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Bounded alternation QCSP on a 3-element domain

Dmitriy Zhuk[(1)] Catarina Carvalho[(2)] Barnaby Martin[(3)]

[(1)] Katedra algebry, Matematicko-fyzikální fakulta, Univerzita Karlova, Sokolovská 49/83, Praha 8, Czech Republic

[(2)] Research Centre for Mathematics and Theoretical Physics, University of Hertfordshire, College Lane Campus, Hatfield AL10 9AB, United Kingdom

[(3)] Department of Computer Science, Durham University, Upper Mountjoy Campus, Stockton Road, Durham DH1 3LE, United Kingdom

Abstract. We consider the complexity of the bounded alternation QCSP, Π_{2k}-QCSP(Γ) (and Σ_{2k+1}-QCSP(Γ)), as Γ ranges over all structures with domain size 3 and constants. We prove that such problems are always in P, NP-complete, coNP-complete or $\mathbf{\Pi}^{\mathsf{P}}_{2k}$-complete (or $\mathbf{\Sigma}^{\mathsf{P}}_{2k+1}$-complete, respectively). Since the corresponding problems Π_{2k+1}-QCSP(Γ) (and Σ_{2k+2}-QCSP(Γ), respectively) are in $\mathbf{\Pi}^{\mathsf{P}}_{2k}$ (and $\mathbf{\Sigma}^{\mathsf{P}}_{2k+1}$, respectively), our work yields a full classification for all bounded alternation fragments. We borrow heavily from the complexity classification for QCSP(Γ), over all domain size 3 structures with constants, but the tetrachotomy is not in the same place.

1 Introduction

The *quantified constraint satisfaction problem*, QCSP(Γ), asks whether some input sentence, whose quantifier-free part is positive conjunctive (i.e. a conjunction of positive atoms) is true on the *constraint language* Γ comprised of a set of relations over the same domain D. The QCSP is itself a generalisation of its better-studied brother, the *constraint satisfaction problem*, CSP(Γ), in which one assumes the quantification of the sentence is entirely existential. By now, a great deal is known about the CSP, while many important questions for the QCSP remain open.

The celebrated Feder–Vardi dichotomy conjecture, which asserts that CSP(Γ) is always either in P or is NP-complete, when Γ is over a finite domain, has been proved independently in [3] and [11; 13].

At present, we are far from a complexity classification for finite domain QCSPs. The 3-element case with constants is settled in [14], where evidence is also given that the classification is more complicated than previously thought.

In this paper, we consider the bounded alternation restrictions of the QCSP, in which there is an *a priori* bound on the quantifier alternations. The corresponding problems are denoted by Σ_k-QCSP(Γ) and Π_k-QCSP(Γ), when the input is restricted to be Σ_k and Π_k, respectively. Such problems arise naturally in the 2-element (Boolean) case, for example propositional *abduction* and *prioritization* (see [8]). The complexity classification for bounded alternation QCSP has been known to display differences from the unbounded case since [6]. Indeed, there are constraint languages Γ_1 and Γ_2, both on three elements, so that QCSP(Γ_1) and QCSP(Γ_2) are PSPACE-complete, yet their bounded alternation versions are coNP-complete and rising in the polynomial hierarchy, respectively. For Γ_1, we may take $(\sigma_0, \sigma_1, b, 2)$ (see Lemma 6), and, for Γ_2, we may take the single binary relation of disequality $\neq$.

Corresponding author: Barnaby Martin, barnaby.d.martin@durham.ac.uk

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In this paper, we settle the complexity of Π_{2k}-QCSP(Γ) for all 3-element constraint languages Γ which include all constants, as well as the complexities of Σ_{2k+1}-QCSP(Γ), Π_{2k+1}-QCSP(Γ) and Σ_{2k+2}-QCSP(Γ), for all 3-element languages with constants. When we say we include constants we mean that constants naming the elements are available to substitute into relations in the same fashion as variables. Some authors prefer to allow singleton unary relations for each element. For the purposes of this exposition, these methods are equivalent.

2 Main result

We need the notion of polymorphisms, so ubiquitous in the study of CSPs, to formulate our tetrachotomy. A *k-ary polymorphism* of a constraint language Γ on domain D is a function $f\colon D^k \to D$ such that, for all m-ary relations R of Γ, if $(x_1^1, \dots, x_1^m), \dots, (x_k^1, \dots, x_k^m) \in R$, then also $(f(x_1^1, \dots, x_k^1), \dots, f(x_1^m, \dots, x_k^m)) \in R$.

A finite *algebra* is a pair $\mathbb{A} = (A, F)$ where A is a finite set and F is a family of operations of finite arity on A. A subset Λ of A^m is a generating set for $\mathbb{A}^m$ exactly if, for every $(a_1, \dots, a_m) \in A^m$, there exists a k-ary term operation $f \in F$ and $(b_1^1, \dots, b_m^1), \dots, (b_1^k, \dots, b_m^k) \in \Lambda$ so that $f(b_1^1, \dots, b_1^k) = a_1$, $\dots$, $f(b_m^1, \dots, b_m^k) = a_m$. We say that $\mathbb{A}$ has the *polynomially generated powers* property (PGP) if there exists a polynomial p such that, for every n, A^n can be generated by $p(n)$ tuples, and we say that it has the *exponentially generated powers* property (EGP) if there exist $b > 1$ and $C > 0$ such that for every n, A^n cannot be generated by less than Cb^n tuples. In [12], it is proved that only the two cases PGP and EGP can arise in finite algebras.

An operation is *idempotent* if $f(x, \dots, x) = x$ (for all x). All operations in this paper are idempotent. An idempotent algebra is an algebra all of whose operations are idempotent. An operation f is an (idempotent) *weak-near-unanimity* (WNU) operation if, for all x and y,

$$f(y, x, \dots, x, x) = f(x, y, \dots, x, x) = \dots = f(x, x, \dots, y, x) = f(x, x, \dots, x, y).$$

Given α, β strict subsets of A with $\alpha \cup \beta = A$, the algebra $\mathbb{A}$ is described as *$\alpha\beta$-projective* if, for each of its k-ary operations f, there is some coordinate i so that: $f(x_1, \dots, x_k) \in \alpha$, if $x_i \in \alpha$; and $f(x_1, \dots, x_k) \in \beta$, if $x_i \in \beta$. It is proved in [12, Theorem 2] that idempotent algebras have EGP if and only if there exist α, β so that all its operations are $\alpha\beta$-projective.

We now present some polymorphisms that will play a key role in our proofs. Let $a, c \in \{0, 1, 2\}$ be constants, and define:

$$f_{a,c}(x, y, z) = \begin{cases} x & \text{if } x = y \text{ or } y = z = a, \\ c & \text{otherwise,} \end{cases} \qquad s_{a,c}(x, y) = \begin{cases} x & \text{if } x = y \text{ or } y = a, \\ c & \text{otherwise,} \end{cases}$$

$$g_{a,c}(x, y) = \begin{cases} x & \text{if } x = a \text{ or } y \neq c, \\ c & \text{otherwise,} \end{cases} \qquad s_c(x, y) = \begin{cases} x & \text{if } x = y, \\ c & \text{otherwise.} \end{cases}$$

If $\{a, b, c\} = \{0, 1, 2\}$, then call any ternary idempotent operation f, such that $f(x, a, b) = x$ and $f(x, c, c) = c$, *ab-stable*. Note that $f(x, y, z) = s_{a,c}(x, y)$ is an *ab*-stable operation. Also note that, for any c, the binary operation s_c is a *semilattice* as it is associate, commutative, and idempotent. This polymorphism has played a key role in the complexity of QCSP in the 3-element case [5].

Theorem 1. Suppose Γ is a finite constraint language on $\{0, 1, 2\}$ with constants. Then Π_{2k}- QCSP(Γ) is

(1) in P if Pol(Γ) has the PGP property and has a WNU operation;
(2) NP-complete if Pol(Γ) has the PGP property and has no WNU operation;
(3) $\mathbf{\Pi}^{\mathsf{P}}_{2k}$-complete if Pol($\Gamma$) has the EGP property and has no WNU operation;
(4) in P if Pol(Γ) contains $s_{a,c}$ and $g_{a,c}$ for some $a, c \in \{0, 1, 2\}$, $a \neq c$;
(5) in P if Pol(Γ) contains $f_{a,c}$ for some $a, c \in \{0, 1, 2\}$, $a \neq c$;
(6) coNP-complete otherwise.

