac{1}{2} (\mathbf{D}^T (\nabla \hat{\mathbf{u}}) + (\nabla \hat{\mathbf{u}})^T \mathbf{D}) \quad (29)$$

Here $\mathbf{J} = \mathbf{H} + \frac{1}{i\omega} \mathbf{D}$, where $\mathbf{H}$ and $\mathbf{D}$ are defined in Appendix B2-B3.

The kinematic Eq. (29) is weighted by an arbitrary function $\mathbf{w}_s$, and the resulting expression is integrated over the elastic PML domain Ω_{sp} . The weak form of the equation can then be written as:

$$\int_{\Omega_{sp}} \mathbf{w}_s \left[i\omega \mathbf{H}^T \hat{\epsilon} \mathbf{H} + \mathbf{D}^T \hat{\epsilon} \mathbf{H} + \mathbf{H}^T \hat{\epsilon} \mathbf{D} + \frac{1}{i\omega} \mathbf{D}^T \hat{\epsilon} \mathbf{D} - \frac{1}{2} i\omega (\mathbf{H}^T (\nabla \hat{\mathbf{u}}) + (\nabla \hat{\mathbf{u}})^T \mathbf{H}) - \frac{1}{2} (\mathbf{D}^T (\nabla \hat{\mathbf{u}}) + (\nabla \hat{\mathbf{u}})^T \mathbf{D}) \right] d\Omega_{sp} = 0 \quad (30)$$

To facilitate the inverse Fourier transform, we now introduce auxiliary variables in the time domain in integral form:

$$\mathbf{I}_e(\mathbf{x}, t) = \int_0^t \epsilon(\mathbf{x}, \tau) d\tau \quad (31a)$$

$$\mathbf{J}_e(\mathbf{x}, t) = \int_0^t \mathbf{I}_e(\mathbf{x}, \tau) d\tau \quad (31b)$$

The differential forms and time-frequency domain relationships of these variables are provided in [36] and are not repeated here.

In addition, we assume that the structure is axisymmetric and the material isotropic, which allows for certain simplifications. Eq. (27) can therefore be rewritten as [46]:

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sigma_{rz} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 \\ C_{12} & C_{11} & C_{12} & 0 \\ C_{12} & C_{12} & C_{11} & 0 \\ 0 & 0 & 0 & C_{44} \end{bmatrix} \begin{bmatrix} \epsilon_{rr} \\ \epsilon_{\theta\theta} \\ \epsilon_{zz} \\ \epsilon_{rz} \end{bmatrix} \quad (32)$$

In compact form, Eq. (32) can be written as $\boldsymbol{\sigma} = \mathbf{c}^E \boldsymbol{\epsilon}$. In the development of the equations in the following sections, both the stress $\boldsymbol{\sigma}$ and strain $\boldsymbol{\epsilon}$ tensors are represented in vector form.

Furthermore, coordinate stretching is applied only in the radial and axial directions, while the circumferential direction remains unstretched; that is, $h_3 = 1$, and $d_3 = 0$.

Substituting Eqs. (31) and (32) into Eq. (26), and applying the inverse Fourier transform, the weak form time-domain wave equation can be written as:

$$\begin{aligned} & \int_{\Omega_{sp}} \mathbf{v}_s [\mathbf{L}_{D2}^R]^T \mathbf{c}^E \dot{\mathbf{I}}_e d\Omega_{sp} + \int_{\Omega_{sp}} \mathbf{v}_s [\mathbf{L}_{D2}^{\varepsilon}]^T \mathbf{c}^E \mathbf{I}_e d\Omega_{sp} + \int_{\Omega_{sp}} \mathbf{v}_s [\mathbf{L}_{D2}^{\Pi}]^T \mathbf{c}^E \mathbf{J}_e d\Omega_{sp} + \int_{\Omega_{sp}} \rho h_1 h_2 \mathbf{v}_s \ddot{\mathbf{u}} d\Omega_{sp} + \\ & \int_{\Omega_{sp}} \rho (h_2 d_1 + h_1 d_2) \mathbf{v}_s \dot{\mathbf{u}} d\Omega_{sp} + \int_{\Omega_{sp}} \rho d_1 d_2 \mathbf{v}_s \mathbf{u} d\Omega_{sp} - \int_{\Gamma_{fsp}} \mathbf{v}_s \left[h_2 \sigma_{rr} n_{sr} + F^{-1} \left(\frac{1}{i\omega} d_2 \sigma_{rr} n_{sr} \right) \right] d\Gamma_{fsp} = 0 \end{aligned} \quad (33)$$

The submatrices in Eq. (33) are provided in Appendix B4-B6. On the acoustic-structure interface Γ_{fsp} , the shear stress σ_{sr} is assumed to be zero, as the fluid does not support shear forces. Here, n_{sr} denotes the radial component of the unit normal vector on the interface, pointing outward from the solid domain.

Similarly, applying the inverse Fourier transform to Eq. (30) yields the time-domain kinematics equation:

$$\int_{\Omega_{sp}} \mathbf{w}_s [\mathbf{N}_H^{\times}] \ddot{\mathbf{I}}_e d\Omega_{sp} + \int_{\Omega_{sp}} \mathbf{w}_s [\mathbf{N}_D^{\pm}] \dot{\mathbf{I}}_e d\Omega_{sp} + \int_{\Omega_{sp}} \mathbf{w}_s [\mathbf{N}_D^{\times}] \mathbf{I}_e d\Omega_{sp} - \int_{\Omega_{sp}} \mathbf{w}_s [\mathbf{L}_H^{-}] \dot{\mathbf{u}} d\Omega_{sp} - \int_{\Omega_{sp}} \mathbf{w}_s [\mathbf{L}_D^{-}] \mathbf{u} d\Omega_{sp} = 0 \quad (34)$$

The submatrices in Eq. (34) are provided in Appendix B7-B11.

The vector form displacement and auxiliary strain variables can now be discretised using high-order spectral elements:

$$\mathbf{u} = \mathbf{N}_u \mathbf{u}_q = \begin{bmatrix} \mathbf{N}_{uq} & 0 \\ 0 & \mathbf{N}_{uq} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{qr} \\ \mathbf{u}_{qz} \end{bmatrix} \quad (35)$$

and

$$\boldsymbol{\varepsilon} = \mathbf{N}_e \boldsymbol{\varepsilon}_k = \begin{bmatrix} \mathbf{N}_{ek} & 0 & 0 & 0 \\ 0 & \mathbf{N}_{ek} & 0 & 0 \\ 0 & 0 & \mathbf{N}_{ek} & 0 \\ 0 & 0 & 0 & \mathbf{N}_{ek} \end{bmatrix} \begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \\ \varepsilon_{rz} \end{bmatrix} \quad (36)$$

Here, $\mathbf{N}_u$ and $\mathbf{N}_e$ are the global spectral element shape functions. $\mathbf{u}_q$ and $\boldsymbol{\varepsilon}_k$ are the nodal values of displacement and auxiliary strain, respectively, expressed in column vector form. In addition, $\mathbf{N}_{uq}$ is a row vector of length m_s , where m_s is the number of nodes in the elastic region. Similarly, $\mathbf{N}_{ek}$ is a row vector of length m_{sp} , where m_{sp} is the number of nodes in the elastic PML region.

The weighting functions are chosen to be identical to the shape functions, such that $\mathbf{v}_s = [\mathbf{N}_u]^T$, and $\mathbf{w}_s = [\mathbf{N}_e]^T$.

Eqs. (35)-(36) are now substituted into Eqs. (33)-(34), and the resulting expressions can be assembled into the following global matrix form:

$$\mathbf{M}_{spml} \ddot{\mathbf{U}} + \mathbf{C}_{spml} \dot{\mathbf{U}} + \mathbf{K}_{spml} \mathbf{U} + \mathbf{G}_{spml} \mathbf{J}_s = \mathbf{F}_s \quad (37)$$

Here

$$\mathbf{U} = \begin{Bmatrix} \mathbf{u}_q \\ \mathbf{I}_e \end{Bmatrix} \quad (38)$$

$$\mathbf{J}_s = \begin{Bmatrix} 0 \\ \mathbf{J}_e \end{Bmatrix} \quad (39)$$

$$\mathbf{M}_{spml} = \begin{bmatrix} \mathbf{M}_u & 0 \\ 0 & \mathbf{M}_e \end{bmatrix} \quad (40)$$

$$\mathbf{C}_{spml} = \begin{bmatrix} \mathbf{C}_u & \mathbf{C}_{ue} \\ \mathbf{C}_{eu} & \mathbf{C}_e \end{bmatrix} \quad (41)$$

$$\mathbf{K}_{spml} = \begin{bmatrix} \mathbf{K}_u & \mathbf{K}_{ue} \\ \mathbf{K}_{eu} & \mathbf{K}_e \end{bmatrix} \quad (42)$$

$$\mathbf{G}_{spml} = \begin{bmatrix} 0 & \mathbf{G}_{ue} \\ 0 & 0 \end{bmatrix} \quad (43)$$

$$\mathbf{F}_s = \int_{\Gamma_{fsp}} [\mathbf{N}_u]^T \left[h_2 \sigma_{rr} n_{sr} + F^{-1} \left(\frac{1}{i\omega} d_2 \hat{\sigma}_{rr} n_{sr} \right) \right] d\Gamma_{fsp} \quad (44)$$

The submatrices in these equations are provided in Appendix B12-B22.

It can be seen that Eq. (37) features a diagonally structured global mass matrix, $\mathbf{M}_{spml}$, which facilitates the implementation of an explicit time-marching scheme within a spectral element framework.

2.4. Acoustic-structure coupling

The standard acoustic-structure interface coupling conditions have been discussed in various publications [11,36]. However, in this paper, the interface Γ_{fsp} is stretched in the axial direction (see Fig. 1). As a result, the acoustic-structure interface integrations involve stretching functions. The implementation of these interface integrations, as shown in Eqs. (22) and (44), is discussed in this section.

It is assumed that the fluid radial particle velocity is equal to the solid radial particle velocity on Γ_{fsp} :

$$\psi_1 = \frac{\partial u_r}{\partial t} \quad (45)$$

The frequency-time domain representation of this equation can be written as:

$$F^{-1} \left(\frac{1}{i\omega} \hat{\psi}_1 \right) = u_r \quad (46)$$

Substituting Eqs. (45) and (46) into Eq. (22) yields:

$$\mathbf{F}_a = \int_{\Gamma_{fsp}} [\mathbf{N}_\phi]^T \left[h_2 \frac{\partial u_r}{\partial t} n_{fr} + d_2 u_r n_{fr} \right] d\Gamma_{fsp} \quad (47)$$

Note that u_r denotes the radial displacement in the solid domain, and thus $\mathbf{F}_a$ represents the coupling between the fluid and solid domains.

It is also assumed that the fluid pressure is equal to the normal elastic stress on Γ_{fsp} , and applying the fluid momentum equation yields:

$$\sigma_{rr} = \rho_0 \frac{\partial \phi}{\partial t} \quad (48)$$

Here, ρ_0 is the fluid density.

The frequency-time domain representation of this equation can be written as:

$$F^{-1} \left(\frac{1}{i\omega} \hat{\sigma}_{rr} \right) = \rho_0 \phi \quad (49)$$

Substituting Eqs. (48) and (49) into Eq. (44) leads to:

$$\mathbf{F}_s = \int_{\Gamma_{fsp}} [\mathbf{N}_u]^T \left[h_2 \rho_0 \frac{\partial \phi}{\partial t} n_{sr} + d_2 \rho_0 \phi n_{sr} \right] d\Gamma_{fsp} \quad (50)$$

Here, the fluid domain velocity potential ϕ must be included in Eq. (50). This equation is then substituted into Eq. (37) as a right-hand-side boundary condition term for solving the solid domain displacement field.

In this paper, the fluid-domain system Eq. (17) and the solid-domain system Eq. (37) are solved separately in their respective domains, while the interface coupling conditions are enforced at the fluid–solid boundary using an explicit time-marching scheme. Specifically, Eq. (17) is applied to the inner and outer fluid regions individually, as shown in Fig. 1, while Eq. (37) is applied to the solid region. The coupling between these regions is enforced through Eqs. (47) and (50) on the relevant interfaces.

