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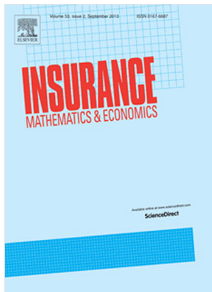

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Ruin Probabilities Under Capital Constraints

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Abstract

In this paper, we generalise the classic compound Poisson risk model, by the introduction of ordered capital levels, to model the solvency of an insurance firm. A breach of the higher capital level, the magnitude of which does not cause further breaches of either the lower level or the so-called intermediate confidence level (of the shareholders), requires a capital injection to restore the surplus to a solvent position. On the other hand, if the confidence level is breached capital injections are no longer a viable method of recapitalisation. Instead, the company can borrow money from a third party, subject to a constant interest rate which is paid back until the surplus returns to the confidence level and subsequently can be restored to a fully solvent position by a capital injection. If at any point the surplus breaches the lower capital level, the company is considered 'insolvent' and is forced to cease trading. For the aforementioned risk model, we derive an explicit expression for the 'probability of insolvency' in terms of the ruin quantities of the classical risk model. Under the assumption of exponentially distributed claim sizes, we show that the probability of insolvency is in fact directly proportional to the classical ruin function. It is shown that this result also holds for the asymptotic behaviour of the insolvency probability, with a general claim size distribution. Explicit expressions are also derived for the moment generating function of the accumulated capital injections up to the time of insolvency and finally, in order to better capture the reality, dividend payments to the companies shareholders are considered, along with the capital constraint levels, and explicit expressions for the probability of insolvency, under this modification, are obtained.

Keywords: Insolvency Probabilities, Capital Injections, Debit Interest, Accumulated Capital, Classical Risk Model.

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1 Introduction

In recent risk theory literature, more and more attention is being paid to risk models with recovery techniques for a surplus process in red. Two of the most prevalent techniques that have been proposed are *debit interest* (lending) and *capital injections*.

Within the the framework of debit interest risk models, it is assumed that an insurance firm does not cease to operate when traditional ruin occurs, i.e the surplus drops below zero for the first time. Instead, the insurer can borrow money at a constant interest rate and then repay the debts continuously from its premium income. The absolute ruin probability, in a debit interest setting, was first introduced by Gerber (1971) for the compound Poisson risk model, see also Dickson and Dos Reis (1997). Following its inception in risk theory, the debit interest recovery technique has been applied within various different risk models, see Dassios and Embrechts (1989) and Embrechts and Schmidli (1994) for piecewise deterministic Markov risk processes, Gerber and Yang (2007) for a jump-diffusion risk process, Yin and Wang (2010) for a perturbed compound Poisson risk process with investment, Zhang et. al. (2011) for a Markov Arrival risk model, Cai (2007) for the Gerber-Shiu function in the compound Poisson model, and references therein.

On the other hand, a more realistic alternative to restore capital is by means of capital injections. Capital injections were first introduced in the risk theory context, by Pafumi (1998) and since then, the ruin probability and other ruin related quantities, such as the distribution of the deficit at ruin or the distribution of the surplus prior to ruin, have been extensively studied for the compound Poisson risk model by many authors, see among others, Nie et al. (2011), Eisenberg and Schmidli (2011), Dickson and Qazvini (2016) and the references therein.

In this paper, we aim to derive explicit expressions for the insolvency probabilities (defined in Section 2), in a risk model that consists of capital levels and a confidence level for the shareholders. In more detail, we show that the insolvency probability, under the aforementioned risk model, can be evaluated in terms of the ruin probability of the classical risk model, for which powerful methodologies, numerical techniques and many applicable results have been derived over the last half century. Additionally, we derive the distribution of the accumulated capital injections up to the time of insolvency.

The risk process we employ consists of the following characteristics:

a) We consider a compound Poisson risk process for which two (positive) capital levels are introduced, namely the upper level c_u and lower level c_l ($\leq c_u$), to model the solvency requirements of an insurance firm. We assume that the insurance firm starts from a solvent position which exceeds c_u. If the level c_u is crossed, due to a claim, the insurance firm is able to recover the capital by means of capital injections (given the level c_l has not been crossed), which are assumed to be provided by the shareholders or transferred from a different line of business .

b) Additionally, we determine an intermediate capital level (between the c_u and c_l), which indicates the confidence level of the shareholders. If the aforementioned intermediate confidence level is crossed, then the shareholders lose confidence and are not prepared to inject the necessary capital to restore the surplus level to c_u. In this case, the company must borrow the funds from a third party, subject to debit interest, which is repaid continuously from the premium income until the confidence level is reached, the shareholders regain confidence and inject the remaining capital to bring the company back to a solvent position. The repayment of the debt, subject to constant debit interest, can be equivalently considered as the continuous payment of a penalty, which is issued to the company for being in an 'insolvent position'.

c) Finally, a breach of the c_l capital level means that the firm is considered as completely insolvent and thus the regulator's strongest actions are enforced (trading ceases).

The reason that capital injections are chosen as the initial recovery mechanism (versus debit borrowing) is confirmed both intuitively and from market evidence. On an intuitive level, insurance firms first look for internal methods of covering capital losses and secondly for external loans, since in general external loans are considered as liabilities for insurance firms). On the other hand, in practise, there is evidence of capital injections being implemented so as to meet the solvency levels required under Solvency II regulations [see for example, among others, the report of the ING group insurance in the Netherlands [17], the case of Liberty Insurance in Ireland, [1], or MOODY'S report of April 2016 [20]].

The paper is organised as follows: In Section 2, we introduce the risk model, with the above characteristics, in terms of the surplus process of an insurance firm. A graphical interpretation of the model is given and the probability of insolvency is defined and explained. In Section 3, we derive an explicit expression of the probability of insolvency. This explicit expression is given in terms of the classical ruin probability, shifted by the level c_u, and the probability of hitting the intermediate confidence level before hitting c_l, in the debit environment. Moreover, the latter 'hitting probability' is analysed and an integro-differential equation is obtained. In the same section, under exponentially distributed claim amounts, we show that the insolvency probability is proportional to the classical ruin probability. Finally, in this section, the asymptotic behaviour for the probability of insolvency is investigated. In Section 4, we derive explicit expressions for the expected accumulated capital injections up to the time of insolvency. In addition, we show that the distribution of the accumulated capital injections is a mixture of a degenerative distribution at zero and a pure continuous distribution, which is explicitly determined. In Section 5, we include a constant dividend barrier strategy where the shareholders can obtain part of the surplus as dividends. Under this modification, we again show that the probability of insolvency is given in terms of the ruin probability of the classical risk model under the same dividend barrier strategy.

2 The risk model

In this section, we will adapt the classical risk model to conform with the framework in **(a)-(c)** described in Section 1. Under this modification, we define the 'probability of insolvency', which corresponds to the probability that the risk process down-crosses the lower level $\mathcal{C}_l$.

In the classical Cramér-Lundberg risk model, the surplus process of an insurance company is defined by $U(t) = u + ct - S(t)$, $t \geqslant 0$, where $u \geqslant 0$ is the insurer's initial capital, $c > 0$ is a constant premium rate, $S(t) = \sum_{i=1}^{N(t)} X_i$ are the aggregate claims with $\{N(t)\}_{t \geqslant 0}$ a Poisson process representing the number of claims that have arrived up to time $t \geqslant 0$, with intensity $\lambda > 0$, and $\{X_k : k \in \mathbb{Z}_+\}$ is a sequence of independent and identically distributed (i.i.d.) random variables, representing the claim sizes, with a common distribution function $F_X(\cdot)$, density function $f_X(\cdot)$, and mean $\mathbb{E}(X) = \mu < \infty$. It is further assumed that $\{N(t)\}_{t \geqslant 0}$ and $\{X_k : k \in \mathbb{Z}_+\}$ are mutually independent.

We assume that if the surplus falls below the $\mathcal{C}_u$ level, due to the occurrence of a claim, then the shareholders in the company inject capital instantaneously to cover this fall, given that the capital level $\mathcal{C}_l$ has not been crossed. The sum of total capital injections, up to time $t \geqslant 0$, is defined by the pure jump process $\{Z(t)\}_{t \geqslant 0}$.

Moreover, the intermediate confidence level up to which the shareholders are prepared to inject capital is denoted by $\mathcal{B}$, where $\mathcal{C}_u \geqslant \mathcal{B} \geqslant \mathcal{C}_l$. A drop, due to a claim, of the surplus below the confidence level $\mathcal{B}$, requires that the insurance firm borrows an amount of money equal to the size of the deficit below $\mathcal{B}$ at a debit force $\delta > 0$, given that the capital level $\mathcal{C}_u$ has not been crossed.

When the surplus is between the levels $\mathcal{C}_l$ and $\mathcal{B}$, debts (or the penalty for the insurance firm) are repaid continuously from the premium income. During this period of time, the insurance firm can either recover back to level $\mathcal{B}$ (where the shareholders have renewed confidence and will instantaneously inject the amount $\mathcal{C}_u - \mathcal{B}$ in order to restore the surplus to level $\mathcal{C}_u$) or becomes insolvent by falling, due to further claims, below the level $\mathcal{C}_l$ [see Fig: 1]. Note that, using similar arguments as in Cai (2007) one can see that the confidence level, $\mathcal{B}$, depends on the debit force of interest (or penalty rate) and lies in the interval $[\mathcal{C}_l, \mathcal{C}_l + \frac{c}{\delta}]$. In order to emphasise the effects of the debit or penalised environment, in the remainder of this paper we consider the case $\mathcal{B} = \mathcal{C}_l + c/\delta$.