Note that in each of the last three cases, Pol(Γ) contains s_c for some c.

This paper proves a tetrachotomy for the 3-element case QCSP with constants under bounded alternation across the classes: P, NP-complete, coNP-complete, and rising in the polynomial hierarchy. The original 3-element case QCSP with constants from [14] was also a tetrachotomy, but across the classes: P, NP-complete, coNP-complete, and PSPACE-complete. Much of the technical development for the present paper was already accomplished in [6; 14]; consequently our discourse mirrors that of the latter. We explain explicitly where we need to deviate from it in the respective places.

The paper is structured as follows. After the preliminaries, we discuss coNP-membership and hardness in Section 4. We then move on to the case of EGP and WNU in Section 5, before giving our main result for Π_{2k}-QCSP in Section 6 and the remaining cases in Section 7.

Oftentimes, we define relations locally in lemmas and proofs. These definitions should be considered local. For example, σ_0 and σ_1 appear as multiple distinct relations within different proofs.

3 Preliminaries

The following results will be needed.

Theorem 2 (Carvalho, Martin & Zhuk, [4])**.** Let Γ be a finite constraint language with all constants such that Pol(Γ) has the PGP property. Then QCSP(Γ) reduces to a polynomial number of instances of CSP(Γ) and is in NP.

For an algebra $\mathbb{A} = (A, F)$ we define $\mathrm{Inv}(\mathbb{A})$ to be the relations over A which are *invariant* under the operations of $\mathbb{A}$; equivalently, have the operations of $\mathbb{A}$ as polymorphisms.

Corollary 3 (Zhuk & Martin, [14, Corollary 32])**.** QCSP(Γ_0) is in P for every finite $\Gamma_0 \subseteq \mathrm{Inv}(\{g_{0,2}, s_{0,2}\})$.

Corollary 4 (Zhuk & Martin, [14, Corollary 41])**.** QCSP(Γ) is in P for every finite $\Gamma \subseteq \mathrm{Inv}(f_{0,2})$.

A *subalgebra* $\mathbb{D} = (D, F|_D)$ of $\mathbb{A}$ consists of a subset $D \subseteq A$ that is preserved by all the operations of F in $\mathbb{A}$; equivalently any operation of F in $\mathbb{A}$ restricted to the domain D is an operation of $\mathbb{D}$.

By abuse of notation we often identify a subalgebra with its domain (since its operations are inherited by restriction from the original algebra). A *congruence* on an algebra $\mathbb{A}$ is an equivalence relation $\sim$ on its domain so that, for each k-ary operation f in $\mathbb{A}$, $f(x_1, \ldots, x_k) \sim f(y_1, \ldots, y_k)$ whenever $x_1 \sim y_1, \ldots, x_k \sim y_k$. We can quotient $\mathbb{A}$ by $\sim$ in the obvious way to obtain a new algebra that we describe as a *homomorphic image* of $\mathbb{A}$. A *factor* of $\mathbb{A}$ is a homomorphic image of a subalgebra of $\mathbb{A}$.

We always may assume that an instance of QCSP(Γ) is of the prenex form $\forall x_1 \exists y_1 \forall x_2 \exists y_2 \ldots \forall x_n \exists y_n \, \Phi$, since, if it is not, it may readily be brought into such a form in LOGSPACE. Then a solution is a sequence of (Skolem) functions $f_1, \ldots, f_n$ such that $(x_1, f(x_1), x_2, f_2(x_1, x_2), \ldots, x_n, f_n(x_1, \ldots, x_n))$ is a solution of Φ for all $x_1, \ldots, x_n$ (i.e., $y_i = f_i(x_1, \ldots, x_i)$). This belies a (Hintikka) game semantics for the truth of a QCSP instance in which a player called Universal (male) plays the universal variables and a player called Existential (female) plays the existential variables, one after another, from the outside in. The Skolem functions above give a strategy for Existential. In our proofs we may occasionally revert to a game-theoretical parlance. Let φ be an instance of of QCSP(Δ) and φ' an instance of QCSP(Γ). If we want to prove that φ is a yes-instance implies φ' is a yes-instance, then it is enough to argue that, if Existential has a winning strategy for φ on Δ, then she also has a winning strategy for φ' on Γ. To prove this, one has to map Universal's plays on φ' in Γ backwards to his plays on φ in Δ.

Let us remind ourselves of some problems that we use for hardness.

(1) (Monotone) *Not-All-Equal* 3-*Satisfiability* (NAE3SAT) is CSP $\big(\{0,1\}; \{0,1\}^3 \backslash \{(0,0,0),(1,1,1)\}\big)$.
(2) (Monotone) 1-*in*-3-*Satisfiability* (1-in-3SAT) is CSP $\big(\{0,1\}; \{(0,0,1),(0,1,0),(1,0,0)\}\big)$.
(3) *Quantified Not-All-Equal* 3-*Satisfiability* (QNAE3SAT) is QCSP $\big(\{0,1\}; \{0,1\}^3 \backslash \{(0,0,0),(1,1,1)\}\big)$.

Here, (1) and (2) are NP-complete according to [10], and (3) is PSPACE-complete and its Π_{2k}-restriction (and Σ_{2k+1}-restriction) is complete for $\mathbf{\Pi}^{\mathsf{P}}_{2k}$ (and $\mathbf{\Sigma}^{\mathsf{P}}_{2k+1}$, respectively) in the polynomial hierarchy [7].

4 Membership and hardness

Before moving on to a collection of coNP-hardness results, let us remind ourselves of a fundamental coNP-membership result from [6] that will play a key role in this classification. We fix the notation $\mathrm{AE}_3 = \{(0,0,0),(1,1,1)\}$ and $\mathrm{NAE}_3 = \{0,1\}^3 \backslash \{(0,0,0),(1,1,1)\}$.

Lemma 5 (Chen, [6])**.** If Γ is a finite constraint language with the polymorphism s_2, then Π_{2k}- QCSP(Γ) is in coNP.

The following is an adaptation of [14, Lemma 17], but proving coNP-hardness and not PSPACE-hardness. The proof is almost identical to that in [14], but the result is formally incomparable (hence we give the proof).

Lemma 6. Suppose $b \in \{0,1\}$, and define:[1]

$$\sigma_0 = \{(a_1,a_2,a_3) : a_1 \in A,\ a_2,a_3 \in \{b,2\},\ (a_1 \in \{0,2\} \vee a_2 = a_3)\},$$
$$\sigma_1 = \{(a_1,a_2,a_3) : a_1 \in A,\ a_2,a_3 \in \{b,2\},\ (a_1 \in \{1,2\} \vee a_2 = a_3)\}.$$

Then Π_2-QCSP($\{\sigma_0,\sigma_1,\{b\},\{2\}\}$) is coNP-hard.

Proof. The reduction will be from the complement of (monotone) NAE3SAT (co-NAE3SAT). Consider an instance I of co-NAE3SAT of the form

$$\forall x_1 \forall x_2 \dots \forall x_n \left(\mathrm{AE}_3(x_{a_1},x_{b_1},x_{c_1}) \vee \dots \vee \mathrm{AE}_3(x_{a_s},x_{b_s},x_{c_s})\right).$$

By $p_k(x_1,\dots,x_k)$, we denote the predicate (relation) on $\{0,1\}$ defined by

$$\forall x_{k+1} \forall x_{k+2} \dots \forall x_n \left(\mathrm{AE}_3(x_{a_1},x_{b_1},x_{c_1}) \vee \dots \vee \mathrm{AE}_3(x_{a_s},x_{b_s},x_{c_s})\right).$$

We inductively define a predicate $\delta_k(x_1,\dots,x_k,y,y')$ for $k = n, n-1, \dots, 1, 0$ satisfying the conditions

(1) if $y,y' \in \{b,2\}$, $y \neq y'$, $x_1,\dots,x_k \in \{0,1\}$, then $\delta_k(x_1,\dots,x_k,y,y') = p_k(x_1,\dots,x_k)$ holds;
(2) if $y = y' \in \{b,2\}$, then $\delta_k(x_1,\dots,x_k,y,y')$ holds;
(3) if $\delta_k(x_1,\dots,x_k,y,y')$ holds, then it holds if we replace some of the values $x_1,\dots,x_k$ by 2;
(4) the predicate δ_k is definable by a Π_2-formula over $\{\sigma_0,\sigma_1,\{b\},\{2\}\}$ that can be efficiently computed.

