

Analytic modelling of CO surface-density profiles in the M31 molecular clouds

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Accepted 2026 June 15. Received 2026 June 10; in original form 2026 March 13

ABSTRACT

We analyse CO observations of molecular clouds in the Andromeda galaxy (M31) with a new method to derive surface-density profiles and compare directly with projected solutions of the isothermal Lane–Emden equation. The observed curvature of the surface-density profiles is consistent with the theoretical solutions. The applicability of the Lane–Emden equation indicates that the time-scales of the turbulent processes responsible for an average force balance within the clouds are shorter than the time-scales for the evolution of the clouds, for example towards collapse or disruption. The M31 profiles resemble those previously reported in Galactic molecular clouds suggesting similar dynamics across different environments.

Key words: ISM: clouds – ISM: kinematics and dynamics – ISM: structure.

1 INTRODUCTION

Three recent studies introduce two new methods to analyse spectral line observations of molecular clouds to determine their internal structure. Applied to clouds in the Milky Way Galactic ring (GR) and in the Andromeda galaxy (M31), the new methods reveal density structure consistent with the solutions of the isothermal Lane–Emden equation (E. Keto 2024; M. R. Krumholz, C. J. Lada & J. Forbrich 2025; C. J. Lada et al. 2025). In the GR clouds, the internal structure is inferred from a direct comparison, whereas in the M31 clouds, the inference is based on the ratios of the virial energies measured as a function of projected radius. In this study, we analyse one of the new methods, differential virial analysis (DVA) introduced by C. J. Lada et al. (2025), and apply it to the M31 clouds to determine the radial dependence of their surface densities. This allows a direct comparison to the Lane–Emden solutions on par with the previous analysis of the GR clouds and confirms the previous results for the M31 clouds.

The DVA method is simple to apply, consisting of repeated application of conventional virial analysis within progressively smaller projected areas defined by a sequence of increasing isophote intensities. Averages within the nested areas result in data pairs of average surface density within an effective circular radius. The intuitive understanding of DVA is verified by application to theoretical polytropes and model clouds from numerical simulations (M. R. Krumholz et al. 2025).

A statistical interpretation of the comparison between the observational results and theoretical models is complicated by correlated statistical uncertainties. Fluctuations in the observed

integrated intensity affect both the measured surface density and the position of each isophote that defines the corresponding area and effective radius. Because the isophotes are nested, these same fluctuations propagate into multiple data pairs, producing correlated uncertainties across the radial profile. In contrast, the method of E. Keto (2024), which relies on azimuthal rather than areal averaging, returns the surface density directly as a function of projected radius without these covariances. Nonetheless, the covariances inherent in the DVA method are tractable, and the method is statistically well defined, as demonstrated in this study.

The benefit of the DVA method is its wider applicability to typical radio-frequency spectral line data. The mathematically simpler method of E. Keto (2024) requires sufficient angular resolution to measure the peak integrated intensity and enough dynamic range to identify the half-width at half-maximum (HWHM). These conditions are not always satisfied, particularly when single-dish resolution is limited or interferometric dynamic range is restricted. The DVA method avoids these limitations. One advantage of both methods is that the differential description of the internal structure of clouds is not sensitive to the absolute calibration of the CO-to-mass ratio (M. R. Krumholz et al. 2025).

Our analysis of the DVA method consists of a mathematical description to compare with the method of E. Keto (2024; Section 2) and a procedure to estimate the propagated errors that includes the covariance of surface density and radius within each data pair (Appendix A). The procedure is simple to apply, requiring the lengths of the isophotes as the only additional quantity that is not already available in traditional observational analysis. We also include a more informal analysis of the validity of the spherical approximation and the effect of the areal averaging (Appendices

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B1 and B2). To estimate the goodness-of-fit of the Lane–Emden model, we derive the generalized least squares χ^2_{GLS} statistic that takes into account the covariances of the nested data pairs.

Section 5 suggests an interpretation of the observational results: Self-gravitational equilibrium may be established within a local region in a turbulent field if the time-scale for the internal dynamics to evolve to a balance of forces is shorter than the time-scale for disruption or collapse of the cloud by self-gravity or the larger scale surrounding turbulence.

Comparison of the analyses of the extragalactic clouds in M31 with the previously analysed GR clouds (Section 5.1) suggests similar dynamical states in both populations.

2 DESCRIPTION OF THE METHOD

E. Keto (2024) defines molecular clouds as regions of higher surface density around a local emission peak. The azimuthally averaged radial profile of the surface density, $\Sigma(r)_R$ is,

$$\Sigma(r)_R = \frac{1}{2\pi r \Delta R} \int_0^{2\pi} \int_{r-\Delta R/2}^{r+\Delta R/2} N(r' \cos \theta, r' \sin \theta) r' dr' d\theta, \quad (1)$$

where N is the observed surface density defined as a function of the sky coordinates (x, y) , and r is the projected radius from the peak surface density that is defined as the centre of the cloud. The subscript R on the observed surface density Σ indicates that this definition is based on the projected radius.