Considering the above features, the surplus process under with capital constraints, denoted by $\{U_\delta^Z(t)\}_{t \geqslant 0}$, has dynamics of the following form

$$dU_\delta^Z(t) = \begin{cases} cdt - dS(t), & U_\delta^Z(t) \geqslant \mathcal{C}_u, \\ \Delta Z(t), & \mathcal{B} \leqslant U_\delta^Z(t) < \mathcal{C}_u, \\ \left[c + \delta(U_\delta^Z(t) - b)\right] dt - dS(t), & \mathcal{C}_l < U_\delta^Z(t) < \mathcal{B}, \end{cases} \tag{2.1}$$

where $\Delta Z(t) = Z(t) - Z(t-)$.

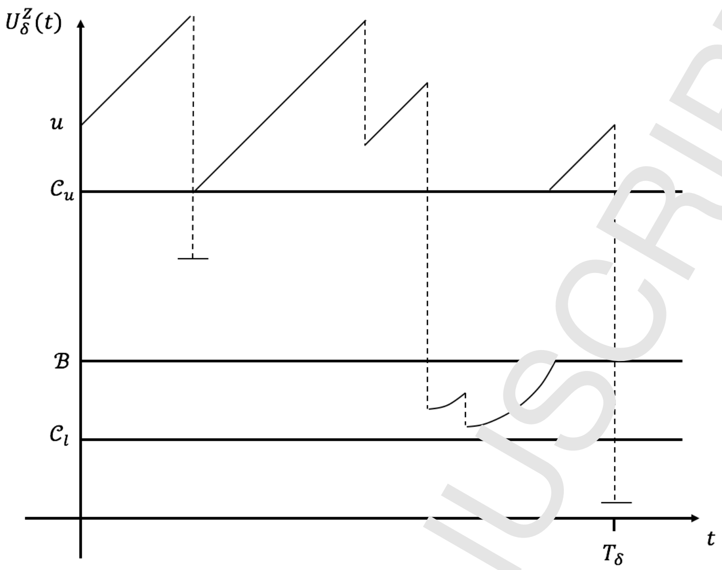

Figure 1: Typical sample path of the surplus process under capital constraints.

The crucial features of the proposed risk model could be interpret as the capital management tools employed to reduce the probability of insolvency. Thus, it follows that for the surplus process $\{U_\delta^Z(t)\}_{t\geqslant 0}$, we should define the time to insolvency, denoted by T_δ, as

$$T_\delta = \inf\left\{t \geqslant 0 : U_\delta^Z(t) \leqslant \mathcal{C}_l \,|U_\delta^Z(0) = u\right\},$$

with $T_\delta = \infty$ if $U_\delta^Z(t) > \mathcal{C}_l$ for all $t \geqslant 0$. Then, the *probability of insolvency*, which we denote $\psi_I(u)$, is given by

$$\psi_I(u) = \mathbb{P}\left(T_\delta < \infty \middle| U_\delta^Z(0) = u\right),$$

with $\psi_I(u) = 1$ for $u \leqslant \mathcal{C}_l$ and $\phi_I(u) = 1 - \psi_I(u)$ is the corresponding probability that the insurance firm never experiences insolvency. As it will be seen in the subsequent subsection, $\psi_I(u)$ will be derived in terms of the ruin probability for the classical risk model, namely $\psi(u)$.

Finally, we point out, similar to Cai (2007), that $\psi_I(u)$ has different sample paths for $u \geqslant \mathcal{C}_u$ and $\mathcal{C}_l < u < \mathcal{B}$. Therefore, we distinguish between the two situations by denoting $\psi_I(u) = \psi_I^+(u)$ for $u \geqslant \mathcal{C}_u$ and $\psi_I(u) = \psi_I^-(u)$ for $\mathcal{C}_l < u < \mathcal{B}$. Now, due to the instantaneous capital injection when the surplus lies within the interval $[\mathcal{B}, \mathcal{C}_u)$ we say that for $\mathcal{B} \leqslant u < \mathcal{C}_u$, $\psi_I(u) = \psi_I^+(k)$. It follows that the corresponding *solvency probabilities* are given by $\phi_I(u) = 1 - \psi_I(u) = \phi_I^+(u)$, for $u \geqslant \mathcal{C}_u$, and $\phi_I(u) = \phi_I^-(u)$ for $\mathcal{C}_l < u < \mathcal{B}$. Finally, we assume the net profit condition holds, i.e.

$$\eta = (c/\lambda\mu) - 1 > 0.$$

3 The probability of insolvency

In this section, we derive a closed form expression for the probability of insolvency in terms of the infinite-time ruin probability of the classical risk model and an exiting (hitting) probability between two capital levels. Note that $\psi_I^+(u)$, is the risk quantity of primary interest as it is assumed that the insurance firm starts from a solvent level i.e. $u \geqslant \mathcal{C}_u$. Ultimately, we show that the probability of insolvency is proportional to the classical ruin function. Corresponding formulae for $\psi_I^-(u)$, $\mathcal{C}_l < u < \mathcal{B}$, are also derived.

Before we proceed, we first define some ruin related quantities that will be extensively used in the following. First, let the time to cross the level $\mathcal{C}_u$, for $u \geqslant \mathcal{C}_u$, be denoted by T, such that

$$T = \inf\{t \geqslant 0 : U_\delta^Z(t) < \mathcal{C}_u | \, U_\delta^Z(0) = u \geqslant \mathcal{C}_u\}, \tag{3.1}$$

with the corresponding probability of down-crossing the level $\mathcal{C}_u$, defined by

$$\xi(u) = \mathbb{P}\left(T < \infty \middle| U_\delta^Z(0) = u \geqslant \mathcal{C}_u\right).$$

Recalling the behaviour of the surplus process $U_\delta^Z(t)$, given in equation (2.1), it is clear that the dynamics above the level $\mathcal{C}_u$ are identical to that of the classical surplus process under a constraint free environment, i.e. for $u \geqslant \mathcal{C}_u$ we have $dU_\delta^Z(t) \equiv d\widetilde{U}(t)$ where

$$\widetilde{U}(t) = \tilde{u} + ct - S(t), \quad t \geqslant 0,$$

with $\widetilde{U}(0) = \tilde{u} := u - \mathcal{C}_u$. Then, it should be clear that T, defined by equation (3.1), is equivalent to the time to ruin in the classical risk model with no capital constraints and initial capital $\tilde{u} \geqslant 0$, given by

$$T = \inf\{t \geqslant 0 : \widetilde{U}(t) < 0 | \, \widetilde{U}(0) = \tilde{u}\}.$$

Hence, the function $\xi(u)$ is identical to the classic ruin probability $\psi(\tilde{u}) = \mathbb{P}(T < \infty | \widetilde{U}(0) = \tilde{u}) = 1 - \phi(\tilde{u})$.

Extending the arguments of Nie et al. (2011), by conditioning on the occurrence and size of the first drop below $\mathcal{C}_u$, for $u \geqslant \mathcal{C}_u$, and using the fact that $dU_\delta^Z(t) \equiv d\widetilde{U}(t)$ above the level $\mathcal{C}_u$, we obtain an expression for the solvency probability, $\phi_I^+(u)$, of the form

$$\begin{aligned}
\phi_I^+(u) &= \phi(\tilde{u}) + \int_0^{\mathcal{C}_u - \mathcal{B}} g(\tilde{u}, y)\phi_I^+(\mathcal{C}_u)\, dy + \int_{\mathcal{C}_u - \mathcal{B}}^{\mathcal{C}_u - \mathcal{C}_l} g(\tilde{u}, y)\phi_I^-(\mathcal{C}_u - y)\, dy \\
&= \phi(\tilde{u}) + G(\tilde{u}, \mathcal{C}_u - \mathcal{B})\phi_I^+(\mathcal{C}_u) + \int_{\mathcal{C}_u - \mathcal{B}}^{\mathcal{C}_u - \mathcal{C}_l} g(\tilde{u}, y)\phi_I^-(\mathcal{C}_u - y)\, dy,
\end{aligned} \tag{3.2}$$

where

$$G(\tilde{u}, y) = \mathbb{P}\left(T < \infty, |\widetilde{U}(T)| \leqslant y \middle| \, \widetilde{U}(0) = \tilde{u}\right),$$

is the joint distribution of down-crossing the level $\mathcal{C}_u$ and experiencing a deficit (below $\mathcal{C}_u$) of at most y, with $g(\tilde{u}, y) = \frac{\partial}{\partial y} G(\tilde{u}, y)$ the corresponding density function. This risk quantity was first introduced and analysed by Gerber et al. (1987) for modelling the 'deficit at ruin'.

Note that, in the above expression, $\phi_I^+(u)$ is given in terms of $\phi_I^-(u)$. In order to derive an analytic expression for $\phi_I^+(u)$, independent of $\phi_I^-(u)$, we introduce the following hitting probability.