Let us define

$$\delta_n(x_1,\dots,x_n,y_0,y_{2s}) = \exists y_1 \dots \exists y_{2s-1} \bigwedge_{1 \le i \le s} \Big(\sigma_0(x_{a_i},y_{i-1},y_i) \wedge \sigma_0(x_{b_i},y_{i-1},y_i) \wedge \sigma_0(x_{c_i},y_{i-1},y_i) \wedge \sigma_1(x_{a_i},y_{s+i-1},y_{s+i}) \wedge \sigma_1(x_{b_i},y_{s+i-1},y_{s+i}) \wedge \sigma_1(x_{c_i},y_{s+i-1},y_{s+i})\Big),$$

$$\delta_{k-1}(x_1,\dots,x_{k-1},y,y') = \forall x_k\ \delta_k(x_1,\dots,x_k,y,y').$$

Let us check that δ_n satisfies the above properties (1)–(4). If $p_n(x_1,\dots,x_n)$ holds, then there exists i such that $x_{a_i} = x_{b_i} = x_{c_i} = d$. If $d = 0$, then to satisfy $\delta_n(x_1,\dots,x_n,y_0,y_{2s})$, we send $y_1,y_2,\dots,y_{i-1}$ to y_0, and $y_i,y_{i+1},\dots,y_{2s-1}$ to y_{2s}. If $d = 1$, then we send $y_1,y_2,\dots,y_{s+i-1}$ to y_0, and $y_{s+i},y_{s+i+1},\dots,y_{2s-1}$ to y_{2s}. Suppose $\delta_n(x_1,\dots,x_n,y_0,y_{2s})$ holds on $y,y' \in \{b,2\}$, $y \neq y'$, $x_1,\dots,x_n \in \{0,1\}$. Then there should be i such that $y_{i-1} \neq y_i$. If $i \le s$, then $x_{a_i} = x_{b_i} = x_{c_i} = 0$; if $i > s$, then $x_{a_{i-s}} = x_{b_{i-s}} = x_{c_{i-s}} = 1$. Thus, $p_n(x_1,\dots,x_n)$ holds, which completes the proof of (1). To prove (2) it is sufficient to send $y_1,\dots,y_{2s-1}$ to $y = y'$. (3) and (4) follow from the definition of δ_n.

Let us prove by induction on k that $\delta_{n-1},\dots,\delta_0$ satisfy the above properties (1)–(4). Suppose they hold for δ_k. Then (4) for δ_{k-1} follows from the definition and (4) for δ_k, (3) for δ_{k-1} follows from the definition and the properties of σ_0 and σ_1, (2) for δ_{k-1} follows in the same fashion as for δ_n, and (1) for δ_{k-1} follows from (1) and (3) for δ_k.

[1] Note that the commas separating terms to the right of the colon indicate a conjunction.

Thus, we proved the properties (1)–(4) for δ_{k-1}, and by the inductive assumption we conclude that they hold for δ_0. Since the quantified formula over $\{\sigma_0, \sigma_1, \{b\}, \{2\}\}$ defining δ_0 is efficiently computable, it remains to check that $\delta_0(b, 2)$ holds if and only if I holds. □

Recall that an operation f is 01-stable if it is idempotent, $f(x, 0, 1) = x$, and $f(x, 2, 2) = 2$. We say that a ternary relation R is an AND-*type* relation if

$$R \cap (\{0,1\} \times \{0,1\} \times \{0,1,2\}) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

that is, it is row-wise the truth table of AND. Note, however, that the tuples of R are columns in the matrix. Recall that $s_2(x, y) = x$ if $x = y$, and $s_2(x, y) = 2$ otherwise, and note that if s_2 preserves R, then R also contains $(2, 2, 0)$ and $(2, 2, 2)$. We say that a ternary relation R is an OR-*type* relation if

$$R \cap (\{0,1\} \times \{0,1\} \times \{0,1,2\}) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix},$$

that is, it is row-wise the truth table of OR. Again, if s_2 preserves R, then it also contains $(2, 2, 1)$ and $(2, 2, 2)$.

The following is an adaptation of [14, Lemma 20]. We follow the proof there, but the result is formally incomparable.

Lemma 7. Suppose Γ contains an AND-type relation and an OR-type relation, and Γ is preserved by s_2. Then Π_2-QCSP(Γ) is coNP-hard.

Proof. By $R_{\mathrm{and},2}$ and $R_{\mathrm{or},2}$, we denote a ternary AND-type and an OR-type relation, respectively. For $n = 2, 3, 4, \ldots$, put

$$R_{\mathrm{and},n+1}(x_1, \ldots, x_n, x_{n+1}, y) = \exists z \left(R_{\mathrm{and},n}(x_1, \ldots, x_n, z) \wedge R_{\mathrm{and},2}(x_{n+1}, z, y)\right),$$
$$R_{\mathrm{or},n+1}(x_1, \ldots, x_n, x_{n+1}, y) = \exists z \left(R_{\mathrm{or},n}(x_1, \ldots, x_n, z) \wedge R_{\mathrm{or},2}(x_{n+1}, z, y)\right).$$

Let us define a relation $\xi_n(x_1, y_1, z_1, \ldots, x_n, y_n, z_n)$ for every n by

$$\begin{aligned} \xi_n = \exists u \exists u_1 \ldots \exists u_n \exists v \exists v_1 \ldots \exists v_n \Big(& R_{\mathrm{and},3}(x_1, y_1, z_1, u_1) \wedge \cdots \wedge R_{\mathrm{and},3}(x_n, y_n, z_n, u_n) \\ & \wedge R_{\mathrm{or},3}(x_1, y_1, z_1, v_1) \wedge \cdots \wedge R_{\mathrm{or},3}(x_n, y_n, z_n, v_n) \\ & \wedge R_{\mathrm{and},2}(u, v, v) \wedge R_{\mathrm{or},n}(u_1, \ldots, u_n, u) \wedge R_{\mathrm{and},n}(v_1, \ldots, v_n, v)\Big). \end{aligned}$$

It follows from the definition that $\xi_n \cap \{0,1\}^{3n}$ is defined by

$$\mathrm{AE}_3(x_1, y_1, z_1) \vee \mathrm{AE}_3(x_2, y_2, z_2) \vee \cdots \vee \mathrm{AE}_3(x_n, y_n, z_n).$$

Suppose that $R(y_1, \ldots, y_t) = \xi_n(y_{i_1}, y_{i_2}, \ldots, y_{i_{3n}})$, where $i_1, \ldots, i_{3n} \in \{1, 2, \ldots, t\}$. Let us show that $\forall y_1 \ldots \forall y_t\ \xi_n(y_{i_1}, y_{i_2}, \ldots, y_{i_{3n}})$ holds if $\xi_n(y_{i_1}, y_{i_2}, \ldots, y_{i_{3n}})$ holds for all $y_1, \ldots, y_t \in \{0, 1\}$. We need to prove that if $\{0,1\}^t \subseteq R$ then $A^t \subseteq R$, which follows from the fact that the semilattice s_2 preserves R.

Now we may encode the complement of NAE3SAT (co-NAE3SAT) using Γ. This complement can be expressed by a formula of the form $\forall y_1 \ldots \forall y_t \left(\mathrm{AE}_3(y_{i_1}, y_{i_2}, y_{i_3}) \vee \cdots \vee \mathrm{AE}_3(y_{i_{3n-2}}, y_{i_{3n-1}}, y_{i_{3n}})\right)$, which is equivalent to $\forall y_1 \ldots \forall y_t\ \xi_n(y_{i_1}, y_{i_2}, \ldots, y_{i_{3n}})$. Thus, we reduced a coNP-complete problem to Π_{2k}-QCSP(Γ), which completes the proof. □

The following is an adaptation of [14, Lemma 21], the proof is identical, but the result is formally incomparable (hence we give the proof).