In acoustic-structure coupled problems, the standard unit for the velocity potential in the fluid is m^2/s , whereas the standard unit for displacement in the solid is m . Due to the differing physical meanings and typical magnitudes of these quantities, their numerical ranges can vary significantly. To facilitate a consistent presentation of results, the velocity potential is normalised by:

$$\bar{\phi} = \frac{\phi}{f_c} \quad (51)$$

Here, f_c is the centre frequency of the sound source, and an overbar denotes normalisation of a variable. Eq. (51) is then substituted into Eqs. (17) and (37) during the numerical implementation of the model.

3. Spectral elements and time marching

The shape functions used to discretise the fluid velocity potential, solid displacement, and auxiliary variables are constructed using high-order spectral elements. These spectral elements are employed throughout both the fluid and solid domains. In this framework, the nodal distribution is non-equidistant. Furthermore, nodal and numerical integration points coincide at the edges of Gauss-Lobatto-Legendre (GLL) elements. This leads to zero entries in the global diagonal mass matrix corresponding to nodes located along the centreline of the fluid domain in an axisymmetric model, which in turn prevents the inversion of the global mass matrix. To address this issue, Gauss-Lobatto-Jacobi (GLJ) functions are used alongside the GLL functions. The spectral element functions and the time-marching scheme are detailed in this section.

3.1. Spectral elements

The n th order Legendre polynomial, denoted by L_n , can be computed using the following recurrence relation [47]:

$$L_0(x) = 1 \quad (52a)$$

$$L_1(x) = x \quad (52b)$$

$$L_{n+1}(x) = \frac{2n+1}{n+1}xL_n(x) - \frac{n}{n+1}L_{n-1}(x) \quad (52c)$$

We then proceed with the definition of GLL points, which are used for meshing. For a polynomial order n , the GLL points are given by:

$$x_i^{n, GLL} = \begin{cases} -1, & i = 0 \\ \text{zeros of } L'_n & i = 1, 2, \dots, n-1 \\ 1 & i = n \end{cases} \quad (53)$$

Here, L'_n denotes the first derivative of the Legendre polynomial L_n .

The shape functions can be constructed at the discrete GLL points using an appropriate interpolation scheme. Numerical integration can also be performed at these collocation points, which is a key feature of the SEM. The Lagrange polynomials, used as interpolation functions, are defined as:

$$l_i^n = \prod_{j=0, j \neq i}^n \frac{x - x_j}{x_i - x_j} \quad (54)$$

The Lagrange polynomials have the characteristic that each l_i^n is equal to 1 at its local coordinate x_i , and zero at all other nodes of the element. The shape functions at the discrete GLL points can thus be expressed using the Kronecker delta:

$$l_i^n(x_j) = \delta_{ij} \quad (55)$$

The Gauss-Lobatto-Legendre quadrature is used for numerical integration, with the weights defined as:

$$w_j^{GLL} = \frac{2}{n(n+1)} \frac{1}{[L'_n(x_j)]^2} \quad (56)$$

Here, x_j , $j = 0, \dots, n$, are the collocation points defined in Eq. (53).

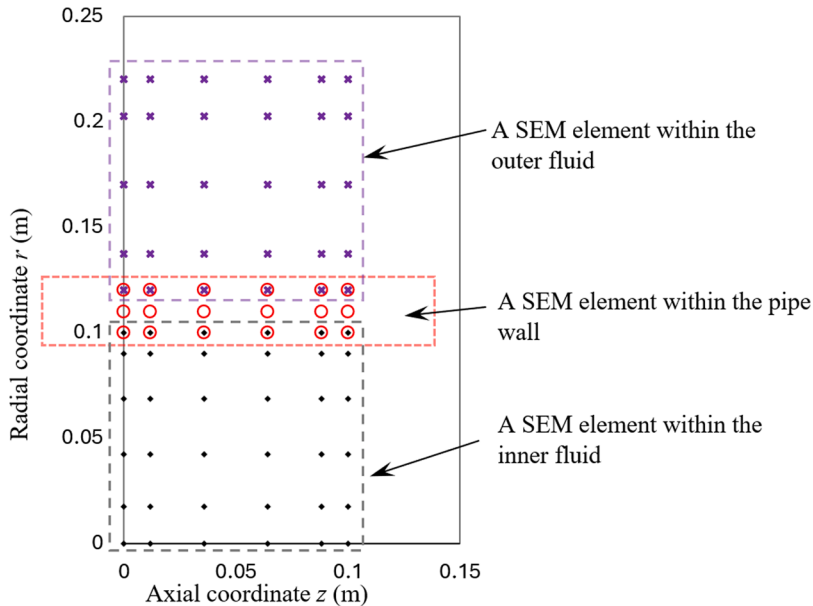


Fig. 2. Example SEM elements for a fluid-filled pipe immersed in a fluid domain. A representative SEM element is highlighted in each of the inner fluid, pipe wall, and outer fluid regions. For the inner fluid element, 5th-order GLJ points are used in the radial direction and 5th-order GLL points in the axial direction. The pipe wall element uses 2nd-order GLL points radially and 5th-order axially. For the outer fluid element, 4th-order GLL points are used in the radial direction and 5th-order in the axial direction.

The GLL nodes include both endpoints, as shown in Eq. (53). If applied to the inner fluid region, this results in nodal points lying on the centreline. In the SEM, collocation-based quadrature is used for integration, and at the centreline, $r = 0$. This leads to zero entries in the global mass matrix for an axisymmetric model, since the volume element $d\Omega_{fp}$ is integrated as $2\pi r \, dz \, dr$, which vanishes when $r = 0$.

To address this issue, Gauss-Lobatto-Jacobi (GLJ) nodes are used in the radial direction for elements adjacent to the centreline in the inner fluid region. We begin with the definition of the Jacobi polynomials, which can be computed using the following recurrence relation [39]:

$$P_0^{(0,1)}(x) = 1 \quad (57a)$$

$$P_1^{(0,1)}(x) = \frac{1}{2}(3x - 1) \quad (57b)$$

$$P_{n+1}^{(0,1)}(x) = \left[\frac{(2n+3)(2n+1)x-1}{(n+2)(2n+1)} \right] P_n^{(0,1)}(x) - \left[\frac{n(2n+3)}{(n+2)(2n+1)} \right] P_{n-1}^{(0,1)}(x) \quad (57c)$$

The GLJ points used for meshing are defined as:

$$x_i^{n, GLJ} = \begin{cases} -1, & i = 0 \\ \text{zeros of } P_n' & i = 1, 2, \dots, n-1 \\ 1 & i = n \end{cases} \quad (58)$$

Here, P_n' denotes the first derivative of the Jacobi polynomial $P_n^{(0,1)}$.

The Lagrangian interpolating polynomial functions based on the GLJ points are defined as follows:

$$h_0^n = \frac{2(-1)^n(x-1)P_n'}{(n+2)(n+1)n} \quad (59a)$$

$$h_i^n = \frac{(x^2-1)P_n'}{n(n+2)P_n^{(0,1)}(x_i)(x-x_i)}, \quad 1 \leq i \leq n-1 \quad (59b)$$

$$h_n^n = \frac{(x+1)P_n'}{n(n+2)} \quad (59c)$$

The shape functions at the discrete GLJ points can also be expressed using the Kronecker delta, such that $h_i^n(x_j) = \delta_{ij}$.

For elements adjacent to the centreline in the inner fluid, meshing is carried out using both GLL and GLJ points. Specifically, GLL nodes and their associated quadrature rule are applied in the axial direction z , while GLJ nodes and the corresponding quadrature rule are used in the radial direction r . An example mesh for a fluid-filled pipe immersed in a fluid domain is shown in Fig. 2.

In the figure, fluid velocity potential nodes in the inner fluid element are represented by diamonds, solid displacement nodes in the pipe wall element by circles, and fluid velocity potential nodes in the outer fluid element by crosses. It can be seen that the elements in the inner fluid, pipe wall, and outer fluid are collocated at the coupling interfaces. However, they have different node numbering due to differing degrees of freedom in the fluid and solid domains.

The spectral function orders are generally different in the radial and axial directions. This variation is primarily dictated by the geometry and the number of nodes per wavelength required by the SEM scheme. In the pipe wall, where the thickness is typically smaller than the wavelength, a lower-order polynomial (2nd order in Fig. 2) is used in the radial direction. This variation in polynomial order not only reduces the total number of DOFs in the system but also improves the time step in explicit time integration schemes.

Note that in the inner fluid element, the GLL and GLJ points have different node spacings in the axial and radial directions, respectively.

Numerical integration in the solid and outer fluid regions is performed using GLL quadrature. For elements adjacent to the centreline in the inner fluid region, L'Hôpital's rule can be applied to the quadrature formulation. The numerical integration in these axis-adjacent elements is then expressed as:

$$\int_{\Omega_{fp}} \phi 2\pi r \, dr \, dz = \sum_{i=0}^n \sum_{j=0}^m \frac{w_i^{GLJ}}{1+\zeta_i} w_j^{GLL} \phi(\zeta_i, \eta_j) 2\pi r(\zeta_i, \eta_j) \mathcal{J}(\zeta_i, \eta_j) \quad (60)$$

Here, ζ is the local element coordinate in the r direction, and η is the local element coordinate in the z direction. A GLJ polynomial of order n is applied in the ζ direction, and a GLL polynomial of order m is applied in the η direction. $\mathcal{J}$ is the determinant of the Jacobian matrix for the element.

On the centreline, where $r = 0$ and $\zeta_0 = -1$, the singular term $\frac{r(\zeta_0, \eta_j)}{1+\zeta_0}$ can be resolved by applying L'Hôpital's rule [40]:

$$\frac{r(\zeta_0, \eta_j)}{1+\zeta_0} = \frac{\partial r(\zeta_0, \eta_j)}{\partial \zeta} \quad (61)$$

The weights for the GLJ quadrature are defined as:

$$w_i^{GLJ} = \begin{cases} \frac{8}{n(n+2)(n+1)^2} & i = 0 \\ \frac{4}{n(n+2)[P_n^{(0,1)}(\zeta_i)]^2} & 1 \leq i \leq n \end{cases} \quad (62)$$

Here, ζ_i , $i = 0, \dots, n$, are the GLJ collocation points defined in Eq. (58).

The formulae used to compute the GLJ weights are not unique. An alternative formulation is provided in Ref. [39], which produces identical results.

3.2. Fourth-order explicit Runge–Kutta time marching

In Section 3.1, we discussed the use of spectral elements to discretise the spatial variables. The wave equation, however, is a partial differential equation in both space and time. In this section, we detail a fourth-order Runge-Kutta method implemented using an explicit predictor-multicorrector scheme for time integration.

In this paper, the fluid system Eq. (17) and the solid system Eq. (37) are not assembled into a single system. Instead, they are implemented separately within their respective fluid and solid regions, and coupled through boundary conditions enforced at the interfaces. Specifically, we begin by executing Eq. (17) in the inner fluid region, where a sound source is typically located. For an explicit time-marching scheme, we assume that the fluid domain variables are known at the current time step t^k , and our objective is to compute the unknown variables at the next time step t^{k+1} . A key challenge arises here: to solve Eq. (17), we require accurate coupling conditions at t^{k+1} to serve as boundary conditions. However, these conditions cannot be readily provided, as the solid domain variables, governed by Eq. (37), have not yet been computed.

There are two main approaches proposed in the literature to address this challenge. The first is a staggered scheme [48,49]. In this approach, predictor values from the time-marching scheme in the solid domain are used at the coupling interface, allowing Eq. (17) to be evaluated in the fluid region. The resulting corrector values from Eq. (17) can then be applied as boundary conditions for solving Eq. (37) in the solid domain. In the solid domain, there are two coupling interfaces: one with the inner fluid and one with the outer fluid. Similarly, predictor values from the outer fluid region can be used at the corresponding interface to complete the boundary conditions for Eq. (37). After solving the solid domain, Eq. (17) can then be implemented for the outer fluid region. Since the predictor values used in this first pass are not accurate solutions, the process must be repeated. In the second iteration, the corrector values are used as boundary conditions, enabling accurate computation across the entire coupled system.