Let $\chi_\delta(u, \mathcal{C}_u, \mathcal{C}_l) \equiv \chi_\delta(u)$ be the probability that the surplus process hits the upper confidence level $\mathcal{B}$, before hitting the lower capital level $\mathcal{C}_l$ from initial capital $\mathcal{C}_l < u < \mathcal{B}$, defined by

$$\chi_\delta(u) = \mathbb{P}\left(T^{\mathcal{B}} < T_\delta \middle| U_\delta^Z(0) = u\right), \tag{3.3}$$

where

$$T^{\mathcal{B}} = \inf\left\{t \geqslant 0 : U_\delta^Z(t) = \mathcal{B} \middle| U_\delta^Z(0) = u\right\}, \quad \mathcal{C}_l < u < \mathcal{B}.$$

Proposition 1. *For $\mathcal{C}_l < u < \mathcal{B}$, the surplus process $\{U_\delta^Z(t)\}_{t\geqslant 0}$, will hit either the capital level $\mathcal{C}_l$ or the confidence level $\mathcal{B}$, over an infinite-time horizon, almost surely (a.s.).*

Proof. Using similar arguments as in Cai (2007), we first note that when the surplus process is within the interval $(\mathcal{C}_l, \mathcal{B})$, it is driven by the debit interest force $\delta > 0$, until the surplus returns to level $\mathcal{B}$ (or experiences insolvency). Therefore, for initial capital $\mathcal{C}_l < u < \mathcal{B}$, the process is immediately subject to debit interest on the amount $\mathcal{B} - u > 0$ and the evolution of the surplus process (assuming no claims appear up to time $t \geqslant 0$), due to the dynamics of the process below the level $\mathcal{B}$, can be expressed by

$$h(t; u, \mathcal{B}) = \mathcal{B} + (u - \mathcal{B})e^{\delta t} + c\int_0^t e^{\delta s}\, ds, \qquad t \geqslant 0. \tag{3.4}$$

Let us further define $t_0 \equiv t_0(u, \mathcal{B})$ to be the solution to $h(t; u, \mathcal{B}) = \mathcal{B}$. Then

$$t_0 = \ln\left(\frac{c}{\delta(u - \mathcal{B}) + c}\right)^{1/\delta} < \infty, \quad \text{for} \quad \mathcal{C}_l < u < \mathcal{B}, \tag{3.5}$$

is the time taken for the surplus to reach the upper level $\mathcal{B}$, i.e. $h(t_0; u, \mathcal{B}) = \mathcal{B}$, in the absence of claims and $h(t; u, \mathcal{B}) \in (u, \mathcal{B})$ for all $t < t_0$. Therefore, it is clear that the surplus process will recover to the upper level $\mathcal{B}$, if no claims occur before time $0 \leqslant t_0 < \infty$.

Now, consider the events $E_n = \{\tau_n > t_0\}$, where $\{\tau_n\}_{n\in\mathbb{N}}$ is a sequence of i.i.d. random variables denoting the inter-arrival time between the $(n-1)$-th and n-th claim and t_0 is as defined above. Then, since the inter-arrival times are i.i.d. and it is assumed that the claims occur according to a Poisson process, it follows that, for all $n \in \mathbb{N}$, the events E_n are independent and we have

$$\mathbb{P}(E_n) = \mathbb{P}(\tau_n > t_0) = e^{-\lambda t_0} > 0.$$

Therefore, it follows that

$$\sum_{n=1}^{\infty} \mathbb{P}(E_n) = \infty,$$

and thus, by the second Borel-Cantelli Lemma [see Feller (1971)], it follows that

$$\mathbb{P}\left(\limsup_{n\to\infty} \{\tau_n > t_0\}\right) = 1.$$

That is, the event $\{\tau_n > t_0\}$ occurs infinitely often with probability 1 (a.s.). □

Now, conditioning on which of the levels the surplus first hits, from initial capital $\mathcal{C}_l < u < \mathcal{B}$, using the result of Proposition 1 and noticing that $\phi_I^-(x) = 0$ for $x \leqslant \mathcal{C}_l$, it follows that

$$\phi_I^-(u) = \chi_\delta(u)\phi_I^+(\mathcal{C}_u). \tag{3.6}$$

Substituting the above expression into equation (3.2), we obtain

$$\phi_I^+(u) = \phi(\tilde{u}) + \phi_I^+(\mathcal{C}_u)\left[G(\mathcal{C}_l, \mathcal{C}_u - \mathcal{B}) + \int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g(\tilde{u}, y)\chi_\delta(\mathcal{C}_u - y)\, dy\right]. \tag{3.7}$$

To complete the above expression for $\phi_I^+(u)$, the boundary condition $\phi_I^+(\mathcal{C}_u)$ and the hitting probability $\chi_\delta(u)$ need to be determined. Setting $u = \mathcal{C}_u$ in equation (3.7), and solving the resulting equation for $\phi_I^+(\mathcal{C}_u)$, we have that

$$\phi_I^+(\mathcal{C}_u) = \frac{\phi(0)}{1 - \left(G(0, \mathcal{C}_u - \mathcal{B}) + \int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g(0, y)\chi_\delta(\mathcal{C}_u - y)\, dy\right)}, \tag{3.8}$$

and thus, equation (3.7) may be re-written, for $u \geqslant \mathcal{C}_u$, as

$$\phi_I^+(u) = \phi(\tilde{u}) + \frac{\phi(0)\left[G(\mathcal{C}_l, \mathcal{C}_u - \mathcal{B}) + \int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g(\tilde{u}, y)\chi_\delta(\mathcal{C}_u - y)\, dy\right]}{1 - \left(G(0, \mathcal{C}_u - \mathcal{B}) + \int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g(0, y)\chi_\delta(\mathcal{C}_u - y)\, dy\right)}.$$

Then, since $\phi_I^+(u) = 1 - \psi_I^+(u)$, for $u \geqslant \mathcal{C}_u$, the probability of insolvency, namely $\psi_I^+(u)$, is given by

$$\psi_I^+(u) = \psi(\tilde{u}) - \frac{\phi(0)\left[G(\mathcal{C}_l, \mathcal{C}_l - \mathcal{B}) + \int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g(\tilde{u}, y)\chi_\delta(\mathcal{C}_u - y)\, dy\right]}{1 - \left(G(0, \mathcal{C}_u - \mathcal{B}) + \int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g(0, y)\chi_\delta(\mathcal{C}_u - y)\, dy\right)}. \tag{3.9}$$

Moreover, from Dickson (2005), we have that the general form for the density of the deficit at ruin, with zero initial capital, is given by

$$g(0, y) = \frac{\lambda}{c}\overline{F}_X(y),$$

and thus, equation (3.9) reduces to

$$\psi_I^+(u) = \psi(\tilde{u}) - \frac{\phi(0)\left[G(\tilde{u}, \mathcal{C}_u - \mathcal{B}) + \int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g(\tilde{u}, y)\chi_\delta(\mathcal{C}_u - y)\,dy\right]}{1 - \frac{\lambda}{c}\left(\mu F_e(\mathcal{C}_u - \mathcal{B}) + \int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} \overline{F}_X(y)\chi_\delta(\mathcal{C}_u - y)\,dy\right)}, \tag{3.10}$$

where $\overline{F}_X(x) = 1 - F_X(x)$ and $F_e(x) = \frac{1}{\mu}\int_0^x \overline{F}_X(y)\,dy$ is the so-called equilibrium distribution.

Finally, by employing equation (3.10), combining equations (3.6) and (3.8) and defining $G_{\tilde{u}}(y) = G(\tilde{u}, y)/\psi(\tilde{u})$, with $g_{\tilde{u}}(y) = g(\tilde{u}, y)/\psi(\tilde{u})$, such that $G_{\tilde{u}}(y) = \mathbb{P}(|\widetilde{U}(T)| \leqslant y \big| T < \infty)$ is a proper distribution function, as in Willmot (2002) (and references therein), we get the following Theorem for the probability of insolvency. Note that similar arguments as above can be applied for $\psi_I^-(u)$.

Theorem 1. *For $u \geqslant \mathcal{C}_u$, the probability of insolvency, $\psi_I^+(u)$, is given by*

$$\psi_I^+(u) = \psi(\tilde{u})\left[1 - \frac{\phi(0)\left[G_{\tilde{u}}(\mathcal{C}_u - \mathcal{B}) + \int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g_{\tilde{u}}(y)\chi_\delta(\mathcal{C}_u - y)\,dy\right]}{1 - \frac{\lambda}{c}\left(\mu F_e(\mathcal{C}_u - \mathcal{B}) + \int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} \overline{F}_X(y)\chi_\delta(\mathcal{C}_u - y)\,dy\right)}\right], \tag{3.11}$$

where $\psi(u)$ is the ruin probability of the classical risk model and $\tilde{u} = u - \mathcal{C}_u$.

For $\mathcal{C}_l < u < \mathcal{B}$, $\psi_I^-(u)$ is given by

$$\psi_I^-(u) = 1 - \frac{\phi(0)\chi_\delta(u)}{1 - \frac{\lambda}{c}\left(\mu F_e(\mathcal{C}_u - \mathcal{B}) + \int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} \overline{F}_X(y)\chi_\delta(\mathcal{C}_u - y)\,dy\right)}. \tag{3.12}$$

Remark 1. *From equations (3.11) and (3.12), it follows that the two types of insolvency probabilities are given in terms of the (shifted) ruin probability and deficit of the classical risk model, as well as the probability of exiting between two levels. Thus, $\psi_I^+(\cdot)$ and $\psi_I^-(\cdot)$ can be calculated by employing the well known results, with respect to $G_{\tilde{u}}(\cdot)$ and $\psi(\cdot)$ (see for example Gerber et al. (1987), Dickson (2005), and the references therein), whilst the latter exiting probability, $\chi_\delta(u)$, can be calculated as follows.*

Proposition 2. *For $\mathcal{C}_l < u < \mathcal{B}$, the probability of the surplus process, $\{U_\delta^Z(t)\}_{t\geqslant 0}$, hitting the upper level $\mathcal{B}$ before hitting the lower level $\mathcal{C}_l$ (under a debit force $\delta > 0$), denoted $\chi_\delta(u)$, satisfies the following integro-differential equation*

$$(\delta(u - \mathcal{B}) + c)\chi_\delta'(u) = \lambda\chi_\delta(u) - \lambda\int_0^{u-\mathcal{C}_l} \chi_\delta(u - x)\,dF_X(x), \tag{3.13}$$

with boundary conditions

$$\lim_{u\uparrow\mathcal{B}} \chi_\delta(u) = 1,$$
$$\lim_{u\downarrow\mathcal{C}_l} \chi_\delta(u) = 0.$$

Proof. Using the notations introduced in the proof of Proposition 1, by conditioning on the time and amount of the first claim, it follows that

$$\chi_\delta(u) = e^{-\lambda t_0} + \int_0^{t_0} \lambda e^{-\lambda t} \int_0^{h(t;u,\mathcal{B})-\mathcal{C}_l} \chi_\delta\big(h(t;u,\mathcal{B}) - x\big)\, dF_X(x)\, dt. \tag{3.14}$$

Employing the change of variable $y = h(t;u,\mathcal{B})$ and recalling the form of t_0 given in equation (3.5), we have that

$$\begin{aligned} \chi_\delta(u) = \left(\frac{\delta(u-\mathcal{B})+c}{c}\right)^{\frac{\lambda}{\delta}} &+ \lambda\left(\delta(u-\mathcal{B})+c\right)^{\frac{\lambda}{\delta}} \int_u^{\mathcal{B}} \left(\delta(y-\mathcal{B})+c\right)^{-\frac{\lambda}{\delta}-1} \\ &\times \int_0^{y-\mathcal{C}_l} \chi_\delta(y-x)\, dF_X(x)\, dy. \end{aligned} \tag{3.15}$$

Differentiating the above equation, with respect to u and combining the resulting equation with equation (3.15), we obtain equation (3.13).