Lemma 8. Suppose $\Gamma \subseteq \mathrm{Inv}(s_2)$ and Γ contains the constant 0, the relation $\delta(x, y, z) = (x \neq 0) \vee (y = z)$ and an AND-type relation R_2. Then Π_2-QCSP(Γ) is coNP-hard.

Proof. Here we will define a reduction from the complement of 1-in-3SAT (co-1-in-3SAT). First, put

$$\delta_1(x_1, x_2, x_3, x_4) = \exists t\,(\delta(x_1, x_3, t) \wedge \delta(x_2, t, x_4)).$$

It is not hard to see that $\delta_1(x_1, x_2, x_3, x_4) = (x_1 \neq 0) \vee (x_2 \neq 0) \vee (x_3 = x_4)$. Put

$$\delta_2(x_1, x_2, x_3, x_4) = \delta_1(x_1, x_2, x_3, x_4) \wedge \delta_1(x_1, x_3, x_2, x_4) \wedge \delta_1(x_2, x_3, x_1, x_4).$$

By the definition, the first three variables of δ_2 are symmetric, as they contain the equivalent of a 3-cycle and a transposition, so we can generate all permutations of x_1, x_2, x_3.

Suppose $(a, b, c, d) \in \delta_2$. It is not hard to see that $a = b = c = 0$ implies $d = 0$. Also $a = 1$, $b = c = 0$ implies $d = 1$. If there are at least two 1s in (a, b, c), then d can be arbitrary. Put $R'(x, y) = R_2(x, x, y)$. By the definition of an AND-type relation, R' contains $(0, 0), (1, 1), (2, 2), (2, 0)$ but does not contain $(0, 1), (0, 2), (1, 0), (1, 2)$. Put

$$\delta_3(x_1, x_2, x_3, x_4) = \exists x_1' \exists x_2' \exists x_3' \left(\delta_2(x_1', x_2', x_3', x_4) \wedge R'(x_1, x_1') \wedge R'(x_2, x_2') \wedge R'(x_3, x_3')\right).$$

Then $(a, b, c, 0) \in \delta_3$ for all tuples (a, b, c) but the tuples from $R_{1\text{-in-}3} \coloneqq \{(0, 0, 1), (0, 1, 0), (1, 0, 0)\}$.

Again, we apply the recursive formula to build an n-ary "and":

$$R_{n+1}(x_1, \dots, x_{n+1}, z) = \exists t\,(R_n(x_1, \dots, x_n, t) \wedge R_2(t, x_{n+1}, z)).$$

Define $\zeta_n(x_1, \dots, x_{3n})$ by

$$\zeta_n = \exists z_1 \dots \exists z_n \exists t \left(R_n(z_1, \dots, z_n, t) \wedge \delta_3(x_1, x_2, x_3, z_1) \wedge \dots \wedge \delta_3(x_{3n-2}, x_{3n-1}, x_{3n}, z_n) \wedge (t = 0)\right).$$

Now we may encode co-1-in-3SAT using Γ. If $\overline{R_{1\text{-in-}3}}$ is the complement of $R_{1\text{-in-}3}$ with respect to $\{0, 1\}^3$, then we use a formula of the form $\forall y_1 \dots \forall y_t \left(\overline{R_{1\text{-in-}3}}(y_{i_1}, y_{i_2}, y_{i_3}) \vee \dots \vee \overline{R_{1\text{-in-}3}}(y_{i_{3n-2}}, y_{i_{3n-1}}, y_{i_{3n}})\right)$, which is equivalent to $\forall y_1 \dots \forall y_t\ \xi_n(y_{i_1}, y_{i_2}, \dots, y_{i_{3n}})$. Thus, we reduced a coNP-complete problem to Π_2-QCSP(Γ), which completes the proof. □

In the next lemma we follow the proof method of [14, Lemma 22].

Lemma 9. Suppose $\sigma = \{0, 2\}^2 \cup \{1, 2\}^2$, and

$$\sigma_0 = \{(a_1, a_2, a_3) : a_1 \in A,\ a_2, a_3 \in \{1, 2\},\ (a_1 \in \{0, 2\} \vee a_2 = a_3)\},$$
$$\sigma_1 = \{(a_1, a_2, a_3) : a_1 \in A,\ a_2, a_3 \in \{0, 2\},\ (a_1 \in \{1, 2\} \vee a_2 = a_3)\}.$$

Then Π_2- QCSP($\{\sigma, \sigma_0, \sigma_1, \{0\}, \{1\}\}$) is coNP-hard.

Proof. Let us encode co-NAE3SAT using $\{\sigma, \sigma_0, \sigma_1, \{0\}, \{1\}\}$. An instance I of this complement can be expressed by a formula of the form $\forall x_1 \dots \forall x_t \left(\mathrm{AE}_3(x_{a_1}, x_{b_1}, x_{c_1}) \vee \dots \vee \mathrm{AE}_3(x_{a_s}, x_{b_s}, x_{c_s})\right)$. Let

$$\Phi_0 = \bigwedge_{1 \leq i \leq s} \left(\sigma_0(x_{a_i}, y_{i-1}, y_i) \wedge \sigma_0(x_{b_i}, y_{i-1}, y_i) \wedge \sigma_0(x_{c_i}, y_{i-1}, y_i)\right),$$
$$\Phi_1 = \bigwedge_{1 \leq i \leq s} \left(\sigma_1(x_{a_i}, z_{i-1}, z_i) \wedge \sigma_1(x_{b_i}, z_{i-1}, z_i) \wedge \sigma_1(x_{c_i}, z_{i-1}, z_i)\right),$$
$$\Psi = \forall x_1 \dots \forall x_t\, \exists y_0 \exists y_1 \dots \exists y_s\, \exists z_0 \exists y_1 \dots \exists z_s\, (y_0 = 1 \wedge \Phi_0 \wedge z_0 = 0 \wedge \Phi_1 \wedge \sigma(y_s, z_s)).$$

Let us show that I is equivalent to Ψ. We first show that if Existential wins in co-NAE3SAT then Existential also wins in Π_2- QCSP($\{\sigma, \sigma_0, \sigma_1, \{0\}, \{1\}\}$), and then show the reverse statement. Note, however, that Existential does not play at all in co-NAE3SAT, as it is purely universal.

Claim 10. $I \in$ co-NAE3SAT if and only if $\Psi \in \Pi_2$- QCSP($\{\sigma, \sigma_0, \sigma_1, \{0\}, \{1\}\}$).

Proof of Claim 10. ($\Rightarrow$). Suppose in Ψ that Universal plays some sequence of 0s, 1s and 2s for the x_is. Let this be duplicated by Universal in I where we additionally consider 2 in Ψ as 1 in I. We may assume that the resulting instance of co-NAE3SAT has a clause j that is either $(0,0,0)$ or $(1,1,1)$.

Suppose first that clause j is $(1,1,1)$, which arises from a setting in Ψ of $(x_{a_j}, x_{b_j}, x_{c_j}) \in \{1,2\}^3$. Existential plays in Ψ as follows: $y_0, \ldots, y_{j-1}, y_j, \ldots, y_s$ are set to 1, $z_0, \ldots, z_{j-1}$ are set to 0, and $z_j, \ldots, z_s$ are set to 2. Let us look at the interesting conjunction in Φ_1, which is $\sigma_1(x_{a_j}, z_{j-1}, z_j) \wedge \sigma_1(x_{b_j}, z_{j-1}, z_j) \wedge \sigma_1(x_{c_j}, z_{j-1}, z_j)$, and note that this is satisfied trivially as $(x_{a_j}, x_{b_j}, x_{c_j}) \in \{1,2\}^3$. The fact that the other parts are satisfied is straightforward, since $y_i = y_{i-1}$ and $z_i = z_{i-1}$.

Suppose now that clause j is $(0,0,0)$, which arises only from a setting in Ψ of $(x_{a_j}, x_{b_j}, x_{c_j}) = (0,0,0)$. Existential plays in Ψ as follows: $z_0, \ldots, z_{j-1}, z_j, \ldots, z_s$ are set to 0, $y_0, \ldots, y_{j-1}$ are set to 1, and $y_j, \ldots, y_s$ are set to 2. This case concludes as the previous.