Assuming that the $^{12}\text{CO}(2-1)$ integrated intensity is proportional to the mass surface density (Appendix B3), the corresponding theoretical surface density, S , for comparison with the solution of the Lane–Emden equation is,

$$S_R(x) = 2 \int_0^{\sqrt{R_0^2 - x^2}} n(\sqrt{x^2 + z^2}) dz, \quad (2)$$

where R_0 is the truncation radius, n is the number density, $y = 0$, and z is coordinate along the line of sight. The observations of the GR clouds indicate that in practice, R_0 is large compared to the characteristic size of the clouds as measured by their half-width at half-maxima (HWHM). The observational and theoretical profiles of surface density may be compared in non-dimensional units by scaling an observed profile by its peak density and its width by its HWHM.

C. J. Lada et al. (2025) define a molecular cloud as a region of higher surface density enclosed within an isophote. For each isophote or threshold N_i , the set of all pixels satisfying

$$\Omega(N_i) = \{(x, y) | N(x, y) > N_i\} \quad (3)$$

defines the region $\Omega(N_i)$. The average surface density within the isophote N_i is,

$$\Sigma_A(N_i) = \frac{1}{A(N_i)} \int_{\Omega(N_i)} N(x, y) dx dy, \quad (4)$$

where the area, $A(N_i)$, is the number of pixels in the set $\Omega(N_i)$ times the pixel area $dx dy$. Each area-based average surface density $\Sigma_A(N_i)$ is associated with a corresponding effective radius $r_i = \sqrt{A(N_i)}/\pi$. We can refer to $S_A(N_i)$ as $S_A(r_i)$. The subscript A on the surface density indicates that this definition is based on the area within an isophote.

The corresponding surface density from the Lane–Emden equation is defined as,

$$S_A(r_i) = \frac{1}{\pi r_i^2} \int_0^{r_i} S_R(r') 2\pi r' dr'. \quad (5)$$

In the DVA method, the scaling procedure to compare the area-based observational and theoretical profiles is necessarily different from the scaling procedure used in E. Keto (2024) because the cloud centre, the peak surface density, and the HWHM are not observationally defined. For each cloud, the set of observed surface densities $\Sigma_A(r_i)$ represents a segment of the complete radial profile. The segment can be matched to the theoretical profile $S_A(r)$ as a function of the position. At any trial position, we have a single vertical and a single horizontal scaling factor to convert the non-dimensional theoretical profile to observational units. This procedure fits the *curvature* of the observed and theoretical profiles. Since the profile of the surface density derived from the Lane–Emden solution can be approximated as two segments with different simple power laws, the fit is unique only if it includes the transition region between these power-law regions where the curvature is changing rather than constant. In contrast, the method of E. Keto (2024) avoids this limitation. It simply compares the normalized profiles without the requirement of fitting.

3 RESULTS

Figure C1 compares projected solutions of the isothermal Lane–Emden equation for hydrostatic equilibrium to the observed radial surface density profiles of 24 of the 26 M31 clouds from C.J. Lada et al (2025) that contain at least 10 radial points. Two clouds (K297A and K301A) lacked sufficient curvature to obtain a unique fit because their observed points lie entirely in the inner or outer power-law regions of the profiles and are not shown here. For each of the remaining 24 clouds the figure shows only every other point for clarity. The error bars on the plots are calculated according to the procedure described in Appendix A. Also shown on the plots are the reduced χ^2_{GLS} values for the model fits.

We evaluate the fits using the generalized least-squares statistic, χ^2_{GLS} , which accounts for the full covariance matrix of the surface-density measurements as described in Appendix C. For 23 out of 24 clouds, the reduced statistic χ^2_{GLS}/ν , where $\nu = N - k$ is the number of degrees of freedom, is of order unity. This indicates that the residuals are consistent in magnitude with the observational uncertainties and that no systematic deviations from the Lane–Emden model are detected. The χ^2_{GLS}/ν values therefore provide a quantitative confirmation of the close agreement between the observed profiles and the predictions of the Lane–Emden equation for the clouds as shown in the figure.

E. Keto (2024) previously found that the surface density profiles of the GR clouds match the projected solutions of the Lane–Emden equation at all observed projected distances. The same result obtained here for a different set of clouds and by a different analytic method confirms that the results are not dependent on the analytic method. The comparison indicates that the two populations share similar dynamics.

C. J. Lada et al. (2025) previously found that virial equilibrium applies throughout the M31 clouds when the projected virial volume is a function of projected radius following the DVA method. At larger radii, the balance of energies is dominated by the kinetic and gravitational energies. At smaller radii where the gravitational energy decreases along with the enclosed mass, the balance shifts to a dominance by the kinetic energy inside the virial volume and the pressure energy at its surface. The balance of forces defined by the Lane–Emden equation is consistent with this shift in dominant energies.