The first boundary condition can be found by letting $u \to \mathcal{B}$ in equation (3.15). Now, for the second boundary condition one can see that if

$$\lim_{u \downarrow \mathcal{C}_l} \int_u^{\mathcal{B}} \left[\big(\delta(y-\mathcal{B})+c\big)^{-\frac{\lambda}{\delta}-1} \int_0^{y-\mathcal{C}_l} \chi_\delta(y-x)\, dF(x)\right] dy < \infty,$$

then

$$\lim_{u \downarrow \mathcal{C}_l} \lambda\big(\delta(u-\mathcal{B})+c\big)^{\frac{\lambda}{\delta}} \int_u^{\mathcal{B}} \left[\big(\delta(y-\mathcal{B})+c\big)^{-\frac{\lambda}{\delta}-1} \int_0^{y-\mathcal{C}_l} \chi_\delta(y-x)\, dF(x)\right] dy = 0,$$

since $\mathcal{B} = \mathcal{C}_l + \frac{c}{\delta}$. Alternatively, if

$$\lim_{u \downarrow \mathcal{C}_l} \int_u^{\mathcal{B}} \left[\big(\delta(y-\mathcal{B})+c\big)^{-\frac{\lambda}{\delta}-1} \int_0^{y-\mathcal{C}_l} \chi_\delta(y-x)\, dF(x)\right] dy = \infty,$$

then, by L'Hopital's rule, we have

$$\lim_{u \downarrow \mathcal{C}_l} \lambda\big(\delta(u-\mathcal{B})+c\big)^{\frac{\lambda}{\delta}} \int_u^{\mathcal{B}} \left[\big(\delta(y-\mathcal{B})+c\big)^{-\frac{\lambda}{\delta}-1} \int_0^{y-\mathcal{C}_l} \chi_\delta(y-x)\, dF(x)\right] dy = 0.$$

Using the above limiting results and taking the limit $u \to \mathcal{C}_l$, in equation (3.15), we obtain the second boundary condition. □

Recalling Remark 1 and Theorem 1, the two types of insolvency probabilities depend heavily on the solution of the integro-differential equation (3.13), which is discussed in the next subsection.

3.1 Explicit expressions for exponential claim size distribution

In this subsection, we derive explicit expressions for the two types of insolvency probabilities, under the assumption of exponentially distributed claim amounts. Then, by comparing the explicit expression of the insolvency probabilities with the classical ruin probability under exponentially distributed claims, we identify that these two probabilities are proportional. To illustrate the applicability of our results (and thus the relationship between $\psi_I^+(u)$ and $\psi(u)$), we finally provide numerical results.

Let us assume the claim sizes are exponentially distributed with parameter $\beta > 0$, i.e. $F_X(x) = 1 - e^{-\beta x}$, $x \geqslant 0$. Then, equation (3.13) reduces to

$$(\delta(u - \mathcal{B}) + c)\chi_\delta'(u) = \lambda\chi_\delta(u) - \lambda \int_{\mathcal{C}_l}^{u} \beta e^{-\beta(u-x)}\chi_\delta(x)\, dx, \qquad \mathcal{C}_l < u < \mathcal{B}. \tag{3.16}$$

The above integro-differential equation can be solved as a boundary value problem, since from Proposition 2 the boundary conditions at $\mathcal{C}_l$ and $\mathcal{B}$ are given. Thus, differentiating the above equation with respect to u, yields a second order homogeneous ODE of the form

$$\chi_\delta''(u) + p(u)\chi_\delta'(u) = 0, \tag{3.17}$$

where

$$p(u) = \frac{\delta - \lambda + \beta[\delta(u - \mathcal{B}) + c]}{\delta(u - \mathcal{B}) + c} = \frac{\delta - \lambda}{\delta(u - \mathcal{B}) + c} + \beta. \tag{3.18}$$

Employing the general theory of differential equations, the above ODE has a general solution of the form

$$\chi_\delta'(u) = Ce^{-\int p(u)\, du},$$

where C is an arbitrary constant that needs to be determined. Recalling the form of $p(u)$, given in equation (3.18), the above solution reduces to

$$\chi_\delta'(u) = Ce^{-\beta u} \left(\delta(u - \mathcal{B}) + c\right)^{\frac{\lambda}{\delta} - 1}.$$

Integrating the above equation from $\mathcal{C}_l + \epsilon$ to u, for some small $\epsilon > 0$ and $\mathcal{C}_l < u < \mathcal{B}$, we have that

$$\chi_\delta(u) - \chi_\delta(\mathcal{C}_l + \epsilon) = C \int_{\mathcal{C}_l+\epsilon}^{u} e^{-\beta w} \left(\delta(w - \mathcal{B}) + c\right)^{\frac{\lambda}{\delta} - 1} dw.$$

Letting $\epsilon \to 0$ and using the second boundary condition of Proposition 2, the general solution of equation (3.17) is given by

$$\begin{aligned} \chi_\delta(u) &= C \int_{\mathcal{C}_l}^{u} e^{-\beta w} \left(\delta(w - \mathcal{B}) + c\right)^{\frac{\lambda}{\delta} - 1} dw \\ &= Cc^{\frac{\lambda}{\delta} - 1} \int_{\mathcal{C}_l}^{u} e^{-\beta w} \left(\frac{\delta(w - \mathcal{B})}{c} + 1\right)^{\frac{\lambda}{\delta} - 1} dw. \end{aligned} \tag{3.19}$$

Finally, in order to complete the solution we need to determine the constant C, which can be obtained by using the first boundary condition for $\chi_\delta(u)$ of Proposition 2 i.e. $\lim_{u\to\mathcal{B}} \chi_\delta(u) = 1$. Letting $u \to \mathcal{B}$ in equation (3.19), we obtain

$$C^{-1} = c^{\frac{\lambda}{\delta}-1} \int_{\mathcal{C}_l}^{\mathcal{B}} e^{-\beta w} \left(\frac{\delta(w-\mathcal{B})}{c} + 1\right)^{\frac{\lambda}{\delta}-1} dw$$
$$= c^{\frac{\lambda}{\delta}-1} C_1^{-1},$$

where $C_1^{-1} = \int_{\mathcal{C}_l}^{\mathcal{B}} e^{-\beta w} \left(\frac{\delta(w-\mathcal{B})}{c} + 1\right)^{\frac{\lambda}{\delta}-1} dw$.

Proposition 3. *For $\mathcal{C}_l < u < \mathcal{B}$ and exponentially distributed claim amounts with parameter $\beta > 0$, the probability of the surplus process $\{U_\delta^Z(t)\}_{t\geq 0}$ hitting the upper level $\mathcal{B}$, before hitting the lower level $\mathcal{C}_l$, under a debit force $\delta > 0$, is given by*

$$\chi_\delta(u) = C_1 \int_{\mathcal{C}_l}^{u} e^{-\beta w} \left(\frac{\delta(w-\mathcal{B})}{c} + 1\right)^{\frac{\lambda}{\delta}-1} dw, \tag{3.20}$$

where

$$C_1^{-1} = \int_{\mathcal{C}_l}^{\mathcal{B}} e^{-\beta w} \left(\frac{\delta(w-\mathcal{B})}{c} + 1\right)^{\frac{\lambda}{\delta}-1} dw. \tag{3.21}$$

Using Theorem 1 and Proposition 3, the two types of insolvency probabilities, namely $\psi_I^+(u)$ and $\psi_I^-(u)$, under exponentially distributed claim amounts, are given in the following Theorem.

Theorem 2. *Let the claim amounts be exponentially distributed with parameter $\beta > 0$. Then, for $u \geqslant \mathcal{C}_u$, the probability of insolvency, $\psi_I^+(u)$, is given by*

$$\psi_I^+(u) = \frac{(1+\eta)e^{\frac{\lambda\eta}{c}\mathcal{C}_u}}{1 + \frac{\lambda\eta}{c} C_1^{-1} e^{\beta\mathcal{C}_u}} \psi(u), \tag{3.22}$$

and, for $\mathcal{C}_l < u < \mathcal{B}$, $\psi_I^-(u)$ is given by

$$\psi_I^-(u) = 1 - \frac{\frac{\lambda\eta}{c} e^{\beta\mathcal{C}_u} \int_{\mathcal{C}_l}^{u} e^{-\beta w} \left(\frac{\delta(w-\mathcal{B})}{c} + 1\right)^{\frac{\lambda}{\delta}-1} dw}{1 + \frac{\lambda\eta}{c} C_1^{-1} e^{\beta\mathcal{C}_u}}, \tag{3.23}$$

where C_1^{-1} is given in Proposition 3.