($\Leftarrow$). Let us consider Universal playing in I and we duplicate this in Ψ (in other words, we only consider Universal plays in Ψ on $\{0,1\}$). We cannot have $y_0, \ldots, y_s = 1$ and $z_0, \ldots, z_s = 0$, because we violate $\sigma(y_s, z_s)$. Thus, at least at some point in one of these sequences, we make a transition to 2. First, let us assume that it is in the y_i at the point $y_{j-1} = 1$ and $y_j = 2$. Then the conjunction $\sigma_0(x_{a_j}, y_{j-1}, y_j) \wedge \sigma_0(x_{b_j}, y_{j-1}, y_j) \wedge \sigma_0(x_{c_j}, y_{j-1}, y_j)$ enforces that $(x_{a_j}, x_{b_j}, x_{c_j}) = (0,0,0)$. Now let us assume that it is in the z_i at the point $z_{j-1} = 0$ and $z_j = 2$. Then the conjunction $\sigma_1(x_{a_j}, z_{j-1}, z_j) \wedge \sigma_1(x_{b_j}, z_{j-1}, z_j) \wedge \sigma_1(x_{c_j}, z_{j-1}, z_j)$ enforces that $(x_{a_j}, x_{b_j}, x_{c_j}) = (1,1,1)$. Thus, in both cases, we deduce that I is a no-instance of NAE3SAT. ■

Thus, we have reduced a coNP-complete problem to Π_2- QCSP($\{\sigma, \sigma_0, \sigma_1, \{0\}, \{1\}\}$), which completes the proof. □

5 Exponentially generated powers and weak-near-unanimity on a 3-element domain

In this section we consider constraint languages Γ on $\{0,1,2\}$ with constants for which Pol(Γ) has the EGP property and contains a WNU operation. It can be shown (see [5, Theorem 7.2] after [12, Theorem 2]) that in this case Pol(Γ) contains a semilattice operation. Since adding pp-definable relations to Γ does not change the complexity of QCSP(Γ), we may assume that k is the maximal arity of the relations in Γ and Γ contains all pp-definable relations of arity at most k. Also, EGP implies that there exists α and β, neither equal to A, but so that $\alpha \cup \beta = A$, so that all operations of Pol(Γ) are $\alpha\beta$-projective [12]. If $\alpha \cap \beta = \varnothing$, then there can be no WNU. Hence, we may assume that $\alpha = \{0,2\}$ and $\beta = \{1,2\}$, which implies that $\{0,2\}$, $\{1,2\}$, and $\{0,2\}^2 \cup \{1,2\}^2$ are in Γ. Thus, in this section we have the following assumptions:

(1) Γ is a finite constraint language on $A = \{0,1,2\}$;
(2) Γ contains the relations $\{0\}, \{1\}, \{2\}, \{0,2\}, \{1,2\}$, $\{0,2\}^2 \cup \{1,2\}^2$, and the equality relation;
(3) Γ is preserved by the semilattice operation s_2;
(4) Pol(Γ) is $\{0,2\}\{1,2\}$-projective;
(5) all relations in Γ are of arity at most $k \geq 4$;
(6) Γ contains all relations of arity at most k that can be pp-defined from Γ.

These assumptions will hold up to, but not including, Theorem 25. Similar to a 01-stable operation, we define 0-stable and 1-stable operations, where, for $c = 0, 1$, f is called *c-stable* if it is idempotent, $f(x,c) = x$, and $f(x,2) = 2$. Note that by adding a dummy variable we can make it a 01-stable operation.

5.1 Finiteness of the language

In this subsection we will derive some properties of Γ based on the facts that Γ is finite and Pol(Γ) is $\{0,2\}\{1,2\}$-projective.

For $B, C \subseteq A$ and $n \geq 3$, we denote by $\mathrm{nu}^n_{B,C}$ the operation of arity n defined on a tuple $(a_1, \dots, a_n)$ as follows. If $a_1 = \dots = a_n$ then it returns a_1. If $a_1 = \dots = a_{i-1} = a_{i+1} = \dots = a_n \in B$ and $a_i \in C$ for some i, then it returns a_j for $j \neq i$. Otherwise, it returns 2. Note that, in order to preserve the notation of [14], α and β remain sets, while α_i and γ are tuples of elements.

Lemma 11 (Zhuk & Martin, [14, Lemma 42])**.** Suppose ϱ is a relation of the minimal arity N in Γ that is not preserved by $\mathrm{nu}^{2k}_{B,C}$. Then, after some permutation of variables, ϱ satisfies one of the following conditions:

(1) There exist $\alpha_1, \alpha_2 \in \varrho$ such that
   (a) $\mathrm{nu}^{2k}_{B,C}(\alpha_1, \alpha_2, \alpha_2, \dots, \alpha_2) = \gamma \notin \varrho$,
   (b) $\alpha_2(i) \in B$ and $\gamma(i) = \alpha_2(i) \neq 2$ for $i \in \{1, \dots, N-1\}$,
   (c) $\gamma(N) = 2$, and
   (d) $\alpha_1(i) \neq \alpha_2(i)$ for every i.

   Moreover, if $C = A$ then $N = 2$, i.e. ϱ is binary, $\alpha_2(N) \notin B$, and $\alpha_1 = (2, 2)$.

(2) There exist $\alpha_1, \alpha_2, \alpha_3 \in \varrho$ such that
   (a) $\mathrm{nu}^{2k}_{B,C}(\alpha_1, \alpha_2, \alpha_3, \alpha_3, \dots, \alpha_3) = \gamma \notin \varrho$,
   (b) $\gamma(i) = \alpha_3(i) \neq 2$ for $i \in \{1, \dots, N-1\}$,
   (c) $\gamma(N) = 2$,
   (d) $\alpha_1(N), \alpha_2(N) \in C$ and $\alpha_3(i) \in B$ for every i, and
   (e) $\alpha_1(i) \neq \alpha_3(i)$ or $\alpha_2(i) \neq \alpha_3(i)$ for every i.

   Moreover, if $2 \in C$, then $\alpha_1(i), \alpha_2(i) \in \{\alpha_3(i), 2\}$ for every i, $\alpha_1(N) = \alpha_2(N) = 2$, and $N = 3$.

Our plan is to apply the above lemma to define relations that will be used to prove hardness results.

Lemma 12 (Zhuk & Martin, [14, Lemma 32])**.** One of the following cases holds.

(1) $(\{0,2\} \times \{0,2\} \times \{0,2\}) \backslash \{(0,0,2)\} \in \Gamma$;
(2) $(\{1,2\} \times \{1,2\} \times \{1,2\}) \backslash \{(1,1,2)\} \in \Gamma$;
(3) $(\{0,2\} \times \{0,2\} \times \{1,2\}) \backslash \{(0,0,2)\} \in \Gamma$;
(4) $(\{1,2\} \times \{1,2\} \times \{0,2\}) \backslash \{(0,0,2)\} \in \Gamma$.

Lemma 13 (Zhuk & Martin, [14, Lemma 44])**.** One of the following cases holds.

(1) $(\{0,2\} \times \{0,2\} \times \{0,2\}) \backslash \{(0,0,2)\} \in \Gamma$;
(2) there exists a relation $\delta \in \Gamma$ such that $(0,1), (2,2) \in \delta$ and $(0,2) \notin \delta$.

Similarly, if we switch 0 and 1, we get the following lemma.

Lemma 14 (Zhuk & Martin, [14, Lemma 45])**.** One of the following cases holds.

(1) $(\{1,2\} \times \{1,2\} \times \{1,2\}) \backslash \{(1,1,2)\} \in \Gamma$;
(2) there exists $\delta \in \Gamma$ such that $(1,0), (2,2) \in \delta$ and $(1,2) \notin \delta$.

Lemma 15 (Zhuk & Martin, [14, Lemma 46])**.** One of the following cases holds.

(1) $(\{0,2\} \times \{0,2\} \times \{0,2\}) \backslash \{(0,0,2)\}, (\{1,2\} \times \{1,2\} \times \{1,2\}) \backslash \{(1,1,2)\} \in \Gamma$;
(2) $(\{1,2\} \times \{0,2\} \times \{0,2\}) \backslash \{(1,0,2)\}, (\{0,2\} \times \{0,2\} \times \{0,2\}) \backslash \{(0,0,2)\} \in \Gamma$;
(3) $(\{0,2\} \times \{1,2\} \times \{1,2\}) \backslash \{(0,1,2)\}, (\{1,2\} \times \{1,2\} \times \{1,2\}) \backslash \{(1,1,2)\} \in \Gamma$.