The second approach involves rewriting the wave equation in the fluid domain using displacement potentials instead of velocity potentials [50,51]. The advantage of this formulation is that displacement terms can be directly used to enforce continuity conditions at the fluid-solid interfaces when solving the fluid wave equation. In this approach, predictor values for the displacement terms are accurate due to the use of an explicit second-order Newmark scheme. Consequently, the wave equations in both the inner and outer fluid regions must be solved first. The resulting corrector values from the fluid solutions can then be fed back into the elastic wave equation. Unlike the staggered scheme, this approach does not require iteration. However, it is not applicable to the current PML formulation, as both displacement and velocity terms are involved at the fluid-solid interfaces within the PML region.

In this paper, we propose an alternative approach based on the fourth-order Runge-Kutta method. Compared to the second-order Newmark scheme [50,51], it offers higher accuracy for the same time step size, thereby enabling the use of larger time steps. The proposed method also eliminates the need for iteration at every time step [48,49]. A four-stage predictor-multicorrector algorithm is introduced, in which the fluid-solid boundary conditions are progressively exchanged at each stage. This stage-wise enforcement of fluid-solid coupling within each time-marching step removes the need for any additional iteration.

The general form of the time-domain wave equation can be expressed as:

$$\mathbf{M}_{sf} \mathbf{a}_{sf} + \mathbf{C}_{sf} \mathbf{v}_{sf} + \mathbf{K}_{sf} \mathbf{u}_{sf} + \mathbf{G}_{sf} \mathbf{J}_{sf} + \mathbf{F}_{sf} = 0 \quad (63)$$

Here, the subscript s denotes the solid domain, to which Eq. (37) applies, while the subscript f refers to the fluid domain, where Eq. (17) is used. In the fluid domain, the terms $\mathbf{G}$ and $\mathbf{J}$ in Eq. (63) can be omitted. The vector $\mathbf{F}$ includes both body force terms and fluid-solid interface coupling terms. Furthermore, the following definitions hold: $\mathbf{v} = \frac{d\mathbf{u}}{dt}$, $\mathbf{a} = \frac{d^2\mathbf{u}}{dt^2}$ and $\mathbf{u} = \frac{d\mathbf{J}}{dt}$.

Explicit Runge-Kutta methods are formulated for first-order differential equations [52]; therefore, Eq. (63) can be rewritten as:

$$\frac{d}{dt} \begin{bmatrix} \mathbf{J} \\ \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{I} & 0 \\ 0 & 0 & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{G} & -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{J} \\ \mathbf{u} \\ \mathbf{v} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\mathbf{M}^{-1}\mathbf{F} \end{bmatrix} \quad (64)$$

The subscript has been omitted in Eq. (64) for brevity. In the SEM framework, the mass matrix $\mathbf{M}$ is diagonal; therefore, its inverse $\mathbf{M}^{-1}$ can be efficiently computed as $\mathbf{M}^{-1} = 1./\mathbf{M}$, where the element-wise division is applied only to the diagonal entries. It should be noted that the matrices $\mathbf{C}$ and $\mathbf{G}$ appear in Eq. (64) only because the PML is employed. In an enlarged SEM model without PML, $\mathbf{C}$ and $\mathbf{G}$ are zero matrices, and Eq. (64) can be simplified accordingly.

Furthermore, in the implementation of the enlarged SEM model, the stiffness matrix $\mathbf{K}$ is not assembled. Instead, the local stiffness

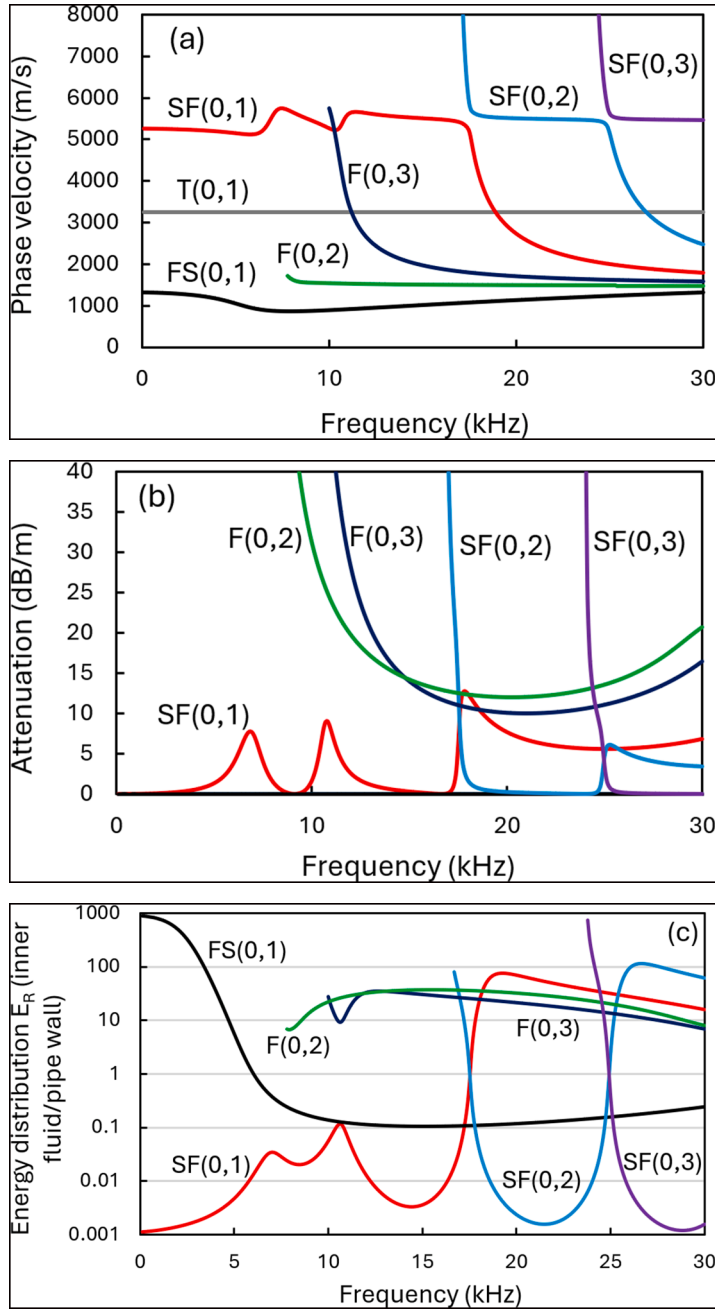


Fig. 3. Dispersion characteristics of guided waves in a water-filled pipe immersed in water: (a) phase velocity; (b) attenuation; (c) energy distribution.

matrix–vector product, $\mathbf{K}\mathbf{u}$, is computed for each element and subsequently assembled into the corresponding global vector at each time step. This matrix-free treatment of the stiffness operator yields substantial memory savings, particularly for problems with a large number of degrees of freedom. It also improves computational efficiency by exploiting highly efficient tensor-product operators based on GLL or GLJ points [53].

For the SEM-PML model proposed in this article, the global matrices $\mathbf{K}$, $\mathbf{C}$, and $\mathbf{G}$ are assembled. The development of efficient tensor-product operators for these matrices remains an open research topic and could potentially further improve the computational efficiency of the proposed SEM-PML model.

We now proceed with an explicit four-stage predictor-multicorrector algorithm. The initial conditions are assumed to be static, i.e., $\mathbf{u}^0 = 0$, $\mathbf{v}^0 = 0$ and $\mathbf{J}^0 = 0$. The coupling term $\mathbf{F}^0$ can be determined from $\mathbf{u}^0$ and $\mathbf{v}^0$; therefore, $\mathbf{F}^0$ is known at the initial time step. Without loss of generality, we assume that the solutions $\mathbf{u}^k$, $\mathbf{v}^k$ and $\mathbf{J}^k$ are known at time step t^k , and our objective is to compute the

solutions at the next time step t^{k+1} . Here, $t^{k+1} = t^k + \Delta t$, where Δt is the constant time step. Suppose the total number of time marching steps is N . The total simulation duration is then given by $T = N \Delta t$.

The four-stage Runge–Kutta predictor–multicorrector algorithm for the k^{th} time step is presented as follows:

$$\left\{ \begin{array}{l}
 \text{Stage 1 :} \\
 \text{Calculate } \mathbf{F}^k \text{ from } \mathbf{u}^k \text{ and } \mathbf{v}^k \text{ in the fluid and solid domains.} \\
 \text{Update the Stage 1 predictor in the fluid and solid domains :} \\
 \mathbf{a}^k = \mathbf{M}^{-1} \left(-\mathbf{C}\mathbf{v}^k - \mathbf{K}\mathbf{u}^k - \mathbf{G}\mathbf{J}^k - \mathbf{F}^k \right) \\
 \text{Stage 2 :} \\
 \mathbf{u}^{s2} = \mathbf{u}^k + \frac{\Delta t}{2} \mathbf{v}^k, \quad \mathbf{v}^{s2} = \mathbf{v}^k + \frac{\Delta t}{2} \mathbf{a}^k \\
 \mathbf{J}^{s2} = \mathbf{J}^k + \frac{\Delta t}{2} \mathbf{u}^k \\
 \text{Calculate } \mathbf{F}^{k+\frac{1}{2}} \text{ from } \mathbf{u}^{s2} \text{ and } \mathbf{v}^{s2} \text{ in the fluid and solid domains.} \\
 \text{Update the Stage 2 predictor in the fluid and solid domains :} \\
 \mathbf{a}^{s2} = \mathbf{M}^{-1} \left(-\mathbf{C}\mathbf{v}^{s2} - \mathbf{K}\mathbf{u}^{s2} - \mathbf{G}\mathbf{J}^{s2} - \mathbf{F}^{k+\frac{1}{2}} \right) \\
 \text{Stage 3 :} \\
 \mathbf{u}^{s3} = \mathbf{u}^k + \frac{\Delta t}{2} \mathbf{v}^{s2}, \quad \mathbf{v}^{s3} = \mathbf{v}^k + \frac{\Delta t}{2} \mathbf{a}^{s2} \\
 \mathbf{J}^{s3} = \mathbf{J}^k + \frac{\Delta t}{2} \mathbf{u}^{s2} \\
 \text{Calculate } \mathbf{F}^{k+\frac{1}{2}} \text{ from } \mathbf{u}^{s3} \text{ and } \mathbf{v}^{s3} \text{ in the fluid and solid domains.} \\
 \text{Update the Stage 3 predictor in the fluid and solid domains :} \\
 \mathbf{a}^{s3} = \mathbf{M}^{-1} \left(-\mathbf{C}\mathbf{v}^{s3} - \mathbf{K}\mathbf{u}^{s3} - \mathbf{G}\mathbf{J}^{s3} - \mathbf{F}^{k+\frac{1}{2}} \right) \\
 \text{Stage 4 :} \\
 \mathbf{u}^{s4} = \mathbf{u}^k + \Delta t \mathbf{v}^{s3}, \quad \mathbf{v}^{s4} = \mathbf{v}^k + \Delta t \mathbf{a}^{s3} \\
 \mathbf{J}^{s4} = \mathbf{J}^k + \Delta t \mathbf{u}^{s3} \\
 \text{Calculate } \mathbf{F}^{k+1} \text{ from } \mathbf{u}^{s4} \text{ and } \mathbf{v}^{s4} \text{ in the fluid and solid domains.} \\
 \text{Update the Stage 4 predictor in the fluid and solid domains :} \\
 \mathbf{a}^{s4} = \mathbf{M}^{-1} \left(-\mathbf{C}\mathbf{v}^{s4} - \mathbf{K}\mathbf{u}^{s4} - \mathbf{G}\mathbf{J}^{s4} - \mathbf{F}^{k+1} \right) \\
 \text{The multicorrector :} \\
 \text{The solutions are advanced to the next time step :} \\
 \mathbf{u}^{k+1} = \mathbf{u}^k + \frac{\Delta t}{6} (\mathbf{v}^k + 2\mathbf{v}^{s2} + 2\mathbf{v}^{s3} + \mathbf{v}^{s4}) \\
 \mathbf{v}^{k+1} = \mathbf{v}^k + \frac{\Delta t}{6} (\mathbf{a}^k + 2\mathbf{a}^{s2} + 2\mathbf{a}^{s3} + \mathbf{a}^{s4}) \\
 \mathbf{J}^{k+1} = \mathbf{J}^k + \frac{\Delta t}{6} (\mathbf{u}^k + 2\mathbf{u}^{s2} + 2\mathbf{u}^{s3} + \mathbf{u}^{s4})
 \end{array} \right. \quad (65)$$

At each stage, the stage-wise predictor–corrector algorithm is applied to the inner fluid, the solid, and the outer fluid regions. The fluid–solid coupling, detailed in Section 2.4, is updated at each stage using the corresponding predicted displacement and velocity values. In this algorithm, the fluid–solid coupling is enforced four times progressively per time-marching step to ensure accuracy.