4 INTERPRETATION

If the M31 clouds are described by the Lane–Emden equation, then we can infer certain conditions on the dynamical state of the turbulent clouds from the physical state described by the equation. The Lane–Emden equation combines the equation for hydrostatic equilibrium with a polytropic equation of state. The hydrostatic condition limits the significance of any acceleration of the gas within the cloud with respect to the inward force from self-gravity and the outward force of the pressure gradient. The limit can be derived from the Euler equation for an inviscid fluid with self-gravity,

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla P - \rho \nabla \Phi, \quad (6)$$

where $D/Dt = \partial_t + \mathbf{v} \cdot \nabla$ is the material derivative and the terms on the right hand side are the forces due to pressure and gravity. For a characteristic length scale L and fluid speed V , the magnitude of the acceleration term is,

$$\left| \rho \frac{D\mathbf{v}}{Dt} \right| \sim \rho \left(\frac{V}{t} + \frac{V^2}{L} \right) \sim \rho \frac{V^2}{L}, \quad (7)$$

where the second equality follows if the flow varies on a characteristic time-scale $\sim L/V$. We can estimate the magnitudes of the pressure and gravitational forces as,

$$|\nabla P| \sim \frac{\Delta P}{L} \quad \text{and} \quad \rho |\nabla \Phi| \sim \rho g. \quad (8)$$

This limits the large-scale or bulk velocities in the clouds,

$$V^2 \ll \frac{\Delta P}{\rho} \quad \text{and} \quad V^2 \ll gL. \quad (9)$$

The comparison in Fig. C1 also indicates that the equation of state is effectively isothermal so that $P \approx \rho a^2$ where a is a constant effective sound speed. Therefore, $V \ll a$. This limits the bulk flow within the clouds to speeds less than the effective sound speed which is approximately the turbulent velocity dispersion. The second condition also limits the bulk flow speed to less than the characteristic dynamical speed $v_{\text{dyn}} \equiv \sqrt{gL}$, for example the free-fall speed or the virial speed.

We can also derive equivalent conditions on the time-scales. Define the sound crossing and free-fall time-scales as,

$$t_x \sim \frac{L}{a} \quad \text{and} \quad t_{\text{dyn}} \sim \sqrt{\frac{L}{g}}. \quad (10)$$

Then $V \ll a$ and $v \ll \sqrt{gL}$ require that $t_{\text{evo}} \gg \max[t_x, t_{\text{dyn}}]$. Therefore, Fig. C1 indicates that the evolutionary time-scales of the large-scale flows, t_{evo} , that lead to the gravitational collapse or disruption of the M31 clouds are long compared the time-scales to establish a balance between pressure and self-gravity.

The conditions required by the Lane–Emden equation are independently supported by the observations. The observed line widths are approximately constant within each cloud indicating that the turbulent kinetic energy is effectively isothermal. The observed mean velocities of the CO spectral lines vary with position in the clouds by less than the observed line widths. This implies that at least along the line-of-sight, the bulk flow speeds are less than the line widths. Spectral line profiles such as asymmetric split lines indicative of collapse or expansion velocities greater than the line width are not observed.

5 DISCUSSION

Consider a cloud defined as in Section 2 by the extent of a high-density region in a field of self-gravitating turbulence. The self-gravitating cloud is continuously disturbed by turbulence on all scales, interior and exterior. The mean dynamical state of the cloud is defined by a spatial average over the fluctuating turbulent substructure on scales up to its characteristic dynamical length, for example a Jeans length. The observed state of the cloud represents a mean state after the averaging that is inherent in the observation and applied in the data analysis smooths out the local fluctuations.

The observational comparison with the Lane–Emden solutions requires the following conditions apply to the M31 clouds.

- (1) The mean state is one of approximate self-gravitational equilibrium.
- (2) The turbulent processes responsible for establishing the force balance between the pressure gradient and self-gravity operate on a shorter time-scale than the turbulent processes responsible for the collapse or disruption of the cloud.

Following E. Keto (2024), we can understand this separation of time-scales from the spatial dependence of the turbulent dynamical time-scales. The turbulent processes within a cloud that are responsible for its internal structure, operate on a range of time-scales with a *maximum* defined by the crossing time of the cloud. The turbulent fluctuations that are of large enough scale to modify the entire cloud operate on a range of time-scales with a *minimum* defined by the same crossing time of the cloud. This separation of time-scales allows a cloud to evolve towards self-gravitational equilibrium within the stochastic but relatively slowly changing boundary conditions of the local environment defined by the surrounding, larger scale turbulence.

Apparent equilibrium is found to describe many observations of molecular clouds. For example, Galactic and extragalactic surveys covering galactic scales find average virial parameters in the range of 1–2 (M. Heyer et al. 2009; J. Roman-Duval et al. 2010; T. Wong et al. 2011; A. Hughes et al. 2013; D. Colombo et al. 2014; I. Evans et al. 2021; M.-A. Miville-Deschênes, N. Murray & E. J. Lee 2017). Occasional values ~ 3 are explainable within a higher pressure environment. These observational surveys reveal mean dynamical states with a high degree of correlation with states of hydrostatic or virial equilibrium over a wide range of cloud scales. This suggests that the equilibrium mean state is an endpoint of a deterministic evolutionary process operating within self-gravitating turbulence.