5.2 Connection with Boolean functions

A mapping $f\colon \{0,1\}^k \to \{0,1\}$ is called a k-ary *Boolean function*. We say that a k-ary Boolean function f is represented by Γ (or that Γ is a representation of f) if there exists a relation $R \subseteq \{0,1,2\}^{k+1}$ pp-definable using Γ such that for all $a_1, \dots, a_n \in \{0,1\}$ and $b \in \{0,1,2\}$, we have

$$(a_1, \dots, a_k, b) \in R \Leftrightarrow f(a_1, \dots, a_k) = b.$$

Note that when we restrict the first n coordinates of R to $\{0,1\}$, its last coordinate cannot be 2. For example, an AND-type relation is a representation of the function conjunction and an OR-type relation is a representation of function disjunction. A k-ary Boolean function f is *monotonic* if $f(x_1,\ldots,x_k) \le f(y_1,\ldots,y_k)$ whenever $x_i \le y_i$ for every $i \in \{1,2,\ldots,k\}$. A k-ary Boolean function is *linear* if it is of the form $c_1x_1 + \ldots + c_kx_k + c_0$ for $c_0, c_1, \ldots, c_k \in \{0,1\}$.

Lemma 16 (Zhuk & Martin, [14, Lemma 47])**.** The set of all Boolean functions represented by Γ is a clone generated by one of the following sets: $\{0,1\}$, $\{+,0,1\}$, $\{\wedge,0,1\}$, $\{\vee,0,1\}$, $\{\wedge,\vee,0,1\}$ or $\{\wedge,\neg,0,1\}$. We represent this clone by BoolClo(Γ).

Theorem 17 (Zhuk & Martin, [14, Theorem 51])**.** Suppose Γ is preserved by a 0-stable operation. Then Γ contains an AND-type relation (equivalently BoolClo(Γ) contains conjunction).

Theorem 18 (Zhuk & Martin, [14, Theorem 52])**.** Suppose Γ is preserved by a 1-stable operation. Then Γ contains an OR-type relation (equivalently BoolClo(Γ) contains disjunction).

5.3 The proof of the exponentially generated powers classification

Recall the assumptions on Γ given at the start of Section 5.

Lemma 19 (Zhuk & Martin, [14, Lemma 53])**.** Suppose Pol(Γ) does not have a 0-stable operation. Then Γ contains one of the two relations $\{(0,0),(1,0),(2,0),(0,2),(2,2)\}$ or $\{(0,1),(1,1),(2,1),(0,2),(2,2)\}$.

Lemma 20 (Zhuk & Martin, [14, Lemma 54])**.** Suppose Pol(Γ) does not have a 1-stable operation. Then Γ contains one of the two relations $\{(0,0),(1,0),(2,0),(1,2),(2,2)\}$ or $\{(0,1),(1,1),(2,1),(1,2),(2,2)\}$.

Lemma 21 (Zhuk & Martin, [14, Lemma 55])**.** Suppose Pol(Γ) does not have a 01-stable operation. Then there exists $b \in \{0,1\}$ such that $\sigma_0, \sigma_1 \in \Gamma$, where

$$\sigma_0 = \{(a_1,a_2,a_3) : a_1 \in A,\ a_2,a_3 \in \{b,2\},\ (a_1 \in \{0,2\} \vee a_2 = a_3)\},$$
$$\sigma_1 = \{(a_1,a_2,a_3) : a_1 \in A,\ a_2,a_3 \in \{b,2\},\ (a_1 \in \{1,2\} \vee a_2 = a_3)\}.$$

We now give a tighter version of [14, Lemma 57].

Lemma 22. Suppose Pol(Γ) does not have a 0-stable or 1-stable operation. Then Π_2-QCSP(Γ) is coNP-hard.

Proof. By Lemmas 19 and 20, we have the relations

$$\delta_0 = \begin{pmatrix} 0 & 1 & 2 & 0 & 2 \\ a_0 & a_0 & a_0 & 2 & 2 \end{pmatrix} \qquad \text{and} \qquad \delta_1 = \begin{pmatrix} 0 & 1 & 2 & 1 & 2 \\ a_1 & a_1 & a_1 & 2 & 2 \end{pmatrix}$$

in Γ for some $a_0, a_1 \in \{0,1\}$. By Lemma 15, for some $b_0, b_1 \in \{0,1\}$ the relations

$$\psi_0 = (\{a_0,2\} \times \{b_0,2\} \times \{b_0,2\}) \backslash \{(a_0,b_0,2)\},$$
$$\psi_1 = (\{a_1,2\} \times \{b_1,2\} \times \{b_1,2\}) \backslash \{(a_1,b_1,2)\}$$

are in Γ. We define

$$\sigma_0(x,y,z) = \exists t\ \delta_0(x,t) \wedge \psi_0(t,y,z) \wedge \psi_0(t,z,y),$$
$$\sigma_1(x,y,z) = \exists t\ \delta_1(x,t) \wedge \psi_1(t,y,z) \wedge \psi_1(t,z,y).$$

If $b_0 = b_1$ then by Lemma 6 the problem is coNP-hard.

Since Γ contains $\{0,2\}^2 \cup \{1,2\}^2$, if $b_0 = 1$ and $b_1 = 0$, then by Lemma 9 the problem is coNP-hard. Assume that $b_0 = 0$ and $b_1 = 1$. Put

$$\sigma_1'(x,y,z) = \exists z'\ \sigma_1(x,1,z') \wedge \sigma_0(z',y,z),$$
$$\sigma_0'(x,y,z) = \exists z'\ \sigma_0(x,0,z') \wedge \sigma_1(z',y,z),$$

and notice that σ_0' and σ_1' are the relations required in Lemma 9 for the proof of coNP-hardness. □

Lemma 23 (Zhuk & Martin, [14, Lemma 58])**.** Suppose Γ is not preserved by $f_{0,2}$, and Γ is preserved by $s_{0,2}$, $\{(0,1)\} \cup \{1,2\} \times \{0,1,2\}$, $\{(0,0)\} \cup \{1,2\} \times \{0,1,2\}$, $\{(0,0),(1,1)\} \cup \{2\} \times \{0,1,2\} \in \Gamma$. Then the relation $\delta(x,y,z) = (x \neq 0) \vee (y = z)$ can be pp-defined from Γ.

Lemma 24 (Zhuk & Martin, [14, Lemma 59])**.** Suppose Γ is preserved by a 0-stable operation, but not preserved by $g_{0,2}$, and $R = \{(0,0),(1,1)\} \cup \{2\} \times \{0,1,2\} \in \Gamma$. Then Γ contains a relation $\delta \in \{\{(0,1),(2,2)\},\{(0,1),(2,1),(2,2)\}\}$.

Now we are ready to prove the classification of the complexity for constraint languages we consider in this section. The following is our version of [14, Theorem 60]. We phrase it as it translates from [14], which means we have two coNP-complete cases. If we had only included the last three cases, the theorem would also be true.

Theorem 25. Suppose Γ is a finite constraint language on $\{0,1,2\}$ with constants, Γ is preserved by s_2, and $\mathrm{Pol}(\Gamma)$ is $\{0,2\}\{1,2\}$-projective. Then Π_2- QCSP(Γ) is

(1) coNP-complete, if $\mathrm{Pol}(\Gamma)$ does not contain a 01-stable operation;
(2) in P, if Γ is preserved by $s_{a,2}$ and $g_{a,2}$ for some $a \in \{0,1\}$;
(3) in P, if Γ is preserved by $f_{a,2}$ for some $a \in \{0,1\}$;
(4) coNP-complete, otherwise.

Proof. Let r be the maximal arity of relations in Γ, let $\ell = \max(4,r)$. Note that if we add to Γ all the relations of arity at most ℓ that can be pp-defined from Γ, then we do not change the complexity of Π_{2k}- QCSP(Γ). Also, adding the equality relation to Γ does not change the complexity, since the equalities can be propagated out. Another important fact is that adding pp-definable relations and the equality relation to a constraint language does not affect the set of polymorphisms $\mathrm{Pol}(\Gamma)$. Thus, it is sufficient to prove the claim only for constraint languages Γ containing all such relations. Additionally, since $\mathrm{Pol}(\Gamma)$ is $\{0,2\}\{1,2\}$-projective, $\{0,2\}$, $\{1,2\}$, and $\{0,2\}^2 \cup \{1,2\}^2$ are from Γ. Therefore, Γ satisfies all the assumptions we formulated in the beginning of this section.