Proof. Let us begin by considering the numerator in equation (3.11) i.e.

$$\phi(0) \left[G_{\tilde{u}}(\mathcal{C}_u - \mathcal{B}) + \int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g_{\tilde{u}}(y)\chi_\delta(\mathcal{C}_u - y)\, dy\right].$$

Assuming that the claim amounts are exponentially distributed, employing the corresponding forms for $G_{\tilde{u}}(y)$ and $g_{\tilde{u}}(y)$, from Dickson(2005) i.e.

$$G_{\tilde{u}}(y) = 1 - e^{-\beta y}$$
$$g_{\tilde{u}}(y) = \beta e^{-\beta y},$$

and using equation (3.20) of Proposition 3, it follows that the above equation may be written as

$$\phi(0)\left[\left(1 - e^{-\beta(\mathcal{C}_u - \mathcal{B})}\right) + C_1\beta \int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} e^{-\beta y} \int_{\mathcal{C}_l}^{\mathcal{C}_u-y} e^{-\beta w}\left(\frac{\delta(w-\mathcal{B})}{c} + 1\right)^{\frac{\lambda}{\delta}-1} dw dy\right]. \tag{3.24}$$

Changing the order of integration, evaluating the resulting inner integral and after some algebraic manipulations, equation (3.24) can be re-written in the form

$$\phi(0)\left[1 - e^{-\beta(\mathcal{C}_u-\mathcal{B})}\left(1 - C_1 \int_{\mathcal{C}_l}^{\mathcal{B}} e^{-\beta w}\left(\frac{\delta(w-\mathcal{B})}{c} + 1\right)^{\frac{\lambda}{\delta}-1} dw\right) - C_1\frac{c}{\lambda}e^{-\beta\mathcal{C}_u}\right],$$

which, after recalling the definition of the constant C_1 given in Proposition 3, reduces to the concise form

$$\phi(0)\left[1 - C_1\frac{c}{\lambda}e^{-\beta\mathcal{C}_u}\right].$$

Now, considering a similar methodology as above, the corresponding denominator in equation (3.11) reduces to

$$1 - \frac{1}{1+\eta}\left(1 - C_1\frac{c}{\lambda}e^{-\beta\mathcal{C}_u}\right).$$

Substituting the above forms of the numerator and denominator of equation (3.11), we have that the insolvency probability, for $u \geqslant \mathcal{C}_u$, is given by

$$\psi_I(u) = \psi(\tilde{u})\left(1 - \frac{\phi(0)A}{1 - \frac{1}{1+\eta}A}\right),$$

where

$$A = \left(1 - C_1\frac{c}{\lambda}e^{-\beta\mathcal{C}_u}\right).$$

Re-arranging the above equation, substituting the forms of both $\phi(0)$ and $\psi(\tilde{u})$, under exponentially distributed claim sizes (see Grandell (1991)) and noticing that $\psi(\tilde{u}) = \psi(u - \mathcal{C}_u) = e^{\frac{\lambda\eta}{c}\mathcal{C}_u}\psi(u)$, since $\psi(u) = \frac{1}{1+\eta}e^{-\frac{\lambda\eta}{c}u}$, we obtain our result. For $\psi_I^-(u)$, given by equation (3.23), one can apply similar arguments and thus the proof is omitted. □

Remark 2. *(i) From equation (3.22), we conclude that the constant* $\frac{(1+\eta)e^{-\frac{\lambda\eta}{c}\mathcal{C}_u}}{1+\frac{\lambda\eta}{c}C_1^{-1}e^{\beta\mathcal{C}_u}}$ *plays the role of a 'measurement of protection' for the insurer. Thus, given a set of parameters, the above factor could lead to lower/higher value of* $\psi_I^+(u)$, *compared to the classical ruin probability* $\psi(u)$, *in the sense that the insurer is more/less protected by the capital constraints.*

(ii) If we set $\mathcal{C}_u = \mathcal{B} = 0$ *such that* $\mathcal{C}_l = -\frac{c}{\delta}$, *then equation (3.22) becomes*

$$\psi_I^+(u) = \frac{e^{-\frac{\lambda\eta}{c}u}}{1+\frac{\lambda\eta}{c}C_1^{-1}} \quad u \geqslant 0,$$

where $C_1^{-1} = \int_{-\frac{c}{\delta}}^{0} e^{-\beta w}\left(\frac{\delta w}{c}+1\right)^{\frac{\lambda}{\delta}-1} dw$ *and thus we retrieve Theorem 12 of Dassios and Embrechts (1989) for the ruin probability in the classical risk model with debit interest, under exponentially distributed claim sizes.*

Example 1 (Comparison of the probability of insolvency versus the classical ruin probability)**.** *In order to compare the insolvency probability* $\psi_I^+(u)$, $u \geqslant \mathcal{C}_u$, *with the classical ruin probability,* $\psi(u)$, *recall that under exponentially distributed claim sizes,* $\psi(u)$ *is given by*

$$\psi(u) = \frac{1}{1+\eta}e^{-\frac{\lambda\eta}{c}u}, \quad u \geqslant 0.$$

In addition, we consider the following set of parameters $\lambda = \beta = 1$, $\eta = 5\%$, *which due to the net profit condition, fixes our premium rate at* $c = 1.05$. *We further set the debit force* $\delta = 0.05$ *and the fixed lower capital level* $\mathcal{C}_l = 3$, *which in turn gives* $\mathcal{B} = 24$, *since* $\mathcal{B} = \mathcal{C}_l + \frac{c}{\delta}$. *Table 1 (below) shows the comparison of the classical and the insolvent ruin probabilities for several values of* u *and the level* $\mathcal{C}_u$ *such that* $u \geqslant \mathcal{C}_u > \mathcal{B} = 24$.

	$\mathcal{C}_u = 25$		$\mathcal{C}_u = 30$		$\mathcal{C}_u = 50$	
u	$\psi(u)$	$\psi_I^+(u)$	$\psi(u)$	$\psi_I^+(u)$	$\psi(u)$	$\psi_I^+(u)$
$\mathcal{C}_u$	0.290	0.509	0.228	6.933×10^{-3}	0.088	1.439×10^{-11}
$\mathcal{C}_u + 5$	0.228	0.401	0.180	5.464×10^{-3}	0.069	1.134×10^{-11}
$\mathcal{C}_u + 10$	0.180	0.316	0.142	4.306×10^{-3}	0.055	8.938×10^{-12}
$\mathcal{C}_u + 15$	0.142	0.249	0.112	3.394×10^{-3}	0.043	7.044×10^{-12}
$\mathcal{C}_u + 20$	0.112	0.196	0.088	2.675×10^{-3}	0.034	5.552×10^{-12}

Table 1. Classical ruin against insolvency probabilities, exponential claims.

Furthermore, in Table 2 (below), numerics for the required initial capital are given in the case of a fixed probability of insolvency and $\mathcal{C}_u$ *level.*

	u		
$\psi_I^+(u)$	$\mathcal{C}_u = 25$	$\mathcal{C}_u = 26$	$\mathcal{C}_u = 27$
0.1	59.17	47.32	31.34
0.05	73.72	61.87	45.90
0.025	88.28	76.43	60.46
0.01	107.52	95.67	79.70

Table 2: Initial capital required for varying insolvency probabilities and $\mathcal{C}_u$ levels

For reasons explained in Section 3, numerics for $\psi_I^-(u)$ are omitted.

3.2 Asymptotics results for the probability of insolvency

In this subsection we derive an asymptotic expression for the probability of insolvency, namely $\psi_I^+(u)$. Note that an asymptotic expression for $\psi_I^-(u)$ cannot be considered since $\mathcal{C}_l < u < \mathcal{B}$.

Hence, using the form for $\psi_I^+(u)$ given in Theorem 1, and the fact it is expressed in terms of $\psi(\cdot)$ and $G.(\cdot)$, we can derive an explicit asymptotic expression for the probability of insolvency, in terms of the ruin probability of the classical risk model.

We begin by deriving asymptotic expressions for $G_{\tilde{u}}(y)$ and $g_{\tilde{u}}(y)$. From Gerber et al. (1987), it follows that the distribution of the deficit at ruin, namely $G(u, y)$, satisfies the following renewal equation

$$G(u,y) = \frac{\lambda}{c}\int_0^u G(u-x,y)\overline{F}_X(x)\,dx + \frac{\lambda}{c}\int_u^{u+y}\overline{F}_X(x)\,dx, \tag{3.25}$$

which is a defective renewal equation since $\frac{\lambda}{c}\int_0^\infty \overline{F}_X(x)\,dx = \frac{\lambda\mu}{c} < 1$, given that the net profit condition holds. Thus, as in Feller (1971) we assume there exists a constant R, known as the Lundberg exponent, such that

$$\frac{\lambda}{c}\int_0^\infty e^{Rx}\overline{F}_X(x)\,dx = 1,$$

then, $\frac{\lambda}{c}e^{Rx}\overline{F}_X(x)$ forms a density of a proper probability function. Multiplying equation (3.25) by e^{Ru}, with R satisfying the above condition, we have

$$e^{Ru}G(u,y) = \frac{\lambda}{c}\int_0^u e^{R(u-x)}G(u-x,y)e^{Rx}\overline{F}_X(x)\,dx + \frac{\lambda}{c}e^{Ru}\int_u^{u+y}\overline{F}_X(x)\,dx, \tag{3.26}$$

which is now in the form of a proper renewal equation. Then, direct application of the Key Renewal Theorem [see Rolski et al. (1999), Thm 6.1.11], gives that

$$\lim_{u\to\infty} e^{Ru}G(u,y) = \frac{\int_0^\infty e^{Rt}\int_t^{t+y}\overline{F}_X(x)\,dxdt}{\int_0^\infty te^{Rt}\overline{F}_X(t)\,dt}.$$

Following a similar argument [see also, Grandell (1999)], we obtain the following asymptotic expression for the classic probability of ruin

$$\lim_{u\to\infty} e^{Ru}\psi(u) = \frac{\int_0^\infty e^{Rt}\int_t^\infty \overline{F}_X(x)\,dxdt}{\int_0^\infty te^{Rt}\overline{F}_X(t)\,dt}.$$

Finally, since $G_u(y) = \frac{G(u,y)}{\psi(u)}$, using a similar argument as in Willmot (2002), we have

$$\lim_{u\to\infty} G_u(y) = \frac{\int_0^\infty e^{Rt}\int_t^{t+y} \overline{F}_X(x)\,dx\,dt}{\int_0^\infty e^{Rt}\int_t^\infty \overline{F}_X(x)\,dxdt}.$$

from which it follows, by differentiating the above equation with respect to y, that

$$\lim_{u\to\infty} g_u(y) = \frac{\int_0^\infty e^{Rt}\overline{F}_X(t+y)dt}{\int_0^\infty e^{Rt}\int_t^\infty \overline{F}_X(x)\,dxdt}.$$

Thus, combining the above asymptotic expressions and using equation (3.11) the asymptotic behaviour of $\psi_I^+(u)$, as $u\to\infty$, is given by the following Proposition.