5.1 Comparison of galactic and extragalactic clouds

The earlier analysis of the GR clouds resulted in four main observational findings (E. Keto 2024).

- (i) The surface densities are consistent with solutions of the isothermal Lane–Emden equation.
- (ii) The clouds are in approximate virial equilibrium with an external pressure that is different around each cloud.
- (iii) The required external pressure matches the turbulent pressure of the surrounding molecular ISM. (In the M31 clouds this pressure is equal to the external pressure in the virial theorem which indicates equilibrium at all radii.)
- (iv) The mid-plane pressure of the multi-phase turbulent ISM acts as an external boundary condition for the larger scale, turbulent molecular ISM.

From the earlier virial analysis in C. J. Lada et al. (2025) and the analysis with the Lane–Emden equation presented here, these four descriptions of the GR clouds apply to the extragalactic M31 clouds as well. The two populations therefore share similar dynamics.

6 CONCLUSIONS

We have analysed the CO emission from molecular clouds in M31 using the area-based DVA method of C. J. Lada et al. (2025) in which effective radii and corresponding surface densities are derived from nested isophotal regions. When the same averaging procedure is applied to the isothermal Lane–Emden solutions, the theoretical and observed profiles can be compared directly. For 24 of 26 clouds with sufficient radial sampling, the observed surface-density profiles agree closely with the Lane–Emden predictions. The observations imply that self-gravitational equilibrium can develop within a turbulent field because the time-scale for the establishment of force balance on the spatial scales within the clouds is shorter than the time-scale for their collapse or disruption by larger scale turbulent fluctuations on spatial scales outside the clouds.

The close similarity between the M31 clouds studied here and those previously analysed in the GR shows that both populations share comparable dynamical states.

We provide a statistical interpretation that accounts for the inherent covariances in the DVA method. We define a simple procedure for the approximate propagation of errors, and we derive the generalized least squares χ^2_{GLS} statistic for the estimation of the goodness-of-fit with theoretical models. We analyse the spherical approximation and find justification if the effective support (thermal plus turbulent plus magnetic) acts isotropically on average, and the gravitational potential is dominated by a centrally condensed mass even if individual isophotes deviate from circularity. We consider how area-based averaging further suppresses local fluctuations, yielding smooth profiles that represent a mean structure on the local scale.

7 DATA AVAILABILITY

This study uses data published with C. J. Lada et al. (2025). No new data were generated in this study.

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APPENDIX A: ESTIMATED UNCERTAINTIES

Here we express the variances and covariance of the surface density and radius in terms of those measured for the mass and area. We follow the method outlined in section 3.2 of C. J. Lada et al. (2025) to derive the variances of the area and mass within each isophote (equations A2 and A3). Because the location of an isophote depends on random fluctuations in the observational noise, we also derive the corresponding covariance.

The mass is defined as the area integral of the observed surface density N over the isophotal region defined by equation (3),

$$M_i = \int_{\Omega(N_i)} N dA. \quad (\text{A1})$$

From equation (3), the index i labels the nested cumulative isophotal regions. The average gradients of area and mass around the isophote are

$$A'_i = \left. \frac{dA}{dN} \right|_{N_i} \approx \frac{A_{i+1} - A_{i-1}}{\Delta N}, \quad (\text{A2})$$

$$M'_i = \left. \frac{dM}{dN} \right|_{N_i} \approx \frac{M_{i+1} - M_{i-1}}{\Delta N}, \quad (\text{A3})$$

where $\Delta N = N_{i+1} - N_{i-1}$. These gradients represent the change in the values of the area and mass due to a shift of the isophote caused by observational noise. Because these estimates are based on the actual shape of the isophote, they empirically include the added uncertainty due to non-circular boundaries. Because the gradients represent a coherent shift whereas random fluctuations along the isophote partially cancel, the variances derived from these estimates include a factor for the correlation length,

$$\lambda_i = \sqrt{\frac{\ell_c}{L_i}}, \quad (\text{A4})$$

where the correlation length is $\ell_c = \sqrt{2}$ FWHM, and L_i is the isophote length. The square root factor follows from the random-walk nature of the fluctuations. The variances are

$$\text{Var}(A_i) = (A'_i)^2 \lambda_i^2 \sigma_b^2, \quad (\text{A5})$$

$$\text{Var}(M_i) = A_i \sigma_b^2 \Omega_b + (M'_i)^2 \lambda_i^2 \sigma_b^2, \quad (\text{A6})$$

$$\text{Cov}(M_i, A_i) = A'_i M'_i \lambda_i^2 \sigma_b^2, \quad (\text{A7})$$

where Ω_b is the beam area, σ_b is the observational noise, and the first term in $\text{Var}(M_i)$ is the contribution from observational noise within the isophote.