For an explicit time-marching scheme, the time step in each region is constrained by the Courant–Friedrichs–Lewy (CFL) stability condition [51]. This condition ensures that information cannot travel more than one grid cell per time step:

$$\Delta t \leq C_t \left(\frac{\Delta h}{c_p} \right) \quad (66)$$

Here, Δh denotes the smallest distance between adjacent grid points, and c_p is the primary wave speed in the fluid or solid. C_t is the

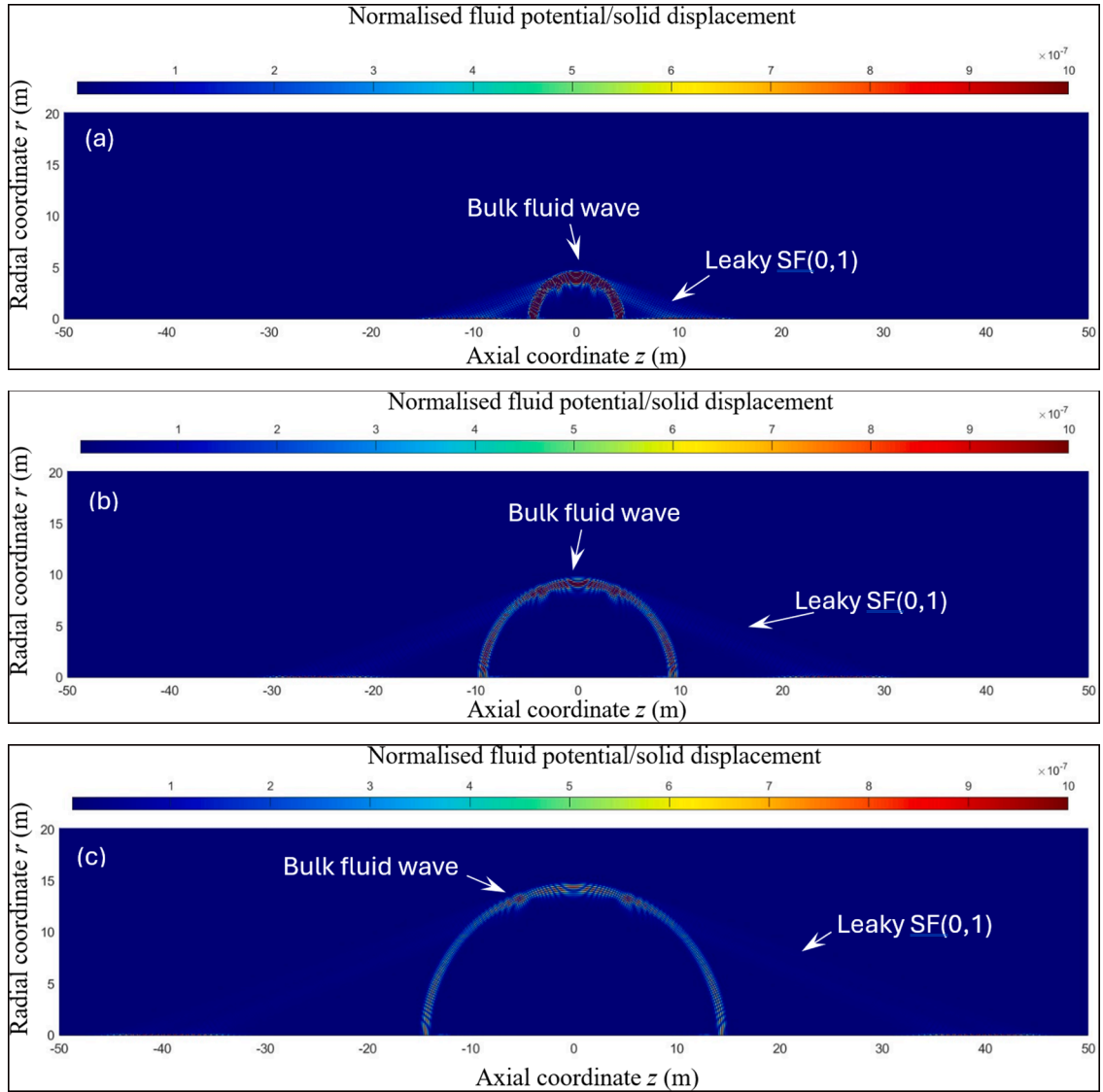


Fig. 4. Sound field distribution of normalised fluid potential and solid displacement excited by a point source: (a) $t = 3.362$ ms; (b) $t = 6.793$ ms; (c) $t = 10.224$ ms.

CFL number, typically ranging from 0.3 to 0.5 [51].

In a time-marching scheme, the time step, Δt , should be chosen as large as possible to minimise the total number of marching steps, N , required for a given simulation duration, T , while maintaining numerical stability. In this paper, a synchronous time-marching scheme is adopted for the fluid and solid domains. However, the grid spacing, Δh , generally differs between these domains. Consequently, the direct application of Eq. (66) results in different values of Δt for the fluid and solid domains. To ensure stability of the coupled system, the time step is chosen as the minimum of these values.

Furthermore, the fourth-order Runge–Kutta method is two orders more accurate than the second-order Newmark scheme [51]. This higher accuracy allows the CFL number C_t to be set to 1 for the applications considered in this study, significantly increasing the permissible time step.

4. Results and discussion

In this section, we present the excitation, propagation, and scattering of guided and radiated wave modes from a point source located in the inner fluid. We begin by characterising the modal properties of guided waves in the system, which serve as a baseline for further analysis. These modal properties are computed using the semi-analytical finite element (SAFE) method [11]. We then validate

the PML formulation proposed in this study using two approaches. The first approach involves an enlarged model, in which leaky and radiated wave modes can be observed before they interact with the outer boundary of the computational domain. A comparison between the results from the enlarged model and those from the PML model is presented and discussed. The second approach involves applying a two-dimensional Fourier transform to time-domain solutions, yielding frequency–wavenumber representations. These can be directly compared with the SAFE solutions for validation. Finally, we investigate the scattering of these waves from a structural defect in the pipeline.

The pipe studied in this work is an 8-inch Schedule 40 steel pipe, with an outer radius of 109.54 mm and an inner radius of 101.361 mm. The material properties of the steel are: longitudinal bulk wave velocity $c_L = 5960$ m/s, shear bulk wave velocity $c_T = 3260$ m/s, and density $\rho = 8030$ kg/m³. The pipe is surrounded internally and externally by water, with properties $c_0 = 1480$ m/s and $\rho_0 = 1000$ kg/m³, where c_0 is the sound speed in water.

4.1. Modal properties of guided waves

The modal properties of guided waves were obtained using a SAFE model [11]. In this model, the thickness of the PML was set to five times the pipe wall thickness. The phase velocity, attenuation ratio, and energy distribution of these modes are shown in Fig. 3.

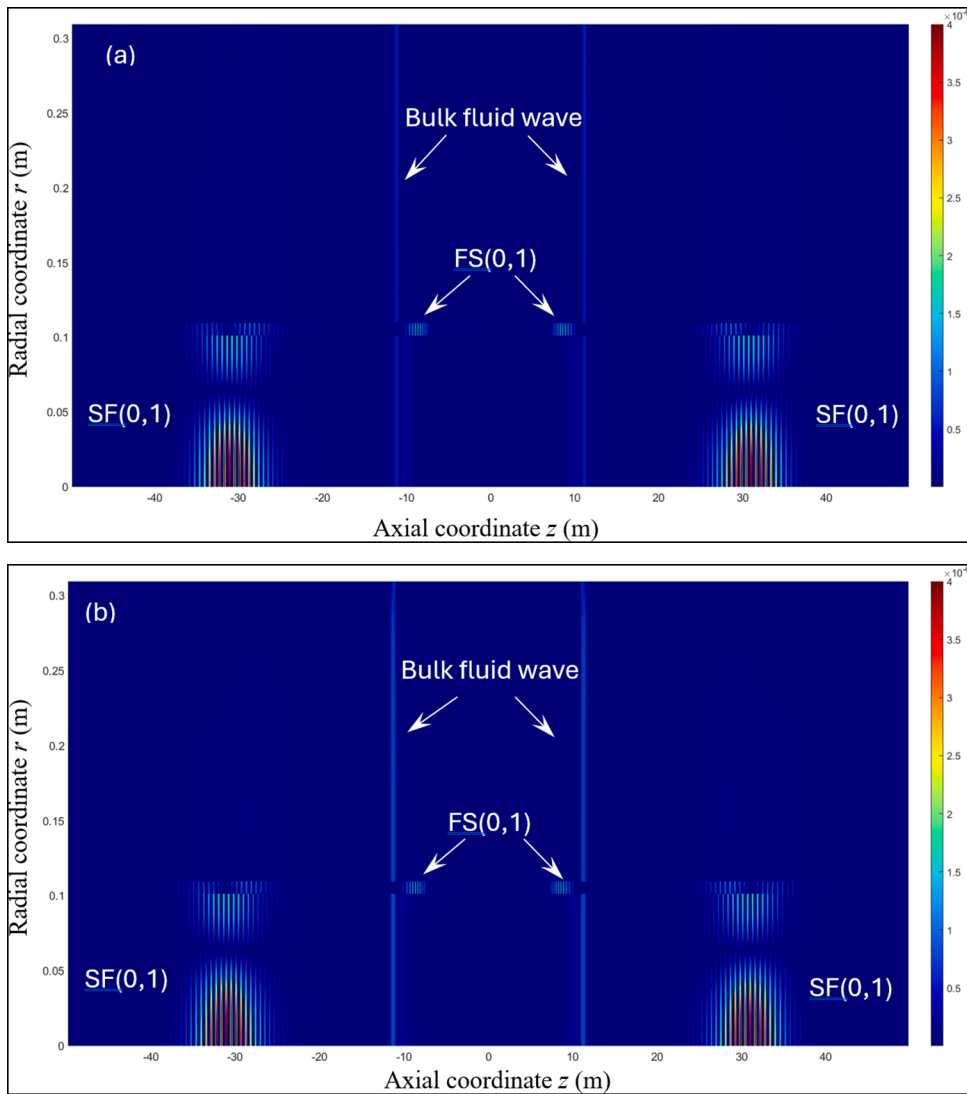


Fig. 5. Comparison of predicted sound field distributions of normalised fluid potential and solid displacement at $t = 8.165$ ms: (a) zoomed-in view of the enlarged model; (b) SEM-PML model with $z \in [-50, 50]$ m, and $r \in [0, 0.30954]$ m; (c) SEM-PML model with $z \in [-5, 50]$ m, and $r \in [0, 0.30954]$ m.

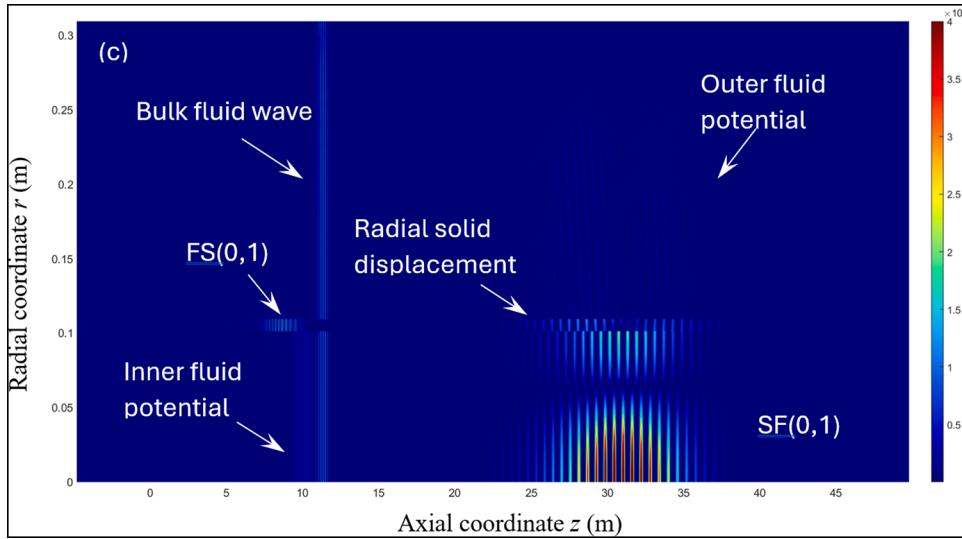


Fig. 5. (continued).