Proposition 4. *The probability of Insolvency, $\psi_I^+(u)$, behaves asymptotically as*

$$\psi_I^+(u) \sim K\psi(u), \quad u\to\infty,$$

where $\psi(u)$ is the classical ruin probability and K is a constant of the form

$$K = 1 - \frac{\phi(0)\left[\int_0^\infty e^{Rt}\int_t^{t+(\mathcal{C}_u-\mathcal{B})} \overline{F}_X(x)\,dxdt + \int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l}\int_0^\infty e^{Rt}\overline{F}_X(t+y)\chi_\delta(\mathcal{C}_u-y)\,dt\,dy\right]}{\frac{\mu\eta}{R}\left(1-\frac{\lambda}{c}\left(\mu F_e(\mathcal{C}_u-\mathcal{B}) + \int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l}\overline{F}_X(y)\chi_\delta(\mathcal{C}_u-y)\,dy\right)\right)}.$$

4 Probability characteristics of the accumulated capital injections

In this section we aim to obtain the probabilistic characteristics of the accumulated capital injections up to the time of insolvency, including an analytic expression for the first moment and an expression for the moment generating function. For the latter, we show that the distribution of the accumulated capital injections up to the time of insolvency is a mixture of a degenerate and continuous distribution.

4.1 Moments of the accumulated capital injections up to time of insolvency

Let the total accumulated capital injections, up to time $t \geqslant 0$, be denoted by the pure jump process $\{Z(t)\}_{t\geq 0}$ and consider $\mathbb{E}(Z_{u,\mathcal{C}_u})$, where $Z_{u,\mathcal{C}_u} = Z(T_\delta)$ is the accumulated capital injections up to the time of insolvency, given the initial capital level u. Due to similar reasons as the insolvency probability, it is necessary to decompose $\mathbb{E}(Z_{u,\mathcal{C}_u})$ depending on the size of the initial capital. Therefore define $\mathbb{E}(Z_{u,\mathcal{C}_u}) = \mathbb{E}(Z^+_{u,\mathcal{C}_u})$ when $u \geqslant \mathcal{C}_u$ and $\mathbb{E}(Z_{u,\mathcal{C}_u}) = \mathbb{E}(Z^-_{u,\mathcal{C}_u})$, when $\mathcal{C}_l < u < \mathcal{B}$. Using a similar argument as in the previous section (that is, conditioning on the amount of the first drop below the capital level $\mathcal{C}_u$), we have that $\mathbb{E}(Z^+_{u,\mathcal{C}_u})$, for $u \geqslant \mathcal{C}_u$, satisfies

$$\begin{aligned}
\mathbb{E}(Z^+_{u,\mathcal{C}_u}) &= \int_0^{\mathcal{C}_u-\mathcal{B}} \left(y + \mathbb{E}\left(Z^+_{\mathcal{C}_u,\mathcal{C}_u}\right)\right) g(\tilde{u},y)\,dy \\
&\qquad + \int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} \left((\mathcal{C}_u-\mathcal{B}) + \mathbb{E}\left(Z^+_{\mathcal{C}_u,\mathcal{C}_u}\right)\right) g(\tilde{u},y)\chi_\delta(\mathcal{C}_u-y)dy \\
&= \int_0^{\mathcal{C}_u-\mathcal{B}} yg(\tilde{u},y)\,dy + (\mathcal{C}_u-\mathcal{B})\int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g(\tilde{u},y)\chi_\delta(\mathcal{C}_u-y)dy \\
&\qquad + \mathbb{E}\left(Z^+_{\mathcal{C}_u,\mathcal{C}_u}\right)\left[G(\tilde{u},\mathcal{C}_u-\mathcal{B}) + \int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g(\tilde{u},y)\chi_\delta(\mathcal{C}_u-y)dy\right].
\end{aligned} \tag{4.1}$$

In order to complete the calculation for $\mathbb{E}\left(Z^+_{u,\mathcal{C}_u}\right)$ given by the above expression, we need to compute the value of $\mathbb{E}(Z^+_{u,\mathcal{C}_u})$ at $u = \mathcal{C}_u$, namely $\mathbb{E}(Z^+_{\mathcal{C}_u,\mathcal{C}_u})$, which can be obtained by setting $u = \mathcal{C}_u$ in equation (4.1). That is,

$$\begin{aligned}
\mathbb{E}\left(Z^+_{\mathcal{C}_u,\mathcal{C}_u}\right) &= \int_0^{\mathcal{C}_u-\mathcal{B}} yg(0,y)\,dy + (\mathcal{C}_u-\mathcal{B})\int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g(0,y)\chi_\delta(\mathcal{C}_u-y)dy \\
&\qquad + \mathbb{E}\left(Z^+_{\mathcal{C}_u,\mathcal{C}_u}\right)\left[G(0,\mathcal{C}_u-\mathcal{B}) + \int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g(0,y)\chi_\delta(\mathcal{C}_u-y)dy\right],
\end{aligned}$$

from which we have that

$$\mathbb{E}\left(Z^+_{\mathcal{C}_u,\mathcal{C}_u}\right) = \frac{\int_0^{\mathcal{C}_u-\mathcal{B}} yg(0,y)\,dy + (\mathcal{C}_u-\mathcal{B})\int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g(0,y)\chi_\delta(\mathcal{C}_u-y)dy}{1 - \left(G(0,\mathcal{C}_u-\mathcal{B}) + \int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g(0,y)\chi_\delta(\mathcal{C}_u-y)dy\right)}. \tag{4.2}$$

On the other hand, in order to compute $\mathbb{E}\left(Z^-_{u,\mathcal{C}_u}\right)$, for $\mathcal{C}_l < u < \mathcal{B}$, note that $\mathbb{E}\left(Z^-_{u,\mathcal{C}_u}\right)$ satisfies

$$\mathbb{E}\left(Z^-_{u,\mathcal{C}_u}\right) = \chi_\delta(u)\left((\mathcal{C}_u-\mathcal{B}) + \mathbb{E}\left(Z^+_{\mathcal{C}_u,\mathcal{C}_u}\right)\right), \quad \mathcal{C}_l < u < \mathcal{B}, \tag{4.3}$$

with $\mathbb{E}\left(Z^+_{\mathcal{C}_u,\mathcal{C}_u}\right)$ given by equation (4.2).

To illustrate the applicability of the results for $\mathbb{E}\left(Z^+_{u,\mathcal{C}_u}\right)$ and $\mathbb{E}\left(Z^-_{u,\mathcal{C}_u}\right)$, we will give explicit expressions for the two types of the expected accumulated capital injections up to the time of insolvency, when the claim amounts are exponentially distributed.

Proposition 5. *Let the claim amounts be exponentially distributed with parameter $\beta > 0$, i.e. $F(x) = 1 - e^{-\beta x}$, $x \geqslant 0$. Then, the expected accumulated capital injections, $\mathbb{E}\left(Z^{+}_{u,\mathcal{C}_u}\right)$ for $u \geqslant \mathcal{C}_u$, is given by*

$$\mathbb{E}\left(Z^{+}_{u,\mathcal{C}_u}\right) = K_1 \psi^{+}_{I}(u), \tag{4.4}$$

where

$$K_1 = \frac{1}{1+\eta}\left(\frac{\lambda}{c\beta} C_1^{-1} e^{\beta \mathcal{C}_u}\left(1 - e^{-\beta(\mathcal{C}_u - \mathcal{B})}\right) - (\mathcal{C}_u - \mathcal{B})\right),$$

and $\psi^{+}_{I}(u)$ is the probability of insolvency, for $u \geqslant \mathcal{C}_u$, given in Theorem 2.

For $\mathcal{C}_l < u < \mathcal{B}$, $\mathbb{E}\left(Z^{-}_{u,\mathcal{C}_u}\right)$ is given by

$$\mathbb{E}\left(Z^{-}_{u,\mathcal{C}_u}\right) = K_2 \phi^{-}_{I}(u) \tag{4.5}$$

where

$$K_2 = \frac{1}{\beta\eta}\left(1 - e^{-\beta(\mathcal{C}_u - \mathcal{B})}\right) + (\mathcal{C}_u - \mathcal{B}),$$

and $\phi^{-}_{I}(u)$ is the solvency probability, for $\mathcal{C}_l < u < \mathcal{B}$, which can be obtained from equation (3.23) *of Theorem 2.*

Proof. The result follows from employing the ruin related quantities, under exponentially distributed claims (see Section 3.1), in equations (4.1), (4.2) and (4.3), making some algebraic manipulations and recalling the forms of $\psi^{+}_{I}(u)$ and $\psi^{-}_{I}(u) = 1 - \phi^{-}_{I}(u)$, from Theorem 2. □

4.2 The distribution of the accumulated capital injections up to the time of insolvency

In this subsection, we show that the distribution of the accumulated capital injections up to the time of insolvency is a mixture of a degenerative distribution at zero and a continuous distribution.