We map the measured uncertainties in area and mass to uncertainties in surface density, $\Sigma = M/A$, and radius, $r = \sqrt{A/\pi}$. The surface density and radius are correlated variables, and the statistical uncertainty in the surface density includes terms arising from the covariance with radius. For the vector of random variables $\mathbf{X} = (A, M)$ and the vector function $\mathbf{Y} = (\Sigma, r)$, the covariance matrix $\mathbf{C}(\mathbf{Y})$ is obtained from $\mathbf{C}(\mathbf{X})$ through

$$\mathbf{C}(\mathbf{Y}) = \mathbf{J} \mathbf{C}(\mathbf{X}) \mathbf{J}^T, \quad (\text{A8})$$

where the Jacobian is

$$\mathbf{J} = \begin{bmatrix} \frac{\partial \Sigma}{\partial A} = -\frac{\Sigma}{A}, & \frac{\partial \Sigma}{\partial M} = \frac{1}{A}, & \frac{\partial r}{\partial A} = \frac{1}{2\sqrt{\pi} r}, & \frac{\partial r}{\partial M} = 0 \end{bmatrix}. \quad (\text{A9})$$

The variances for Σ and r are then

$$\text{Var}(\Sigma) = \frac{1}{A^2} \text{Var}(M) + \frac{\Sigma^2}{A^2} \text{Var}(A) - \frac{2\Sigma}{A^2} \text{Cov}(M, A), \quad (\text{A10})$$

$$\text{Var}(r) = \frac{1}{4\pi A} \text{Var}(A). \quad (\text{A11})$$

The covariance and correlation of Σ and r are

$$\text{Cov}(\Sigma, r) = \frac{1}{2\sqrt{\pi A}} [\text{Cov}(M, A) - \Sigma \text{Var}(A)], \quad (\text{A12})$$

$$\rho = \frac{\text{Cov}(\Sigma, r)}{\sigma_\Sigma \sigma_r}. \quad (\text{A13})$$

The uncertainty of r_i (equation A11) depends on the gradient of the area (equation A2), rather than the localization error associated with the finite beam width. Because the gradient is measured from the beam-smoothed image, the observed variance of the area includes the effect of the finite resolution.

As is often the case in radio-frequency observations, particularly interferometric observations, the peak signal-to-noise ratio exceeds the spatial dynamic range. Consequently, isophotes separated in value by the observational noise typically lie closer together than the beam width ($2.3 \times 1.9 \text{ arcsec}^2$ or $8.5 \times 7 \text{ pc}^2$ HWHM). In the M31 sample, the estimated error in the area-based radius is smaller by a factor of order 10 than the localization uncertainty set by the finite resolution. Intuitively, the error in the area-based radius reflects the average over many independent beams within an isophote, whereas the resolution-based uncertainty derives from a single measurement. The uncertainties from the covariance depend on the variance of the area or radius and are small compared to the variance in the surface density due to observational noise interior to the isophote.

APPENDIX B: SMOOTH, SPHERICAL CLOUDS

B1 Averaging of non-circular geometries to circular

The CO integrated intensities of clouds in the images of C. J. Lada et al. (2024) show fluctuations of the isodensity contours (isophotes) that are not evident in the smoothness of the observed surface density profiles. The smoothness in the profiles is a result of three averages that apply in the analytic procedure described in Section 2.

(1) The surface density at an effective radius is the average over all pixels within the corresponding isophote rather than the average of values at a single radius.

(2) The effective radius depends only on the area and not on the perimeter geometry.

(3) Successive values of the surface density and effective radius are not independent since each value incorporates the areas and surface densities of all nested higher threshold isophotes.

B2 Validity of spherical approximation for non-circular clouds

The Lane–Emden equation is spherically symmetric while some of the isophotes in the images of the M31 molecular clouds are non-circular (C. J. Lada et al. 2024). The validity of the spherical approximation requires two conditions.

(1) The effective support (thermal + turbulent + magnetic) must be approximately isotropic or radial.

(2) The gravitational potential must be dominated by the cloud's own mass distribution, and the distribution must be centrally condensed.

These two conditions generally hold when the spectral lines show a constant or radially dependent velocity dispersion, and the surface density has a single maximum, explained further below.

(1) With the internal energy dominated by turbulence and the approximately constant velocity dispersion of the CO lines within each cloud, the effective pressure depends on the local scalar ρa^2 , where a is an effective sound speed that includes non-thermal contributions. The pressure gradient then follows the density gradient. In the images, the isophotes mark thresholds in the continuous field of the surface density rather than discontinuities. Therefore, the pressure varies smoothly across the isophotes and on average is dominated by its radial component even if an isophote is not everywhere perpendicular to the radial direction.

(2) The gravitational potential is derived from the Poisson integral of the density. The integration effectively smooths over perturbations of the density structure. Higher order multipoles from non-spherical shapes, evidenced by elongated or locally fluctuating isophotes, contribute at order unity while the monopole from the enclosed mass remains dominant.