Material damping was not included in the model; therefore, the attenuation ratio of a mode is solely associated with the leakage of energy from the pipe wall into the theoretically infinite outer fluid region. The modes are labelled according to the nomenclature introduced in [11], using the forms $F(m,p)$, $FS(m,p)$, $SF(m,p)$, and $T(m,p)$. Here, F indicates that the modal energy is predominantly concentrated in the inner fluid over the frequency range considered in this paper. FS indicates that the modal energy is predominantly concentrated in the inner fluid at low frequencies and gradually shifts to the solid at higher frequencies. SF indicates that the modal energy is predominantly concentrated in the solid at low frequencies and gradually shifts to the inner fluid at higher frequencies. T denotes an uncoupled torsional wave mode, with the energy entirely confined to the solid pipe wall. The first index, m , denotes the circumferential mode order, while the second index, p , denotes the sequential mode number within the same circumferential mode family. For example, $m = 0$ corresponds to axisymmetric wave modes.

The energy distribution of the wave modes in the coupled pipeline system is shown in Fig. 3(c). Specifically, the energy distribution is defined as the ratio of the sound power carried by a mode in the inner fluid to that carried in the pipe wall. The sound power in the outer fluid region is excluded from this formulation because it depends on the PML thickness and is therefore not constant. For example, the $SF(0,1)$ mode has most of its energy concentrated in the solid pipe wall at low frequencies (below 17.5 kHz), whereas at frequencies above 17.5 kHz the dominant energy lies in the inner fluid (see Fig. 3c). In contrast, the $FS(0,1)$ mode has most of its energy in the inner fluid at low frequencies (below 6 kHz), while at higher frequencies (above 6 kHz) the dominant energy is in the pipe wall.

There are two trapped wave modes in this case: $T(0,1)$ and $FS(0,1)$. The $T(0,1)$ mode is an uncoupled torsional wave mode confined to the pipe wall, exhibiting only torsional displacement. Because fluids do not support shear motion, $T(0,1)$ is uncoupled from both the inner and outer fluid regions. The $FS(0,1)$ mode has its dominant energy either in the inner fluid or in the pipe wall, depending on the frequency. However, its motion decays exponentially in the outer fluid region, and it does not leak energy into the surrounding medium. Both trapped modes have zero attenuation in this case, as material damping is not included in the model.

In contrast, the other modes shown in Fig. 3 are leaky wave modes, exhibiting strong frequency-dependent dispersion and attenuation characteristics. The SAFE model also produces radiation wave modes; however, these are standing waves confined within the PML region, and their properties depend on the PML parameters. As such, radiation modes are not included in the figure.

4.2. Validation of the time domain SEM-PML model

We now validate the time-domain SEM-PML model using two approaches and different model parameters. A full enlarged model without PML has been incorporated as the reference solution. This model has a pipeline length of 100 m and an outer fluid radius of 20 m, resulting in a large computational domain with transient wave propagation in both space and time.

A brief overview of the contents covered in this section is provided here. The validation begins with an introduction to the parametric and implementation details of the enlarged SEM model, which serves as the full-domain reference model, followed by snapshots of wave propagation in the full model at different time instants, as shown in Fig. 4. Due to the large dimensions of the model, the fine spatial variations within the inner fluid and the pipe wall are not clearly visible in these full-domain plots. Therefore, Fig. 5 presents enlarged views comparing the full model with the SEM-PML model using different truncation lengths. These results demonstrate that the SEM-PML model can accurately truncate the computational domain at different locations while maintaining excellent agreement in the wavefield distributions.

The temporal response is then examined in Fig. 6 by comparing the wave propagation histories at an observation point. The

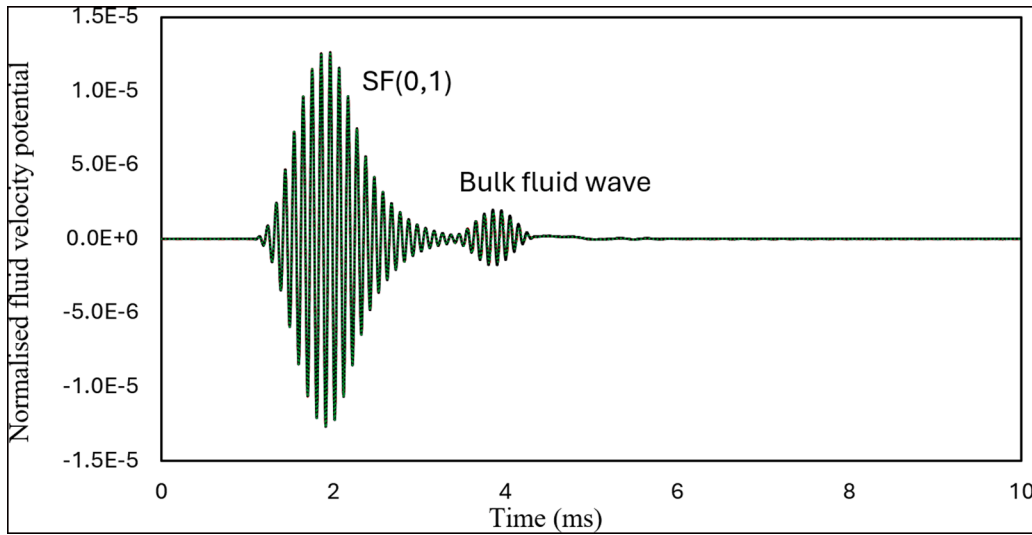


Fig. 6. Normalised fluid velocity potential at $z = 5$ m and $r = 0$. —: enlarged model; - - -: SEM-PML model with $z \in [-50, 50]$ m, and $r \in [0, 0.30954]$ m;: SEM-PML model with $z \in [-5, 50]$ m, and $r \in [0, 0.30954]$ m.

responses obtained from the full model and the SEM-PML model with two different truncation lengths are plotted together and are virtually indistinguishable, further confirming the accuracy of the SEM-PML model in the time domain. The space-time wavefield is subsequently transformed into the wavenumber-frequency domain. The resulting modal characteristics are compared with the SAFE predictions, and excellent agreement is observed, as shown in Fig. 7. Finally, the energy conservation properties of the fourth-order Runge–Kutta scheme are illustrated in Fig. 8.

In the enlarged model, each of the inner fluid, pipe wall, and outer fluid regions is discretised along the axial direction using 1000 GLL elements with a polynomial degree of 8. Nodal points are collocated at the fluid–solid interfaces to facilitate the exchange of fluid–solid boundary conditions (see Fig. 2 for an illustration). In the radial direction, the inner fluid is meshed by a single GLJ element of polynomial order 8, the pipe wall by a single GLL element of polynomial order 2, and the outer fluid by 375 GLL elements of polynomial order 8. The total number of DoFs in the coupled system exceeds 24 million. The time step is $\Delta t = 6.861577 \times 10^{-7}$ s, the total number of marching steps is $N = 17489$, and the total simulation duration is $T = 12$ ms.

The enlarged SEM model is implemented in MATLAB and executed on a workstation equipped with eighteen 3.00 GHz Intel® Xeon® CPUs and 256 GB of memory. Parallel computation is employed, with element-level numerical integration evenly distributed among 17 CPU cores, ensuring that the workload assigned to any two cores differs by no more than one element. The average computational time is 2.6 s per time-marching step. The total computational time, including model loading and initialisation, is approximately 13 hours.

The sound field excited by a point source for the enlarged model is shown in Fig. 4 at different propagation times. The source is a fluid-potential point source located at the pipe centre ($r = 0, z = 0$), generating a 10-cycle Hanning-windowed pulse centred at 10 kHz. In the fluid, the field is expressed in terms of the scaled fluid velocity potential, while in the pipe wall it is expressed as the radial displacement. Although these represent different physical quantities, their amplitudes are normalised (see Section 2.4) so they can be directly compared and shown in the same figure.

Guided waves produce strong motion in both the inner fluid and the pipe wall. However, because the pipe radius is small relative to the outer fluid's radius, this motion is not easily visible in Fig. 4. The detailed guided-wave patterns are therefore presented separately in Fig. 5 using a truncated view.

Figure 4 mainly shows the wave motion in the outer fluid. Two distinct waves are present. The dominant feature is the bulk fluid wave, travelling at the constant speed c_0 . The wavefront of this wave is oriented in the radial direction; that is, it has a phase dependence of $e^{i(\omega t - kr)}$, where r is the radial coordinate in a spherical coordinate system. In three dimensions, this mode has a spherical wavefront that radiates outward from the source. In classical SAFE theory, guided wave modes, including propagating, leaky, trapped, and radiation modes, are all assumed to take the form $e^{i(\omega t - kz)}$, where z is the axial coordinate along the waveguide. This suggests that the conventional SAFE formulation may not be the most suitable approach for investigating the characteristics of radiation wave modes.

It should be noted that this radiation mode is also coupled with the pipe wall and the inner fluid. However, the presence of the pipe wall and the inner fluid has little influence on either its wave speed or its waveform. To the best of our knowledge, the existence of such a radiation wave mode in the external domain of an embedded waveguide has not been previously reported.

In the present study, the outer domain is a fluid medium. Consequently, the radiation mode appears as a single fluid bulk wave, as the wavefield is scalar. If the outer domain were an elastic medium, the corresponding radiation modes could consist of bulk longitudinal and shear waves.