Extending the arguments of Nie et al. (2011), we first consider the case where $u = \mathcal{C}_u$. Then, the probability that there is a first capital injection is; the probability that the surplus process drops, due to a claim, between $\mathcal{C}_u$ and $\mathcal{B}$, which happens with probability $G(0, \mathcal{C}_u - \mathcal{B})$, or the surplus process drops, due to a claim, between $\mathcal{B}$ and $\mathcal{C}_l$ and then recovers back up to the level $\mathcal{B}$ before crossing $\mathcal{C}_l$, which happens with probability $\int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g(0,y)\chi_{\delta}(\mathcal{C}_u - y)\,dy$.

Given that there exists a first capital injection, the process restarts from the level $\mathcal{C}_u$. Hence, if we let N denote the number of capital injections up to the time of insolvency, then by the above reasoning, N has a geometric distribution with p.m.f., for $n = 0, 1, 2, \ldots$

$$\mathbb{P}(N=n)=\left(G(0,\mathcal{C}_u-\mathcal{B})+\int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g(0,y)\chi_\delta(\mathcal{C}_u-y)\,dy\right)^n \times\left(1-\left[G(0,\mathcal{C}_u-\mathcal{B})+\int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g(0,y)\chi_\delta(\mathcal{C}_u-y)\,dy\right]\right),$$

and thus, a probability generating function given by

$$\mathbb{E}\left(z^N\right)=P_N(z)=\frac{1-\left(G(0,\mathcal{C}_u-\mathcal{B})+\int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g(0,y)\chi_\delta(\mathcal{C}_u-y)\,dy\right)}{1-z\left(G(0,\mathcal{C}_u-\mathcal{B})+\int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g(0,y)\chi_\delta(\mathcal{C}_u-y)\,dy\right)}.$$

Then, the accumulated amount of the capital injections up to the time of insolvency starting from $u=\mathcal{C}_u$, namely $Z^+_{\mathcal{C}_u,\mathcal{C}_u}$, has a compound geometric distribution of the form

$$Z^+_{\mathcal{C}_u,\mathcal{C}_u}=\sum_{i=1}^{N} V_i,$$

where $\{V_i\}_{i=1}^{\infty}$ are i.i.d. random variables, denoting the size of the i-th injection, with p.d.f.

$$f_V(y)=\begin{cases}\dfrac{g(0,y)}{G(0,\mathcal{C}_u-\mathcal{B})+\int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g(0,x)\chi_\delta(\mathcal{C}_u-x)\,dx} & 0<y<\mathcal{C}_u-\mathcal{B},\\[2ex] \dfrac{\int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g(0,x)\chi_\delta(\mathcal{C}_u-x)\,dx}{G(0,\mathcal{C}_u-\mathcal{B})+\int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g(0,x)\chi_\delta(\mathcal{C}_u-x)\,dx} & y=\mathcal{C}_u-\mathcal{B},\end{cases}$$

and thus the moment generating function of $Z^+_{\mathcal{C}_u,\mathcal{C}_u}$ can be expressed as

$$M_{Z^+_{\mathcal{C}_u,\mathcal{C}_u}}(z)=P_N(M_V(z)),$$

where

$$M_V(z)=\mathbb{E}\left(e^{zV}\right)=\frac{\int_0^{\mathcal{C}_u-\mathcal{B}} e^{zy}g(0,y)\,dy+e^{z(\mathcal{C}_u-\mathcal{B})}\int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g(0,x)\chi_\delta(\mathcal{C}_u-x)\,dx}{G(0,\mathcal{C}_u-\mathcal{B})+\int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g(0,x)\chi_\delta(\mathcal{C}_u-x)\,dx}.$$

Now, in order to find the moment generating functions of the accumulated capital injections up to the time of insolvency with general initial capital, namely $Z^+_{u,\mathcal{C}_u}$ when $u\geqslant\mathcal{C}_u$ and $Z^-_{u,\mathcal{C}_u}$, when $\mathcal{C}_l<u<\mathcal{B}$ we first note that $Z^+_{u,\mathcal{C}_u}$ and $Z^-_{u,\mathcal{C}_u}$ are equivalent in distribution to $\left(Y^+_u+Z^+_{\mathcal{C}_u,\mathcal{C}_u}\right)\mathbb{I}_{\{A^+\}}$ and $\left(Y^-_u+Z^+_{\mathcal{C}_u,\mathcal{C}_u}\right)\mathbb{I}_{\{A^-\}}$, respectively, where Y^+_u is the amount of the first capital injection, starting from initial capital $u>\mathcal{C}_u$, Y^-_u from initial capital $\mathcal{C}_l<u<\mathcal{B}$ and $\mathbb{I}_{\{\cdot\}}$ is the indicator function with respect to the event that a capital injections occurs from initial capital u. Note that the event that a capital injections occurs from initial capital u can be decomposed to the sub events depending the value of the initial capital and thus we denote A^+ and A^- the events that a capital injections occurs from initial capital $u > \mathcal{C}_u$ and $\mathcal{C}_l < u < \mathcal{B}$, respectively, with probabilities

$$\mathbb{P}(A^+) = G(\tilde{u}, \mathcal{C}_u - \mathcal{B}) + \int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g(\tilde{u}, y)\chi_\delta(\mathcal{C}_u - y)\, dy,$$

and

$$\mathbb{P}(A^-) = \chi_\delta(u).$$

Based on the above notation, for $\tilde{u} = u - \mathcal{C}_u$, the density of Y_u^+ is given by

$$f_{Y_u^+}(y) = \begin{cases} \frac{g(\tilde{u},y)}{G(\tilde{u},\mathcal{C}_u-\mathcal{B})+\int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g(\tilde{u},x)\chi_\delta(\mathcal{C}_u-x)\,dx} & 0 < y < \mathcal{C}_u - \mathcal{B}, \\ \frac{\int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g(\tilde{u},x)\chi_\delta(\mathcal{C}_u-x)\,dx}{G(\tilde{u},\mathcal{C}_u-\mathcal{B})+\int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g(\tilde{u},x)\chi_\delta(\mathcal{C}_u-x)\,dx} & y = \mathcal{C}_u - \mathcal{B}, \end{cases}$$

whilst Y_u^- has a probability mass function of the following form

$$\mathbb{P}(Y_u^- = i) = \begin{cases} 1, & i = \mathcal{C}_u - \mathcal{B} \\ 0 & \text{otherwise.} \end{cases}$$

Then, since Y_u^+ and $Z^+_{\mathcal{C}_u,\mathcal{C}_u}$ are independent, the moment generating function of $Z^+_{u,\mathcal{C}_u}$ is given by

$$M_{Z^+_{u,\mathcal{C}_u}}(z) = \left(M_{Y^+}(z) M_{Z^+_{\mathcal{C}_u,\mathcal{C}_u}}(z) \right) \mathbb{P}(A^+) + \mathbb{P}((A^+)^c), \tag{4.6}$$

where

$$M_{Y_u^+}(z) = \mathbb{E}\left(e^{zY_u^+}\right) = \frac{\int_0^{\mathcal{C}_u-\mathcal{B}} e^{zy} g(\tilde{u}, y)\, dy + e^{z(\mathcal{C}_u-\mathcal{B})} \int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g(\tilde{u}, x)\chi_\delta(\mathcal{C}_u - x)\, dx}{G(\tilde{u}, \mathcal{C}_u - \mathcal{B}) + \int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g(\tilde{u}, x)\chi_\delta(\mathcal{C}_u - x)\, dx},$$

whilst, following a similar argument as above, the moment generating function of $Z^-_{u,\mathcal{C}_u}$ is given by

$$M_{Z^-_{u,\mathcal{C}_u}}(z) = \left(M_{Y_u^-}(z) M_{Z^+_{\mathcal{C}_u,\mathcal{C}_u}}(z) \right) \mathbb{P}(A^-) + \mathbb{P}((A^-)^c), \tag{4.7}$$

where

$$M_{Y_u^-}(z) = \mathbb{E}\left(e^{zY_u^-}\right) = e^{z(\mathcal{C}_u - b)}.$$

From equations (4.6) and (4.7), it should be clear that the distribution of the accumulated capital injections up to the time of insolvency, is mixture of a degenerative distribution at zero and a continuous distribution.

5 Constant dividend barrier strategy with capital constraints

In reality the surplus of a company will not be left to grow indefinitely as a proportion of the profits are paid out as dividends to its shareholders. As mentioned in the previous section, the shareholders can contribute to the capital of the firm, by means of capital injections, for which they would expect financial incentives and therefore the consideration of dividend payments is important when analysing a firms portfolio and insolvency probabilities. Dividend strategies have been extensively studied in the risk theory literature since their introduction by De Finetti (1957), with a main focus on optimisation of the companies utility, see also Avanzi (2009) and references therein for a comprehensive review.