B3 Proportionality of the CO luminosity and the mass surface density of clouds

Observations of ^{12}CO have provided most of the estimates of the surface density and mass of the molecular ISM over the past 50 yr. The empirically based proportionality of the ^{12}CO luminosity or integrated intensity and the mass or mass surface density has been extensively studied and verified as generally correct over a wide range of cloud masses and column densities (A. D. Bolatto, M. Wolfire & A. K. Leroy 2013). We use the conversion factor of $\alpha_{^{12}\text{CO}(2-1)} = 10 \pm 4.5 \alpha_{^{12}\text{CO}(2-1)} = 10 \pm 4.5 \text{ M}_\odot (\text{K km s}^{-1} \text{ pc}^2)^{-1}$ determined specifically for $^{12}\text{CO}(2-1)$ in M31 by direct comparison with dust emission (C. J. Lada et al. 2024). As noted in the introduction, our results depend only on the proportionality rather than the value of the scaling factor.

APPENDIX C: GENERALIZED LEAST-SQUARES FIT TO THE SURFACE DENSITY PROFILE

The surface-density profile derived from nested isophotes produces a sequence of values

$$\Sigma = (\Sigma_1, \Sigma_2, \dots, \Sigma_N)^T, \quad (\text{C1})$$

where

$$\Sigma_i = \frac{M_i}{A_i}, \quad r_i = \sqrt{\frac{A_i}{\pi}}. \quad (\text{C2})$$

Because each isophote encloses all higher threshold regions, successive values are not statistically independent. Therefore, the appropriate goodness-of-fit statistic for comparison to a model profile $\Sigma_{\text{model}}(r)$ is the generalized least-squares (GLS) chi-square,

$$\chi_{\text{GLS}}^2 = \Delta^T \mathbf{C}_\Sigma^{-1} \Delta \quad (\text{C3})$$

The elements of the residual vector Δ are

$$\Delta_i = \Sigma_i - \Sigma_{\text{model}}(r_i). \quad (\text{C4})$$

The full covariance matrix of the surface-density vector Σ is,

$$(\mathbf{C}_\Sigma)_{ij} = \text{Cov}(\Sigma_i, \Sigma_j). \quad (\text{C5})$$

Using standard first-order error propagation, for any two contour levels i and j ,

$$\text{Cov}(\Sigma_i, \Sigma_j) = \sum_{\alpha, \beta} \frac{\partial \Sigma_i}{\partial X_{i, \alpha}} \frac{\partial \Sigma_j}{\partial X_{j, \beta}} \text{Cov}(X_{i, \alpha}, X_{j, \beta}), \quad (\text{C6})$$

where

$$X_{i,\alpha} \in \{M_i, A_i\}. \quad (C7)$$

The required derivatives are

$$\frac{\partial \Sigma_i}{\partial M_i} = \frac{1}{A_i}, \quad \frac{\partial \Sigma_i}{\partial A_i} = -\frac{M_i}{A_i^2} = -\frac{\Sigma_i}{A_i}. \quad (C8)$$

Thus,

$$\begin{aligned} \text{Cov}(\Sigma_i, \Sigma_j) &= \frac{1}{A_i A_j} \text{Cov}(M_i, M_j) \\ &\quad - \frac{\Sigma_j}{A_i A_j} \text{Cov}(M_i, A_j) \\ &\quad - \frac{\Sigma_i}{A_i A_j} \text{Cov}(A_i, M_j) \\ &\quad + \frac{\Sigma_i \Sigma_j}{A_i A_j} \text{Cov}(A_i, A_j). \end{aligned} \quad (C9)$$

C1 Covariance of nested isophote quantities

Because the isophotes are nested, the enclosed quantities satisfy

$$A_i = \sum_{k=0}^i \delta A_k, \quad M_i = \sum_{k=0}^i \delta M_k, \quad (C10)$$

where δA_k and δM_k represent the area and mass in the disjoint annular region between successive contour levels. Assuming independent noise in disjoint annuli, the covariances of the cumulative quantities are

$$\text{Cov}(A_i, A_j) = \sum_{k=0}^{\min(i,j)} \text{Var}(\delta A_k), \quad (C11)$$

$$\text{Cov}(M_i, M_j) = \sum_{k=0}^{\min(i,j)} \text{Var}(\delta M_k), \quad (C12)$$

$$\text{Cov}(M_i, A_j) = \sum_{k=0}^{\min(i,j)} \text{Cov}(\delta M_k, \delta A_k). \quad (C13)$$

These expressions explicitly encode the correlation introduced by nested isophotes. Substituting these into equation (C9) yields the full covariance matrix $\mathbf{C}_\Sigma$.

C2 Statistical interpretation

The analytic methods of error estimation in Appendices A and C are approximate methods developed specifically for the DVA technique. The most reliable method of error estimation is a Monte Carlo simulation embedded in the data analysis pipeline.

Appendix A derives the estimated uncertainties that apply to each individual (r, Σ) pair. Appendix C estimates the goodness-of-fit that applies to the comparison of the model with all the data points for each cloud including the effect of their covariances.

The weighted least squares statistic, χ_{diag}^2 , from the diagonal terms of the covariance matrix, can be predicted from Fig. C1 according to how well the model lies within the error bars of the data. Although, the χ_{GLS}^2 similarly assesses the likelihood of the data given the model, the effect of the covariances is not obvious from the figure.