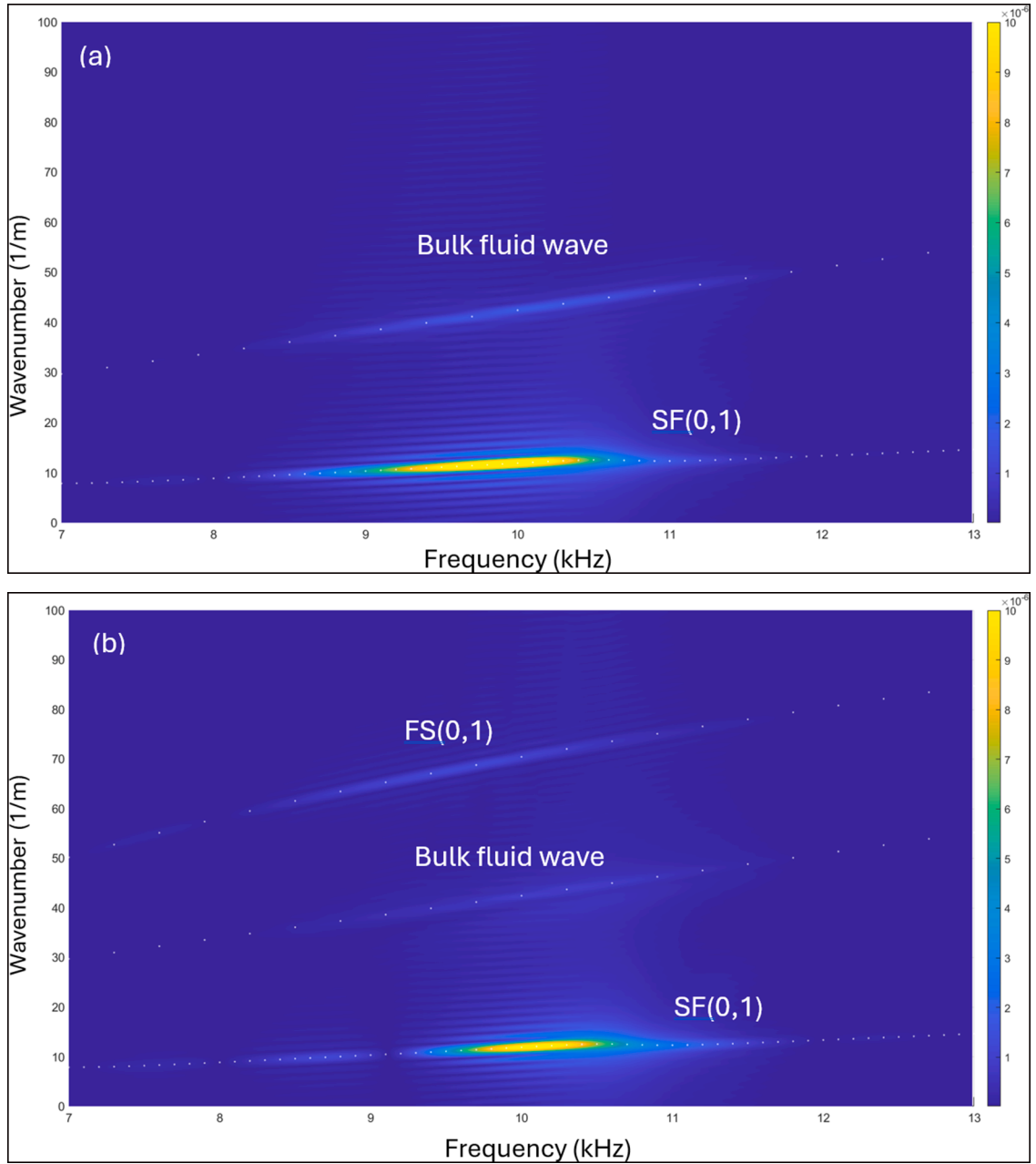


Fig. 7. Modal properties in the frequency-wavenumber domain. Colormap: 2D Fourier transform of time-domain signals; white dotted line: SAFE prediction [11]. (a) Fluid potential sampled in the inner fluid at $r = 0$, $z = [1, 1.03, 1.06, 1.09 \dots, 4.81]$ m. (b) Radial solid displacement sampled on the inner pipe wall at $r = 0.101361$ m, $z = [1, 1.03, 1.06, 1.09 \dots, 4.81]$ m.

Figure 4 also shows the SF(0,1) leaky mode. At 10 kHz, this mode is primarily confined to the pipe wall but continuously radiates energy into the surrounding fluid, resulting in modal attenuation. In the outer fluid, the radiated field forms a conical wavefront emanating from the outer surface of the pipe and propagating in both axial directions when viewed in three dimensions. Interference between the bulk radiation wave and the field radiated by the SF(0,1) mode is visible in the outer fluid region.

The predicted sound field distributions obtained from the SEM-PML model and the enlarged model are shown in Fig. 5. The PML thickness in the radial direction is 0.2 m, and its axial length is also 0.2 m. In Fig. 5(b), the pipeline length is set from -50 m to 50 m, while in Fig. 5(c) it extends from -5 m to 50 m. The GLL polynomial degree is set to 2, with 16 elements assigned in the radial direction within the outer PML. For the left and right axial PMLs, the GLL polynomial degree is also 2, and 12 elements are placed in both the fluid and solid regions. The reduced polynomial degree in the PML regions is chosen to increase the number of nodes per wavelength

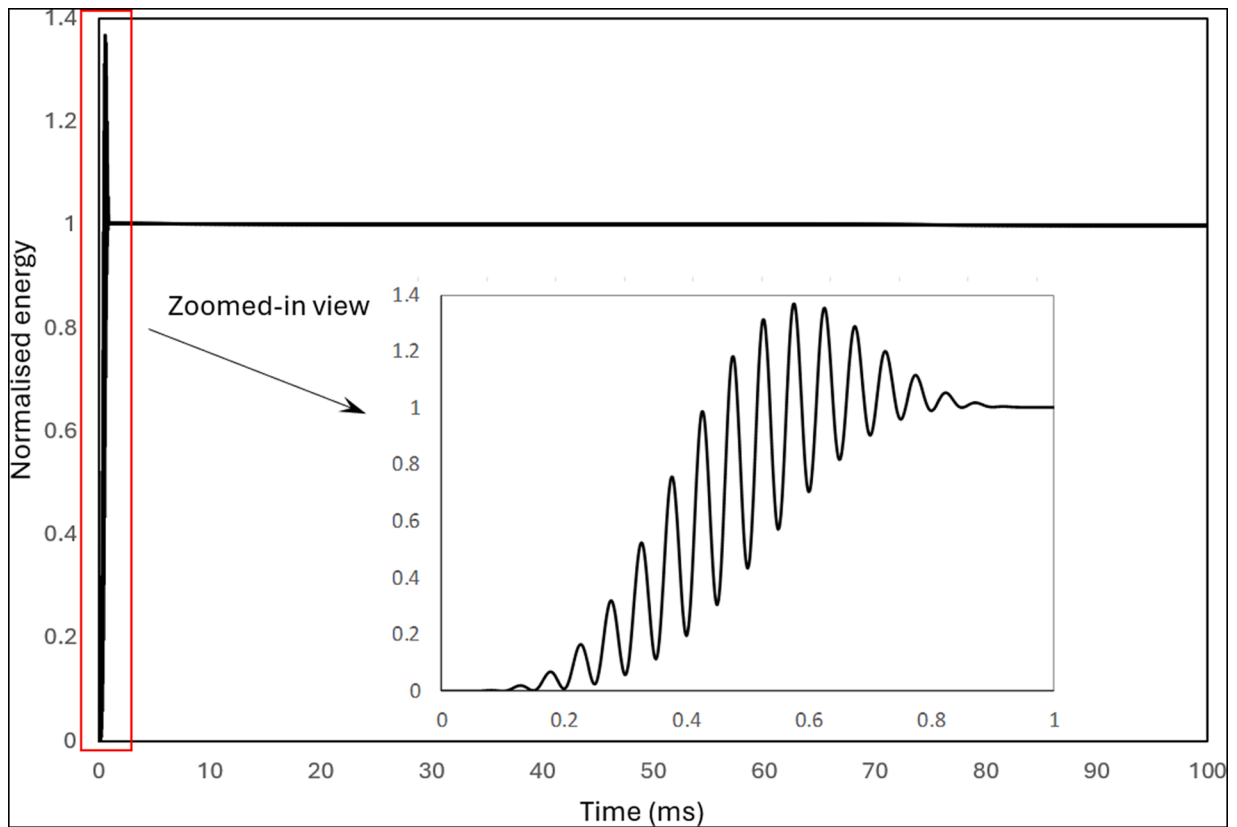


Fig. 8. Energy evolution of the fourth-order Runge–Kutta scheme for acoustic wave propagation in a water-filled pipe with rigid wall boundary conditions.

while avoiding excessively small nodal spacing. This results in six nodes per wavelength in the PML region. It should be noted that the spectral convergence property of the SEM is observed only in the unstretched regions, not in the PML regions [11].

For the SEM–PML model with a total pipe length of 100 m and a PML thickness of 0.2 m, the total number of DoFs is approximately 0.97 million. The time step is $\Delta t = 6.861577 \times 10^{-7}$ s, which is the same as that used in the enlarged SEM model. This is because the pipe wall is very thin, and a three-noded element is employed in the radial direction of the pipe wall, resulting in a nodal spacing of 4.0895 mm. According to the CFL condition given in Eq. (66), the allowable time step is determined by the minimum value of $\Delta h / c_p$ in the computational domain. For the present model, this minimum value occurs in the radial direction of the pipe wall and is unchanged regardless of whether the PML is included. The computational cost is 0.91 s per time-marching step. For the same total of 17,489 time-marching steps, the overall computation time is approximately 4.5 hours.

For the length-truncated SEM–PML model with a total pipe length of 55 m and a PML thickness of 0.2 m, the total number of DoFs is approximately 0.54 million. The time step and the total number of time-marching steps remain unchanged. The computational cost is 0.64 s per time-marching step, resulting in a total computation time of approximately 3.1 hours.

The radial and axial coordinates in Fig. 5 are not drawn to scale. The comparison focuses on the inner fluid and the pipe wall, as the outer PML region is distorted by damping. The pipe wall displacement shown corresponds to the radial component. It can be seen that both PML models match the enlarged model very well in the inner fluid and pipe wall regions. In the outer PML, however, the bulk fluid wave and the SF(0,1) mode exhibit different wave characteristics, making it challenging for a single set of PML parameters to effectively absorb both—a phenomenon also noted in frequency-domain studies [11]. In the inner fluid, the dominant mode is SF(0,1), while in the pipe wall both SF(0,1) and FS(0,1) are clearly present. Interestingly, the bulk fluid wave couples into both the pipe wall and the inner fluid. Although its motion in the pipe wall is weak, it produces a distinct response in the inner fluid. In Fig. 5(c), the left-propagating wave modes are absorbed by the PML, leaving only right-propagating waves visible. This demonstrates that the proposed SEM–PML model can effectively damp waves travelling in both the radial and axial directions.

Fig. 6 shows the time-domain fluid velocity potential in the inner fluid at $z = 5$ m and $r = 0$. The waveforms predicted by the enlarged model and the two SEM–PML models are in close agreement, with maximum differences of less than 0.2% for SF(0,1) and less than 3.7% for the bulk fluid wave. SF(0,1) is the dominant mode in the inner fluid, but it is highly dispersive around 10 kHz. The initial source is a 10-cycle Hanning-windowed pulse, and as a result SF(0,1) extends over more than 20 cycles. In contrast, the bulk fluid wave is non-dispersive and therefore preserves its waveform. Together with Fig. 5, these results demonstrate that the SEM–PML models accurately capture the time-domain sound field in both the inner fluid and the pipe wall.

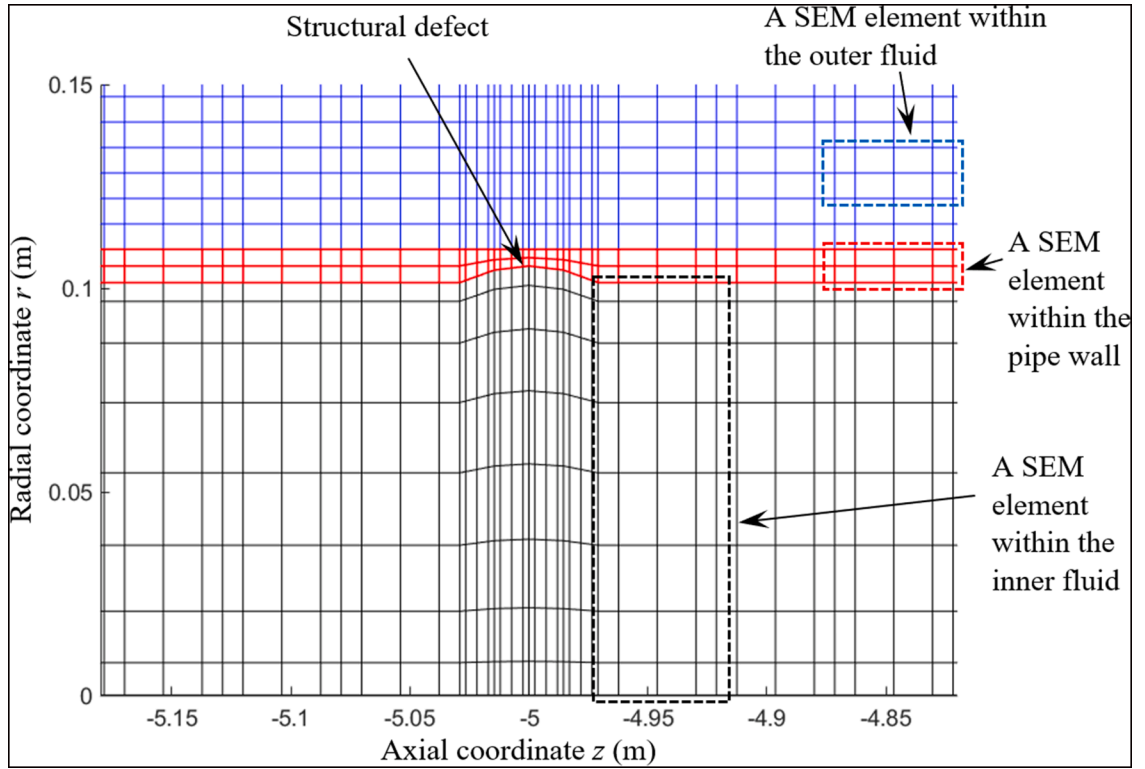


Fig. 9. Partial view of the mesh for a fluid-filled pipe immersed in a fluid domain with a structural defect on the pipe wall. A representative SEM element is highlighted in each region: the inner fluid, pipe wall, and outer fluid. For the inner fluid element, 8th-order GLJ points are used radially and 4th-order GLL points axially. The pipe wall element uses 2nd-order GLL points radially and 4th-order axially. For the outer fluid element, 2nd-order GLL points are used radially and 4th-order axially.