In this section we derive an explicit expression for the insolvency probability to the risk model under the framework in Section 2, with the addition of a constant dividend barrier $b \geqslant \mathcal{C}_u$, such that when the surplus reaches the level b dividends are paid continuously at rate c until a new claim appears (see Fig: 2). The amended surplus process, denoted $U^Z_{\delta,b}(t)$, has dynamics of the following form

$$dU^Z_{\delta,b}(t) = \begin{cases} -dS(t), & U^Z_{\delta,b}(t) = b, \\ cdt - dS(t), & \mathcal{C}_u \leqslant U^Z_{\delta,b}(t) < b, \\ \Delta Z(t), & \mathcal{B} \leqslant U^Z_{\delta,b}(t) < \mathcal{C}_u, \\ \left[c + \delta(U^Z_\delta(t) - r)\right] dt - dS(t), & \mathcal{C}_l < U^Z_{\delta,b}(t) < \mathcal{B}. \end{cases}$$

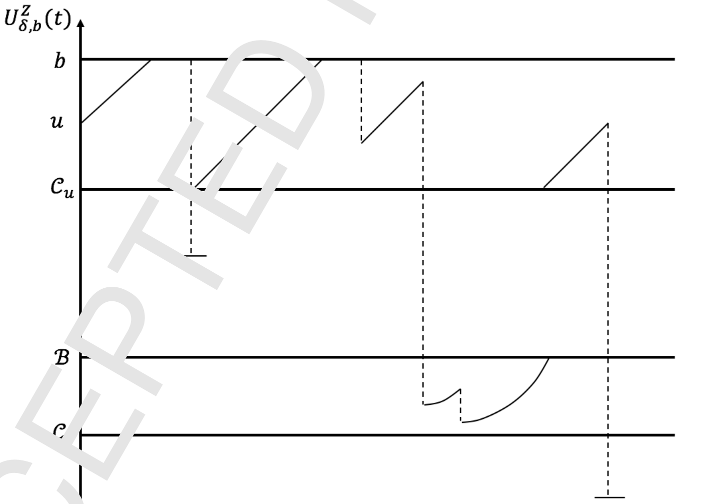

Figure 2: Typical sample path of the surplus process under capital constraints with constant dividend barrier.

The time to insolvency, in the dividend amended model, can be defined by

$$T_{\delta,b} = \inf\left\{t \geqslant 0 : U^Z_{\delta,b}(t) \leqslant \mathcal{C}_l \,|U^Z_{\delta,b}(0) = u\right\}$$

and the probability of insolvency, which we denote by $\psi_{I,b}(u)$, is defined as

$$\psi_{I,b}(u) = \mathbb{P}\left(T_{\delta,b} < \infty \middle| U^Z_{\delta,b}(0) = u\right),$$

with the corresponding solvency probability defined by $\phi_{I,b}(u) = 1 - \psi_{I,b}(u)$.

We once again note that the insolvency probability, as in the previous sections, can be decomposed for $\mathcal{C}_u \leqslant u \leqslant b$ and $\mathcal{C}_l < u < \mathcal{C}_u$, for which we define $\psi_{I,b}(u) = \psi^+_{I,b}(u)$ and $\psi_{I,b}(u) = \psi^-_{I,b}(u)$, for the two separate cases with corresponding solvency probabilities $\phi^+_{I,b}(u)$ and $\phi^-_{I,b}(u)$, respectively.

In order to derive an expression for the solvency probability for $\mathcal{C}_u \leqslant u \leqslant b$, namely $\phi^+_{I,b}(u)$, (or equivalently the insolvency probability $\psi^+_{I,b}(u)$) we will need to define the crossing probability of the surplus below the level $\mathcal{C}_u$ (as we did in Section 3), given by

$$\xi_b(u) = \mathbb{P}(T_b < \infty \big| \mathcal{C}_u \leqslant U^Z_{\delta,b}(0) = u \leqslant b),$$

where $T_b = \inf\{t \geqslant 0 : U^Z_{\delta,b}(t) < \mathcal{C}_u \big| \mathcal{C}_u \leqslant U^Z_{\delta,b}(0) = u \leqslant b\}$ is the first time the process down crosses the level $\mathcal{C}_u$.

Using a similar argument as in Section 3, it follows that the dynamics of the surplus process $U^Z_{\delta,b}(t)$ above the level $\mathcal{C}_u$ are equivalent to that of the classic surplus process with a constant dividend barrier $\tilde{b} = b - \mathcal{C}_u$ (i.e. no capital constraint levels). That is, for $\mathcal{C}_u \leqslant U^Z_{\delta,b}(t) \leqslant b$, we have $dU^Z_{\delta,b}(t) \equiv d\widetilde{U}_{\tilde{b}}(t)$ where

$$\widetilde{U}_{\tilde{b}}(t) = \tilde{u} + ct - S(t), \qquad 0 \leqslant \widetilde{U}_{\tilde{b}}(0) = \tilde{u} \leqslant \tilde{b},$$

with dynamics

$$d\widetilde{U}_{\tilde{b}}(t) = \begin{cases} -dS(t), & \widetilde{U}_{\tilde{b}}(t) = \tilde{b}, \\ c\,dt - dS(t), & 0 \leqslant \widetilde{U}_{\tilde{b}}(t) < \tilde{b}. \end{cases}$$

Thus, it is clear that T_b, defined above, is equivalent to the time of ruin in the classical risk model with a constant dividend barrier strategy and initial capital $0 \leqslant \tilde{u} \leqslant \tilde{b}$, given by

$$T_b = \inf\{t \geqslant 0 : \widetilde{U}_{\tilde{b}}(t) < 0 \big| 0 \leqslant \widetilde{U}_{\tilde{b}}(0) = \tilde{u} \leqslant \tilde{b}\},$$

and the probability $\xi_b(u)$ is identical to the probability of ruin, namely $\psi_{\tilde{b}}(\tilde{u}) = \mathbb{P}(T_b < \infty \big| \widetilde{U}_{\tilde{b}}(0) = \tilde{u}) = 1 - \phi_{\tilde{b}}(\tilde{u})$, for the classical risk model with a constant dividend barrier strategy.

To obtain an expression for the insolvency probability under a constant dividend barrier strategy, recall the fact that $dU^Z_{\delta,b}(t) \equiv d\widetilde{U}_{\tilde{b}}(t)$ when the surplus is above the level $\mathcal{C}_u$ and condition on the occurrence and amount of the first drop below the capital level $\mathcal{C}_u$. Then for $\mathcal{C}_u \leqslant u \leqslant b$, the respective solvency probability $\phi^+_{I,b}(u)$, is given by

$$\begin{aligned}\phi^+_{I,b}(u) &= \phi_{\tilde{b}}(\tilde{u}) + \int_0^{\mathcal{C}_u-\mathcal{B}} g_{\tilde{b}}(\tilde{u},y)\phi^+_{I,b}(\mathcal{C}_u)\,dy + \int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g_{\tilde{b}}(\tilde{u},y)\phi^-_{I,b}(\mathcal{C}_u-y)\,dy \\ &= \phi_{\tilde{b}}(\tilde{u}) + G_{\tilde{b}}(\tilde{u},\mathcal{C}_u-\mathcal{B})\phi^+_{I,b}(\mathcal{C}_u) + \int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g_{\tilde{b}}(\tilde{u},y)\phi^-_{I,b}(\mathcal{C}_u-y)\,dy,\end{aligned}$$

where

$$G_{\tilde{b}}(\tilde{u},y) = \mathbb{P}\left(T_b<\infty, |\widetilde{U}_b(T_b)| \leqslant y \middle| \widetilde{U}_b(0)=\tilde{u}\right)$$

is the distribution of the deficit below $\mathcal{C}_u$ at the time of crossing the capital level, under the constant dividend barrier strategy, and $g_{\tilde{b}}(\tilde{u},y) = \frac{\partial}{\partial y}G_{\tilde{b}}(\tilde{u},y)$ its corresponding density.

For $\mathcal{C}_l < u < \mathcal{B}$, we have

$$\phi^-_{I,b}(u) = \chi_\delta(u)\phi^+_{I,b}(\mathcal{C}_u),$$

where $\chi_\delta(u)$ is the probability of hitting the upper confidence level $\mathcal{B}$ before the lower level $\mathcal{C}_l$, in a debit environment, as studied in Section 3. We point out that the function $\chi_\delta(u)$ is unaffected by the addition of the dividend barrier and therefore the integro-differential equation given in Proposition 3 still holds, along with the corresponding boundary conditions. Following similar arguments as in Section 3 we obtain the following Theorem.

Theorem 3. *For $\mathcal{C}_u \leqslant u \leqslant b$,the probability of insolvency under a constant dividend barrier strategy, $\psi^+_{I,b}(u)$, satisfies*

$$\psi^+_{I,b}(u) = \psi_{\tilde{b}}(\tilde{u}) - \frac{\phi_{\tilde{b}}(0)\left[G_{\tilde{b}}(\tilde{u},\mathcal{C}_u-\mathcal{B}) + \int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g_{\tilde{b}}(\tilde{u},y)\chi_\delta(\mathcal{C}_u-y)\,dy\right]}{1-\left(G_{\tilde{b}}(0,\mathcal{C}_u-\mathcal{B}) + \int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g_{\tilde{b}}(0,y)\chi_\delta(\mathcal{C}_u-y)\,dy\right)}. \tag{5.1}$$

For $\mathcal{C}_l < u < \mathcal{B}$, $\psi^-_{I,b}(u)$ is given by

$$\psi^-_{I,b}(u) = 1 - \frac{\phi_{\tilde{b}}(0)\chi_\delta(u)}{1-\left(G_{\tilde{b}}(0,\mathcal{C}_u-\mathcal{B}) + \int_{\mathcal{C}_u-\mathcal{B}}^{\mathcal{C}_u-\mathcal{C}_l} g_{\tilde{b}}(0,y)\chi_\delta(\mathcal{C}_u-y)\,dy\right)}. \tag{5.2}$$

Remark 3. *Similarly to Remark 1, we point out that from equations* (5.1) *and* (5.2)*, that the two types of insolvency probabilities for the risk model under capital constraints with the addition of a constant dividend barrier, are given in terms of the (shifted) ruin probability and deficit of the classical risk model with constant dividend barrier, as well as the probability of exiting between two capital levels. Thus, $\psi^+_{I,b}(\cdot)$ and $\psi^-_{I,b}(\cdot)$ can be calculated by employing known results, with respect to $G_b(\cdot,\cdot)$ and $\psi_b(\cdot)$ (see Lin et al. (2003), among others), whilst the latter exiting probability, $\chi_\delta(u)$, can be evaluated by Propositions 2 and 3 .*

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