For example, consider data that is consistently above or below the model curve. If the covariances are strong implying that the data points are tied together rather than independent, then the likelihood of the data given the model may be higher than would be the case if the data points were more independent, with weaker covariances. This is because the mismatch may be due to a single observational error that affects all the data points. Conversely, in the case of data that provide a good fit on average but wander above and below the model curve, strong covariances might imply a lower likelihood if the observational errors must be arranged in the particular pattern of the data.

In the ideal limit, the χ_{GLS}^2 follows a χ^2 distribution with ν degrees of freedom where $\nu = n - k$ for a model with n data points and k free parameters. However, this assumes the following conditions: (1) the residual vector must be approximately multivariate Gaussian; (2) the covariance matrix C_Σ must be known, rather than estimated from the data; (3) the fitted model must be sufficiently regular, so that subtracting k fitted parameters is the right degrees-of-freedom correction; and (4) the clouds pooled into a sample must all share the same ν and the same covariance quality.

Given these considerations, the statistics support the conclusion that the Lane–Emden model provides a satisfactory description of the observed profiles. For most clouds, 22 out of 24, $\chi_{\text{GLS}}^2 > \chi_{\text{diag}}^2$. This may reflect subtle curvature mismatches between model and data, or limitations in the approximate covariance model. Nevertheless, $\langle \sqrt{\chi_{\text{GLS}}^2/\nu} \rangle = 0.44 \pm 0.20$, and $\langle \sqrt{\chi_{\text{diag}}^2/\nu} \rangle = 0.37 \pm 0.22$, both below the ideal value of unity for an exactly calibrated uncertainty model. This indicates that the fits are satisfactory but that the adopted uncertainties are conservative, with the dominant overestimate likely originating in the uncertainties assigned to the individual (r, Σ) data pairs and then propagating into the estimated covariances.