Having examined the time-domain fluid potential at a single point in Fig. 6, we next sample the time-domain fluid potential and solid displacement signals at 128 points, from $z = 1$ m to $z = 4.81$ m, with an equal spacing of 0.03 m in the inner fluid and on the pipe wall. These signals are transformed into the frequency-wavenumber domain using a 2D Fourier transform, as shown in Fig. 7. To improve resolution, the data were zero-padded to eight times their original length prior to transformation. Fig. 7(a) shows the transform of the fluid potential samples, while Fig. 7(b) corresponds to the radial displacement samples. The white dotted lines indicate SAFE predictions from [11]. In Fig. 7(a), the mode SF(0,1) is dominant on the fluid centreline, with the bulk fluid wave also clearly visible. In Fig. 7(b), SF(0,1) remains dominant on the pipe wall. The bulk fluid wave, being primarily a fluid mode, appears only weakly because of fluid-solid coupling. The trapped mode FS(0,1) is also clearly visible on the pipe wall, with most of its energy confined to the wall at 10 kHz.

Fig. 8 illustrates the energy evolution of the classical fourth-order Runge–Kutta (RK4) scheme. The classical RK4 method has a global truncation error of $O(\Delta t^4)$; however, exact preservation of the total discrete energy is not generally expected. In the full SEM–PML model, the total energy of the computational domain is expected to decrease because the PML introduces artificial damping and absorbs outgoing waves.

To isolate the influence of the time integration scheme, the energy evolution was investigated for acoustic wave propagation in a fluid-filled pipe with rigid wall boundary conditions. Fluid–solid coupling and the PML were excluded from this verification study. The material properties, waveguide dimensions, and mesh configuration are identical to those presented at the beginning of Sections 4 and 4.2.

The total discrete energy is defined as the sum of the kinetic and potential energies. For the spectral element discretization considered herein, the discrete energy is given by

$$H_h(t) = \frac{1}{2} \dot{\phi}^T \mathbf{M} \dot{\phi} + \frac{1}{2} \sum \phi_e^T \mathbf{K}_e \phi_e \quad (67)$$

where ϕ denotes the nodal values of the fluid velocity potential, and the overdot represents differentiation with respect to time. Here, $\mathbf{M}$ and $\mathbf{K}$ are the mass and stiffness matrices, respectively, while the subscript (e) denotes the local element index.

For the numerical model, the CFL coefficient was set to 1. Using Eq. (66), the time step was computed as $\Delta t = 2.0737423 \times 10^{-6}$ s. A total of $N = 48222$ time steps were performed, corresponding to a simulation duration of $T = 100$ ms.

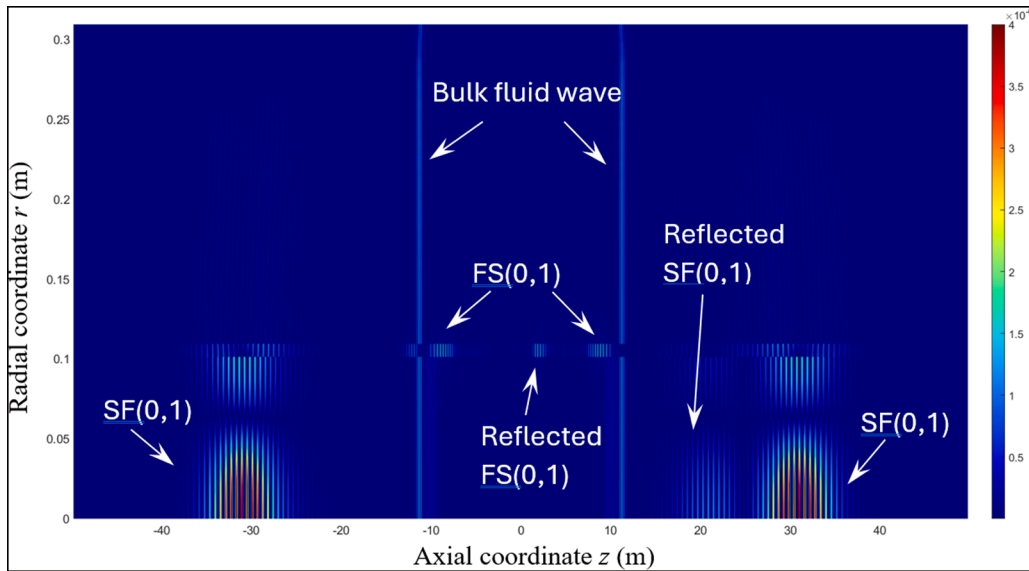


Fig. 10. Distribution of the normalised fluid potential and solid displacement in the coupled structure containing the embedded defect at $t = 8.165$ ms.

The discrete energy defined in Eq. (67) was evaluated at every time step and normalized by its value at $t = 100$ ms for presentation in Fig. 8. Energy oscillations are observed during the first millisecond because the transient 10-cycle Hanning-windowed pulse introduces energy into the system during this period. Once the source vanishes (approximately $t = 1$ ms), no additional energy enters the domain, and the total energy remains nearly constant over the remainder of the simulation.

The relative energy drift between $t = 1$ ms and $t = 100$ ms is approximately 0.31%, indicating that the classical RK4 scheme maintains good long-time energy behaviour for the wave propagation problems considered in this study.

4.3. Scattering from a structural defect on the pipe wall

The time-domain SEM-PML model has been validated using different approaches in the previous section for a uniform water-filled pipeline immersed in water. Here, we introduce a structural defect on the pipe wall and theoretically investigate the scattering of different wave modes from this defect. The dimensions of the SEM-PML model are $z \in [-50, 50]$ m, and $r \in [0, 0.30954]$ m. The defect is modelled as a partially circular notch in the transverse plane. Its circular arc is centred at $r = 0$, $z = -5$ m, with a radius of 0.1054505, extending halfway into the pipe wall at the centre. The mesh around the defect is shown in Fig. 9. The GLL polynomial degree is set to 4 in the axial direction across all regions, with nodal points collocated on both the inner and outer pipe wall surfaces, including the defect region. Owing to the small size of the defect, the element spacing is locally refined in this region, as shown in Fig. 9. In general, the polynomial degree may differ between the radial and axial directions, depending on the structural dimensions under investigation.

Fig. 10 shows the predicted fluid potential and solid displacement distribution for the coupled structure at $t = 8.165$ ms. The defect is included in the model, as shown in Fig. 9. Fig. 10 can be compared with Fig. 5(b), where the structure is identical except that no defect is present. Since the defect is located at $r = 0$, $z = -5$ m, the right-propagating incident wave modes are unaffected, as they do not encounter the defect. In contrast, the left-propagating wave modes pass through the defect, and their waveforms are perturbed from $z = -5$ m to the left boundary. However, because the defect is very small, this perturbation remains minor. In addition to the incident wave modes, reflected echoes are observed from two wave modes: SF(0,1) and FS(0,1). These echoes propagate only to the right of the defect, since the sound source is placed at $r = 0$, $z = 0$, i.e., to the right of the defect. Despite the small size of the defect, these echoes are clearly visible in Fig. 10. The SF(0,1) mode dominates in the inner fluid, whereas the FS(0,1) mode dominates in the pipe wall. This demonstrates the potential of using guided waves to detect defects in the coupled system, as the defect can be identified both in the inner fluid and in the pipe wall, depending on the application.

Fig. 11 shows the pulse-echo responses predicted at two positions along the centre axis within the inner fluid. For reference, the fluid velocity potential responses of the structure without a defect are also plotted in Fig. 11 for comparison. In Fig. 11(a), the receiver is located at the same position as the sound source. In this case, the incident SF(0,1) mode and the bulk fluid wave are superimposed. Although these two waves travel at different speeds, some time and distance are required for them to separate. Note that in Fig. 11(a), the figure has been zoomed in to show the reflected pulse clearly; consequently, the full waveform of the incident pulse has been truncated. The reflected SF(0,1) wave originates from the defect and can be used to estimate its location following the standard pulse-echo principle. In Fig. 11(b), the receiver is positioned 5 m to the right of the sound source. Here, the incident SF(0,1) mode and the bulk fluid wave are separated. The reflected SF(0,1) wave overlaps with the incident bulk fluid wave. The SF(0,1) mode spans more than 10 cycles (the incident sound is a 10-cycle Hanning-windowed pulse), indicating the dispersive nature of this wave mode in this

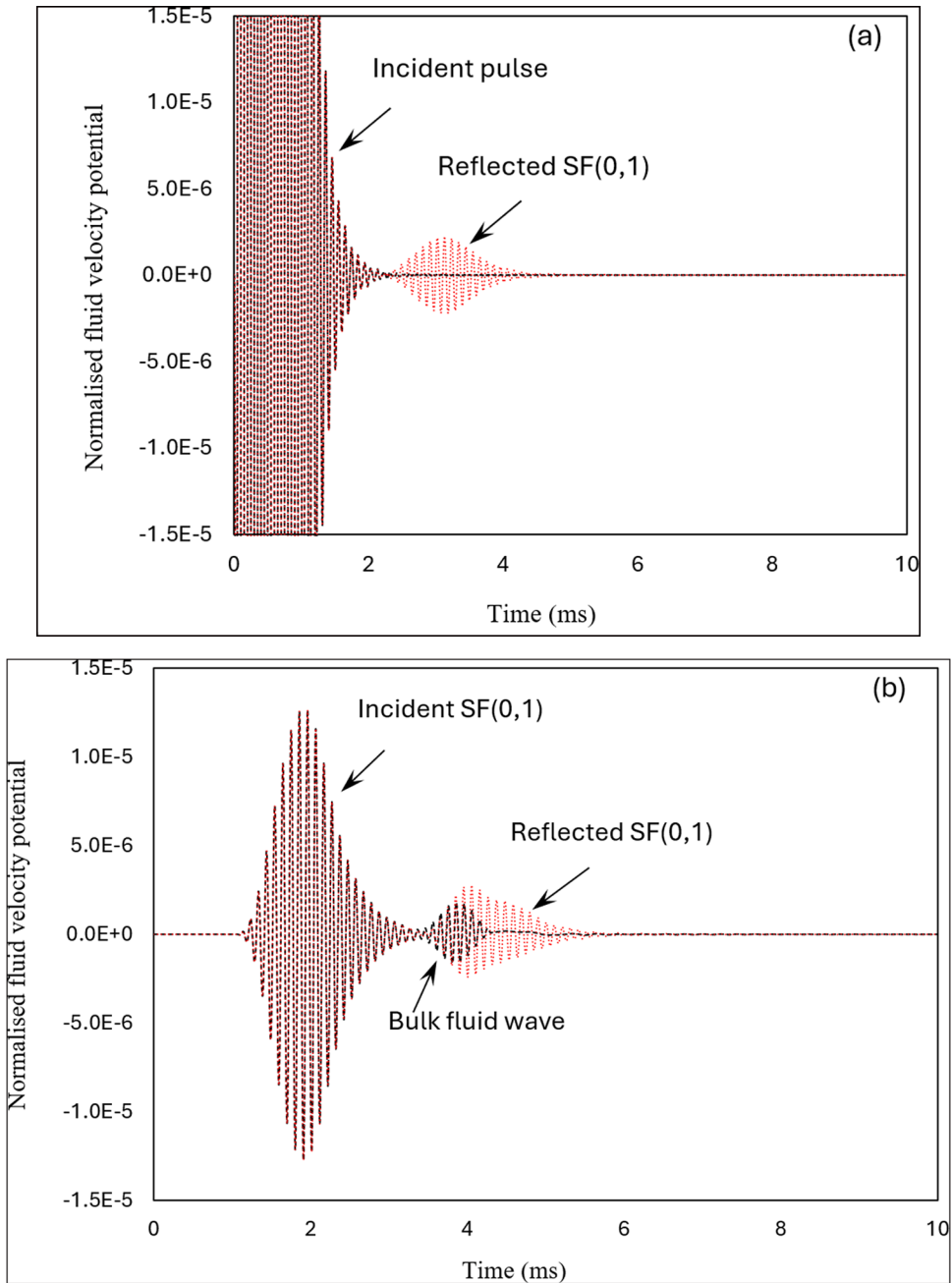


Fig. 11. Normalised fluid velocity potential pulse-echo responses at (a) $z = 0$ and $r = 0$; and (b) $z = 5$ m and $r = 0$. - - -: the predicted result with the defect included in the SEM-PML model; —: the reference result without a defect.

frequency range.