Let us note that Π_{2k}- QCSP(Γ) is in coNP by Lemma 5. The first case follows from Lemmas 6 and 21. The proof continues by branching on whether Γ is preserved by a 0-stable or 1-stable operation. If Γ is not preserved by a 0-stable or 1-stable operation, then by Lemma 22, Π_{2k}- QCSP(Γ) is coNP-complete.

Then we may assume that Γ is preserved by a 0-stable operation (the 1-stable case can be considered in the same way). If Γ is preserved by $f_{0,2}$, then by Corollary 4, QCSP(Γ) is in P. Similarly, if Γ is preserved by $s_{0,2}$ and $g_{0,2}$, then by Corollary 3, QCSP(Γ) is in P. Thus we assume that $f_{0,2}$ does not preserve Γ, and $s_{0,2}$ or $g_{0,2}$ does not preserve Γ.

By Theorem 17, Γ contains an AND-type relation δ. We can check that $\delta_1(x,y) = \delta(x,x,y)$ is equal to $\{(0,0),(1,1)\} \cup \{2\} \times \{0,1,2\}$ (to derive $(2,1)$ we apply a 0-stable operation to $(1,1)$ and $(2,0)$). If Γ is also preserved by a 1-stable operation, then by Theorem 18, Γ contains an OR-type relation, and by Lemma 7, Π_{2k}- QCSP(Γ) is coNP-complete.

Thus, we assume that Γ is not preserved by a 1-stable operation. By Lemma 20, Γ contains a relation $\delta_2 = \{0,1,2\} \times \{c\} \cup \{1,2\} \times \{2\}$ with c equal to 0 or 1. Define $\delta_3(x,y) = \exists x' \exists y' \left(\delta_2(x,x') \wedge \delta(x,x',y') \wedge \delta_1(y',y)\right)$. We can check that $(0,a) \in \delta_3$ implies $a = 0$. Applying s_2 to the tuples $(1,1,1)$, $(1,0,0)$, and to the tuples

$(1,1,1)$, $(0,0,0)$, we obtain that $(1,2,2),(2,2,2) \in \delta$. Then $(1,a),(2,a) \in \delta_3$ for every a (put $x' = y' = 2$ in the definition of δ_3), which means that $\delta_3 = \{(0,0)\} \cup \{1,2\} \times \{0,1,2\}$.

Assume that a 0-stable operation we have, is h_0. Since h_0 is $\{0,2\}\{1,2\}$-projective, it returns the first variable or 2. If $h_0(0,1) = 0$, then h_0 is also a 1-stable operation, which contradicts our assumptions. Thus $h_0(0,1) = 2$. If $h_0(0,2) = 0$, then we get a contradiction with the fact that h_0 preserves δ_3. Thus $h_0 = s_{0,2}$, which means that Γ is not preserved by $g_{0,2}$. By Lemma 24, Γ contains a relation $\delta_4 \in \{\{(0,1),(2,2)\},\ \{(0,1),(2,1),(2,2)\}\}$.

Define $\delta_5(x,y) = \exists z_1 \exists z_2 \left(\delta_3(x,z_1) \wedge \delta_4(z_1,z_2) \wedge \delta_1(z_2,y)\right)$ and check that $\delta_5 = \{(0,1)\} \cup \{1,2\} \times \{0,1,2\} \in \Gamma$. Applying Lemma 23 to δ_5, δ_3, and δ_1, we derive that the required relation $(x \neq 0) \vee (y = z)$ can be pp-defined from Γ. Then by Lemma 8, Π_{2k}- QCSP(Γ) is coNP-complete. □

6 Main result for a 3-element domain

We just need one more lemma and a definition before presenting the proof of Theorem 1. A k-ary operation f is a *projection* if $f(x_1, x_2, \ldots, x_k) = x_i$, for some $i = 1, \ldots, k$. It is known that if Γ is only preserved by projections, then QCSP(Γ) is PSPACE-complete [1].

Lemma 26. Suppose Γ is a finite constraint language on $\{0,1,2\}$ with constants, such that Pol(Γ) has the EGP property and has no WNU. Then Π_{2k}- QCSP(Γ) is $\mathbf{\Pi}^{\mathsf{P}}_{2k}$-complete.

Proof. Since Pol(Γ) has the EGP property, every operation of Pol(Γ) is $\alpha\beta$-projective for some $\alpha, \beta \subsetneq \{0,1,2\}$, and $\alpha \cup \beta = \{0,1,2\}$. If $\alpha \cap \beta = \varnothing$, then $(\alpha \times \alpha) \cup (\beta \times \beta)$ is an equivalence relation preserved by Pol(Γ) and any operation of Pol(Γ) modulo this congruence is a projection.

Let Δ be the homomorphic image of Γ born of the quotient by this congruence (equivalence relation). Let ε be a pp-definition over Γ of the congruence. We reduce from Π_{2k}-QCSP(Δ) to Π_{2k}-QCSP(Γ). Let φ be an instance of the former. We build φ', an instance of the latter, in the following fashion. We reduce instances of $\exists x$ to $\exists x, x'$ while substituting all instances of x by x' and adding the pp-definition of $\varepsilon(x,x')$ to the quantifier-free part. Then we reduce instances of $\forall x$ to $\forall x \cdots \exists x'$ (where there are only two quantified variables and the existential quantifier is brought to the innermost block), while substituting all instances of x by x' and adding the pp-definition of $\varepsilon(x,x')$ to the quantifier-free part. To argue that the reduction is correct, it is enough to observe that nothing is lost by the moving of the existential quantification of x' innermost, since the equivalence class of x' is set by the choice of x, and it is the equivalence class that is significant in Γ for the reduction from Δ.

Otherwise, w.l.o.g. we assume that $\alpha = \{0,2\}$ and $\beta = \{1,2\}$. Since there does not exist a WNU operation, it is known from [2; 9] that there should be a factor of size at least two whose operations are all projections. We consider two cases.

First suppose that there exists a congruence σ such that Pol(Γ)/σ has only projections. This case we have already covered earlier in this proof since we could have chosen α and β disjoint.

Thus, we may assume that there exists a subset $B \subsetneq \{0,1,2\}$ of size two such that all operations of Pol(Γ) on B are projections and B is a subalgebra of Pol(Γ). Hereon, in this proof, we need to depart from the line of argument in [14], as those results would not turn out strong enough for what we need.

Assume that $B = \{0,2\}$. Since all polymorphisms on B are projections, $\delta = \{(0,2),(2,0)\}$ is pp-definable. Let $\sigma_1(x,y)$ be the relation $(\alpha \times \alpha) \cup (\beta \times \beta)$, and $\sigma_2(x,y,u,v)$ be the relation $\sigma_1(x,y) \vee \sigma_1(u,v)$. These relations play a key role in the conservative case as discussed in [14] and are pp-definable because Pol(Γ) has the EGP (see [12]). Now, $S_1(x,y) \coloneqq \exists y' \left(\sigma_2(x,1,y',1) \wedge \delta(y',y)\right)$ pp-defines $\{0,1,2\}^2 \backslash \{(0,2)\}$, and $S_2(x,y) \coloneqq \sigma_2(x,0,y,1) \wedge (y \in B)$ pp-defines all possibilities with $y \in B$ except $(1,0)$. Thus, $T = S_1 \cap S_2$ pp-defines $\{(0,0),(1,2),(2,0),(2,2)\}$. Let $\gamma = \{0,2\}^3 \backslash \{(0,0,0),(2,2,2)\}$ and note that this is pp-definable, as

all polymorphisms on B are projections. We can reduce from QNAE3SAT in the following fashion. From a Π_{2k} instance

$$\varphi = \forall x_1^1, \dots, x_1^{r_1} \exists y_1^1, \dots, y_1^{r_1} \dots \forall x_k^1, \dots, x_k^{r_k} \exists y_k^1, \dots, y_k^{r_k} \left(NAE_3(z_1^1, z_1^2, z_1^3) \wedge \dots \wedge NAE_3(z_k^1, z_k^2, z_k^3) \right)$$