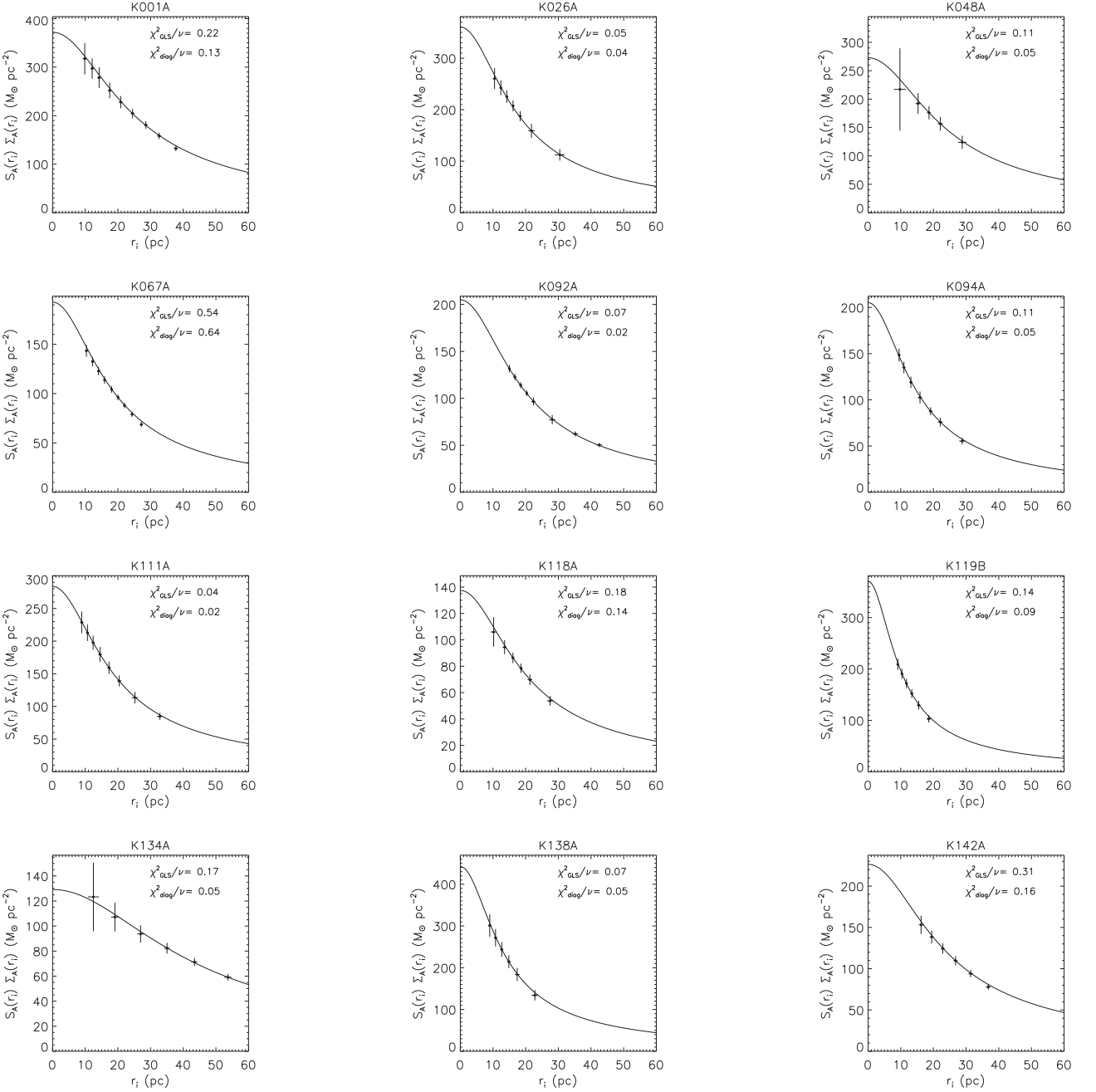
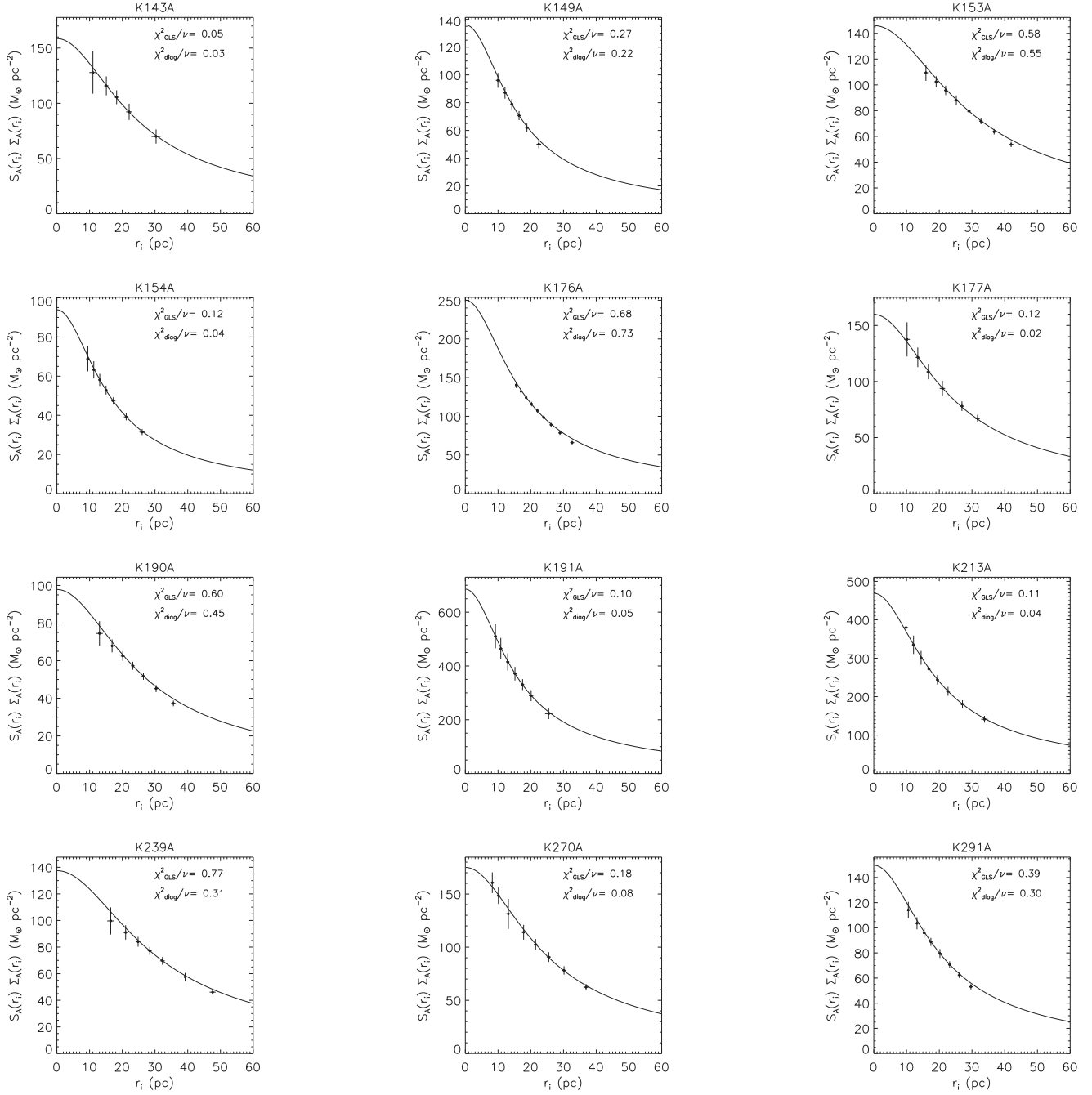


Figure C1. Comparison of the observed area-based surface density profiles, $\Sigma_A(r_i)$ (points) with the surface density profiles $S_A(r_i)$ (line) of the Lane-Emden equation converted to physical units. The plot shows only every other data point for clarity. Error bars show the 1σ statistical uncertainties. The reduced generalized least squares χ^2_{GLS}/ν value and the weighted least squares χ^2_{diag}/ν from the diagonal elements of the covariance matrix are shown for each cloud.

**Figure C1.** *Continued.*

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