5. Conclusions

In this study, two new sets of multi-axis PML formulations were developed for acoustic and elastic wave propagation in fluid and solid regions. These formulations are compatible across fluid–solid interfaces and integrate seamlessly with the unstretched regions. Importantly, the proposed multi-axis PML formulations produce a diagonal global mass matrix, making them directly implementable within standard spectral element frameworks. Exponential stretching functions were employed, which typically outperform polynomial stretching functions in wave propagation problems.

The numerical model was applied to investigate wave excitation, propagation, and scattering in an infinitely long fluid-filled

pipeline immersed in an external fluid. The axisymmetric nature of the pipeline was exploited, and both high order Gauss-Lobatto-Legendre (GLL) and Gauss-Lobatto-Jacobi (GLJ) nodal points and quadrature rules were employed in the inner fluid to avoid zero entries in the mass matrix along the centreline. The degrees of freedom of the high order functions were varied in the radial and axial directions, as well as across the fluid, solid, and PML regions, to optimise the efficiency and accuracy of the model.

The model demonstrated that a fluid-dominated wave mode and a structure-dominated wave mode are both excited by a point sound source at the inner-fluid centreline. In addition, a radiation wave mode was identified, appearing as a bulk fluid wave with a non-dispersive, constant wave speed. This radiation mode has not previously been included in published guided-wave dispersion curves, nor has its presence been documented in time-domain wave excitation models. Its characteristics suggest potential use for pipeline localisation with receivers placed in the outer fluid.

The model was used to study the scattering of guided waves from a structural defect in the pipe wall. Echoes from both the fluid and structural wave modes were observed: the fluid-wave echo was apparent within the inner fluid, whereas the structural-wave echo was primarily confined to the pipe wall. The structural wave mode, being a trapped mode, concentrates energy within the structure in this frequency range. Note that guided-wave mode characteristics are frequency-dependent, as evident from dispersion curves. In contrast, the radiation wave mode was not perturbed by the defect and is therefore unlikely to be suitable for defect detection in this configuration.

Declaration of competing interest

The author declares that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A

$$\mathbf{D}_3^R = \begin{bmatrix} h_2 h_3 & 0 & 0 \\ 0 & h_1 h_3 & 0 \\ 0 & 0 & h_1 h_2 \end{bmatrix} \quad (\text{A1})$$

$$\mathbf{D}_3^E = \begin{bmatrix} h_2 d_3 + h_3 d_2 & 0 & 0 \\ 0 & h_1 d_3 + h_3 d_1 & 0 \\ 0 & 0 & h_1 d_2 + h_2 d_1 \end{bmatrix} \quad (\text{A2})$$

$$\mathbf{D}_3^I = \begin{bmatrix} d_2 d_3 & 0 & 0 \\ 0 & d_1 d_3 & 0 \\ 0 & 0 & d_1 d_2 \end{bmatrix} \quad (\text{A3})$$

$$\mathbf{D}_3^T = h_1 h_2 h_3 + \frac{h_2 h_3 d_1 + h_1 h_3 d_2 + h_1 h_2 d_3}{i\omega} - \frac{h_3 d_1 d_2 + h_2 d_1 d_3 + h_1 d_2 d_3}{\omega^2} - \frac{d_1 d_2 d_3}{\omega^2 i\omega} \quad (\text{A4})$$

$$\mathbf{D}_2^R = \begin{bmatrix} h_2 & 0 \\ 0 & h_1 \end{bmatrix} \quad (\text{A5})$$

$$\mathbf{D}_2^E = \begin{bmatrix} d_2 & 0 \\ 0 & d_1 \end{bmatrix} \quad (\text{A6})$$

$$\mathbf{M}_\phi = \int_{\Omega_{fp}} \left\{ \frac{1}{c^2} [h_1 h_2] [\mathbf{N}_\phi]^T [\mathbf{N}_\phi] \right\} d\Omega_{fp} \quad (\text{A7})$$

$$\mathbf{M}_\psi = \int_{\Omega_{fp}} [\mathbf{N}_\psi]^T \mathbf{H}^2 [\mathbf{N}_\psi] d\Omega_{fp} \quad (\text{A8})$$

$$\mathbf{C}_\phi = \int_{\Omega_{fp}} \left\{ \frac{1}{c^2} [(h_2 d_1 + h_1 d_2)] [\mathbf{N}_\phi]^T [\mathbf{N}_\phi] \right\} d\Omega_{fp} \quad (\text{A9})$$

$$\mathbf{C}_{\psi\phi} = - \int_{\Omega_{fp}} [\mathbf{N}_\psi]^T [\mathbf{L}_\phi] d\Omega_{fp} \quad (\text{A10})$$

$$\mathbf{C}_{\phi\psi} = \int_{\Omega_{fp}} [\mathbf{L}_\phi]^T \mathbf{D}_2^R [\mathbf{N}_\psi] d\Omega_{fp} \quad (\text{A11})$$

$$\mathbf{C}_\psi = \int_{\Omega_{fp}} [\mathbf{N}_\psi]^T \mathbf{D}^2 [\mathbf{N}_\psi] d\Omega_{fp} \quad (\text{A12})$$

$$\mathbf{K}_\phi = \int_{\Omega_{fp}} \left\{ \frac{1}{c^2} [d_1 d_2] [\mathbf{N}_\phi]^T [\mathbf{N}_\phi] \right\} d\Omega_{fp} \quad (\text{A13})$$

$$\mathbf{K}_{\phi\psi} = \int_{\Omega_{fp}} [\mathbf{L}_\phi]^T \mathbf{D}_2^\epsilon [\mathbf{N}_\psi] d\Omega_{fp} \quad (\text{A14})$$

$$\mathbf{H}^2 = \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix} \quad (\text{A15})$$

$$\mathbf{D}^2 = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \quad (\text{A16})$$

$$\mathbf{L}_\phi = \begin{bmatrix} \frac{\partial \mathbf{N}_\phi}{\partial x_1} \\ \frac{\partial \mathbf{N}_\phi}{\partial x_2} \end{bmatrix} \quad (\text{A17})$$

Appendix B

$$\mathbf{J} = \begin{bmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{bmatrix} \quad (\text{B1})$$

$$\mathbf{H} = \begin{bmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{bmatrix} \quad (\text{B2})$$

$$\mathbf{D} = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \quad (\text{B3})$$

$$\mathbf{L}_{D2}^R = \begin{bmatrix} \frac{\partial}{\partial r} h_2 & 0 \\ \left(\frac{1}{r}\right) h_1 h_2 & 0 \\ 0 & \frac{\partial}{\partial z} h_1 \\ \frac{\partial}{\partial z} h_1 & \frac{\partial}{\partial r} h_2 \end{bmatrix} \quad (\text{B4})$$

$$[\mathbf{L}_{D2}^\epsilon] = \begin{bmatrix} \frac{\partial}{\partial r} d_2 & 0 \\ \left(\frac{1}{r}\right) (h_1 d_2 + h_2 d_1) & 0 \\ 0 & \frac{\partial}{\partial z} d_1 \\ \frac{\partial}{\partial z} d_1 & \frac{\partial}{\partial r} d_2 \end{bmatrix} \quad (\text{B5})$$

$$[\mathbf{L}_{D2}^{\Pi}] = \begin{bmatrix} 0 & 0 \\ \left(\frac{1}{r}\right)d_1d_2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (\text{B6})$$

$$[\mathbf{N}_H^{\times}] = \begin{bmatrix} h_1h_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & h_2h_2 & 0 \\ 0 & 0 & 0 & h_1h_2 \end{bmatrix} \quad (\text{B7})$$

$$[\mathbf{N}_D^+] = \begin{bmatrix} 2h_1d_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2h_2d_2 & 0 \\ 0 & 0 & 0 & h_1d_2 + h_2d_1 \end{bmatrix} \quad (\text{B8})$$

$$[\mathbf{N}_D^{\times}] = \begin{bmatrix} d_1d_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & d_2d_2 & 0 \\ 0 & 0 & 0 & d_1d_2 \end{bmatrix} \quad (\text{B9})$$

$$[\mathbf{L}_H^-] = \begin{bmatrix} \frac{\partial}{\partial r}h_1 & 0 \\ \left(\frac{1}{r}\right) & 0 \\ 0 & \frac{\partial}{\partial z}h_2 \\ \frac{\partial}{\partial z}h_1 & \frac{\partial}{\partial r}h_2 \end{bmatrix} \quad (\text{B10})$$

$$[\mathbf{L}_D^-] = \begin{bmatrix} \frac{\partial}{\partial r}d_1 & 0 \\ 0 & 0 \\ 0 & \frac{\partial}{\partial z}d_2 \\ \frac{\partial}{\partial z}d_1 & \frac{\partial}{\partial r}d_2 \end{bmatrix} \quad (\text{B11})$$

$$\mathbf{M}_u = \int_{\Omega_{sp}} \rho h_1 h_2 [\mathbf{N}_u]^T [\mathbf{N}_u] d\Omega_{sp} \quad (\text{B12})$$

$$\mathbf{M}_e = \int_{\Omega_{sp}} [\mathbf{N}_e]^T [\mathbf{N}_H^{\times}] [\mathbf{N}_e] d\Omega_{sp} \quad (\text{B13})$$

$$\mathbf{C}_u = \int_{\Omega_{sp}} \rho (h_2d_1 + h_1d_2) [\mathbf{N}_u]^T [\mathbf{N}_u] d\Omega_{sp} \quad (\text{B14})$$

$$\mathbf{C}_{ue} = \int_{\Omega_{sp}} [\mathbf{N}_u]^T [\mathbf{L}_{D2}^R]^T [\mathbf{C}^E] [\mathbf{N}_e] d\Omega_{sp} \quad (\text{B15})$$

$$\mathbf{C}_e = \int_{\Omega_{sp}} [\mathbf{N}_e]^T [\mathbf{N}_D^+] [\mathbf{N}_e] d\Omega_{sp} \quad (\text{B16})$$

$$\mathbf{C}_{eu} = - \int_{\Omega_{sp}} [\mathbf{N}_e]^T [\mathbf{L}_H^-] [\mathbf{N}_u] d\Omega_{sp} \quad (\text{B17})$$

$$\mathbf{K}_u = \int_{\Omega_{sp}} \rho d_1 d_2 [\mathbf{N}_u]^T [\mathbf{N}_u] d\Omega_{sp} \quad (\text{B18})$$

$$\mathbf{K}_{ue} = \int_{\Omega_{sp}} [\mathbf{N}_u]^T [\mathbf{L}_{D2}^e]^T [\mathbf{c}^E] [\mathbf{N}_e] d\Omega_{sp} \quad (\text{B19})$$

$$\mathbf{K}_e = \int_{\Omega_{sp}} [\mathbf{N}_e]^T [\mathbf{N}_D^e] [\mathbf{N}_e] d\Omega_{sp} \quad (\text{B20})$$

$$\mathbf{K}_{eu} = - \int_{\Omega_{sp}} [\mathbf{N}_e]^T [\mathbf{L}_D^-] [\mathbf{N}_u] d\Omega_{sp} \quad (\text{B21})$$

$$\mathbf{G}_{ue} = \int_{\Omega_{sp}} [\mathbf{N}_u]^T [\mathbf{L}_{D2}^u]^T [\mathbf{c}^E] [\mathbf{N}_e] d\Omega_{sp} \quad (\text{B22})$$

Data availability

Data will be made available on request.

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