of QNAE3SAT, where $z_1^1, z_1^2, z_1^3, \dots, z_k^1, z_k^2, z_k^3$ are of the form x_i^j or y_i^j, we build an instance φ' of Π_{2k}- QCSP(Γ) as follows

$$\begin{aligned}\varphi' = \forall x_1^1, \dots, x_1^{r_1} \exists y_1^1, \dots, y_1^{r_1} \dots \forall x_k^1, \dots, x_k^{r_k} \exists y_k^1, \dots, y_k^{r_k} \\ \exists x_1'^1, \dots, x_1'^{r_1} \dots \exists x_k'^1, \dots, x_k'^{r_k} \Big(\gamma(z_1^{1'}, z_1^{2'}, z_1^{3'}) \wedge \dots \wedge \gamma(z_k^{1'}, z_k^{2'}, z_k^{3'}) \\ \wedge T(x_1^1, x_1'^1) \wedge \dots \wedge T(x_1^{r_1}, x_1'^{r_1}) \\ \wedge \dots \wedge T(x_k^1, x_k'^1) \wedge \dots \wedge T(x_k^{r_k}, x_k'^{r_k}) \Big)\end{aligned}$$

where $z_i^{j'}$ is y_i^j if z_j^i was y_i^j, and $z_i^{j'}$ is $x_i'^j$ if z_i^j was x_i^j. Let us argue that φ is a yes-instance of QNAE3SAT iff φ' is a yes-instance of Π_{2k}- QCSP(Γ).

($\varphi \to \varphi'$). The values 0 and 2 play the role in φ' that false (0) and true (1) play in φ. When Universal plays x_i^j to $0, 1, 2$ in φ', then $x_i'^j$ becomes $0, 2, \{0, 2\}$, respectively, in φ'. Thus, we consider that Universal played x_i^j to $0, 1, \{0, 1\}$, respectively, in φ (the last doesn't matter). Existential then wins according to her winning strategy on φ since γ encodes $\{0, 2\}^3 \backslash \{(0, 0, 0), (2, 2, 2)\}$ on $\{0, 2\}$.

($\varphi' \to \varphi$). Note that a Universal play of 0 to x_i^j forces $x_i'^j$ to be 0 and a play of 1 to x_i^j forces $x_i'^j$ to be 2 (furthermore, we can always make $x_i'^j \in \{0, 2\}$). Thus, when Universal plays x_i to 0 and 1 in φ, then we consider that Universal played x_i^j to 0 and 1, respectively, in φ'. Existential then wins according to her winning strategy on φ' reading 0 as false (0) and 2 as true (1).

If $B = \{1, 2\}$, then we are in a symmetric case to the last. If $B = \{0, 1\}$, then we can complete the argument in a similar fashion, with σ_1 in place of T (this case is somewhat simpler). □

Now, we are ready to prove Theorem 1.

Theorem 1. Suppose Γ is a finite constraint language on $\{0, 1, 2\}$ with constants. Then Π_{2k}- QCSP(Γ) is

(1) in P if Pol(Γ) has the PGP property and has a WNU operation;
(2) NP-complete if Pol(Γ) has the PGP property and has no WNU operation;
(3) $\mathbf{\Pi}_{2k}^{\mathsf{P}}$-complete if Pol($\Gamma$) has the EGP property and has no WNU operation;
(4) coNP-complete if Pol(Γ) has the EGP property and Pol(Γ) does not contain an ab-stable operation;
(5) in P if Pol(Γ) contains $s_{a,c}$ and $g_{a,c}$ for some $a, c \in \{0, 1, 2\}$, $a \neq c$;
(6) in P if Pol(Γ) contains $f_{a,c}$ for some $a, c \in \{0, 1, 2\}$, $a \neq c$;
(7) coNP-complete otherwise.

Proof. Suppose Pol(Γ) has the PGP property. We know from Theorem 2 that QCSP(Γ) can be reduced to a polynomial collection of instances of CSP(Γ). If Γ additionally has a WNU, then it follows from [3; 11] that CSP(Γ), and therefore also QCSP(Γ), is in P. If Pol(Γ) does not have a WNU operation, then NP-hardness follows from the identity reduction from CSP(Γ), whose NP-hardness has long been known (e.g. see again [3; 11]).

Suppose Pol(Γ) has the EGP property. If Pol(Γ) has no WNU operation, then by Lemma 26 Π_{2k}- QCSP(Γ) is $\mathbf{\Pi}_{2k}^{\mathsf{P}}$-complete.

Suppose Pol(Γ) contains a WNU operation. Since Pol(Γ) has the EGP property, all operations of Pol(Γ) are $\alpha\beta$-projective for some α and β. If $\alpha \cap \beta = \varnothing$, then Pol($\Gamma$) cannot have a WNU operation, which contradicts our assumption.

Suppose $\alpha \cap \beta \neq \varnothing$. If Pol($\Gamma$) contains an ab-stable operation f, that is $f(x,a,b) = x$ and $f(x,c,c) = c$ for $\{a,b,c\} = \{0,1,2\}$, then, since f is $\alpha\beta$-projective, we have $c \in \alpha \cap \beta$. Similarly, if Pol(Γ) contains $s_{a,c}, g_{a,c}$, or $f_{a,c}$, then $c \in \alpha \cap \beta$. Since both variables of the operation $g(x,y) = w(x,\dots,x,y)$ should be $\alpha\beta$-projective, g is the semilattice operation s_c. W.l.o.g. we assume that $c = 2$, whereupon the remaining part of the classification follows from Theorem 25. □

7 The remaining cases

The following has the same proof as Theorem 1, bearing in mind that Σ_{2k+1}-NAE3SAT is $\mathbf{\Sigma}^{\mathsf{P}}_{2k+1}$-complete.

Theorem 27. Suppose Γ is a finite constraint language on $\{0,1,2\}$ with constants. Then Σ_{2k+1}-QCSP(Γ) is

(1) in P if Pol(Γ) has the PGP property and has a WNU operation;
(2) NP-complete if Pol(Γ) has the PGP property and has no WNU operation;
(3) $\mathbf{\Sigma}^{\mathsf{P}}_{2k+1}$-complete if Pol($\Gamma$) has the EGP property and has no WNU operation;
(4) in P if Pol(Γ) contains $s_{a,c}$ and $g_{a,c}$ for some $a, c \in \{0,1,2\}$, $a \neq c$;
(5) in P if Pol(Γ) contains $f_{a,c}$ for some $a, c \in \{0,1,2\}$, $a \neq c$;
(6) coNP-complete otherwise.

Note that in each of the last three cases Pol(Γ) contains s_c for some c.

Now, it is easy to see that, for any finite Γ, Π_{2k+1}-QCSP(Γ) is in $\mathbf{\Pi}^{\mathsf{P}}_{2k}$ and Σ_{2k+2}-QCSP(Γ) is in $\mathbf{\Sigma}^{\mathsf{P}}_{2k+1}$. This is because the innermost universal quantification commutes with the atoms of the quantifier-free part, which is itself conjunctive. It follows, for $k \geq 1$, that the classification for Π_{2k+1}-QCSP(Γ) may be read from that for Π_{2k}-QCSP(Γ) and the classification for Σ_{2k+2}-QCSP(Γ) may be read from that for Σ_{2k+1}-QCSP(Γ).

Finally, there are the cases in which the innermost block of quantifiers is universal.

Theorem 28. Π_{2k+1}-QCSP(Γ) is polynomial time equivalent with Π_{2k}-QCSP(Γ) and Σ_{2k+2}-QCSP(Γ) is polynomial time equivalent with Σ_{2k+1}-QCSP(Γ).

Proof. The non-trivial forward direction proceeds by the fact that the innermost universal quantification commutes with conjunction. Then a (partially) universally quantified atom can be replaced by a conjunction of atoms with constants (and here we use that constants are always available). For example, if we have a (partially) universally quantified atom $\forall x R(x,y,z)$, then we would replace this with the conjunction $R(0,y,z) \wedge R(1,y,z) \wedge R(2,y,z)$. Since the signature is finite, this reduction is guaranteed to be polynomial and the result follows. □

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